ORIE 630 - HW2

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1. Let $\mathcal{C}$ be the collection of convex sets $C \subseteq \mathbb{R}^n$ containing S. Then our two definitions of the convex hull of S are the sets X and Y, where X is the set of all convex combinations of elements of S and $Y = \bigcap_{C \in \mathcal{C}} C$. We'll show that $X \subseteq Y$ and $Y \subseteq X$.

   First, suppose $x \in X$. Then x is a convex combination of elements of S. For an arbitrary $C \in \mathcal{C}$, $S \subseteq C$, so x is a convex combination of elements of C, and C is convex, so $x \in C$. Hence $x \in C$ for all $C \in \mathcal{C}$, that is, $x \in \bigcap_{C \in \mathcal{C}} C = Y$.

   Next, note that X is a convex set containing S. Hence $X \in \mathcal{C}$, so $Y = X \cap \bigcap_{C \in \mathcal{C}} C \subseteq X$.

2. By definition, we have $x \in \mathbb{R}^n$ is a convex combination of $v_1, v_2, \ldots, v_k$ if and only if there is some $\lambda \in \mathbb{R}^k$ such that

$$x = \sum_{j=1}^{k} \lambda_j v_j, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad \lambda_j \geq 0, j = 1, 2, \ldots, k$$

   We can rewrite these requirements as a set of $n + 1$ equality constraints and k inequality constraints on the weight vector $\lambda \in \mathbb{R}^k$. Let $V = [v_1, \ldots, v_k]$ be the matrix whose columns are the points v_j, and let $v_{i\cdot}^T$ be the rows of V. We have

$$\begin{aligned}
v_{i\cdot}^T \lambda &= x_i \quad i = 1, \ldots, n \\
e^T \lambda &= 1 \\
e_i^T \lambda &\geq 0 \quad i = 1, \ldots, k
\end{aligned}$$

   Here $e \in \mathbb{R}^k$ is the vector of all ones, and $e_i \in \mathbb{R}^k$ is the vector with 1 in the ith position, and all other entries 0. The set of all λ satisfying the above is a nonempty pointed polyhedron. It is pointed because $\lambda \geq 0$, and every line must cross some coordinate axis, and nonempty due to our assumption that x is a convex combination of $v_1, \ldots, v_k$, meaning there is some feasible solution λ.

   We showed in class (part 1 of the Fundamental Theorem of LP) that since there is a feasible solution λ, there must be a *basic* feasible solution λ'. Since λ' is a basic solution, $ra(\lambda') = k$ and λ' satisfies each of the $n + 1$ equality constraints. To have rank k, it must meet at least $k - (n + 1)$ inequality constraints with equality, i.e. at most $n + 1$ of the coordinates λ'_j are nonzero. This means x is a convex combination of at most $n + 1$ of the points $v_1, \ldots, v_k$, corresponding to the positive components of λ'.

3. The proof of this is analagous to the proof that if $b^T y$ is bounded above on a pointed polyhedron Q, then $\max\{b^T y : y \in Q\}$ is attained at a verted of Q. (This is part 3 of the Fundamental theorem of LP from 9/8).

   First we show that the definition of concavity extends in the natural way to convex combinations of any finite number of points.

   Claim 1 *Let f be concave, and let $x_1, \ldots, x_k \in \mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\lambda_i \geq 0$ for all $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$. Then*

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \geq \sum_{i=1}^{k} \lambda_i f(x_i)$$

Proof: by induction on k. For $k = 2$, the claim reduces to the definition of concavity, therefore it is true. For the induction step, assume the claim is true for all $j < k$, $k \geq 3$. Then choose $x_1, \ldots, x_k$ and $\lambda_1, \ldots, \lambda_k$ to satisfy the conditions of the claim. There must be some i such that $\lambda_i < 1$. Assume WLOG that $\mu = \lambda_k < 1$. Now we have

$$
\begin{aligned}
f\Big(\sum_{i=1}^{k} \lambda_i x_i\Big) &= f\Big((1-\mu)\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\mu} x_i + \mu x_k\Big) \\
&\geq (1-\mu) f\Big(\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\mu} x_i\Big) + \mu f(x_k) \qquad \text{(since f is concave)}
\end{aligned}
$$

But $\frac{\lambda_i}{1-\mu} \geq 0$ for $i = 1, \ldots, k-1$ and $\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\mu} = \frac{1-\mu}{1-\mu} = 1$, so we can invoke our induction hypothesis on the first term, and we obtain

$$
\begin{aligned}
f\Big(\sum_{i=1}^{k} \lambda_i x_i\Big) &\geq (1-\mu)\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\mu} f(x_i) + \mu f(x_k) \\
&= \sum_{i=1}^{k} \lambda_i f(x_i)
\end{aligned}
$$

Next we prove the main result.

Claim 2 *If* $\inf\{f(x) : x \in Q\}$ *is attained, it is attained by a vertex of* Q.

Proof: We can write $Q = P + K$ where P is the set of convex combinations of the vertices $v_1, \ldots, v_k$ of Q, and K is the recession cone of Q. Suppose $\inf\{f(x) : x \in Q\}$ is attained at a point $x^* \in Q$. We can write $x^* = p + d$ where $p \in P$ and $d \in K$. But $x^* = \frac{1}{2}p + \frac{1}{2}(p + 2d)$, so

$$
f(x^*) \geq \frac{1}{2}f(p) + \frac{1}{2}f(p + 2d)
$$

However $p, p + 2d \in Q$ since $0, d \in K$, and the infimum is achieved at x^* so we know that $f(x^*) \leq f(p)$ and $f(x^*) \leq f(p+2d)$. Combined with the above inequality, we can conclude by an averaging argument that $f(x^*) = f(p) = f(p + 2d)$. In particular, the infimum is also achieved at $p = \sum_i \lambda_i v_i$. Using our generalized concavity,

$$
f(p) \geq \sum_{i=1}^{k} \lambda_i f(v_i)
$$

But again, $f(p) \leq f(v_i)$ for all i since it is the infimum. Since $\lambda_i \geq 0$, we must have $f(p) = f(v_i)$ for all i where $\lambda_i > 0$ by an averaging argument. In particular, there is some v_i such that $\lambda_i > 0$, so the infimum is achieved at this vertex v_i.

4. (a) Since $a \geq 0$ and $y \geq 0$, $a_i y_i \geq 0$ for $i = 1, \ldots, m$. For a particular i,

$$
a_i y_i \leq \sum_{i=1}^{m} a_i y_i = a^T y \leq \gamma
$$

$$
\Rightarrow y_i \leq \frac{\gamma}{a_i} \leq \frac{\gamma}{\min_i a_i}
$$

$$
\Rightarrow \|y\|_1 = \sum_{i=1}^{m} y_i \leq \frac{m\gamma}{\min_i a_i}
$$

Therefore the feasible region is bounded. There are $m + 1$ inequality constraints: $e_i^T y \geq 0$ for $i = 1, \ldots, m$, and $a^T y \leq \gamma$. It's easy to see that any m of these are linearly independent since

the e_i are the standard basis and $a > 0$. In a basic feasible solution, the constraints achieving equality must have rank m, so all but one constraint is achieved with equality. One basic feasible solution is the origin. In each of the others, there is one j such that $y_j > 0$. Then

$$y_i = \begin{cases} \frac{\gamma}{a_i} & i = j \\ 0 & i \neq j \end{cases}$$

(b) An optimal solution is obtained by examining the objective function at each basic feasible solution, i.e. finding $\max_i \{\frac{b_i \gamma}{a_i}\}$. The economic interpretation is that we have a fixed amount of money γ to spend on some combination of i items with costs a_i and benefits b_i. The best solution is to find the single product with the best benefit to cost ratio and spend all of our money on that product, i.e. buy the product with the "biggest bang for the buck".

(c) The dual of this problem is

$$\min_{\alpha, x} \alpha \gamma, \quad \alpha a - x = b, \quad \alpha \geq 0, \quad x \geq 0$$

Since $x = \alpha a - b \geq 0$, we need to choose

$$\alpha = \min\{\alpha : \alpha a \geq b\} = \min\{\alpha : \alpha \geq \frac{b_i}{a_i}, \forall i = 1, \ldots, m\} = \max_i \frac{b_i}{a_i}$$

This gives an objective function value of $\max_i \frac{b_i \gamma}{a_i}$, matching our solution to the primal problem, which confirms that is was optimal.

(d) This problem is clearly bounded if there is any feasible solution to the dual, which we can rewrite as $\min\{\alpha \geq 0 : \alpha a \geq b\}$, thinking of $x \geq 0$ as surplus variables. Then we get certain restrictions on α based on the signs of a_i and b_i:

a_i	b_i	
$+$	$+$	$\alpha \geq \frac{b_i}{a_i}$
$+$	$-$	$\alpha \geq 0$
$-$	$+$	$\alpha \leq 0$
$-$	$-$	$\alpha \leq \frac{b_i}{a_i}$

So there is a feasible solution to the dual whenever there is no i such that $a_i \leqslant 0 < b_i$, and $\max\{\frac{b_i}{a_i} : a_i, b_i > 0\} \leqslant \min\{\frac{b_i}{a_i} : a_i, b_i \leq 0\}$. If there is a feasible solution, the optimal solution (to the primal) is $\frac{b_j}{a_j} e_j$ where $j \in \operatorname{argmax}\{\frac{b_i}{a_i} : a_i, b_i > 0\}$, or the origin if no such i exists.

Note that if there is no feasible solution to the dual, then the primal can be seen unbounded by sending x_j to infinity if $a_j \leq 0 < b_j$, and by sending x_i and x_j to infinity in the ratio $\frac{-a_j}{a_i}$ if $a_i, b_i > 0$, $a_j < 0$, $b_j \leq 0$ and $\frac{b_i}{a_i} > \frac{b_j}{a_j}$.

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