The story of the Pessimist vs. Optimist (Robust optimization)

We’ll consider linear programming problems where some or all of the data are uncertain. We want a feasible solution with good objective value for any data in a specified set. Consider

$$\max b^T y$$
$$A^T y \leq c.$$ 

This is equivalent to

$$\max \eta$$
$$\eta - b^T y \leq 0$$
$$A^T y - c \xi \leq 0$$
$$\xi \leq 1$$
$$-\xi \leq -1,$$

where all of the data is in the constraint matrix. So we have

$$\max b^T y$$
$$A^T y \leq c,$$

with only uncertainty in $A$. We’ll assume each column $a_j$ of $A$ lies in an uncertainty set $E_j$. Conservative thinking leads to the problem

$$\max b^T y$$
$$a_j^T y \leq c, \quad \forall a_j \in E_j, j = 1, \ldots, n.$$ 

This is a so-called semi-infinite LP problem and can model nonlinear problems: an infinite number of linear constraints can model a quadratic constraint, for instance (see the example below).

We’ll model the $E_j$’s as ellipsoids, possibly degenerate (e.g., each $a_j$ may be sparse and zero coefficients are probably “certain”). Let $E_j = \{ \bar{a}_j + D_j w_j : \|w_j\|_2 \leq 1 \}$ for some $\bar{a}_j \in \mathbb{R}^m$, $D_j \in \mathbb{R}^{m \times p_j}$. Fix $j$; then we want:

$$\max \{ a_j^T y : a_j \in E_j \} \leq c_j,$$

a single constraint. But

$$\max \{ a_j^T y : a_j \in E_j \}$$
$$= \max \{ a_j^T y + (D_j^T y)^T w_j : \|w_j\|_2 \leq 1 \}$$
$$= \bar{a}_j^T y + \| D_j^T y \|_2.$$
So we get the deterministic equivalent of our robust LP program:

$$\max b^T y$$
$$\bar{a}_j^T y + \| D_j^T y \|_2 \leq c_j, \quad j = 1, \ldots, n.$$ 

Write this as the conic programming problem:

$$\max b^T y$$
$$\bar{a}_j^T y + \xi_j = c_j, \quad j = 1, \ldots, n$$
$$D_j^T y + z_j = 0, \quad j = 1, \ldots, n$$
$$y \in \mathbf{R}^m, \left( \begin{array}{c} \xi_j \\ z_j \end{array} \right) \in K_{2^{1+p_j}}, \quad j = 1, \ldots, n.$$ 

This is a second-order cone problem. If all the $p_j$’s are small, this is almost as easy to solve as a comparable LP problem. The dual of this conic problem is

$$\min c^T x$$
$$\sum_{j=1}^n (\bar{a}_j x_j + D_j v_j) = b$$
$$\left( \begin{array}{c} x_j \\ v_j \end{array} \right) \in (K_{2^{1+p_j}})^* = K_{2^{1+p_j}}, \quad j = 1, \ldots, n.$$ 

Write $v_j = x_j w_j$, $w_j \in \mathbf{R}^{p_j}$, so $\|w_j\|_2 \leq 1$. Then we get

$$\min_{x, w_1, \ldots, w_n} c^T x$$
$$\sum_{j=1}^n (\bar{a}_j + D_j w_j) x_j = b$$
$$x \geq 0, \|w_j\| \leq 1, \forall j.$$ 

This is equivalent to

$$\min c^T x$$
$$\sum_{j=1}^n a_j x_j = b$$
$$x \geq 0,$$

for some $a_j \in E_j, j = 1, 2, \ldots, n$. This is the optimist’s problem: $x$ is feasible as long as it is feasible for some data in the uncertainty set. A nice example of designing an antenna via robust LP
Remarks on semi-infinite programming:

\[
\begin{align*}
\max \quad & b^T y \\
(P) \quad & a(t)^T y \leq c(t), \, t \in T,
\end{align*}
\]

where \( T \) is compact.

**Example 1** \( T = [0, 2\pi] \), \( a(t) = (\cos(t); \sin(t)) \), \( c(t) = 1 \); then the feasible region is the unit ball in \( \mathbb{R}^2 \).

Dual:

\[
\begin{align*}
\min \quad & \sum_{t \in T} c(t) x(t) = \int c(t) x(t) dt \\
(D) \quad & \sum_{t} a(t) x(t) = \int a(t) x(t) dt = b \\
& x(t) \geq 0, \, t \in T.
\end{align*}
\]

Think of the dual simplex algorithm for \((P)\), which chooses just \( m \) of the \( a(t) \)'s with \( b \) a nonnegative combination. So \( x \) would be a discrete measure concentrating on just \( m \) points. Then somehow find a violated constraint \( t \) for the corresponding \( y \). As the iterations proceed, \( a(t_1), \ldots, a(t_m) \) may become degenerate, e.g., for \( m = 2 \), \([a(t_1), a(t_2)]\) becomes ill-conditioned as \( |t_1 - t_2| \to 0 \). Instead, we could consider a basis of \([a(t_1), a(t_2)]; t_2 - t_1 \) or in the limit \([a(t_1), a'(t_1)]\). See [2] for details.

**References**
