Suppose we are given \((f, G) \in \mathcal{F}\), where \(f\) is convex on \(C := [-1, 1]^n\), \(G\) is a convex subset of \(C\) with either \(G = \emptyset\) or \(\text{vol}(G) \geq \delta^n\), and max \(f(C) - \min f(C) \leq 1\).

**Definition 1** A pair \((H, z)\) is a **localizer** for \((f, G)\) if either \(z = (*)\) and \(G \subseteq H\) or \(z \in G\) and \(f(x) \leq f(z), x \in G \implies x \in H\). So \(\{x \in G : f(x) \leq f(z)\} \subseteq H\).

For simplicity, define \(f(*) := \infty\).

**Proposition 1** If \((H, z)\) is a localizer for \((f, G)\) with \(\theta := \left(\frac{\text{vol}(H)}{\text{vol}(C)}\right)^{1/n}\), and \(\theta < \frac{\delta}{2}\), then

a) if \(z = (*)\), then \(G = \emptyset\);
b) if \(z \in G\), then \(\epsilon(z, f, G) \leq \frac{2\theta}{\delta}\).

**Proof:** \(\text{vol}(H) = \theta^n \text{vol}(C) = (2\theta)^n < \delta^n\), where \(\text{vol}(G) \geq \delta^n\) if \(G \neq \emptyset\), so a) follows.

Now suppose \(z \in G\). Let \(z_*\) be any minimizer of \(f\) over \(G\), and consider \(G(\epsilon) := \{(1 - \epsilon)z_* + \epsilon x : x \in G\}\) for any \(\epsilon > \frac{2\theta}{\delta}\). \(\text{vol}(G(\epsilon)) = \epsilon^n \text{vol}(G) \geq (\epsilon\delta)^n\) while \(\text{vol}(H) = (2\theta)^n < \text{vol}(G(\epsilon))\). So there is some \(x \in G(\epsilon) \setminus H\). So \(f(x) \geq f(z)\). Hence, for some \(\hat{x} \in G\), we have

\[
f(z) \leq f(x) = f((1 - \epsilon)z_* + \epsilon \hat{x}) \leq (1 - \epsilon)f(z_*) + \epsilon f(\hat{x}) = \min f(G) + \epsilon(f(\hat{x}) - f(z_*)) \leq \min f(G) + \epsilon.
\]

Since \(\epsilon > \frac{2\theta}{\delta}\) was arbitrary, \(f(z) \leq \min f(G) + \frac{2\theta}{\delta}\). \(\square\)


Start with the localizer \((H, z) = (C, (*))\). At iteration \(k\), let \(x_k\) be the center of gravity of \(H_k\), where \((H_k, z_k)\) is the current localizer:

\[
x_k = \frac{\int_{H_k} x d\lambda}{\int_{H_k} d\lambda}.
\]

Call the oracle at \(x_k\). If \(x_k \notin G \cap \text{int}(C)\), and the oracle gives a separating hyperplane \(G \subseteq \{x \in C : v^T x \leq v^T_k x_k\}\), then set \(z_{k+1} = z_k\) and \(a_k := v_k\). If \(x_k \in G \cap \text{int}(C)\) and the origin gives \(f(x_k)\) and \(g(x_k) \in \partial f(x_k)\), then set \(z_{k+1} = \arg\min\{f(x_k), f(z_k)\}\) and \(a_k := g(x_k)\).

In either case, \(H_{k+1} := \{x \in H_k : a_k^T x \leq a_k^T x_k\}\). Stop if \(\left(\frac{\text{vol}(H_{k+1})}{\text{vol}(C)}\right)^{1/n} \leq \frac{\delta}{2}\), or \(\left(\frac{\text{vol}(H_{k+1})}{\text{vol}(C)}\right)^{1/n} < \frac{\delta}{2}\) and \(z_{k+1} = (*)\).

**Proposition 2** In MCG, each \((H_k, z_k)\) is a localizer.
Proof: By induction on \( k \); trivial for \( k = 0 \).

Assume true for \( k \). If \( x_k \not\in G \cap \text{int}(C) \), then \( z_{k+1} = z_k \) and \( a_k = v_k \) with \( G \subseteq \{ x \in C : v_k^T x \leq v_k^T x_k \} \). So \( G \setminus H_{k+1} = G \setminus H_k \). If \( x \in G \setminus H_{k+1} \), \( x \in G \setminus H_k \), so \( f(x) \geq f(z_k) = f(z_{k+1}) \).

If \( x_k \in G \cap \text{int}(C) \), then we get \( f(x_k) \) and \( g(x_k) = a_k \). Then take any \( x \in G \setminus H_{k+1} \); either \( x \in G \setminus H_k \) so \( f(x) \geq f(z_k) \geq f(z_{k+1}) \), or \( x \in H_k \) and \( g(x_k)^T x \geq g(x_k)^T x_k \), so \( f(x) \geq f(x_k) + g(x_k)^T (x-x_k) \geq f(x_k) \geq f(z_{k+1}) \). \( \square \)

**Proposition 3** (Grübaum, Mityagin) If \( D \subseteq \mathbb{R}^n \) is a convex compact set with center of gravity \( x \), then for any \( 0 \neq a \in \mathbb{R}^n \),

\[
\text{vol}(\{ y \in D : a^T y \leq a^T x \}) \leq \left( 1 - \frac{n}{n+1} \right)^n \text{vol}(D) \leq \frac{e-1}{e} \text{vol}(D).
\]

**Theorem 1** (Yudin and Nemirovski) If the method of centers of gravity performs 2.2n \( \ln \frac{2}{\delta} \) iterations and still has \( z_k = (\ast), G = \emptyset \). If it produces \( z_k \in G \), then within 2.2n \( (\ln \frac{2}{\delta} + \ln \frac{1}{\epsilon}) \) steps it produces \( z_k \) with \( \epsilon(z_k, f, G) \leq \epsilon \).

**Proof:** After \( k \) steps, we have localizer \( (H_k, z_k) \) with \( \left( \frac{\text{vol}(H_k)}{\text{vol}(C)} \right)^{1/n} \leq \left( \frac{e-1}{e} \right)^{k/n} \). Note \( \frac{1}{\ln \frac{1}{\epsilon}} < 2.2 \). Then \( \left( \frac{\text{vol}(H_k)}{\text{vol}(C)} \right)^{1/n} < \frac{2}{\delta} \) within \( \frac{n \ln \frac{2}{\delta}}{e} < 2.2n \ln 2/\delta \) steps. Similarly, within 2.2n \( (\ln \frac{2}{\delta} + \ln \frac{1}{\epsilon}) \) steps, \( \left( \frac{\text{vol}(H_k)}{\text{vol}(C)} \right)^{1/n} < \delta \epsilon \), so we have an \( \epsilon \)-optimal \( z_k \). \( \square \)

In general, computing the center of gravity is “hard,” so how can we circumvent it?

a) Use only nice sets \( H_k \) for which the center of gravity is easy to compute (e.g., the ellipsoid method).

b) Use a different notion of center, which is “easy to compute.” (a) leads to the ellipsoid method of Yudin and Nemirovski (1976) and Shor (1977). This is a simple modification of MCG: at every iteration, we have a localizer \( (E_k, z_k) \), with \( E_k \) an ellipsoid. We call the oracle at \( x_k \), the center of the ellipsoid, and update \( z_k \) as above, but then set \( E_{k+1} \) to be the minimum volume ellipsoid containing

\[
E_k^{1/2} := \{ x \in E_k : a_k^T x \leq a_k^T x_k \}.
\]

Questions: 1) How do we represent \( E_k \)?

2) How fast do the volumes of \( E_k \)’s shrink?

1) \( E_k = E(B_k, x_k) := \{ x \in \mathbb{R}^n : (x - x_k)^T B_k^{-1} (x - x_k) \leq 1 \} \) with some symmetric, positive definite \( B_k \).

2) \( \frac{\text{vol}(E_{k+1})}{\text{vol}(E_k)} < \exp \left\{ -\frac{1}{2(n+1)} \right\} \).