Diameters of Polyhedra
So far we have seen some bad news regarding bounds for diameters of polyhedra:

- The number of vertices can be super-exponential;
- The Hirsch conjecture fails.

In this lecture we will have some good news:

- The Hirsch conjecture holds for some polyhedra;
- There is a subexponential bound on $\Delta(d, n)$ (but not polynomial).

**Lemma 1** Let $P$ be a d-polyhedron with $n$ facets. Choose $0 \neq a \in \mathbb{R}^d$ and $a_0 \in \mathbb{R}$ so that

$$ P \subseteq \{ x \in \mathbb{R}^d : a^T x \geq a_0 \} \quad \text{and} \quad P' = \{ x \in P : a^T x = a_0 \} $$

is nonempty. Then $P'$ is a d'-polyhedron with n' facets, d' < d, n' < n, and all vectors of $P'$ are vertices of $P$, with two adjacent in $P'$ if and only if they are adjacent in $P$.

![Figure 1: Example of $P$ and $P'$ polyhedra for a 3-dim hyperrectangle.](image)

**Proof:** Exercise.

**Definition 1** A (0,1)-polytope in $\mathbb{R}^d$ is the convex hull of a subset of the (0,1)-vectors in $\mathbb{R}^d$.

**Theorem 1** (D. Naddef, 1989) If $P$ is a (0,1)-polytope in $\mathbb{R}^d$ with $n$ facets, then $\delta(P) \leq \min\{d,n-d\}$.

**Proof:** (of Theorem 1) (i) We need $\delta(P) \leq d$. We proceed by induction: true for $d = 1$. Suppose it is true for dimension less than $d$, and consider a (0,1)-polytope $P$ of dimension $d$. Let $v$ and $w$ be vertices of $P$. If $v_i = w_i = 0$ for some $i$, then choose $a = e_i$ and $a_0 = 0$ in the lemma, and note that $v$ and $w$ are vertices of the (0,1)-polytope $P'$ of lower dimension, so $d_P(v,w) \leq d_{P'}(v,w) \leq d' < d$, so we are good. Similarly, if $v_i = w_i = 1$ for some $i$. So,
assume \( v = 0 \) and \( w = e \). Then any edge from \( v \) goes to a vertex \( u \) of \( P \) with some \( u_i = 1 \). So,
\[
d_P(v, w) \leq d_P(v, u) + d_P(u, w) \leq 1 + (d - 1) = d.
\]

(ii) We now show \( \delta(P) \leq n - d \) by induction on \( d \).

If \( v \) and \( w \) both lie on the same facet \( F \), say defined by \( a^T x = a_0 \), of \( P \), then \( a \neq 0 \) implies, say, \( a_d \neq 0 \), and then
\[
F = \{ x \in P : a^T x = a_0 \} = \left\{ (x_1; \ldots; x_d) \in P : x_d = \frac{a_0 - a_1 x_1 - \cdots - a_d x_{d-1}}{a_d} \right\}.
\]

Look at \( P' = \{ \tilde{x} : (x_1; \ldots; x_d) \in \mathbb{R}^{d-1} : (x_1; \ldots; x_{d-1}; \frac{a_0 - a_1 x_1 - \cdots - a_d x_{d-1}}{a_d}) \in F \} \), a \((0, 1)\)-polytope in \( \mathbb{R}^{d-1} \) with at most \( n-1 \) facets: then \( d_P(v, w) \leq d_{P'}(\tilde{v}, \tilde{w}) \leq (n-1) - (d-1) = n - d \), by the induction hypothesis.

If \( v \) and \( w \) do not lie on a common facet, there must be at least \( 2d \) facets (\( d \) for \( v \), \( d \) for \( w \)) and \( d_P(v, w) \leq d = 2d - d \leq n - d \).

There is an alternative proof for \( \delta(P) \leq d \) that goes as follows.

Take any vertices \( v \) and \( w \), and without loss of generality assume \( v = 0 \). Consider minimizing \( e^T x \) over \( P \) by the simplex method with some anti-cycling rule, starting at \( w \). Since the objective is integer on vertices, with initial value at most \( d \) and final value 0, this means at most \( d \) nondegenerate steps.

Now we prove the following theorem. We will use the lemma above several times. Also, we use the easily established fact that \( \Delta(d, n) \) is monotonic in \( n \) for fixed \( d \).

**Theorem 2** (Basically Kalai-Kleitman) For \( 1 \leq d \leq n \), \( \Delta(d, n) \leq d \log n = n \log d \).

Here, the logarithms are to base 2. Note that the log of both \( d \log n \) and \( n \log d \) is \((\log d)(\log n)\), polynomial in \( \log d, \log n \). So this bound is quasipolynomial. The proof uses the following lemma.

**Lemma 2** (Kalai-Kleitman) For \( 1 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor \), \( \Delta(d, n) \leq (d - 1, n - 1) + 2\Delta(d, \left\lfloor \frac{n}{2} \right\rfloor) + 2 \).

**Proof:** (of Lemma 2) Choose a \( d \)-polyhedron \( P \) with \( n \) facets and two vertices \( v \) and \( w \) so that \( d_P(v, w) = \Delta(d, n) \). Without loss of generality, we can assume \( P \) is simple, so that all vertices lie on exactly \( d \) facets. If \( v \) and \( w \) both lie on a common facet \( P' \), then \( d_P(v, w) \leq \Delta(d - 1, n - 1) \). Suppose not. Let \( k_v \) denote the largest \( k \) so that there is a set \( \mathcal{F}_v \) of at most \( \left\lfloor \frac{n}{2} \right\rfloor \) facets with all paths from \( v \) of length at most \( k \) meeting only facets in \( \mathcal{F}_v \). This makes sense since paths of length 0 meet only \( d \) facets, while paths of length \( \delta(P) \) meet all \( n \) facets. Define \( k_w \) and \( \mathcal{F}_w \) similarly.

**Claim 1** \( k_v \leq \Delta(d, \left\lfloor \frac{n}{2} \right\rfloor) \).

**Proof:** (of Claim 1) Let \( P_v \) denote the \( d \)-polyhedron defined by the \( m \) (\( = |\mathcal{F}_v| \leq \left\lfloor \frac{n}{2} \right\rfloor \)) inequalities defining the facets in \( \mathcal{F}_v \). Choose a shortest path in \( P \) from \( v \) of length \( k_v \) to a vertex of \( P \), say \( t \).
Claim 2 This is also the shortest path in $P_v$ from $v$ to $t$.

Indeed, any shorter path cannot be a path in $P$, so it would have to meet a facet of $P$ not in $F_v$. But this is a contradiction.

So $k_v = d_P(v, t) = d_{P_v}(v, t) \leq \Delta(d, m) \leq \Delta(d, \lceil \frac{n}{2} \rceil)$, establishing Claim 1. Similarly, $k_w \leq \Delta(d, \lceil \frac{n}{2} \rceil)$.

By definition, if we allow ourselves to go at most $k_v + 1$ steps from $v$, we can reach a set $G_v$ of facets with $|G_v| > \lfloor \frac{n}{2} \rfloor$. Similarly, if we allow ourselves to go at most $k_w + 1$ steps from $w$, we can reach a set $G_w$ of facets with $|G_w| > \lfloor \frac{n}{2} \rfloor$. So, there is a facet, say $G$, in both $G_v$ and $G_w$, and a vertex $t$ in $G$ with $d_{P_v}(v, t) \leq k_v + 1$ and a vertex $u$ in $G$ with $d_{P_w}(w, u) \leq k_w + 1$. Then,

$$d_P(v, w) \leq d_P(v, t) + d_P(t, u) + d_P(w, u) \leq d_P(v, t) + d_G(t, u) + d_P(w, u) \leq k_v + 1 + \Delta(d - 1, n - 1) + k_w + 1 \leq \Delta(d - 1, n - 1) + 2\Delta(d, \lfloor \frac{n}{2} \rfloor) + 2.$$ 

□

Proof: (of Theorem 2) By induction on $d + n$.

- $d = 1$: $LHS = RHS = 1$.
- $d = 2$: $LHS = n - 2 < n = RHS$.
- $d = 3$: If $n < 6$, then any two vertices are on a common facet, so $\Delta(3, n) \leq \Delta(2, n - 1) \leq n - 3 < n^{\log 3}$. If $n \geq 6$, by the lemma,

$$\Delta(3, n) \leq \Delta(2, n - 1) + 2\Delta(3, \lfloor \frac{n}{2} \rfloor) + 2 = (n - 3) + 2(3^{\log \lfloor \frac{n}{2} \rfloor}) + 2 \leq n - 1 + 2(3^{\log n}) = n - 1 + \frac{2}{3} \cdot 3^{\log n},$$

so we want $n - 1 \leq \frac{1}{3} \cdot n^{\log 3}$. You can check that this is true for $n = 6$. The derivative of its left-hand side is 1, and the derivative of its right-hand side is $\frac{\log 3}{3} \cdot n^{\log 3 - 1} > 1$ for $n \geq 6$, and thus it is true for all $n > 6$.

- $d \geq 4$: If $n < 2d$, any two vertices lie on a common facet, so their distance is at most $\Delta(d - 1, n - 1)$.

- $d \geq 4$ and $n \geq 2d$: If $n = 8$ (so $d = 4$), two vertices not sharing a facet can be joined by

$$\frac{1}{a} + \frac{\Delta(3, 7)}{b} \leq 1 + 3^{\log 7} \leq 4^{\log 7} \leq 4^{\log 8}$$

steps, where the term $a$ represents any bounded edge from one vertex and the term $b$ represents the steps from the resulting vertex to the other vertex in some facet.
\[ \Delta(d, n) \leq \Delta(d - 1, n - 1) + 2\Delta(d, \left\lceil \frac{n}{2} \right\rceil) + 2 \]

\[ \leq (d - 1)^{\log(n-1)} + 2d^{\log n-1} + 2 \]

\[ = \left( \frac{d - 1}{d} \right)^{\log(n-1)} d^{\log(n-1)} + 2 \cdot d^{\log n} + 2 \]

\[ \leq \left( \frac{d - 1}{d} \right)^3 d^{\log n} + \frac{2}{d} \cdot d^{\log n} + 2 \]

\[ = d^{\log n} - \frac{3}{d} \cdot d^{\log n} + \frac{3}{d^2} \cdot d^{\log n} - \frac{1}{d^3} \cdot d^{\log n} + 2 \cdot d^{\log n} + 2 \]

\[ = \left( 1 - \frac{1}{d} + \frac{3}{d^2} - \frac{1}{d^3} \right) d^{\log n} + 2 \]

\[ \leq \left( 1 - \frac{1}{d} + \frac{3}{4d} - \frac{1}{d^3} \right) d^{\log n} + 2 \]

\[ = d^{\log n} - \frac{1}{4d} \cdot d^{\log n} - \frac{1}{d^3} \cdot d^{\log n} + 2 \]

\[ \leq d^{\log n}, \]

since each of the subtracted terms is at least 1. \qed