\[ w = dz_0 + Mz + q \]  
\[ 0 \leq w \perp z \geq 0, \quad z_0 \geq 0 \]  

We showed last time:

**Theorem 1** Lemke’s Algorithm for a monotone LCP will either produce a complementary solution or show that the LCP is infeasible.

In particular, if the LCP comes from (QP) and (QD) with \( H \) positive semidefinite, \( M = \begin{bmatrix} H & -A^T \\ A & 0 \end{bmatrix}, \quad q = \begin{bmatrix} c \\ -b \end{bmatrix} \), and then any secondary ray has \( \bar{z} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \neq 0 \), with either \( \bar{x} \) showing (QD) infeasible or \( \bar{y} \) showing (QP) infeasible. (See the end of the last notes.)

**Definition 1** \( M \) is **copositive plus** if \( z^T Mz \geq 0 \) for \( z \geq 0 \), with equality only if \( (M + MT)z = 0 \).

In fact, our proof of the theorem showed the conclusion holds if \( M \) is just copositive plus (Lemke). Also, if \( M \) is strictly monotone \( (z^T Mz > 0 \text{ if } z \neq 0) \), there cannot be any secondary ray, so the algorithm gives a complementary solution. Finally, it can be shown that the algorithm always gives a complementary solution if \( M \) is a \( P \)-matrix.

**How are complementary pivot algorithms 1 and 2 related?**

In fact, Lemke’s algorithm can be reduced to the Lemke-Howson algorithm. We add the constraint:

\[ w_0 = -z_0 - e^T z + \omega \geq 0 \]

to the system to get the LCP:

\[
\begin{bmatrix}
  w_0 \\
  w
\end{bmatrix} = 
\begin{bmatrix}
  -1 & -e^T \\
  d & M
\end{bmatrix}
\begin{bmatrix}
  z_0 \\
  z
\end{bmatrix}
+ 
\begin{bmatrix}
  \omega \\
  q
\end{bmatrix}
\]

\[ 0 \leq \begin{bmatrix}
  w_0 \\
  w
\end{bmatrix} \perp \begin{bmatrix}
  z_0 \\
  z
\end{bmatrix} \geq 0, \quad z_0 \geq 0 \]  

Choose \( \omega \) so that \( z_0 + e^T z < \omega \) for all basic feasible solutions (BFS) to the system (1). Then the constraint we added just intersects rays going to infinity, and thus makes the unbounded polyhedron bounded. We have a complementary solution to this:

\[ ((w_0; w) = (0; d\omega + q), (z_0; z) = (\omega; 0)) \]

Now start complementary pivot algorithm 1 for this augmented LCP starting from this complementary solution.
E.g., search $G_0$, so start by increasing $w_0$, meaning come down the primary ray and then pivot as in complementary pivot algorithm 2. Continue until finding a complementary solution to the original LCP, a desirable complementary solution ($w_0 > 0$, $z_0 = 0$) to the augmented LCP, or a secondary ray, which leads to an undesirable complementary solution ($w_0 = 0$, $z_0 > 0$) to the augmented LCP.

**Computational complexity of complementary pivot algorithms**

Consider the following LCP:

\[
M := M_n := \begin{bmatrix}
1 & 0 \\
2 & 1 \\
2 & 2 \\
2 & 2 & 2 \\
. & . & . & . \\
2 & 2 & 2 & 1
\end{bmatrix} \in \mathbb{R}^{n \times n}, \quad q := q^{(n)} := \begin{bmatrix}
-2^n \\
-2^n & -2^{n-1} \\
. & . & . \\
-2^n & -2^{n-1} & -2^{n-2} & . & -2
\end{bmatrix} \in \mathbb{R}^n.
\]

Note: $M + M^T = 2ee^T$, so $z^T(M + M^T)z = 2(e^T z)^2 \geq 0$. Thus, $M$ is monotone and also a $P-$matrix.

We’ll use $d := d^{(n)} := e := e^{(n)} := \begin{bmatrix} 1 \\
. \\
1 \end{bmatrix} \in \mathbb{R}^n$.

Besides, $M_n = \begin{bmatrix} 1 & 0 \\
2e^{(n-1)} & M_{n-1} \end{bmatrix}$, $q^{(n)} = -2^n e^{(n)} + \begin{bmatrix} 0 \\
q^{(n-1)} \end{bmatrix}$.

Therefore, $w = dz_0 + Mz + q$ becomes

\[
\begin{bmatrix} w_1 \\
. \\
w_n \end{bmatrix} = (z_0 - 2^n)e^{(n)} + \begin{bmatrix} z_1 \\
2z_1 \\
2z_1 \end{bmatrix} + \begin{bmatrix} 0 \\
M_{n-1} \end{bmatrix} \begin{bmatrix} z_2 \\
. \\
z_n \end{bmatrix} + \begin{bmatrix} 0 \\
q^{(n-1)} \end{bmatrix}.
\]

(5)

**Theorem 2** Lemke’s algorithm applied to this LCP takes $2^n-1$ pivots starting with $z_0, w_1, \ldots, w_{n-1}$ basic and ending with $z_1, w_2, \ldots, w_n$ basic with $z_0$ decreasing from $2^{n+1} - 2$ to 0 in steps of 2.

**Proof:** By induction (check that it’s true for $n = 1$).

Assume it is true for dimension at most $n - 1$. Consider $M_n$ and $q^{(n)}$. Start with $z_0 = \max(-\frac{q^{(n)}}{e^T}) = 2^n + 2^{n-1} + \ldots + 2 = 2^{n+1} - 2$ and $w_n = 0$.

Rewrite (5) as

\[
\begin{aligned}
w_1 &= z_0' + z_1 \\
\begin{bmatrix} w_2 \\
. \\
w_n \end{bmatrix} &= e^{(n-1)}z_0' + 2e^{(n-1)}z_1 + M_{n-1} \begin{bmatrix} z_2 \\
. \\
z_n \end{bmatrix} + q^{(n-1)}
\end{aligned}
\]

(6) (7)

with $z_0' := z_0 - 2^n$. 

2
While \( z'_0 \) remains positive, so does \( w_1 \), so \( z_1 \) stays nonbasic, and by the inductive hypothesis the algorithm will perform \textit{exactly} like Lemke’s algorithm on the \((n-1)\)-dimensional problem taking \( 2^{n-1} - 1 \) pivots to reach the solution with \( z'_0 = 0 \), so \( z_0 = 2^n \) and then \( w_1 \) hits 0. When \( w_1 \) hits zero, the next step is to increase \( z_1 \) and subsequent solutions will have \( z_1 \) basic and \( w_1 \) nonbasic. So we rewrite the system above, interchanging \( w_1 \) and \( z_1 \): (5) is also equivalent to:

\[
    z_1 = z''_0 + w_1 \\
    \begin{bmatrix}
        w_2 \\
        \vdots \\
        w_n
    \end{bmatrix} = e^{(n-1)}z''_0 + 2e^{(n-1)}w_1 + M_{n-1} \begin{bmatrix}
        z_2 \\
        \vdots \\
        z_n
    \end{bmatrix} + q^{(n-1)}
\]

with \( z''_0 := 2^n - z_0 \).

Again by the induction hypothesis there is a path of \textit{a.c.} solutions to the system with \( z''_0 \) decreasing from \( 2^n - 2 \) to 0, i.e., \( z_0 \) increasing from 2 to \( 2^n \) in steps of 2.

Reverse this path and add it to the first:

**Table 1: Basic variables change**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>( z_0 &gt; 0, )</th>
<th>( w_1, )</th>
<th>( w_2, ..., w_{n-1} )</th>
<th>( z_0 = 2^{n+1} - 2, ) ( z'_0 = 2^n - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( 2^{n-1} )</td>
<td>( z_0 ) , 0, ( \vdots ) , ( w_1 )</td>
<td>( z_2, ..., w_n )</td>
<td>( z_0 = 2^n, ) ( z'_0 = 0, ) ( z''_0 = 0, ) ( (w_1 = 0) )</td>
<td></td>
</tr>
<tr>
<td>( 2^n-1 )</td>
<td>( z_1 ) , ( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( 2^n )</td>
<td>( \text{compl soln} ) ( z_1, )</td>
<td>( w_2, ..., w_{n-1}, w_n )</td>
<td>( z_0 = 0, ) ( z'_0 = 2^n )</td>
<td></td>
</tr>
</tbody>
</table>

A little way out on the primary ray of the problem above

Note that we have added an extra BFS at the end, corresponding to \( z''_0 = 2^n \), so a little way out on the primary ray of this \((n-1)\)-dimensional problem, but also corresponding to \( z_0 = 0 \), so a complementary solution to the original LCP. We also have the correct set of basic variables (we used the inductive hypothesis to make sure the BFS in the middle coincided).

The number of BFS in total is \( 2^{n-1} \) (the first set) plus \( 2^{n-1} - 1 \) (the second set, but subtracting one because of the overlap in the middle), plus another one for the last one. Number of BFS: \( 2^{n-1} + 2^{n-1} - 1 + 1 = 2^n \). \( z_0 \) is decreased by 2 in each step.

This concludes the inductive step and thus the proof of the theorem.