Sensitivity Analysis in LP and SDP Using Interior-Point Methods

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Part I: Linear Programming (LP)

LP in standard form:

\[
LPP(b, c) \quad \min_x \quad c^T x \\
\text{s.t.} \quad Ax = b, \\
\quad x \geq 0,
\]

\(c\) and \(x \in IR^n\), \(b \in IR^m\), and \(A \in IR^{m \times n}\).

Associated dual LP:

\[
LPD(b, c) \quad \max_{y, s} \quad b^T y \\
\text{s.t.} \quad A^T y + s = c, \\
\quad s \geq 0,
\]

\(y \in IR^m\) and \(s \in IR^n\).

- The coefficient matrix \(A\) is fixed.
Assumptions:

1. Both $LPP(b, c)$ and $LPD(b, c)$ have strictly feasible points.
2. $A$ has full row rank.

Central Path

Solution to:

\[
\begin{align*}
Ax &= b, \\
A^T y + s &= c, \\
XSe &= \mu e,
\end{align*}
\]

with $x > 0$ and $s > 0$ for some $\mu > 0$.

Under Assumptions 1 and 2, such a solution exists and is unique for each positive $\mu$. 
• Let \((x, y, s)\) be the current iterate.

• Seek an approximation to the point on the central path with parameter \(\mu\).

Using Newton’s method:

\[
\begin{align*}
A \Delta x &= r_p \\
A^T \Delta y + \Delta s &= r_d \\
S \Delta x + X \Delta s &= r_{xs},
\end{align*}
\]

where

• \(r_p = b - Ax\),

• \(r_d = c - A^T y - s\), and

• \(r_{xs} = \mu e - XSe\).
**Definition.** A Newton step from \((x, y, s)\) **targeting** the feasible pair of points \((x', y', s')\) of LPP\((b', c')\) and LPD\((b', c')\) that satisfies \(X'S'e = v\) is the triple \((\Delta x, \Delta y, \Delta s)\) solving (1) for \(r_p = b' - Ax\), \(r_d = c' - A^Ty - s\), and \(r_{xs} = v - XSe\).

**Remark 1.** Such a pair of points \((x', y', s')\) might not exist, but the Newton step is still well-defined.
Proposition 1. Assume:

- \((x, y, s)\) is a strictly feasible pair of points for \(LPP(b, c)\) and \(LPD(b, c)\),

- \(b\) is replaced by \(b' = b + \Delta b\), where \(\Delta b \in IR^m\), and

- a full Newton step is taken from \((x, y, s)\) targeting the feasible pair of points \((x', y', s')\) of \(LPP(b', c)\) and \(LPD(b', c)\) that satisfies \(X'S'e = XSe\).

If, and only if,

\[ \|S^{-1}AT(AS^{-1}XAT)^{-1}\Delta b\|_\infty \leq 1, \]

- then resulting iterate will be feasible for the new problem.

Moreover, duality gap will decrease.
Proof:

\( r_p = \Delta b, \ r_d = 0, \) and \( r_{xs} = 0. \)

The third equation defining the Newton step:

\[ S\Delta x + X\Delta s = 0. \]

Rewriting it componentwise,

\[ s_i \Delta x_i + x_i \Delta s_i = 0 \quad \text{so} \quad \frac{\Delta x_i}{x_i} + \frac{\Delta s_i}{s_i} = 0, \quad i = 1, \ldots, n. \]
Proof (Cont’d):

The next iterate will be feasible iff

$$x_i + \Delta x_i \geq 0, \quad \text{and} \quad s_i + \Delta s_i \geq 0, \quad i = 1, \ldots, n.$$ 

Combining two observations, it is necessary and sufficient to have:

$$\left\| S^{-1} \Delta s \right\|_{\infty} \leq 1.$$

The result follows from the solution of the Newton step.
Proposition 2. Assume:

- $(x, y, s)$ is a strictly feasible pair of points for $LPP(b, c)$ and $LPD(b, c)$,

- $c$ is replaced by $c' = c + \Delta c$, where $\Delta c \in IR^n$, and

- a full Newton step is taken from $(x, y, s)$ targeting the feasible pair $(x', y', s')$ of $LPP(b, c')$ and $LPD(b, c')$ that satisfies $X'S'e = XS'e$.

If, and only if,

$$\|S^{-1}(I - AT(AS^{-1}XA^T)^{-1}AS^{-1}X)\Delta c\|_\infty \leq 1,$$

- then resulting iterate will be feasible for the new problem.

Moreover, duality gap will decrease.
Simplex Approach

- Partition $A$ as $B$ and $N$ corresponding to the basic and nonbasic columns, respectively.

- Let $(x^*, y^*, s^*)$ be a pair of optimal solutions for $LPP(b, c)$ and $LPD(b, c)$.

- Partition $x^*$ as $x^*_B$ and $x^*_N$; $s^*$ as $s^*_B$ and $s^*_N$; $c$ as $c_B$ and $c_N$ accordingly.
1. Let $b$ be replaced by $b + \Delta b$. The optimal basis remains optimal iff

$$B^{-1}(b + \Delta b) \geq 0 \quad \text{or} \quad B^{-1} \Delta b \geq -B^{-1}b = -x_B^*.$$ 

2. Let $c$ be replaced by $c + \Delta c$. The optimal basis remains optimal iff

$$c_N^T + \Delta c_N^T - c_B^T B^{-1} N - \Delta c_B^T B^{-1} N \geq 0 \quad \text{or} \quad \Delta c_N - N^T B^{-T} \Delta c_B \geq -s_N^*.$$
Assumption: The optimal solution is unique and non-degenerate.

Symmetrized simplex bounds

1. Change in $b$

\[ B^{-1} \Delta b \geq -B^{-1}b = -x^*_B \quad \text{so} \quad (X^*_B)^{-1}B^{-1} \Delta b \geq -e. \]

The largest $L_\infty$-box around the origin:

\[ \|(X^*_B)^{-1}B^{-1} \Delta b\|_\infty \leq 1. \]
Symmetrized simplex bounds (Cont’d):

2. Change in $c$

$$\Delta c_N - N^T B^{-T} \Delta c_B \geq -s_N^*$$

so

$$(S_N^*)^{-1}(\Delta c_N - N^T B^{-T} \Delta c_B) \geq -e.$$  

The largest $L_\infty$-box around the origin:

$$\|(S_N^*)^{-1}(\Delta c_N - N^T B^{-T} \Delta c_B)\|_\infty \leq 1.$$
Comparison

**Theorem 1.** Let \((x, y, s)\) be any pair of strictly feasible solutions with small duality gap \(\mu n\). The bounds arising from Propositions 1 and 2 evaluated at these points are given by

\[
\left\| \begin{bmatrix} (X_B^*)^{-1}B^{-1} + O(\mu) \\ O(\mu) \end{bmatrix} \Delta b \right\|_\infty \leq 1,
\]

and

\[
\left\| \begin{bmatrix} O(\mu) \\ -(S_N^*)^{-1}N^TB^{-T} + O(\mu) \\ (S_N^*)^{-1} + O(\mu) \end{bmatrix} \begin{bmatrix} \Delta c_B \\ \Delta c_N \end{bmatrix} \right\|_\infty \leq 1.
\]

Moreover, these bounds are asymptotically the same as the symmetrized simplex bounds.
Conclusion of Part I

Under the assumption of unique, non-degenerate solution, the interior-point approach yields asymptotically exactly the same bounds as those arising from the simplex approach that keep the current basis optimal (after symmetrization with respect to the origin).
Part II: Semidefinite Programming (SDP)

SDP in standard form:

\[
\text{SDP}(b, C) \quad \min_X \quad C \cdot X \\
\quad AX = b, \\
\quad X \succeq 0,
\]

where

\[
AX = (A_i \cdot X)_{i=1}^m,
\]

\(A_i \in SIR^{n \times n}, \ b \in IR^m, \ C \in SIR^{n \times n} \) are given, and \(X \in SIR^{n \times n}\).

\(SIR^{n \times n} : n \times n \) symmetric matrices.

\(X \succeq 0: \) \(X\) is symmetric positive semidefinite.

\(P \cdot Q = \text{Trace}(P^T Q) = \sum_{i,j} P_{ij}Q_{ij}\)
Associated dual problem:

\[
SDD(b, C) \quad \max_{y, S} \quad b^T y \\
A^* y + S = C, \\
S \succeq 0,
\]

where \( y \in IR^m, \ S \in SIR^{n \times n}, \) and

\[
A^* y = \sum_{i=1}^{m} y_i A_i.
\]

Assumptions:

1. Both \( SDP(b, C) \) and \( SDD(b, C) \) have strictly feasible solutions.

2. \( A \) is surjective, i.e. \( A_i \)'s are linearly independent.
Central Path

Solution to:

\[ \sum_{i=1}^{m} y_i A_i \cdot X + S = C, \]
\[ XS = \mu I, \]

(2)

together with \( X > 0 \) and \( S > 0 \) for \( \mu > 0 \).

Remark. Newton’s method cannot be applied directly to (2) since the system is not square.

Remedy: Symmetrize the third equation so that the residual lies in \( SIR^{n \times n} \).
Let \((X, y, S)\) be the current iterate.

The directions we will examine will be Newton steps for nonlinear systems of the form

\[
\begin{align*}
\mathcal{A} \tilde{X} + \mathcal{A}^* \tilde{y} + \tilde{S} &= b, \\
\Theta(\tilde{X}, \tilde{S}) &= C, \\
\end{align*}
\]

where \(\Theta(X, S)\) is some symmetrization of \(XS\) and where \(X'\) and \(S'\) are the targeted points.

More than 20 different symmetrizations!
The Newton step \((\Delta X, \Delta y, \Delta S)\) is the solution to

\[
\begin{align*}
\mathcal{A} \Delta X &= r_p, \\
\mathcal{A}^* \Delta y + \Delta S &= R_d, \\
\mathcal{E} \Delta X + \mathcal{F} \Delta S &= R_{EF},
\end{align*}
\]

where

- \(r_p = b - \mathcal{A} X\) is the primal residual,
- \(R_d = C - \mathcal{A}^* y - S\) is the dual residual,
- the operators \(\mathcal{E} = \mathcal{E}(X, S)\) and \(\mathcal{F} = \mathcal{F}(X, S)\) are the derivatives of \(\Theta\) with respect to \(\tilde{X}\) and \(\tilde{S}\) respectively, evaluated at \((X, S)\), and
- \(R_{EF} = R_{EF}(X, S) = \Theta(X', S') - \Theta(X, S)\).
The solution to the Newton step is unique if

- \( \mathcal{E} \) is nonsingular, and

- \( m \times m \) Schur complement matrix \( A\mathcal{E}^{-1}FA^* \) is nonsingular.

In this case, Newton step is given by:

\[
\begin{align*}
(\mathcal{A}\mathcal{E}^{-1}FA^*)\Delta y &= r_p - \mathcal{A}\mathcal{E}^{-1}(R_{EF} - \mathcal{F}R_d), \\
\Delta S &= R_d - A^*\Delta y, \\
\Delta X &= \mathcal{E}^{-1}(R_{EF} - \mathcal{F}\Delta S).
\end{align*}
\]
We consider the following search directions:

- AHO (Alizadeh, Haeberly, Overton)
- H..K..M (Helmberg et. al., Kojima et. al., Monteiro)
- NT (Nesterov, Todd)
Notation:

\[(P \odot Q)K := \frac{1}{2}(PKQ^T + QKP^T),\]

where \(P, Q \in IR^{n \times n}\) and \(K \in SIR^{n \times n}\), and we will regard it as an operator from \(SIR^{n \times n}\) to itself.
Observations:

1. For the three directions,

\[ \mathcal{E} = S \odot M, \quad \mathcal{F} = MX \odot I, \]

where \( M \) is a symmetric positive definite matrix.

\( \mathcal{E} \) is invertible.

- \( M = I \) for AHO
- \( M = S \) for H..K..M
- \( M = W^{-1} \) for NT where \( WSW = X \).

2. For all \((X, y, S)\) strictly feasible, \( \mathcal{E}^{-1} \mathcal{F} \) is positive definite for H..K..M and NT. This also holds for AHO under additional assumptions on \((X, y, S)\).
Proposition 3. Assume

- \((X, y, S)\) is a strictly feasible pair of points for \(\text{SDP}(b, C)\) and \(\text{SDD}(b, C)\),
- \(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*\) is nonsingular,
- \(b\) is replaced by \(b' = b + \Delta b\),
- a full Newton step is taken from \((X, y, S)\) targeting the feasible pair \((X', y', S')\) of \(\text{SDP}(b', C)\) and \(\text{SDD}(b', C)\) that satisfies \(\Theta(X', S') = \Theta(X, S)\).

If

\[
\left\| X^{-\frac{1}{2}} \left( \mathcal{E}^{-1}\mathcal{F}\mathcal{A}^* \left[ \left( \mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^* \right)^{-1}\Delta b \right] \right) X^{-\frac{1}{2}} \right\|_2 \leq 1,
\]

\[
\left\| S^{-\frac{1}{2}} \left( \mathcal{A}^* \left[ \left( \mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^* \right)^{-1}\Delta b \right] \right) S^{-\frac{1}{2}} \right\|_2 \leq 1,
\]

then the resulting iterate will be feasible for the new problem. Moreover, the duality gap of the new iterate will decrease if \(\mathcal{E}^{-1}\mathcal{F}\) is positive definite.
Proof:

The next iterate will be feasible for the new problem if and only if $X + \Delta X \succeq 0$ and $S + \Delta S \succeq 0$. But,

$$X + \Delta X \succeq 0 \quad \text{holds iff} \quad I + X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}} \succeq 0.$$ 

Therefore, all eigenvalues of $X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}$ should be $\geq 1$. A sufficient condition is

$$\|X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\|_2 \leq 1.$$
Proof (Cont’d):

Similarly,

\[ S + \Delta S \succeq 0 \quad \text{holds iff} \quad I + S^{-\frac{1}{2}} \Delta SS^{-\frac{1}{2}} \succeq 0. \]

A sufficient condition is

\[ \|S^{-\frac{1}{2}} \Delta SS^{-\frac{1}{2}}\|_2 \leq 1. \]

The result follows from the solution to the Newton system.
Proposition 4. Assume:

- $(X, y, S)$ is a strictly feasible pair of points for $SDP(b, C)$ and $SDD(b, C)$,
- $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ is nonsingular,
- $C$ is replaced by $C' = C + \Delta C$, where $\Delta C \in SIR^{n \times n}$, and
- a full Newton step is taken from $(X, y, S)$ targeting the feasible pair $(X', y', S')$ of $SDP(b, C')$ and $SDD(b, C')$ that satisfies $\Theta(X', S') = \Theta(X, S)$.
If

\[
\left\| X^{-\frac{1}{2}} \left( -\mathcal{E}^{-1} \mathcal{F} \Delta C + \mathcal{E}^{-1} \mathcal{F} A^* (A \mathcal{E}^{-1} \mathcal{F} A^*)^{-1} A \mathcal{E}^{-1} \mathcal{F} \Delta C \right) X^{-\frac{1}{2}} \right\|_2 \leq 1,
\]

\[
\left\| S^{-\frac{1}{2}} \left( \Delta C - A^* (A \mathcal{E}^{-1} \mathcal{F} A^*)^{-1} A \mathcal{E}^{-1} \mathcal{F} \Delta C \right) S^{-\frac{1}{2}} \right\|_2 \leq 1.
\]

then the resulting iterate will be feasible for the new problem. Moreover, the duality gap of the new iterate will decrease if $\mathcal{E}^{-1} \mathcal{F}$ is positive definite.
Concluding Remarks

- For LP, the bounds arising from the interior-point approach compares nicely with those from the simplex approach under the assumption of unique, non-degenerate solution.

- The comparison in the degenerate case remains unanswered.

- For SDP, our preliminary computational tests indicate that the three directions perform similarly if the iterates lie in a relatively wide neighborhood of the central path.

- NT direction yields similar results even when the iterates get too close to the boundary whereas the other two directions deteriorate significantly.