1. This question and the next are concerned with central cuts. Suppose we have an ellipsoid $E := E(B, y)$, and we add two cuts symmetrically placed with respect to the center $y$. Consider

$$
\bar{E}_\alpha := \{ x \in E : a^T y - \alpha \sqrt{a^T B a} \leq a^T x \leq a^T y + \alpha \sqrt{a^T B a} \}
$$

for some nonzero $a \in \mathbb{R}^n$ and some $0 \leq \alpha \leq 1$.

a) Write the condition for $x$ to lie in $\bar{E}_\alpha$ as two quadratics.

b) By combining these two quadratics suitably, find an ellipsoid $E(B_+, y_+)$ that contains $\bar{E}_\alpha$, depending on a scalar parameter $\sigma$.

c) Find the value of $\sigma$ that minimizes the volume of the resulting ellipsoid as a function of $\alpha$. Show that for $\alpha = n^{-1/2}$ this ellipsoid is just $E$, while for $\alpha$ smaller than this it gives an ellipsoid of smaller volume than $E$. (In fact, this is the minimum-volume ellipsoid among all those containing $\bar{E}_\alpha$, not just those obtained this way.)

a) For simplicity, define $\bar{a}$ as $a/(a^T B a)^{1/2}$, so that the two-sided constraint can be written as $-\alpha \leq a^T (x - y) \leq \alpha$. Then the two quadratic inequalities are

$$(x - y)^T B^{-1} (x - y) \leq 1,$$

$$(x - y)^T \bar{a} \bar{a}^T (x - y) \leq \alpha^2.$$

b) Multiply the first inequality by $1 - \sigma$ and the second by $\sigma$ for $0 \leq \sigma \leq 1$ and add, to get

$$(x - y)^T ((1 - \sigma)B^{-1} + \sigma \bar{a} \bar{a}^T)(x - y) \leq 1 - \sigma + \sigma \alpha^2,$$

or, using the Sherman-Morrison-Woodbury formula, $(x - y)^T B_+^{-1} (x - y) \leq 1$, where

$$B_+ := \left( \frac{1 - \sigma + \sigma \alpha^2}{1 - \sigma} \right) (B - \sigma \bar{a} \bar{a}^T).$$

(This assumes $\sigma < 1$, since otherwise, the quadratic inequality defines a slab of infinite volume for $n > 1$. For $n = 1$, the problem is trivial and the optimal $\sigma$ is 1. We assume $n > 1$ in what follows.) With $\sigma < 1$, $B_+$ is positive definite, and this defines the ellipsoid $E(B_+, y)$ (note that $y$ is unchanged). Since its defining inequality is derived from those for $\bar{E}_\alpha$, $E(B_+, y)$ contains it.

c) The volume of this ellipsoid is the square root of the determinant of $B_+$ times that of the unit ball, so we want to minimize the determinant of $B_+$, which is easily seen to be $f(\sigma)$ times that of $B$, with

$$f(\sigma) := \frac{(1 - \sigma + \sigma \alpha^2)^n}{(1 - \sigma)^{n-1}}.$$  

It is easy to check that the derivative of $f$, for all $0 \leq \sigma < 1$, is a positive multiple of

$$g(\sigma) := -n(1 - \alpha^2)(1 - \sigma) + (n - 1)(1 - \sigma + \sigma \alpha^2).$$

Note that $g$ has a unique root at $\sigma^* = (1 - n\alpha^2)/(1 - \alpha)$, and is negative to the left and positive to the right. Hence for $0 \leq \alpha \leq n^{-1/2}$, $f$ is minimized at $\sigma^*$, while if $\alpha > n^{-1/2}$, $\sigma^* < 0$ and $f$ is increasing for $\sigma$ between 0 and 1, and so is minimized by $\sigma = 0$ (and so the minimum-volume ellipsoid of this form is the original ellipsoid $E$). For $0 \leq \alpha < n^{-1/2}$, $f$ is
decreasing from 0 to \( \sigma^* \), so the resulting ellipsoid has volume strictly smaller than that of \( E \).

2. Consider a centrally symmetric polytope, a bounded polyhedron of the form \( P := \{ x \in \mathbb{R}^n : -b \leq Ax \leq b \} \) for some \( A, b \).
   
a) Show that there is a minimum-volume ellipsoid \( E = E(B, y) \) containing \( P \).
   
b) Show that any such must have \( y = 0 \), i.e., it must be centrally symmetric also.
   
c) Show that, if \( \bar{E}(B, 0) \) is a (the) minimum-volume ellipsoid containing \( P \), then \( \{ n^{-1/2}x : x \in E(B, 0) \} \) is contained in \( P \).

   (Hence such polytopes can be rounded to a factor \( \sqrt{n} \), not \( n \) as in the general case. In fact, this holds for any centrally symmetric convex body.)

   a) If \( P \) is a convex body, i.e., has a nonempty interior, this follows from Proposition 1 of Lecture 17. If not, \( P \) lies in a lower-dimensional affine set. In fact, the affine hull of \( P \) is a lower-dimensional affine set, and \( P \) has a nonempty interior relative to this; then \( P \) can be enclosed in a minimum-volume ellipsoid in this lower-dimensional set, and hence in a degenerate ellipsoid in \( \mathbb{R}^n \) with \( n \)-dimensional volume 0, hence clearly minimum! Henceforth assume \( P \) is a convex body; otherwise, the following arguments can all be applied within its lower-dimensional affine hull.

   b) Suppose \( E(B, y) \) is a minimum-volume ellipsoid containing \( P \), with \( y \neq 0 \). Then \( E(B, -y) \) contains \(-P = P\). So for any \( x \in P \), \((x-y)^T B^{-1} (x-y) \leq 1 \) and \((x+y)^T B^{-1} (x+y) \leq 1 \). Averaging, we find \( x^T B^{-1} x \leq 1 - y^T B^{-1} y \). Thus \( P \) is contained in \( E(\alpha B, 0) \) with \( \alpha := 1 - y^T B^{-1} y < 1 \), an ellipsoid of strictly smaller volume. This is a contradiction.

   c) Suppose not, so that there is an \( x \) in \( E(B, 0) \) with \( n^{-1/2}x \notin P \). Thus it can be separated from \( P \), so there is an inequality \( a^T z \leq \beta \) valid for \( P \) with \( a^T x > n^{1/2} \beta \). Since \( a^T x \leq \sqrt{a^T B a} \), \( \beta = \alpha \sqrt{a^T B a} \) for some \( \alpha < n^{-1/2} \). But if \( a^T z \leq \alpha \sqrt{a^T B a} \) for all \( z \in P \), then \( a^T z \geq -\alpha \sqrt{a^T B a} \) for all \( z \in P \) also, since \( P \) is centrally symmetric. So \( P \subseteq \bar{E}_\alpha \), implying that \( P \) can be enclosed in an ellipsoid of smaller volume by \( Q_1 \), a contradiction.

3. Suppose that \( P := \{ x \in \mathbb{R}^n : A^T x \leq e \} \) is bounded (where \( e \) is the vector of ones as usual). Assume that the function \( B \rightarrow -\ln \det B \) is convex as a function of the entries of the symmetric matrix \( B \).

   a) Show how the problem of finding the maximum volume ellipsoid with center \( y \) contained in \( P \) can be written as an optimization problem with a finite number of constraints. (Argue that the positive semidefiniteness constraint can be eliminated.)

   b) Exhibit a feasible solution to this problem.

   c) Show that if the center \( y \) is restricted to be 0, your optimization problem can be converted to one with linear constraints on \( B \).

   d) Now return to the general case, where \( y \) is a variable. Try to rewrite the optimization problem with convex constraints (you may want to consider the symmetric square root).

   a) We use the ellipsoid \( E(B, y) \), whose volume is proportional to \( \sqrt{\det B} \). We want every point in the ellipsoid to satisfy each constraint \( a_j^T x \leq 1 \), where \( a_j \) is the \( j \)th column of \( A \). This holds as long as \( a_j^T y + \sqrt{a_j^T B a_j} \leq 1 \). So we can write the problem as

\[
\min_{B, y} \quad -\ln \det B \\
\quad a_j^T y + \sqrt{a_j^T B a_j} \leq 1, \quad \text{for all } j, \\
B \text{ positive definite.}
\]
We can eliminate the last constraint as follows: Define the function \( lndet(M) \) on symmetric \( n \times n \) matrices as \( \ln \det(M) \) if \( M \) is positive definite, \( -\infty \) otherwise. Then replace the objective function above by \( -lndet(B) \); this objective function implicitly requires \( B \) to be positive definite. Moreover, we do not have to worry about this non-real-valued function, since if \( \{B_k\} \) is a sequence of positive definite matrices converging to a non-positive definite matrix \( B \), then \( \ln \det(B_k) \) converges to \( -\infty \). It can be shown that \( -lndet \) is a convex function on symmetric matrices.

b) By Cauchy-Schwarz, any \( x \) with \( \|x\| \leq 1/\|a_j\| \) for all \( j \) lies in \( P \), so we can take the ball of radius \( \epsilon := 1/\max\{\|a_j\|\} \) as \( E \), i.e., \( B = \epsilon^2I \), \( y = 0 \) is feasible in the optimization problem above, as is easily checked.

c) If \( y = 0 \), the constraints become \( \sqrt{a_j^TBa_j} \leq 1 \) for all \( j \), which is equivalent to \( a_j^TBa_j \leq 1 \), which is linear in \( B \).

d) If \( y \) is a variable, the problem above is not convex in \( B \) (think of the 1-dimensional case: \( B \) is a number, and \( \sqrt{B} \) is not convex). If we square as above, we get \( a_j^TBa_j \leq (1 - a_j^Ty)^2 \), which is convex in \( B \) but now not in \( y \). But if we use instead the symmetric square root \( D \) of \( B \) as our variable, the objective becomes \( -lndet(D^2) = -2lndet(D) \), and the constraints \( a_j^Ty + \sqrt{a_j^TD^2a_j} = a_j^Ty + \|Da_j\| \leq 1 \) for all \( j \), and this is convex since \( Da_j \) is linear in \( D \) and the norm is convex!

(A paper by Khachiyan and Todd suggests instead solving a sequence of problems with linear constraints on \( B \) and \( y \) instead of this one with nonlinear constraints.)