1. This question and the next are concerned with central cuts. Suppose we have an ellipsoid $E := E(B, y)$, and we add two cuts symmetrically placed with respect to the center $y$. Consider $\bar{E}_\alpha := \{ x \in E : a^T y - \alpha \sqrt{a^T B a} \leq a^T x \leq a^T y + \alpha \sqrt{a^T B a} \}$ for some nonzero $a \in \mathbb{R}^n$ and some $0 \leq \alpha \leq 1$.

a) Write the condition for $x$ to lie in $\bar{E}_\alpha$ as two quadratics.

b) By combining these two quadratics suitably, find an ellipsoid $E(B_+, y_+)$ that contains $\bar{E}_\alpha$, depending on a scalar parameter $\sigma$.

c) Find the value of $\sigma$ that minimizes the volume of the resulting ellipsoid as a function of $\alpha$. Show that for $\alpha = n^{-1/2}$ this ellipsoid is just $E$, while for $\alpha$ smaller than this it gives an ellipsoid of smaller volume than $E$. (In fact, this is the minimum-volume ellipsoid among all those containing $\bar{E}_\alpha$, not just those obtained this way.)

2. Consider a centrally symmetric polytope, a bounded polyhedron of the form $P := \{ x \in \mathbb{R}^n : -b \leq A x \leq b \}$ for some $A$, $b$.

a) Show that there is a minimum-volume ellipsoid $E = E(B, y)$ containing $P$.

b) Show that any such must have $y = 0$, i.e., it must be centrally symmetric also.

c) Show that, if $E(B, 0)$ is a (the) minimum-volume ellipsoid containing $P$, then $\{ n^{-1/2} x : x \in E(B, 0) \}$ is contained in $P$.

(Hence such polytopes can be rounded to a factor $\sqrt{n}$, not $n$ as in the general case. In fact, this holds for any centrally symmetric convex body.)

3. Suppose that $P := \{ x \in \mathbb{R}^n : A^T x \leq e \}$ is bounded (where $e$ is the vector of ones as usual). Assume that the function $B \mapsto -\ln \det B$ is convex as a function of the entries of the symmetric matrix $B$.

a) Show how the problem of finding the maximum volume ellipsoid with center $y$ contained in $P$ can be written as an optimization problem with a finite number of constraints. (Argue that the positive semidefiniteness constraint can be eliminated.)

b) Exhibit a feasible solution to this problem.

c) Show that if the center $y$ is restricted to be 0, your optimization problem can be converted to one with linear constraints on $B$.

d) Now return to the general case, where $y$ is a variable. Try to rewrite the optimization problem with convex constraints (you may want to consider the symmetric square root).