An improved Kalai-Kleitman bound for the diameter of a polyhedron

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Abstract

Kalai and Kleitman [6] established the bound $n \log(d) + 2$ for the diameter of a $d$-dimensional polyhedron with $n$ facets. Here we improve the bound slightly to $(n - d) \log(d)$.

1 Introduction

A $d$-polyhedron $P$ is a $d$-dimensional set in $\mathbb{R}^d$ that is the intersection of a finite number of half-spaces of the form $H := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$. If $P$ can be written as the intersection of $n$ half-spaces $H_i, i = 1, \ldots, n$, but not fewer, we say it has $n$ facets and these facets are the faces $F_i = P \cap H_i, i = 1, \ldots, n$, each affinely isomorphic to a $(d - 1)$-polyhedron with at most $n - 1$ facets. We then call $P$ a $(d, n)$-polyhedron.

We say $v \in P$ is a vertex of $P$ if there is a half-space $H$ with $P \cap H = \{v\}$. (A polyhedron is pointed if it has a vertex, or equivalently, if it contains no line.) Two vertices $v$ and $w$ of $P$ are adjacent (and the set $[v, w] := \{(1 - \lambda)v + \lambda w : 0 \leq \lambda \leq 1\}$ an edge of $P$) if there is a half-space $H$ with $P \cap H = [v, w]$. A path of length $k$ from vertex $v$ to vertex $w$ in $P$ is a sequence $v = v_0, v_1, \ldots, v_k = w$ of vertices with $v_{i-1}$ and $v_i$ adjacent for $i = 1, \ldots, k$. The distance from $v$ to $w$ is the length of the shortest such path and is denoted $\rho_P(v, w)$, and the diameter of $P$ is the largest such distance,

$$\delta(P) := \max\{\rho_P(v, w) : v \text{ and } w \text{ vertices of } P\}.$$

We define

$$\Delta(d, n) := \max\{\delta(P) : P \text{ a } (d, n)\text{-polyhedron}\}$$

and seek an upper bound on $\Delta(d, n)$. It is not hard to see that $\Delta(d, \cdot)$ is monotonically non-decreasing. Also, the maximum above can be attained by a simple polyhedron, one where each vertex lies in exactly $d$ facets. See, e.g., Klee and Kleinschmidt [12] or Ziegler [18]. A related paper, Ziegler [19], gives the history of the Hirsch conjecture that $\Delta_b(d, n) \leq n - d$, where $\Delta_b(d, n)$ is defined as above but for bounded polyhedra.

In the last few years, there has been an explosion of papers related to the diameters of polyhedra and related set systems. Santos [14] found a counterexample to the Hirsch
conjecture, later refined by Matschke, Santos, and Weibel [13]. Eisenbrand, Hähnle, Razborov, and Rothkoss [3] showed that a slightly improved Kalai-Kleitman bound, $n^{\log(d)+1}$, held for a very general class of set families abstracting properties of the vertices of $d$-polyhedra with $n$ facets, which included the ultracconnected set families considered earlier by Kalai [4]. (This improved bound, for polyhedra, was presented first in Kalai [5].) Another class of set families was introduced by Kim [7]; adding various properties gave families for which this bound held, or other families where the maximum diameter grew exponentially. The latter result is due to Santos [15]. Earlier combinatorial abstractions of polytopes include the abstract polytopes of Adler and Dantzig [1] (these satisfy the exponentially. The latter result is due to Santos [15]. Earlier combinatorial abstractions of polytopes include the abstract polytopes of Adler and Dantzig [1] (these satisfy the Hirsch conjecture for $n - d \leq 5$) and the duoids of [16, 17] (these have a lower bound on their diameter growing quadratically with $n - d$). We also mention the nice overview articles of Kim and Santos [8] (pre-counterexample) and De Loera [2] and Santos [15] (post-counterexample).

Our bound $(n-d)^{\log(d)}$ fits better with the Hirsch conjecture and is tight for dimensions 1 and 2. Also, more importantly, it is invariant under linear programming duality. A pointed $d$-polyhedron with $n$ facets can be written as $\{x \in \mathbb{R}^d : Ax \leq b\}$ for some $n \times d$ matrix $A$ of full rank and some $n$-vector $b$. Choosing an objective function $c^T x$ for $c \in \mathbb{R}^d$ gives the linear programming problem $\max \{c^T x : Ax \leq b\}$, whose dual is $\min \{b^T y : A^T y = c, y \geq 0\}$. The feasible region for the latter is affinely isomorphic to a pointed polyhedron of dimension at most $n-d$ with at most $n$ facets, and equality is possible. Hence duality switches the dimensions $d$ and $n-d$.

## 2 Result

We prove

**Theorem 1** For $1 \leq d \leq n$, $\Delta(d, n) \leq (n-d)^{\log(d)}$, with $\Delta(1, 1) = 0$.

(All logarithms are to base 2; note that $(n-d)^{\log(d)} = d^{\log(n-d)}$ as both have logarithm log($d$)·log($n-d$). We use this in the proof below.)

The key lemma is due to Kalai and Kleitman [6], and was used by them to prove the bound $n^{\log(d)+2}$. We give the proof for completeness.

**Lemma 1** For $2 \leq d \leq \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the largest integer at most $n/2$,

$$\Delta(d, n) \leq \Delta(d-1, n-1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2.$$  

**Proof:** Let $P$ be a simple $(d, n)$-polyhedron and $v$ and $w$ two vertices of $P$ with $\delta_P(v, w) = \Delta(d, n)$. We show there is a path in $P$ from $v$ to $w$ of length at most the right-hand side above. If $v$ and $w$ both lie on the same facet, say $F$, of $P$, then since $F$ is affinely isomorphic to a $(d-1, m)$-polyhedron with $m \leq n-1$, we have $\rho_P(v, w) \leq \rho_F(v, w) \leq \Delta(d-1, m) \leq \Delta(d-1, n - 1)$ and we are done.

Otherwise, let $k_v$ be the largest $k$ so that there is a set $\mathcal{F}_v$ of at most $\lfloor n/2 \rfloor$ facets with all paths of length $k$ from $v$ meeting only facets in $\mathcal{F}_v$. This exists since all paths of length 0 meet only $d$ facets (those containing $v$), whereas paths of length $\delta(P)$ can meet all $n$ facets of $P$. Define $k_w$ and $\mathcal{F}_w$ similarly. We claim that $k_v \leq \Delta(d, \lfloor n/2 \rfloor)$ and similarly for $k_w$. Indeed, let $P_v \supseteq P$ be the $(d, m_v)$-polyhedron ($m_v = |\mathcal{F}_v| \leq \lfloor n/2 \rfloor$) defined by just those linear inequalities corresponding to the facets in $\mathcal{F}_v$. Consider any vertex $t$ of
\( P \) a distance \( k_v \) from \( v \), so there is a shortest path from \( v \) to \( t \) of length \( k_v \) meeting only facets in \( F_v \). But this is also a shortest path in \( P_v \), since if there were a shorter path, it could not be a path in \( P \), and thus must meet a facet not in \( F_v \), a contradiction. So

\[
k_v = \delta_{P_v}(v, t) \leq \Delta(d, m_v) \leq \Delta(d, \lfloor n/2 \rfloor).
\]

Now consider the set \( G_v \) of facets that can be reached in at most \( k_v + 1 \) steps from \( v \), and similarly \( G_w \). Since both these sets contain more than \( \lfloor n/2 \rfloor \) facets, there must be a facet, say \( G \), in both of them. Thus there are vertices \( t \) and \( u \) in \( G \) and paths of length at most \( k_v + 1 \) from \( v \) to \( t \) and of length at most \( k_w + 1 \) from \( v \) to \( u \). Then

\[
\Delta(d, n) = \rho_P(v, w) \\
\leq \rho_P(v, t) + \rho_G(t, u) + \rho_P(w, u) \\
\leq k_v + 1 + \Delta(d - 1, n - 1) + k_w + 1 \\
\leq \Delta(d - 1, n - 1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2,
\]

since, as above, \( G \) is affinely isomorphic to a \((d - 1, m)\)-polyhedron with \( m \leq n - 1 \). \( \square \)

**Proof of the theorem:** This is by induction on \( d + n \). The result is trivial for \( n = d \), since there can be only one vertex. Next, the right-hand side gives 1 for \( d = 1 \) \((n = 2)\) and \( n - 2 \) for \( d = 2 \), which are the correct values. For \( d = 3 \), it gives \((n - 3)\log(3)\), which is greater than the correct value \( n - 3 \) established by Klee [9, 10, 11]. (We could make the proof more self-contained by establishing the \( d = 3 \) case from the lemma: a general argument deals with \( n \geq 13 \), but then there are seven more special cases to check.) Below we will give a general inductive step for the case \( d \geq 4 \), \( n - d \geq 8 \). Also, the result clearly holds by induction if \( n < 2d \), since then any two vertices lie on a common facet, so their distance is at most \( \Delta(d - 1, n - 1) \). The remaining cases are \( d = 4, 8 \leq n \leq 11; d = 5, 10 \leq n \leq 12; d = 6, 12 \leq n \leq 13; \) and \( d = 7 \), \( n = 14 \). All these cases can be checked easily using the lemma, the equation \( \Delta(d, d) = 0 \), and the equations \( \Delta(5, 6) = \Delta(4, 5) = \Delta(3, 4) = \Delta(2, 3) = 1 \).

Now we deal with the case \( d \geq 4, n - d \geq 8 \). For this, \( \log(n - d) \geq 3 \), so we have

\[
\Delta(d, n) \leq \Delta(d - 1, n - 1) + 2 \cdot \Delta(d, \lfloor n/2 \rfloor) + 2
\]

\[
\leq (d - 1)^{\log(n-d)} + 2 \cdot d^{\log(n/2-d)} + 2
\]

\[
\leq \left( \frac{d - 1}{d} \right)^{\log(n-d)} d^{\log(n-d)} + 2 \cdot d^{\log((n-d)/2)} + 2
\]

\[
\leq \left( \frac{d - 1}{d} \right)^{3} d^{\log(n-d)} + 2 \cdot d^{\log(n-d)} + 2
\]

\[
= \left( 1 - \frac{3}{d} \right)^{3} + \left( 1 - \frac{1}{d^2} \right) d^{\log(n-d)} + 2
\]

\[
\leq \left( 1 - \frac{1}{d} \right)^{\log(n-d)} - \frac{1}{4d} \cdot d^{\log(n-d)} - \frac{1}{d^3} \cdot d^{\log(n-d)} + 2
\]

\[
\leq d^{\log(n-d)}
\]

since each of the subtracted terms is at least one. This completes the proof. \( \square \)

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References


