Computing a Cournot Equilibrium in Integers

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December 6, 2013

Abstract

We give an efficient algorithm for computing a Cournot equilibrium when the producers are confined to integers, the inverse demand function is linear, and costs are quadratic. The method also establishes existence, which follows in much more generality because the problem can be modelled as a potential game.

1 Introduction

Consider $n$ producers who produce a single good. If each producer $i$ produces $q_i$, we assume the price at which the good can be sold is

$$p = (a - bQ)_+, \text{ where } Q := \sum_{i=1}^{n} q_i,$$

here and below, denotes the total production, $a$ and $b$ are positive, and $z_+$ denotes the positive part $\max\{0, z\}$ of $z$. Producer $i$ also faces a production cost of $c_i q_i + d_i q_i^2$, with $c_i$ and $d_i$ nonnegative with positive sum. Then if the production vector is $q = (q_i)$, producer $i$ makes profit

$$\pi_i(q) := (a - bQ)_+ q_i - c_i q_i - d_i q_i^2.$$

With a slight abuse of notation, we also write $\pi_i(q)$ as $\pi_i(q_i, q_{-i})$ or $\pi_i(q_i, Q_{-i})$, where $q_{-i} = (q_j)_{j \neq i}$ and $Q_{-i} := \sum_{j \neq i} q_j$, to highlight its dependence on producer $i$'s decision as well as those of the other producers. Note that

$$\pi_i(q_i, Q_{-i}) = (a - bQ_{-i})_+ q_i - c_i q_i - d_i q_i^2.$$

Our main interest is in the case that each producer chooses her decision $q_i$ from the nonnegative integers, $\mathbb{Z}_+$, but we also consider the simpler case where $q_i$ is chosen from the nonnegative reals, $\mathbb{R}_+$. In either case, a Cournot equilibrium is a production profile where each producer $i$ chooses her best response to the decisions of the other producers.

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Definition 1 $q \in \mathbb{R}_+^n$ ($q \in \mathbb{Z}_+^n$) is a real (integer) Cournot equilibrium for $\{\pi_i\}$ if for each $i$ $q_i$ maximizes $\pi_i(\cdot, Q_{-i})$ over $\mathbb{R}_+(\mathbb{Z}_+)$.  

Cournot equilibria, with the related Bertrand equilibria (where the producers choose prices, not quantities) and Stackelberg (leader-follower) variants of these, form a standard topic in oligopoly theory: see, for example Mas-Colell, Whinston, and Green [4], pp. 389–394, or Varian [11], pp. 285–291. Kreps and Scheinkman [2] show how a Cournot equilibrium arises from a two-stage game: first the producers choose capacities, and then, knowing these capacities, each chooses a price as in the Bertrand model. Integer or discrete versions are much less studied, although existence is proved in for example Kukushkin [3] using Tarski’s fixed-point theorem, and Dubey, Haimanko, and Zapechelnyuk [1] using a potential game, following a less general treatment of Shapley [7].

This paper is concerned with the efficient computation of real and integer Cournot equilibria, and as a byproduct, their existence, in the case of a linear inverse demand function and convex quadratic costs as above. While this case is quite restrictive, we note that it is prototypical, and that our technique can be used in an iterative scheme where at each step, the inverse demand function and the cost functions are approximated by a linear function, respectively quadratic functions, at the current iterates.

In the next section we recall the proof of the existence of integer Cournot equilibria using a standard potential function argument, which is non-constructive. Then we show how they can be computed (and incidentally shown to exist) using an efficient direct or binary search on the total production $Q$. Computation of Cournot equilibria in much more general settings is discussed for example in Sherali, Soyster, and Murphy [8] and Thorlund-Petersen [10].

2 Existence

Here we prove existence of both real and integer Cournot equilibria using a potential function approach. This replaces the multi-player criterion for an equilibrium with a single objective whose maximizers yield equilibria; games which permit such a substitution were introduced by Monderer and Shapley [5] and termed potential games.

First we simplify the payoff functions. While negative prices have no interpretation in our setting, we can define payoff functions as if the price were given by $a - bQ$ instead of its positive part. Consider (again with a slight abuse of notation)

$$\bar{\pi}_i(q_i, q_{-i}) := (a - bq_i - bQ_{-i})q_i - cq_i - dq_i^2.$$

Proposition 1 The real (integer) Cournot equilibria for $\{\pi_i\}$ coincide with those for $\{\bar{\pi}_i\}$.

Proof: Note that in any real or integer Cournot equilibrium $q$ for either set of payoff functions, every producer gets a nonnegative payoff, since producing nothing is always an alternative. But then we cannot have
$a - bQ$ nonpositive, for then any producer with positive $q_i$ (and there must be one) gets negative payoff. Since the two payoff functions only differ for $a - bQ$ negative, and no producer has an incentive to move to such a point, an equilibrium for one set of payoff functions is also one for the other. □

We can now use the payoff functions $\{\bar{\pi}_i\}$ in our equilibrium arguments. Consider the potential function

$$f(q) := (a - \frac{1}{2}bQ)Q - \sum_{i=1}^{n}(c_i q_i + (d_i + \frac{1}{2}b)q_i^2).$$

Viewed as a function of $q_i$ and $q_{-i}$, this can alternatively be written as

$$f(q_i, q_{-i}) = a q_i + aQ_{-i} - \frac{1}{2}b(q_i + Q_{-i})^2 - c_i q_i - (d_i + \frac{1}{2}b)q_i^2$$

$$- \sum_{j \neq i}(c_j q_j + (d_j + \frac{1}{2}b)q_j^2)$$

$$= a q_i - b(q_i + Q_{-i})q_i - c_i q_i - d_i q_i^2$$

$$+ aQ_{-i} - \frac{1}{2}bQ_{-i}^2 - \sum_{j \neq i}(c_j q_j + (d_j + \frac{1}{2}b)q_j^2)$$

$$= \bar{\pi}_i(q_i, Q_{-i}) + f_{-i}(q_{-i}),$$

where (as the notation indicates) $f_{-i}$ is independent of $q_i$.

This gives the existence of Cournot equilibria. The proof below is due to Monderer and Shapley [5] in the real case, and Shapley [7] for integers. It seems worthwhile to give it for the sake of completeness.

**Theorem 1** Any maximizer of $f$ over $\mathbb{R}_+^n$ ($\mathbb{Z}_+^n$) is a real (integer) Cournot equilibrium for $\{\bar{\pi}_i\}$, and hence such equilibria exist.

**Proof:** Indeed, any maximizer $q$ of $f$ has the property that $q_i$ is a maximizer of $\bar{\pi}_i(q_i, Q_{-i})$ by the equation above, and so provides a Cournot equilibrium for $\{\bar{\pi}_i\}$ and hence for $\{\pi_i\}$ by Proposition 1. This holds over both the nonnegative reals and the nonnegative integers, and proves the first part. For the existence, we note $f(0) = 0$, and $f(q) < 0$ if $Q > 2a/b$, whence we can confine our search for the maximizer of the continuous function $f$ to the compact set of nonnegative reals or integers with $Q \leq 2a/b$, so that existence of a maximizer is assured. □

The existence proof above is basically non-constructive (although there are algorithms to maximize the strictly concave quadratic function $f$ over the nonnegative orthant, and one could enumerate the exponentially large set of points in $\{q \in \mathbb{Z}_+^n : Q \leq 2a/b\}$, and so we develop efficient algorithms to compute a Cournot equilibrium (and incidentally prove its existence) in the next section.

### 3 Computation

The strictly concave quadratic

$$\bar{\pi}_i(q_i, Q_{-i}) = (a - c_i - bQ_{-i})q_i - (b + d_i)q_i^2$$

+}^{n}(c_i q_i + (d_i + \frac{1}{2}b)q_i^2).$$

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$$- \sum_{j \neq i}(c_j q_j + (d_j + \frac{1}{2}b)q_j^2)$$

$$= a q_i - b(q_i + Q_{-i})q_i - c_i q_i - d_i q_i^2$$

$$+ aQ_{-i} - \frac{1}{2}bQ_{-i}^2 - \sum_{j \neq i}(c_j q_j + (d_j + \frac{1}{2}b)q_j^2)$$

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for any nonnegative $Q_{-i}$ is maximized in $q_i$ over the reals uniquely at
\[
\frac{a - c_i - bQ_{-i}}{2(b + d_i)}.
\]
and over the nonnegative reals at
\[
q_i^* := \frac{(a - c_i - bQ_{-i})_+}{2(b + d_i)}.
\]
(1)
as is easily seen from the first-order conditions or directly.

Since quadratics are symmetric about their maximizers, the maximum
over the integers is attained at any integer in $[\hat{q}_i, \check{q}_i]$, where
\[
\begin{align*}
\hat{q}_i &:= \left( \frac{a - c_i - bQ_{-i}}{2(b + d_i)} \right)_+ + \frac{1}{2}, \\
\check{q}_i &:= \left( \frac{a - c_i - bQ_{-i}}{2(b + d_i)} \right)_+ - \frac{1}{2}.
\end{align*}
\]
(2)

In both cases above, the optimal value or bounds for $q_i$ are given in
terms of $Q_{-i}$, but it would be much more useful if they were defined (in
a somewhat circular way) as functions of the total production $Q$. Such
functions were first considered by Szidarovszky and Yakowitz [9] (see also
Novshek [6], who called them “backward reaction mappings”). For this
reformulation the following result is helpful:

**Lemma 1** For real $\lambda$ and $0 < \mu < 1$,
\[
x R (\lambda - \mu y)_+ \iff x R \left( \frac{\lambda}{1 - \mu} - \frac{\mu}{1 - \mu} (y + x) \right)_+,
\]
where $R$ stands for less than or equal to, greater than or equal to, or equal
to.

**Proof:** We prove the lemma when $R$ is “$\leq$” and “$\geq$”; the third case
then follows. For the first case, the left-hand side is true iff $x \leq 0$ or
$0 < x \leq \lambda - \mu y$, while the right hand-side is true iff
\[
x \leq 0 \text{ or } 0 < x \leq \frac{\lambda}{1 - \mu} - \frac{\mu}{1 - \mu} (y + x).
\]
It is easy to see that these are equivalent.

For the second case, the left-hand side is true iff $x \geq 0$ and $x \geq \lambda - \mu y$,
while the right hand-side is true iff
\[
x \geq 0 \text{ and } x \geq \frac{\lambda}{1 - \mu} - \frac{\mu}{1 - \mu} (y + x),
\]
and once again these are easily seen to be equivalent. $\Box$

We now apply the lemma to the conditions for optimal $q_i$’s above. It
is convenient to define
\[
e_i := a - c_i
\]
for each $i$. We may also assume that these are all positive, since any
producer with a nonpositive $e_i$ will choose to produce nothing no matter
what the others do.
Theorem 2  a) A production profile \( q \in \mathbb{R}^n \) is a real Cournot equilibrium iff
\[
q_i = q^*_i(Q) := \frac{(e_i - bQ)_+}{b + 2d_i}
\]
for each \( i \).

b) A production profile \( q \in \mathbb{Z}^n \) is an integer Cournot equilibrium iff
\[
q_i \in [\hat{q}_i(Q), \check{q}_i(Q)],
\]
where
\[
\hat{q}_i(Q) := \frac{(e_i - b - d_i - bQ)_+}{b + 2d_i},
\]
\[
\check{q}_i(Q) := \frac{(e_i + b + d_i - bQ)_+}{b + 2d_i}
\]
for each \( i \).

Proof: For part (a), we apply the lemma to (1) with \( R = \cdot \) and \( \lambda = e_i/[2(b + d_i)], \mu = b/[2(b + d_i)] \).

For part (b), we apply the lemma to (2) with \( R \geq \cdot \) and \( \lambda = [e_i - b - d_i]/[2(b + d_i)], \mu = b/[2(b + d_i)], \) and with \( R \leq \cdot \) and \( \lambda = [e_i + b + d_i]/[2(b + d_i)], \mu = b/[2(b + d_i)] \). \( \square \)

We can now investigate efficient algorithms to compute Cournot equilibria. In the real case, it suffices to find \( Q \in \mathbb{R}_+ \) satisfying
\[
Q = \phi(Q) := \sum_i \frac{(e_i - bQ)_+}{b + 2d_i}.
\]
This provides an alternative proof of existence, since \( \phi \) is continuous and nonincreasing, positive at zero, and zero for large enough \( Q \), and hence has a fixed point by the intermediate-value theorem. It seems that to find the fixed point we need to do a binary search on \( Q \) to determine for which indices the numerator is positive, but if the \( e_i \)'s are sorted (which takes \( O(n \log n) \) time), this can be avoided by computing the desired sums incrementally. Let us assume the \( e_i \)'s are in nonincreasing order. Then for some \( 1 \leq j \leq n \),
\[
Q = \frac{\sum_{i=1}^j e_i - bQ}{b + 2d_i} = \frac{\sum_{i=1}^j e_i/(b + 2d_i)}{1 + \sum_{i=1}^j b/(b + 2d_i)}
\]
and \( e_j/b \geq Q \geq e_{j+1}/b \), where we take \( e_{n+1} := 0 \). This gives our algorithm, where we successively compute the numerator \( (\nu) \) and the denominator \( (\delta) \), terminating when the ratio exceeds \( e_{j+1}/b \).

Algorithm 1 (Computing a real Cournot equilibrium)

Initialize \( \nu = 0 \) and \( \delta = 1 \).

For \( j = 1, 2, \ldots, n \),
\[
\nu \leftarrow \nu + \frac{e_j}{b + 2d_j}, \quad \delta \leftarrow \delta + \frac{b}{b + 2d_j}.
\]
If \( \frac{\nu}{\delta} \geq \frac{e_{j+1}}{b} \), set \( Q = \frac{\nu}{\delta} \) and \( q_i = q^*_i(Q), i = 1, \ldots, n \), and STOP.

Next \( j \).
It is easy to see inductively that $\nu/\delta \leq e_j/b$. Indeed, the previous $\nu/\delta$ was less than $e_j/b$ either because it is initially zero or because the test at the previous $j$ failed, and the current value is a convex combination of the old value and $e_j/b$. Hence Algorithm 1 finds a real Cournot equilibrium in $O(n)$ time (or $O(n \log n)$ time if the $e_i$’s need to be sorted).

The situation is more complicated in the integer case. We can proceed as in the real case to obtain bounds on $Q$ for any integer Cournot equilibrium. Indeed, if $(e|bd[\bar{n}]| - j|)/b$ denotes the components of $(e_j - b - d_j)$ arranged in nonincreasing order, and $(d|_{(i)})$ denotes the components of $(d_j)$ in the same order, then for some $j = 1, \ldots, n$,

$$Q \geq \sum_{i=1}^{j} \frac{e|bd[\bar{n}]| - j|}{b + 2d_{(i)}}, \quad \text{so} \quad Q \geq \sum_{i=1}^{j} \frac{e|bd[\bar{n}]| - j|}{b + 2d_{(i)}}$$

where $e|bd[\bar{n}]| - j|/b \geq Q \geq e|bd[\bar{n}]| - j|+1|/b$ and again we take $e|bd[\bar{n}]| - j|+1| := 0$. Similarly, if $(e|bd[\bar{n}]| + j|)/b$ denotes the components of $(e_j + b + d_j)$ arranged in nonincreasing order, and $(d|_{(i)})$ denotes the components of $(d_j)$ in the same order, then for some $j = 1, \ldots, n$,

$$Q \leq \sum_{i=1}^{j} \frac{e|bd[\bar{n}]| + j|}{b + 2d_{(i)}}, \quad \text{so} \quad Q \leq \sum_{i=1}^{j} \frac{e|bd[\bar{n}]| + j|}{b + 2d_{(i)}}$$

where $e|bd[\bar{n}]| + j|/b \geq Q \geq e|bd[\bar{n}]| + j|+1|/b$ with $e|bd[\bar{n}]| + j|+1| := 0$.

The difficulties are several. First, these conditions are necessary but not sufficient, since for ease of analysis we have neglected the requirement that each $q_i$ be integer. Second, while both $\sum_1^j [\hat{q}_i(Q)]$ and $\sum_1^n [\hat{q}_i(Q)]$ are nonincreasing in $Q$, and it is not hard to see that this range always includes an integer (see the proof of Theorem 4 below), it is a priori possible that the range jumps over $Q$ as it moves from one integer value to the next. Hence existence does not follow directly.

We deal with these problems individually and sequentially. We use the necessary bounds above to restrict the range of possible $Q$’s. Then we perform a binary search over this range, using the correct bounds with integer parts for the individual $q_i$’s to find the equilibrium, and relying on our earlier existence proof. Finally, we provide a direct proof of existence based on integer changes in $Q$.

The algorithm below provides lower and upper bounds on $Q$ in any integer Cournot equilibrium:

**Algorithm 2 (Computing bounds for an integer Cournot equilibrium)**

Initialize $\nu = 0$ and $\delta = 1$.

For $j = 1, 2, \ldots, n$,

$$\nu \leftarrow \nu + \frac{e|bd[\bar{n}]|}{b + 2d_{(j)}}, \quad \delta \leftarrow \delta + \frac{b}{b + 2d_{(j)}}$$
If \( \nu \geq \frac{ebd_{j+1}}{b} \), set \( \hat{Q} = \lceil \frac{\nu}{\delta} \rceil \) and BREAK.

Next \( j \).

Initialize \( \nu = 0 \) and \( \delta = 1 \).

For \( j = 1, 2, \ldots, n \),

\[
\nu \leftarrow \nu + \frac{ebd_j}{b-2d_j}, \quad \delta \leftarrow \delta + \frac{b}{b-2d_j}.
\]

If \( \nu \geq \frac{ebd_{j+1}}{b} \), set \( \hat{Q} = \lfloor \frac{\nu}{\delta} \rfloor \) and STOP.

Next \( j \).

This algorithm requires some justification. Consider first the lower bound \( \hat{Q} \). While the test fails, the bound in (3) is weaker than the bounds on \( Q \) (between \( ebd_{j+1}/b \) and \( ebd_j/b \)). However, when it first applies, it gives a bound above the lower of these numbers. But why can we ignore higher values of \( j \)? Note that, as above, the ratio \( \nu/\delta \) is updated as a convex combination of the old ratio and \( ebd_j/b \), so the bound for the next \( j \) is either the same or outside the appropriate range, and similarly for higher \( j \)'s. Next consider the upper bound \( \hat{Q} \). While the test fails, the bound in (4) contradicts the bounds on \( Q \), so the current range is impossible. When the test holds, the bound lies within the appropriate range (using again the convex combination property), so provides a valid upper bound if \( Q \) is in this range. For subsequent \( j \)'s, the range for \( Q \) is lower, thus showing that \( \hat{Q} \) is a valid upper bound for all ranges. Hence Algorithm 2 provides correct bounds.

We now provide a simple binary search to obtain an integer Cournot equilibrium based on Theorems 1 and 2.

**Algorithm 3** (Computing an integer Cournot equilibrium)

Use Algorithm 2 to find \( \hat{Q} \) and \( \check{Q} \).

Set \( l := \hat{Q}, \ a := \check{Q} \).

While \( l - u > 0 \),

Set \( Q = \lfloor (l + u)/2 \rfloor \).

If \( \sum_i \hat{q}_i(Q) > Q \),

\( l := Q + 1 \);

Else if \( \sum_i \hat{q}_i(Q) < Q \),

\( u := Q - 1 \);

Else \( l := Q \) and BREAK.

Endwhile

\( Q := l \), set \( q_i \) between \( \lceil \hat{q}_i(Q) \rceil \) and \( \lfloor \check{q}_i(Q) \rfloor \) so that \( \sum_i q_i = Q \).

**Theorem 3** Algorithm 3 correctly obtains an integer Cournot equilibrium in \( O(n \log(na/b)) \) time.

**Proof:** By Theorem 1, there exists an integer Cournot equilibrium \( q \), and with \( Q := \sum_i q_i \), we then have \( \sum_i \hat{q}_i(Q) \leq Q \leq \sum_i \check{q}_i(Q) \). Since these two sums are nonincreasing functions of \( Q \), it follows that the binary search will successfully find such a \( Q \) given that the initial bounds are valid. But this was established below Algorithm 2.

For the time complexity, we require \( O(n \log n) \) time to order the \( \epsilon_i - b - d_i \)'s and \( \epsilon_i + b + d_i \)'s, and then \( O(n) \) time for Algorithm 2. Since \( 0 \leq Q \leq \hat{Q} \leq a/b \), we need \( O(\log(a/b)) \) steps in the binary search, each requiring \( O(n) \) time. \( \square \)
Before presenting examples, we show directly using a method similar to Algorithm 3 that an integer Cournot equilibrium exists. For this, we show that iteratively trying \( Q = 0, 1, \ldots \) will give a value for which the necessary and sufficient conditions of Theorem 2 are satisfied.

**Theorem 4** For some nonnegative integer \( Q \),
\[
\sum_i [\hat{q}_i(Q)] \leq Q \leq \sum_i [\hat{q}_i(Q)],
\]
implying that an integer Cournot equilibrium exists.

**Proof:** Note first that both \( [\hat{q}_i(Q)] \) and \( [\hat{q}_i(Q)] \) are nonincreasing in \( Q \), and also that there is always an integer between them. This is trivial if \( [\hat{q}_i(Q)] = 0 \), and if not, \( \hat{q}_i(Q) = \hat{q}_i(Q) + 2(b + d_i)/(b + 2d_i) > \hat{q}_i(Q) + 1 \), from which the claim follows.

Next, for \( Q = 0 \), \( P_i [\hat{q}_i(Q)] \) is either zero, in which case this \( Q \) satisfies our inequalities, or positive. In the latter case, we know that \( P_i [\hat{q}_i(Q)] \) is zero for \( Q \geq a/b \). Hence there is a first \( Q \) for which
\[
X_i [\hat{q}_i(Q)] \leq Q,
\]
so that
\[
\sum_i [\hat{q}_i(Q)] \leq Q, \quad (5)
\]
Now for any \( i \), if \( \hat{q}_i(Q - 1) \) is positive,
\[
\hat{q}_i(Q - 1) + 1 = \frac{e_i - b - d_i - b(Q - 1)}{b + 2d_i} + 1 = \frac{e_i + b + d_i - bQ}{b + 2d_i} = \hat{q}_i(Q),
\]
and so \( [\hat{q}_i(Q - 1)] \leq \hat{q}_i(Q) - 1 = \hat{q}_i(Q) \), whence \( [\hat{q}_i(Q - 1)] \leq \hat{q}_i(Q) \). On the other hand, if \( \hat{q}_i(Q - 1) \) is zero, then this inequality holds trivially. Hence
\[
\sum_i [\hat{q}_i(Q)] \geq \sum_i [\hat{q}_i(Q - 1)] \geq Q. \quad (6)
\]
But now (5) and (6) imply that the inequalities in the theorem are satisfied, and hence an integer Cournot equilibrium \( q \) exists. Indeed, we can start with \( q_i = [\hat{q}_i(Q)] \) for all \( i \) and then increase \( q_i \)'s within their range until a sum of \( Q \) is achieved.

\( \Box \)

**Example 1** Let us illustrate our algorithms with a small example. Suppose \( a = 5 \), \( b = 1 \), and there are ten producers with costs given by
\[
c = (0, 0, 0, 1, 1, 1, 1, 2, 2, 2), \quad d = (1, 1, 1, 1, 1, 1, 0, 0, 0).
\]

Then
\[
e = (5, 5, 5, 4, 4, 4, 3, 3, 3),
\]

Note that these are in nonincreasing order. Also,
\[
(b + 2d_i) = (3, 3, 3, 3, 3, 3, 3, 1, 1, 1).
\]
For the real case, Algorithm 1 increases \( j \) to 7, and then \( \nu = 31/3 \) and \( \delta = 10/3 \), so \( \nu/\delta = 31/10 > \varepsilon_* \), so the loop ends. Thus \( Q^* = 31/10 \), and we have found the real Cournot equilibrium

\[
q = q^*(Q) = \left( \frac{19}{30}, \frac{19}{30}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, 0, 0, 0 \right).
\]

It is easy to check that this is indeed an equilibrium, either from Theorem 2 or directly from (1).

In the integer case, we first compute

\[
(e_i - b - d_i) = (3, 3, 3, 2, 2, 2, 2, 2)
\]

and

\[
(e_i + b + d_i) = (7, 7, 7, 6, 6, 6, 4, 4).
\]

Note that these are both in nonincreasing order. We first apply Algorithm 2. For the lower bound, \( j \) increases to 10, and then \( \nu = 35/3 \) and \( \delta = 19/3 \), so \( Q = \lfloor 35/19 \rfloor = 2 \). For the upper bound, \( j \) increases to 7, with \( \nu = 45/3 \) and \( \delta = 10/3 \), so \( \nu/\delta = 9/2 > \varepsilon_* \). So the loop ends, and \( Q = \lfloor 9/2 \rfloor = 4 \).

Now we do binary search. The first trial \( Q \) is 3. We find

\[
\hat{q}(Q) = (0, 0, 0, 0, 0, 0, 0, 0), \quad \hat{q}(Q) = \left( \frac{4}{3}, \frac{4}{3}, 1, 1, 1, 1, 1, 1 \right).
\]

So the lower bounds are all zero, and the upper bounds all 1. It follows that any \((0,1)\) vector with three ones is an integer Cournot equilibrium. In particular, \((0,0,0,0,0,0,1,1,1)\) is an integer equilibrium, whereas the last three producers always produce zero in the (unique) real Cournot equilibrium. Again, it is easy to confirm that this is indeed an integer Cournot equilibrium, either from Theorem 2 or directly from (2).

We could also try \( Q \) equal to 2 or 4, but in both cases the inequalities in Theorem 4 fail, so these do not lead to equilibria.

To show the value of obtaining bounds in Algorithm 2, note that if we keep the data above except that we change \( a \) to 50, then the algorithm gives bounds of 40 and 42, so that only at most two values of \( Q \) need to be checked (and 41 and 42 both turn out to yield equilibria, whereas 40 does not); by contrast, a binary search on the range \([0,50]\) would require up to six tests.

A last point illustrated by this example is the following: since the pay-off functions are strictly concave quadratics, one might expect that the output chosen by a producer in a Cournot equilibrium would either be a unique integer, or perhaps one of two consecutive integers. However, in this example, for \( Q = 41 \), we obtain

\[
\hat{q}(Q) = (3, 3, 3, 2, 2, 2, 6, 6), \quad \hat{q}(Q) = (3, 3, 3, 3, 3, 3, 8, 8).
\]

Hence there are equilibria where the last three producers choose either 6, or 7, or 8, for example for producer 8, \( q = (3, 3, 3, 3, 3, 3, 6, 8, 6) \), or \( q = (3, 3, 3, 3, 3, 3, 7, 7, 6) \), or \( q = (3, 3, 3, 3, 3, 3, 8, 6, 6) \).
References


