SDP Relaxation of General Integer Programming Problems

Consider $P_I = \text{conv}\{x : Ax \geq b, x \in \{0,1\}^n\}$ and its LP relaxation $P = \{x \in \mathbb{R}^n : Ax \geq b, 0 \leq x \leq e\}$. Then $P_I$ is contained in $P$, but this relaxation might not be very good.

We aim to get a tighter relaxation of $P_I$ by projecting from a higher-dimensional convex set. This idea is called Lift and Project. The method we describe below is based on the work of Lovász and Schrijver. There are also related work by Sherali-Adams and Balas-Ceria-Cornuejols. See Goeman’s survey paper for details.

Note: As a simple example to illustrate the idea of lift-and-project: $\{x \in \mathbb{R}^n : ||x||_1 \leq 1\}$ has dimension $n$, but $2^n$ linear inequalities are required to represent the given constraint, one for each possible combination of ‘+’ and ‘-’ for each component of $x$. However, consider $x = y - z$ where $y$ and $z$ are nonnegative and satisfy $e^T y + e^T z \leq 1$. This representation has $2n$ dimensions, but only $(2n + 1)$ linear inequalities are required to represent the vector norm constraint.

We will now consider formulations that lift to $SR_{+}^{(1+n)(1+n)}$. Suppose $c^T x - \delta \geq 0$ is a valid inequality; then we can multiply this by $x_j \geq 0$ or by $1 - x_j \geq 0$ to get a quadratic inequality which can be written as a linear inequality in

$$
\begin{pmatrix}
1 \\
x
\end{pmatrix}^T
\begin{pmatrix}
1 \\
x
\end{pmatrix} =
\begin{pmatrix}
1 & x^T \\
x & xx^T
\end{pmatrix}.
$$

Let $M(P)$ denote the set of all $(1+n) \times (1+n)$ symmetric matrices satisfying all these inequalities together with the constraints:

$$Xe_0 = \text{diag}(X), \quad x_{00} = 1,$$

where $e_0 = (1; 0; \ldots; 0)$. This comes from $x_j^2 = x_j$ if $x_j \in \{0,1\}$.

**Definition 1** $N(P) := \{x \in \mathbb{R}^n : \begin{pmatrix}1 \\ x\end{pmatrix} = Xe_0, X \in M(P)\}$.

It follows that $P_I \subseteq N(P) \subseteq P$.

**Definition 2** $N^1(P) := N(P)$ and $N^j(P) := N(N^{j-1}(P))$, for $j > 1$.

**Theorem 1** (Lovász - Schrijver) $N^n(P) = P_I$.

Unfortunately $N^n(P)$ is exponentially large.

$N(P)$ is just defined by linear inequalities. How does this relate to SDP?
**Definition 3** \( N_+(P) := \{ x \in \mathbb{R}^n : (1)_x \} = X_{e_0}, X \in M(P), X \succeq 0 \}.

It follows that \( N_+(P) \subseteq N(P) \), and \( N^+_n(P) = P_I \).

For large problems doing this operation once is hard enough, but can give surprisingly tight relaxations. We now move on to the last application that we will describe in this class.

**Global Minimization of Polynomials (Lasserre and Parrilo)**

**Notation** For polynomials in \( z \in \mathbb{R}^n \), let \( \alpha = (\alpha_1; \alpha_2; \ldots; \alpha_n) \in Z^n_+ \) be the exponent of \( z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}, \quad |\alpha| := \sum_{j=1}^n \alpha_j \).

Let \( p(z) := \sum_{|\alpha| \leq d} p_\alpha z^\alpha \) be a polynomial of total degree \( d \).

If we write a summation over \( |\alpha| \leq d \), we implicitly assume \( \alpha \in Z^n_+ \). Also we won’t distinguish between \( p(.) \) and \( p = (p_\alpha)_{|\alpha| \leq d} \).

Our goal is to find (or approximate) the global minimizer of \( p(z) \), where \( z \in \mathbb{R}^n \) and \( p \) has total degree \( 2m \) (we exclude odd degree polynomials as they do not have global minimizers)

\[
\min_{z \in \mathbb{R}^n} p(z) = \sum_{|\alpha| \leq 2m} p_\alpha z^\alpha.
\]

Even for quartics (\( m = 2 \)) this is NP-hard. This is not a convex problem, but we will describe equivalent problems that are linear (but infinite-dimensional) and convex (and finite-dimensional). We will describe Lasserre’s treatment below.

First, instead of minimizing over \( z \), we minimize the expected value over all probability measure on \( \mathbb{R}^n \):

\[
\min \{ \int p(z) d\mu(z) : \mu \text{ is a probability measure on } \mathbb{R}^n \}.
\]

This is the linear infinite-dimensional formulation.

Next note that:

\[
\int p(z) d\mu(z) = \int (\sum_{\alpha} p_\alpha z^\alpha) d\mu(z) = \sum_{|\alpha| \leq 2m} p_\alpha y_\alpha
\]

where \( y_\alpha := \int z^\alpha d\mu(z) \), the \( \alpha \)-moment of \( \mu \).

Hence this is a linear objective in the finite-dimensional variable \( y = (y_\alpha)_{|\alpha| \leq 2m} \). \( y \) has to correspond to the moments of some Borel measure (this gives the moment cone, which is convex) and the total mass of the measure has to be 1, so \( y_0 = 1 \). Subject to these constraints, we want to minimize the linear objective function:

\[
\sum_{|\alpha| \leq 2m} p_\alpha y_\alpha.
\]
Since the moment cone is intractable (unless $P = NP$), we will find necessary conditions for $y$ to lie in the moment cone.

Notice $y_{(2,0,...,0)} = E_{\mu}(z^2) \geq 0$. More generally, consider the polynomial $q(z) = \sum_{|\alpha| \leq m} q_{\alpha} z^\alpha$. Then:

$$0 \leq E_{\mu}(q(z))^2 = \int (\sum_{\beta, \gamma} q_{\beta, \gamma} z^{\beta+\gamma} d\mu(z)$$

$$= \sum_{|\alpha| \leq 2m} \sum_{|\beta, \gamma| \leq m} q_{\beta, \gamma} |\alpha| = \beta + \gamma q_{\beta, \gamma} d\mu(z)$$

$$= \sum_{y_{\alpha}} \sum_{|\beta, \gamma| \leq m} q_{\beta, \gamma} = q^T M(y) q,$$

where $q = (q_{\beta})_{|\beta| \leq m}$. $M(y)$ has rows and columns indexed by all monomials $z^\beta$, where $|\beta| \leq m$, and $y_{\beta+\gamma}$ in the position corresponding to row $z^\beta$ and column $z^\gamma$. Hence we have the following relaxation:

$$(D): \quad \min_{y_{0}} \sum_{\alpha} p_{\alpha} y_{\alpha} \quad y_{0} = 1, \quad M(y) \succeq 0.$$

Assume without loss of generality that $p_{0} = 0$. Then if we substitute 1 for $y_{0}$ we get a “regular” dual-form SDP with $(2m+n)-1$ variables and a matrix of order $\left(\begin{array}{c}m+n \end{array}\right)$.

Note: the number of $|\alpha| \leq d$ is $\left(\begin{array}{c}d+n \end{array}\right)$.

Example: If $m = n = 2$, there are 14 variables and a matrix $M(y)$ of order 6 is used in the problem formulation. The constraint $M(y) \succeq 0$ written out explicitly looks like:

$$\begin{pmatrix}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{pmatrix} \succeq 0.$$

Next time, we will look at the dual of $(D)$ to get a feel for how good a relaxation this is.