Fact 5: The following are norms on $S\mathbb{R}^{n \times n}$:

- $\|U\|_2 := \max\{\|Uz\|_2 : \|z\|_2 \leq 1\} = \|\lambda(U)\|_\infty$,
- $\|U\|_F := (U \bullet U)^{1/2} = \|\lambda(U)\|_2$, the Frobenius norm of $U$, and
- $\|U\|_{tr} := \|\lambda(U)\|_1 := \sum |\lambda_j(U)|$.

Note that $\sum u_{jj} = \text{trace } U = U \bullet I = I \bullet \Lambda(U) = \sum \lambda_j(U)$, by Fact 3.

Fact 6:

Theorem 1 For $U \in S\mathbb{R}^{n \times n}$, the following are equivalent:

a. $U \succeq 0$ ($U \succ 0$),

b. $z^T U z \geq 0$, $\forall z \in \mathbb{R}^n$ ($z^T U z > 0$, $\forall z \neq 0 \in \mathbb{R}^n$),

c. $\lambda(U) \geq 0$ ($\lambda(U) > 0$),

d. $U = P^T P$ for some $r \times n$ matrix $P$ ($U = P^T P$ for some nonsingular $n \times n$ matrix $P$).

Proof: We have $(a) \Leftrightarrow (b)$ by definition. Let $U = Q \Lambda Q^T$; then $z^T U z = z^T Q \Lambda Q^T z = \hat{z}^T \Lambda \hat{z} = \sum \lambda_j(\hat{z}_j)^2$ where $\hat{z} = Q^T z$. Therefore, we have $(b) \Leftrightarrow (c)$. If $U = P^T P$, then for any $z \in \mathbb{R}^n$, $z^T U z = \hat{z}^T \hat{z} \geq 0$ where $\hat{z} = P z$ ($z^T U z = \hat{z}^T \hat{z} > 0$ for $z \neq 0$ if $P$ is nonsingular), hence we have $(d) \Rightarrow (a)$. Suppose $U \succeq 0$ ($U \succ 0$). Then $P = \Lambda^{1/2} Q^T$ shows $U = P^T P$ where

$\Lambda^{1/2} = \text{Diag}(\sqrt{\text{diag}(\Lambda)}) = \begin{bmatrix} \sqrt{\Lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\Lambda_n} \end{bmatrix}$ and hence $(a) \Rightarrow (d)$. \qed

Corollary 1 If $u \in \mathbb{R}^n$, then $uu^T \succeq 0$.

The corollary follows from the theorem by setting $P = u^T$. Note that, using the eigenvalue decomposition, we can express any matrix $U$ as a sum of $n$ rank-one matrices, i.e. $U = \sum \lambda_j(q_j q_j^T)$.

Corollary 2 Every psd (pd) matrix $U$ has a unique psd (pd) square root $U^{1/2}$ such that $U^{1/2} U^{1/2} = U$ and every pd matrix is nonsingular.

Proof: Given $U = Q \Lambda Q^T \succeq 0$, choose $U^{1/2} = Q \Lambda^{1/2} Q^T \succeq 0$. The uniqueness follows from the uniqueness of the eigenvalue decomposition, since the decomposition of any $U^{1/2}$ provides a decomposition of $U$. If $U \succ 0$, then $Q \Lambda^{-1} Q^T$ gives the inverse of $U$, hence $U$ is nonsingular. \qed
Corollary 3 \( SR_{+}^{n,n} \) and \( SR_{++}^{n,n} \) are convex cones. \( SR_{+}^{n,n} \) is closed and pointed, i.e. \((SR_{+}^{n,n}) \cap (-SR_{+}^{n,n}) = \{0\}\), and \( \text{int} \ SR_{+}^{n,n} = SR_{+}^{n,n} \).

**Proof:** \( SR_{+}^{n,n} = \cap_{z \in R^n} \{ U : z^TUz \geq 0 \} \) is an intersection of (an infinite number of) closed halfspaces (since \( z^TUz \geq 0 \) is linear in \( U \) and defines a halfspace for fixed \( z \)), hence it is closed and convex. Similarly, \( SR_{++}^{n,n} \) is also convex. \( SR_{+}^{n,n} \) is pointed since for any \( U \in (SR_{+}^{n,n}) \cap (-SR_{+}^{n,n}) \), we have \( \lambda(U) \geq 0 \) and \( -\lambda(U) \geq 0 \), which implies that \( \lambda(U) = 0 \), i.e. \( U = 0 \). Now, assume that \( U \) is psd but not pd; then there exist a vector \( z \neq 0 \in R^n \) such that \( z^TUz = 0 \). For any \( \epsilon > 0 \), \( z^T(U - \epsilon I)z = -\epsilon z^Tz < 0 \), hence \( U \notin \text{int} \ SR_{+}^{n,n} \). For any \( U \geq 0 \), say \( \lambda_{\text{min}}(U) := \epsilon > 0 \), let \( V \in SR_{+}^{n,n}, \|V\|_2 \leq \epsilon \) and \( \|z\| = 1 \); then \( z^T(U + V)z = z^TUz + z^TVz \geq \epsilon - \epsilon \geq 0 \). Therefore, we have that \( U + V \geq 0 \) and hence \( U \in \text{int} \ SR_{+}^{n,n} \). □

**Fact 7:** If \( U \geq 0 \) (\( U \succ 0 \)), then \( u_{jj} \geq 0 \) (\( u_{jj} > 0 \)) and if \( u_{jj} = 0 \) for some \( j \), then \( u_{jk} = 0 \) for all \( k \).

**Fact 8:** If \( U \geq 0 \), \( PUP^T \succeq 0 \) for all \( P \in R^{r \times n} \). If \( U \succ 0 \) and \( P \) has full row rank, then \( PUP^T \succ 0 \). Hence, using permutation matrices, every principal rearrangement of a psd (pd) matrix is psd (pd). If \( U = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} \succeq 0 \) (\( U = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} \succ 0 \)), then \( U_{11} \succeq 0 \) (\( U_{11} \succ 0 \)).

So every principal submatrix of a psd (pd) matrix is psd (pd).

**Fact 9:** \( U \in SR_{+}^{n,n} \) is psd (pd) iff every principal minor (determinant of a principal submatrix) is nonnegative (positive) (but there are \( 2^n \) matrices to check!). \( U \) is pd iff every leading principal minor (determinant of a top-left principal submatrix) is positive (now there are only \( n \) matrices to check). \( U \) is pd iff \( U = LL^T \) for some nonsingular lower triangular matrix \( L \). \( L \) is called the Cholesky factor of \( U \) and can be computed in \( O(n^3) \) operations.

**Fact 10:** Suppose \( U = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in SR^{n,n} \) and \( A \succ 0 \), then \( U \succeq (U \succ 0) \) iff \( C - B^T A^{-1} B \succeq 0 \) (\( C - B^T A^{-1} B \succ 0 \)). \( C - B^T A^{-1} B \) is the Schur complement of \( A \) in \( U \).

**Proof:** If \( A \) is nonsingular, \( U = \begin{bmatrix} I & A \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \) and the fact follows. □

Here is some useful notation for the homework: If \( X_i, i \in \{1, \ldots, p\} \), are square matrices, then \( \text{Diag}(X_1, \ldots, X_p) := \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \). If \( X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \), then \( \text{diag}(X) := [x_{11}; \ldots; x_{nn}] \).