A conic program has the following form

$$\min_x c^T x$$

$$Ax = b,$$

$$x \in K,$$

where $A \in \mathbb{R}^{m,n}$ is the constraint matrix, $x \in \mathbb{R}^n$ the decision variable, $c \in \mathbb{R}^n$ the objective function cost coefficient vector, $b \in \mathbb{R}^m$ the right hand side vector, and $K \subset \mathbb{R}^n$ a convex cone.

This is a universal form for convex optimization problems since any arbitrary convex problem,

$$\min_x f(x)$$

$$(CP) \quad x \in C,$$

where $f$ is a convex function and $C$ a convex set, can be expressed as a conic program as follows:

i. We can assume that the objective function is linear, since $(CP)$ is equivalent to

$$\min_{(x, \xi)} \xi$$

$$f(x) - \xi \leq 0,$$

$$(x, \xi) \in \bar{C},$$

where $\bar{C} := C \times \mathbb{R}$.

ii. We can assume that the constraints are in conic form, since

$$\min_x c^T x$$

$$x \in C,$$

is equivalent to

$$\min_{(x, \tau)} c^T x$$

$$\tau = 1,$$

$$(x, \tau) \in K = \{(x, \tau) | \tau > 0, \frac{x}{\tau} \in C\}. $$
Example 1 (SDP problem in dual form) Suppose $M(y)$ is a symmetric $n \times n$ matrix depending linearly (affinely) on $y \in \mathbb{R}^m$, i.e. $M(y) = M_0 + \sum_i y_i M_i$. We want to choose $y$ to minimize the maximum eigenvalue of $M(y)$ which is denoted by $\lambda_{\max}(M(y))$.

Introduce a new variable $\eta$ as an upper bound on $\lambda_{\max}(M(y))$, so we have

$$
\lambda_{\max}(M(y)) \leq \eta \\
\iff \lambda_{\max}(M(y) - \eta I) \leq 0 \\
\iff \lambda_{\min}(\eta I - M(y)) \geq 0 \\
\iff \eta I - M(y) \succeq 0.
$$

Hence the problem becomes

$$
\max_{(\eta, y)} -\eta \eta I - M(y) \succeq 0,
$$

which is an SDP problem in dual form.

Some facts about (symmetric) matrices which will be useful are as follows:

Fact 1: If $P$ and $Q$ are $m \times n$ and $n \times m$ matrices, respectively, then

$$
\text{trace } PQ = \text{trace } QP = \text{trace } P^T Q^T = \text{trace } Q^T P^T = \sum_j \sum_k p_{jk} q_{kj}.
$$

Fact 2: $A$ and $A^*$ are adjoint mappings, i.e.

$$(AX)^T y = (A^* y) \cdot X.
$$

Fact 3: If $Q$ is an orthogonal $n \times n$ matrix, then

$$(QUQ^T) \cdot (QVQ^T) = U \cdot V,
$$

for any $U$ and $V \in S \mathbb{R}^{n \times n}$. Note that, there is a similar property for orthogonal operations on vectors, such as $(Qu)^T (Qv) = u^T v$ for any $u$ and $v \in \mathbb{R}^n$.

Proof: [of fact 3]

$$(QUQ^T) \cdot (QVQ^T) = \text{trace } QUQ^T QVQ^T = \text{trace } QUVQ^T = \text{trace } UQV^T Q = U \cdot V.
$$

Fact 4: If $U \in S \mathbb{R}^{n \times n}$, then there are an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ and a diagonal matrix

$$
\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}
$$

such that $U = Q\Lambda Q^T$. If we let $Q = [q_1, \ldots, q_n]$, then $U = Q\Lambda Q^T$ is equivalent to $UQ = Q\Lambda$, which implies $Uq_j = \lambda_j q_j$, i.e. $\lambda_j$ is an eigenvalue of $U$ with corresponding eigenvector $q_j$. If we order the components of $\lambda$ and the columns of $Q$ so that $\lambda_1 \geq \ldots \geq \lambda_n$, we write $\lambda =: \lambda(U)$, $\Lambda =: \Lambda(U)$. 

2