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ON COMBINED PHASE 1-PHASE 2 PROJECTIVE METHODS FOR LINEAR PROGRAMMING

by

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Abstract

We compare the projective methods for linear programming due to de Ghellinck and Vial, Anstreicher, Todd and Fraley. These algorithms have the feature that they approach feasibility and optimality simultaneously, rather than requiring an initial feasible point. We compare the directions used in these methods and the lower bound updates employed. In many cases the directions coincide and two of the lower bound updates give the same result. It appears that Todd's direction and Fraley's lower bound update have slight advantages, and this is borne out in limited computational testing.

1. Introduction

By definition, interior-point methods for linear programming generate sequences of points that are (relatively) interior to the feasible polyhedron. Thus techniques have to be devised to deal with the frequent case that an initial interior point is not known. From a theoretical point of view, there is no difficulty, since Karmarkar showed in his original paper [16] that primal and dual problems can be combined into one system with a known interior solution using an artificial variable, which was then minimized. To avoid increasing the dimension this way, a number of researchers suggested attacking an artificial problem including an artificial variable and/or an artificial bounding constraint, which then requires setting a high cost and/or high right-hand side so that the original problem is correctly solved. See, for example, [17] or [20].

De Ghellinck and Vial [11] proposed an interior method for linear programming that generates a sequence of not necessarily feasible points that approach feasibility and optimality simultaneously. Anstreicher [3] developed another method with the same property, but which explicitly introduced an artificial variable. This variable does not appear in the objective function, but a constraint is added (which is satisfied only in the limit) equating the artificial variable to zero. Both de Ghellinck and Vial's and Anstreicher's methods are based on primal projective methods, but whereas de Ghellinck and Vial derive their algorithm from their earlier feasibility method [10], Anstreicher's is motivated by the standard-form variants of Karmarkar's method due to Anstreicher [2], Gay [9], Gonzaga [14], Jensen and Steger [21], and Ye and Kojima [27]. De Ghellinck and Vial's method [11] does not explicitly introduce an artificial variable, although they show in [10] that their feasibility algorithm is equivalent to minimizing an artificial variable using Karmarkar's algorithm. Todd [24] proposed a modification of Anstreicher's approach.

Standard-form primal projective methods use lower bound estimates on the optimal value. Both de Ghellinck and Vial's and Anstreicher's papers gave methods to update such lower bounds, and Todd showed that Anstreicher's update is the optimal value of a linear programming problem with just two constraints, which is easily obtained using the dual. De Ghellinck and Vial proposed a different technique that requires the solution of n quadratic equations, where n is the number of variables.

Later, Fraley [5] devised an update technique based on linear programming that could be used in de Ghellinck and Vial's algorithm.

Our aim in this paper is to compare these various combined phase 1-phase 2 projective methods. In section 2 we consider the search directions generated by the algorithms, and show in what circumstances they coincide (when viewed in a common space). The basic determinant is which constraints are tight in the direction-finding subproblem of Todd [24]. In section 3 we address the lower-bound update schemes. When applied to a feasible problem, de Ghellinck and Vial's lower bound is the weakest, followed by that of Anstreicher and Todd, and finally by that of Fraley, which is the strongest. We give an example to show that all inequalities can be strict. When the problem is not feasible, it is possible that de Ghellinck and Vial's method will establish this (by generating a lower bound of $+\infty$), while the others fail to. We suggest a simple remedy.

Finally, section 4 gives the results of limited computational testing on small randomly-generated problems. On these problems, all three directions require roughly the same number of iterations, although Todd's can obtain feasible solutions early. De Ghellinck and Vial's lower bound update has difficulties on problems with unbounded feasible regions. The combination of Todd's direction with Fraley's update seems preferable theoretically, and this is consistent with the numerical results. Fraley and Vial [6,7] also give more extensive computational results for the combination of de Ghellinck and Vial's direction and Fraley's lower bound update on the NETLIB problems; they also test a two-phase primal projective algorithm with encouraging results.

We close this section by giving references to several papers that have considered the difficulties of obtaining an initial interior point solution, either theoretically or practically. The initialization of the dual affine method is discussed in Adler et al. [1], and of the primal barrier method in Gill et al. [12]. For the primal-dual path-following method, a theoretical analysis is provided, for instance, in Kojima et al. [17], while McShane et al. [20] give proposals for implementation. Lustig has now proposed [18] an elegant way to deal with the problem in this approach, and Lustig et al. [19] have shown that his technique reduces to Newton's method on a natural system of equations involving no artificial variables or constraints. Shifted barrier methods, especially suited to warm starts, have been discussed by Freund [8] and by Gill et al. [13].

2. The search directions of the algorithms

All the algorithms under consideration reformulate the linear programming problem

$$\text{(LP)} \qquad \begin{aligned} \min\ & \tilde{c}^T\tilde{x} \\ \tilde{A}\tilde{x} &= \tilde{b} \\ \tilde{x} &\geq 0, \end{aligned}$$

where $\tilde{A} \in \mathbb{R}^{m x n}$, $\tilde{b} \in \mathbb{R}^m$, and $\tilde{c} \in \mathbb{R}^n$, into the homogeneous form

$$\text{(P)} \qquad \begin{aligned} \min\ & c^T x \\ Ax &= 0 \\ d^T x &= 1 \\ x &\geq 0, \end{aligned}$$

where $A := [\tilde{A}, -\tilde{b}]$, $c^T = [\tilde{c}^T, 0]$ and $d^T = [0^T, 1]$. We assume that $\{x: c^T x \leq 0,\ Ax = 0,\ d^T x = 0,\ x \geq 0\} = \{0\}$, so that, if (P) is feasible, it has a bounded set of optimal solutions. Given a point $x > 0$ with $d^T x = 1$ but usually not $Ax = 0$, all the algorithms first try to improve the current lower bound on the optimal value of (P). In the next section we discuss the updating techniques used by de Ghellinck and Vial, Anstreicher and Todd, and Fraley. The algorithms then choose a search direction to improve the current iterate x.

This section is concerned with comparing the search directions of the three algorithms. We therefore assume that all algorithms have the same point x at the start of an iteration. We assume that $x = e$, since otherwise all the algorithms scale the data to transform x to e. We also assume that, after updating, all the algorithms have the same finite lower bound, which we denote by z. (The algorithms of Anstreicher and Todd can also deal with the case that $z = -\infty$, but we ignore this here.)

For most of this section, we suppose that $Ae \neq 0$, so that the current iterate is infeasible. We make a few remarks at the end of the section about the case that e is feasible.

De Ghellinck and Vial's algorithm is based on applying one step of their feasibility method [10] to the system

$$\begin{aligned} u^T x &= 0 \\ Ax &= 0 \\ x &\geq 0, x \neq 0, \end{aligned} \tag{1}$$

where

$$u := c - zd. \tag{2}$$

(If x is feasible in (1), then $\bar{x} = x/d^T x$ is feasible in (P) with objective value z, so is optimal.) Their search direction is then

$$q := q_{GV} := P_B e, \tag{3}$$

where

$$B := \begin{bmatrix} u^T \\ A \end{bmatrix} \tag{4}$$

is the matrix in (1) and, for any matrix M, P_M denotes the orthogonal projector into the null space of M. The update that de Ghellinck and Vial use for z assures that $q \not< e$, so that in particular $q \neq 0$. The next point is then

$$x_+(\lambda) = \frac{e + \lambda q}{1 + \lambda d^T q} \tag{5}$$

for some $\lambda > 0$. Note that if q is nonnegative, it is feasible in (1), and $x_+(\lambda)$ approaches a multiple of q as λ tends to infinity; if not, a search in direction q can be shown to decrease an appropriate potential function suitably [11].

Anstreicher [3] and Todd [24] embed (P) in a larger problem for which $x = e$ gives a solution that is feasible in all but one constraint. Introducing an artificial variable yields

$$
(\hat{P}) \qquad \begin{aligned} \min \quad & \hat{c}^T \hat{x} \\ & \hat{A}\hat{x} = 0 \\ & \hat{d}^T \hat{x} = 1 \\ & \hat{\xi}^T \hat{x} = 0 \\ & \hat{x} \geq 0, \end{aligned}
$$

where

$$
\hat{A} := [A, - Ae], \ \ \hat{c}^T := [c^T, 0], \ \ \hat{d}^T := [d^T, 0], \text{ and } \ \ \hat{\xi}^T := [0^T, 1].
$$

Then $\hat{e} := [e^T, 1]^T \in \mathbb{R}^{n+2}$ is feasible in $(\hat{P})$ except for the constraint $\hat{\xi}^T\hat{x} = 0$. Given the lower bound z, Todd's algorithm then finds a search direction $\hat{g}$ by solving the direction-finding subproblem

$$
(\text{DFSP}) \qquad \begin{aligned} \min \quad & ||\hat{g}|| \\ & \hat{A}\hat{g} = 0 \\ & \hat{e}^T\hat{g} = 0 && (6) \\ & (\hat{c} - z\hat{d})^T\hat{g} \leq -(\hat{c} - z\hat{d})^T\hat{e} && (7) \\ & \hat{\xi}^T\hat{g} \leq - \hat{\xi}^T\hat{e} && (8) \end{aligned}
$$

and if $v \geq z$,

$$
(\hat{c} - v\hat{d})^T\hat{g} \leq 0, \tag{9}
$$

where $v := \hat{c}^T\hat{e}$ is the current objective function value. The motivation for (DFSP) is discussed in detail in [24]. Roughly speaking, if it were possible to take a unit step in the direction $\hat{g}$ from $\hat{e}$ and remain nonnegative, then

$$
\hat{x}_+ = \frac{\hat{e} + \hat{g}}{1 + \hat{d}^T\hat{g}}
$$

would be in the null space of $\hat{A}$, satisfy $\hat{\xi}^T\hat{x}_+ \leq 0$ by (8), and $\hat{c}^T\hat{x}_+ \leq z$ by (7). Thus it would be optimal. Usually, a unit step is impossible, and then (7) assures progress toward the lower bound z, (8) progress toward feasibility, and (9) monotonicity in the objective function where possible. With the constraint (6), the objective of (DFSP) can be viewed as minimizing the angle between $\hat{e}$ and $\hat{e} + \hat{g}$ (or $\hat{x}_+$) subject to $\hat{A}(\hat{e} + \hat{g}) = 0$, $(\hat{c} - z\hat{d})^T(\hat{e} + \hat{g}) \leq 0$, $\hat{\xi}^T(\hat{e} + \hat{g}) \leq 0$, and if $v \geq z$, $(\hat{c} - v\hat{d})^T(\hat{e} + \hat{g}) \leq 0$.

Our aim now is to show to what extent the direction $\hat{g}$ from (DFSP) generates a next point x_+ which can be written in the form (5). We will show that this occurs if the solution to (DFSP) has constraints (7) and (8) tight and constraint (9) not binding. Thus let (DFSP$'$) be (DFSP) with (7) and (8) replaced by equalities and (9) removed:

$$
(\text{DFSP}') \qquad \begin{aligned} &\min \ \|\hat{g}\| \\ &\hat{A}\hat{g} = 0 \\ &\hat{e}^T\hat{g} = 0 \\ &(\hat{c} - z\hat{d})^T\hat{g} = -(\hat{c} - z\hat{d})^T\hat{e} \\ &\hat{\xi}^T\hat{g} = -\hat{\xi}^T\hat{e} \end{aligned}
$$

<u>Theorem 2.1.</u> If (DFSP$'$) is feasible, its unique optimal solution is

$$
\hat{g} = \begin{pmatrix} \dfrac{n+2}{e^TP_Be} P_Be - e \\ -1 \end{pmatrix} \tag{10}
$$

where B is as in (4).

<u>Proof</u>. Let us write (DFSP$'$) in terms of $\hat{q} = \binom{q}{\delta}$, where $\hat{q} = \hat{g} + \hat{e}$.

Let

$$
\hat{u}^T := (\hat{c} - z\hat{d})^T = (u^T, 0).
$$

Then we see that (DFSP$'$) is

$$\begin{aligned} \min\ & \|\hat{q}-\hat{e}\| \\ & \hat{A}\hat{q} = 0 \\ & \hat{e}^T\hat{q} = n + 2 \\ & \hat{u}^T\hat{q} = 0 \\ & \hat{\xi}^T\hat{q} = 0. \end{aligned} \tag{11}$$

But since $\hat{\xi}^T\hat{q} = \delta$, we must have $\delta = 0$ so that (11) reduces to

$$\begin{aligned} \min\ & \|q - e\| \\ & u^Tq = 0 \\ & Aq = 0 \\ & e^Tq = n + 2. \end{aligned}$$

If P_Be were 0, then e would be in the column space of $[u, A^T]$, so that this last problem and hence (DFSP$'$) would be infeasible. Thus P_Be is not zero, and $e^TP_Be = \| P_Be \|^2 > 0$. Let us denote

$$\gamma := \frac{n + 2}{e^TP_Be}.$$

Now $(q\text{-}e)^Te = 1$, so

$$\| q - \gamma e \|^2 = \| q - e + (1\text{-}\gamma) e\|^2 = \|q\text{-}e\|^2 + (1\text{-}\gamma)^2 \|e\|^2 + 2(1\text{-}\gamma).$$

Therefore the problem above is equivalent to

$$\begin{aligned} \min\ & \|q - \gamma e \| \\ & Bq = 0 \\ & e^Tq = n+2 \end{aligned} \tag{12}$$

The solution to this problem, with the constraint $e^T q = n + 2$ removed, is $q = \gamma P_B e$. But this vector also satisfies $e^T q = n + 2$, so that it must solve (12). Since the objective function is strictly convex, the solution to (12) is unique. It follows that the solution to (11) is

$$\hat{q} = \begin{pmatrix} \gamma P_B e \\ 0 \end{pmatrix} \tag{13}$$

so that $\hat{g}$ in (10) is the unique solution to (DFSP$'$). □

Corollary 2.2. If $\hat{g}$ is the optimal solution to (DFSP$'$) and

$$\hat{x}_+(\mu) := \frac{\hat{e} + \mu \hat{g}}{1 + \mu \hat{d}^T \hat{g}}, \tag{14}$$

then for each $\lambda > 0$ there is some $\mu > 0$ with

$$\hat{x}_+(\mu) = \begin{pmatrix} x_+(\lambda) \\ \nu \end{pmatrix} \tag{15}$$

for some positive ν, where $x_+(\lambda)$ is the next iterate produced by de Ghellinck and Vial's algorithm as in (5).

Proof. We find

$$\hat{e} + \mu \hat{g} = \begin{pmatrix} (1-\mu) e + \mu\gamma P_B e \\ 1-\mu \end{pmatrix}$$

so that

$$\hat{x}_+(\mu) = \frac{1}{1 + \frac{\mu \gamma}{1 - \mu} d^T P_B e} \begin{pmatrix} e + \frac{\mu \gamma}{1 - \mu} P_B e \\ 1 \end{pmatrix}.$$

If we choose $\mu = \lambda / (\lambda + \gamma)$, we obtain (15). □

The corollary shows that, modulo appropriate step sizes and lower bound updates, the iterates of de Ghellinck and Vial's algorithm can be simulated by those of an algorithm operating in $\mathbb{R}^{n+2}$ and using (DFSP$'$) to generate search directions. Basically, the corollary follows because the search direction $\hat{q}$ in (13) is equivalent to $\hat{g}$ in (10) in the sense of Gonzaga [15].

It remains to discuss the direction chosen by Anstreicher's algorithm. The original Anstreicher direction is the solution to

$$\text{(DFSP}''\text{)} \qquad \begin{aligned} \min\ & \hat{u}^T\hat{g} \\ & \hat{A}\hat{g} = 0 \\ & \hat{e}^T\hat{g} = 0 \\ & \hat{\xi}^T\hat{g} = -\ \hat{\xi}^T\hat{e} \\ & \|\hat{g}\| \leq R, \end{aligned}$$

where $R := [(n+1)\ (n+2)]^{\frac{1}{2}}$ is the circumradius of the simplex $\hat{S} = \{\hat{x} \geq 0 : \hat{e}^T\hat{x} = n + 2\}$. (DFSP$''$) is analogous to the direction-finding subproblem in the standard-form variants of Karmarkar's algorithm (see, e.g., [2]). However, at the end of section 4 of [3], Anstreicher suggests using instead the solution to

$$\text{(DFSP}'''\text{)} \qquad \begin{aligned} \min\ & \|\hat{g}\| \\ & \hat{A}\hat{g} = 0 \\ & \hat{e}^T\hat{g} = 0 \\ & \hat{u}^T\hat{g} \leq -\ \hat{u}^T\hat{e} \qquad (16) \\ & \hat{\xi}^T\hat{g} = -\ \hat{\xi}^T\hat{e} \end{aligned}$$

as long as the constraint (16) is tight at the optimal solution. This problem is again very close to (DFSP).

Let $\hat{v}_q$ denote the projection of $\hat{v}$ into $\{\hat{x} : \hat{A}\hat{x} = 0, \hat{e}^T\hat{x} = 0\}$. Then the solutions to (DFSP$'$), (DFSP$''$) and (DFSP$'''$) all lie in the plane spanned by $\hat{\xi}_q$ and $\hat{u}_q$, where $\hat{u} = \hat{c} - z\hat{d}$. The same is true for the optimal solution to (DFSP) as long as the monotonicity contraint (9) is not present or not binding at the solution. Figure 1 illustrates these directions in several cases. Here $\hat{g}_T$ denotes the solution to (DFSP) assuming (9) is not present or not binding, $\hat{g}_{GV}$ denotes the solution to (DFSP$'$) (of corollary 2.2), and $\hat{g}_A$ denotes the solution to (DFSP$'''$) if (16) is tight and otherwise to (DFSP$''$). In all cases, the horizontal constraint is $\hat{\xi}^T\hat{g} \leq -\hat{\xi}^T\hat{e}$ and the slanted one $\hat{u}^T\hat{g} \leq -\hat{u}^T\hat{e}$.

Note that moving in the direction $\hat{g}_{GV}$ or $\hat{g}_A$ one always hits the optimality constraint at the same time or before one hits the feasibility constraint. Thus, unless their algorithms terminate in a finite number of iterations with an optimal solution, they never achieve feasibility. (However, Anstreicher discusses the possibility of taking a partial step (the solution to (DFSP) with (7) and (9) removed) when this attains feasibility; no complexity analysis is given for this variant.) On the other hand, Todd's algorithm can achieve feasibility if case (c) in figure 1 obtains.

If feasibility is attained in Todd's algorithm, it is maintained from then on either by reverting to the standard-form variant of Karmarkar's algorithm [2, 9, 14, 21, 27] or by replacing (8) in (DFSP) by an equality. In fact, these two approaches give the same iterates. If the current iterate is feasible, all three algorithms are equivalent to the standard-form variant; see Vial [26].

To conclude the section, we note that de Ghellinck and Vial [11] never add a monotonicity constraint (although monotonicity can be maintained in their feasibility algorithm, see [10]). Anstreicher does just in "phase 0" (before a finite lower bound is generated), although his analysis is unchanged if monotonicity is required throughout, and Todd uses it throughout (as long as $v \geq z$). The addition of the monotonicity constraint is necessary theoretically if the initial lower bound is $-\infty$ and the feasible region is unbounded. However, this constraint is inconsistent with the way de Ghellinck and Vial update their lower bound.

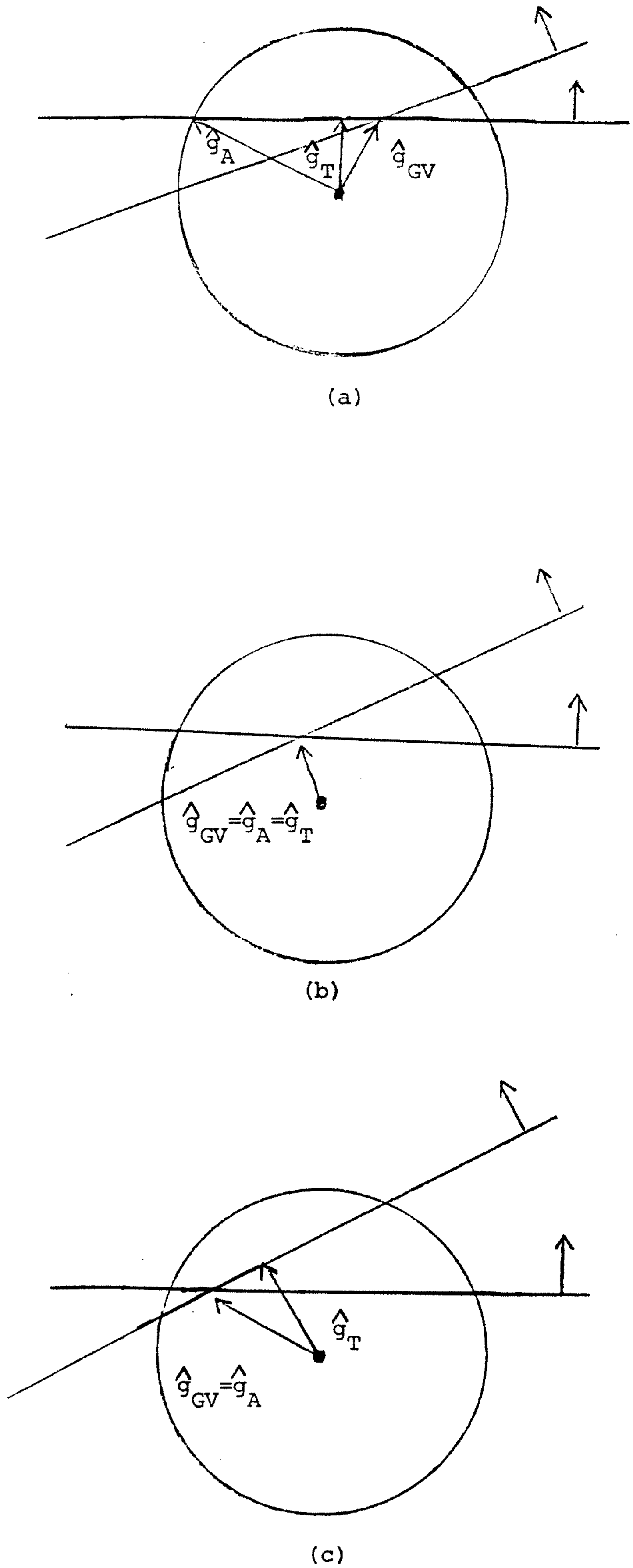

Figure 1: Comparison of search directions

3. Updating the lower bounds

Here we describe the methods used to update the lower bound z in the algorithms of de Ghellinck and Vial [11], Anstreicher [3], Todd [24] and Fraley [5]. We assume that the current finite lower bound is $\hat{z}$ for all methods and that the current iterate is $x = e$. While derived in a different manner, the updates of Anstreicher and Todd are identical. Let us denote the updated lower bounds of de Ghellinck and Vial, Anstreicher and Todd, and Fraley by z_{GV}, z_{AT} and z_F respectively. We aim to show

Theorem 3.1. If (P) is feasible, then $z_{GV} \le z_{AT} \le z_F$. Moreover, both inequalities may be strict.

First we define Fraley's update. For any vector w, let w_p denote its projection into the null space of A. Then, since $w^T x = w^T x_p = w_p^T x$ if $Ax = 0$, (P) can be written as

$$\begin{aligned} \min \ & c_p^T x \\ & Ax = 0 \\ & d_p^T x = 1 \\ & x \ge 0. \end{aligned}$$

If we omit the constraints $Ax = 0$, we obtain a relaxed problem whose objective value is at most that of (P), which we denote by z_*. The dual of this relaxed problem is

$$\begin{aligned} \max \ & z \\ & d_p z \le c_p \end{aligned} \tag{17}$$

whose optimal value is therefore at most z_*. This is the basis for the lower-bound updates for the feasibility-maintaining algorithms of Anstreicher [2], Gay [9], Gonzaga [14], Jensen and Steger [21] and Ye and Kojima [27], based on the method of Todd and Burrell [22].

However, it is easy to strengthen this bound. Indeed, for any vectors f, g,..., one could instead replace the constraints $Ax = 0$ by the constraints $(f - f_p)^T x = 0$, $(g - g_p)^T x = 0, \ldots,$ which are of the form $y^T A x = 0$ and hence define a relaxed problem. Let us replace $Ax = 0$ with the single constraint $(e - e_p)^T x = 0$. The dual of the resulting relaxed problem will be denoted (FD), for Fraley's dual, and can be written as

$$\text{(FD)} \qquad \begin{array}{l} \max \ z \\ d_p z + (e - e_p)\tau \le c_p \end{array}$$

Note that $d_p = d - A^T y_d$ for some y_d, and similarly for c_p and e_p, so that (FD) is equivalent to

$$\begin{array}{l} \max \ z \\ dz + A^T(y_c - z y_d - \tau y_e) \le c, \end{array}$$

which is clearly a restriction of the dual

$$\begin{array}{l} \max \ z \\ dz + A^T y \le c \end{array}$$

of (P). Todd [23] suggested a similar restriction where y was of the form $y_c - \sigma y_d$, with σ not necessarily equal to z.

If the current lower bound $\hat{z}$ is greater than the optimal value of (FD), say z_{FD}, then it remains unchanged; otherwise, it is increased to z_{FD}. Hence Fraley's lower bound is

$$z_F := \max \{\hat{z}, z_{FD}\}. \tag{18}$$

Here we have used Fraley's problem (7.3) [5] to improve the bound; she also suggests two further improvements that require the solution of three-variable linear programming problems.

Next we turn to the lower bound of Anstreicher and Todd. This is based on the problem $(\hat{P})$ of the previous section. For any vector $\hat{w}$, let $\hat{w}_{\hat{p}}$ denote its projection onto the null space of $\hat{A} = [A, -Ae]$. Then $(\hat{P})$ can be written as

$$\begin{aligned} \min \ \hat{c}_{\hat{p}}^{T}\hat{x} & \\ \hat{A}\hat{x} &= 0 \\ \hat{d}_{\hat{p}}^{T}\hat{x} &= 1 \\ \hat{\xi}_{\hat{p}}^{T}\hat{x} &= 0 \\ \hat{x} &\geq 0, \end{aligned}$$

Again, we may relax this problem by omitting the constraints $\hat{A}\hat{x} = 0$. The dual of this relaxation will be denoted (ATD), for Anstreicher and Todd's dual, and can be written as

$$\text{(ATD)} \qquad \begin{aligned} &\max \quad z \\ &\hat{d}_{\hat{p}} z + \hat{\xi}_{\hat{p}} \lambda \leq \hat{c}_{\hat{p}} \,. \end{aligned}$$

In order to relate this to (FD), we need to express $\hat{d}_{\hat{p}}$, etc., in terms of d_p, etc...

<u>Lemma 3.2.</u> If $\hat{w} = (w^T, 0)^T$, then

$$\hat{w}_{\hat{p}} = \begin{pmatrix} w_p + \alpha\,(e - e_p) \\ \alpha \end{pmatrix} \tag{19}$$

where $\alpha := (e - e_p)^T\, w \,/\, (1 + ||e - e_p||^2)$. Also

$$\hat{\xi}_{\hat{p}} = \begin{pmatrix} \beta\,(e - e_p) \\ \beta \end{pmatrix} \tag{20}$$

where $\beta := 1/\,(1 + \|e - e_p\|^2)$.

Proof. We assume without loss of generality that A (and hence $\hat{A}$) has full row rank. Then $P_{\hat{A}} = I - \hat{A}^T (\hat{A}\hat{A}^T)^{-1}\hat{A}$. Now $\hat{A}\hat{A}^T = AA^T + Aee^TA$, so by the Sherman-Morrison formula,

$$(\hat{A}\hat{A}^T)^{-1} = (AA^T)^{-1} - \frac{(AA^T)^{-1}Aee^TA(AA^T)^{-1}}{1 + e^TA(AA^T)^{-1}Ae},$$

and hence $A^T(\hat{A}\hat{A}^T)^{-1}A = A^T(AA^T)^{-1}A - \beta(e - e_p)(e - e_p)^T$. Now it is easy to show that

$$P_{\hat{A}} = \begin{bmatrix} P_A + \beta\,(e - e_p)(e - e_p)^T & \beta\,(e - e_p) \\ \beta\,(e - e_p)^T & \beta \end{bmatrix}, \tag{21}$$

from which the lemma follows. □

Using the lemma, we can express Anstreicher and Todd's dual as

$$\text{(ATD)} \qquad \begin{aligned} &\max \quad z \\ &d_pz + (e - e_p)(\alpha_dz + \lambda\beta - \alpha_c) \leq c_p \\ &\alpha_dz + \lambda\beta - \alpha_c \leq 0, \end{aligned}$$

where $\alpha_d = \beta(e - e_p)^Td$ and $\alpha_c = \beta(e - e_p)^Tc$. Now since $\beta > 0$ and λ is unrestricted, we can express (ATD) in terms of the new unrestricted variable $\tau := \alpha_dz + \lambda\beta - \alpha_c$ and z to get

$$\text{(ATD)} \qquad \begin{aligned} &\max \quad z \\ &d_P z + (e - e_P)\tau \le c_P \\ &\qquad\qquad\qquad \tau \le 0. \end{aligned}$$

If the optimal value of this problem is denoted by z_{ATD}, then Anstreicher and Todd's lower bound update is

$$z_{AT} := \max \{ \hat{z}, z_{ATD} \}. \tag{22}$$

Since this last form of (ATD) is clearly more restricted than (FD), we immediately have $z_{FD} \ge z_{ATD}$ and hence

$$z_F \ge z_{AT}. \tag{23}$$

However, note that (ATD) is a linear programming problem with $n+2$ constraints and only two variables. At a nondegenerate optimal solution, only two constraints will be active, and if neither is the last constraint then this solution also solves (FD) and (23) holds with equality.

Finally, we define de Ghellinck and Vial's update. This is based on q in (3). Since B depends on u which depends on z, q is a function of z. Indeed, we can write

$$q = q(z) = P_B e = P_{u_P^T} P_A e = P_{u_P^T} e_P, \tag{24}$$

where $u_P = c_P - zd_P$.

Suppose first that $u_P = 0$ for some z_0, which is a degenerate case. Then every feasible solution to (P) has objective function value z_0 (see, e.g., [11]). It is possible that z_F and/or z_{AT} is infinite ((FD) and (ATD) are homogenous in $z - z_0$ and τ), in which case they show that (P) is infeasible. Otherwise, $z_F = z_{AT} = z_0$. De Ghellinck and Vial reduce the optimization problem to a feasibility problem in this case, which is equivalent to setting $z = z_0$. Henceforth we assume that this exceptional case does not occur. If $u_p \neq 0$,

$$P_{u_p^T} = I - u_p u_p^T / \|u_p\|^2, \text{ and hence}$$

$$q(z) = e_p - \nu(z)\,(c_p - zd_p), \tag{25}$$

where
$$\nu(z) := (c_p - zd_p)^T e_p / \|c_p - zd_p\|^2.$$

De Ghellinck and Vial define

$$z_{GV} = \begin{cases} \hat{z} & \text{if } \; q(\hat{z}) \nless e \\ \max\{\bar{z} : q(z) < e \;\text{ for } \hat{z} \leq z < \bar{z}\} & \text{otherwise.} \end{cases} \tag{26}$$

It follows that $q(z_{GV}) \nless e$, so that in particular $\|q(z_{GV})\| \geq 1$. From this it is easy to see that $\hat{g}$ in (10) has $\|\hat{g}\| \leq R$, which is a sufficient condition for progress to be made. De Ghellinck and Vial [11] also suggest other lower bounds based on linear programming duality; however, these cannot assure that $q(z) \nless e$, so we confine ourselves to z_{GV} in (26).

It remains to show that z_{GV} is a valid lower bound whenever $\hat{z}$ is. Fraley [5] establishes the following result. For completeness, we give a proof.

<u>Lemma 3.3.</u> Suppose (P) is feasible and has optimal value z_*, and $q(z) < e$. Then $\nu(z) \neq 0$. Moreover, $z \leq z_*$ implies $z < z_*$ and $\nu(z) > 0$, and $\nu(z) > 0$ implies $z < z_*$.

<u>Proof.</u> Let x^* be optimal in (P). Then $e_p^T x^* = e^T x^*$, $c_p^T x^* = c^T x^* = z_*$, $d_p^T x^* = d^T x^* = 1$, and $x^* \geq 0$. From

$$q(z) = e_p - \nu(z)(c_p - zd_p) < e$$

we deduce that

$$-\nu(z)(z_* - z) < 0$$

from which all parts of the lemma follow. □

Corollary 3.4 ([11]). z_{GV} is a valid lower bound if $\hat{z}$ is.

Proof. There is nothing to prove if (P) is infeasible. Otherwise, if $z_{GV} > \hat{z}$, then $q(\hat{z}) < e$ so that $\hat{z} < z_*$ and $\nu(\hat{z}) > 0$. We must have $\nu(z) > 0$ for all $\hat{z} \leq z < z_{GV}$, otherwise some such z would have $q(z) < e$ and $\nu(z) = 0$, a contradiction. Hence all such z, and also z_{GV}, are valid lower bounds. □

When (P) is feasible and $z_{GV} > \hat{z}$, z_{GV} is at most the optimal value of

$$\text{(GVD)} \qquad \begin{array}{ll} \sup & z \\ & e_p - \nu(z)\,(c_p - z d_p) < e \\ & \nu(z) \qquad\qquad > 0\ , \end{array}$$

which can be written

$$\text{(GVD)} \qquad \begin{array}{ll} \sup & z \\ & d_p z \ + (e - e_p)\left(-\frac{1}{\nu(z)}\right) < c_p \\ & \left(-\frac{1}{\nu(z)}\right) < 0. \end{array}$$

Comparing this with (ATD), we conclude that

$$z_{GV} \leq z_{AT} \text{ if (P) is feasible,} \tag{27}$$

since (GVD) is more constrained.

We have now established the inequalities in Theorem 3.1. To finish the proof, we give an example demonstrating that both inequalities may be strict. Let

$$\begin{aligned} A &= (3, -1, -1), \\ c_p &= c = (1, 3, 0)^T, \text{ and} \\ d &= (0, 0, 1)^T. \end{aligned}$$

Then $AA^T = 11$ and an easy computation gives

$$e_P = \frac{1}{11}(8, 12, 12)^T \text{ and } d_P = \frac{1}{11}(3, -1, 10)^T,$$

so that (FD) is

$$\begin{aligned} \max \quad & z \\ & 3z + 3\tau \leq 11 \\ & -z - \tau \leq 33 \\ & 10z - \tau \leq 0, \end{aligned}$$

with optimal solution $z = \frac{1}{3}$, $\tau = \frac{10}{3}$. Let $\hat{z} = -2$. Then $z_F = \max\{\frac{1}{3}, -2\} = \frac{1}{3}$. If the constraint $\tau \leq 0$ is added to get (ATD), the optimal solution becomes $z = 0$, $\tau = 0$, so that $z_{AT} = 0$. Finally, the constraints $q(z) < e$ can be written as

$$\begin{aligned} 3z^2 + 132z - 407 &< 0 \\ z^2 + 16z - 61 &< 0 \\ 11z^2 + 12z - 11 &> 0 \end{aligned}$$

so that $q(\hat{z}) < e$ for $\hat{z} = -2$. It is easily seen that z_{GV} is the negative root of the last quadratic, which is $-(6+\sqrt{157})/11$. (This example shows that the complexity of (26) is necessary in defining z_{GV}; it is not sufficient to choose any z for which $q(z) \leq e$ with at least one equality. Indeed, $q(1) < e$, but 1 is not a valid lower bound.) Hence in this case

$$z_{GV} < z_{AT} < z_F.$$

In fact, z_F is the optimal value of (P), since (FD) is equivalent to its dual. □

Theorem 3.1 includes the hypothesis that (P) is feasible. Is it possible that when (P) is infeasible, z_{GV} will be larger than z_{AT} or z_F, or even that z_{GV} will be $+\infty$, proving infeasibility, while $z_F < \infty$? Again, the answer is affirmative, as shown by the following example.

Let

$$A = \begin{bmatrix} 19 & 1 & 0 \\ 31 & 0 & 1 \end{bmatrix}$$

$$c = c_P = (-1, 19, 31)^T, \text{ and}$$

$$d = (0, 0, 1)^T.$$

It is clear that (P) is infeasible, since $Ax = 0$, $x \geq 0$ implies $x = 0$. It is easy to see that the null space of A is spanned by c, and that

$$e_P = \frac{1}{27}(-1, 19, 31)^T, \text{ and } \quad d_P = \alpha(-1, 19, 31)^T$$

for $\alpha = 31/1325 > 0$. The dual of (FD) is

$$\begin{aligned} \min \quad & (-1, 19, 31)^T x \\ & \tfrac{1}{27}(28, 8, -4)^T x = 0 \\ & \alpha(-1, 19, 31)^T x = 1 \\ & x \geq 0, \end{aligned}$$

which clearly has a feasible solution $x = \beta(0, 1, 2)^T$ for some $\beta > 0$, with (optimal) objective value $\gamma = \frac{1}{\alpha} = 1325/31$.

Suppose $\hat{z} > \gamma$. (Any $\hat{z}$ is a lower bound since (P) is infeasible.) Then $z_{AT} = z_F = \hat{z}$. However, for $z > \gamma$, $c_P - zd_P \neq 0$ so that $c_P - zd_P$ is a nonzero multiple of e_P and

$$q(z) = e_P - \frac{(c_P - zd_P)^T e}{\|c_P - zd_P\|^2}(c_P - zd_P) = 0 < e.$$

Hence $z_{GV} = +\infty$.

It is easy to guard against this possibility. One must compute $q(\hat{z})$ and $\nu(\hat{z})$. If $q(\hat{z}) < e$ and $\nu(\hat{z}) < 0$, then set $z_{AT} = z_F = +\infty$ and stop, since lemma 3.3 implies that (P) is infeasible. Otherwise, continue. If there is some z with $q(z) < e$ and $\nu(z) = 0$, then $e - e_P > 0$ and it is clear that (ATD) and (FD) are unbounded, so again $z_{AT} = z_F = +\infty$ and we can stop. If no such z exists, then $\nu(z) > 0$ for all $\hat{z} \leq z < z_{GV}$, and the argument above shows that $z_{GV} \leq z_{AT} \leq z_F$, even if (P) is infeasible.

To conclude this section we note an important distinction between de Ghellinck and Vial's lower bound update and the others. From (26), z_{GV} depends critically on $\hat{z}$, while according to (18) and (22), z_{AT} and z_F depend on $\hat{z}$ only to ensure that they are no less. Indeed, z_{GV} can only improve $\hat{z}$ if $q(\hat{z}) < e$, which requires n strict inequalities. It appears that in practice, it takes longer to generate the first lower bound with de Ghellinck and Vial's update, and if monotonicity is imposed the initial lower bound is hardly ever improved.

4. Computational Results

Here we give the results of limited computational testing of various algorithms applied to the random test problems studied in Todd [25]. Algorithm B of that paper is Anstreicher's phase 1 - phase 2 method as modified by Todd [24], applied to the problem $(\hat{P})$ of section 2. It is easy to modify the code to use the direction $\hat{g}_{GV}$ or $\hat{g}_A$ instead of $\hat{g}_T$ (see section 2). We also impose the monotonicity constraint (if (9) is not satisfied, project all vectors orthogonal to $\hat{c} - v\hat{d}$ and recompute $\hat{g}$), except where noted below. Besides its theoretical justification, the monotonicity constraint appears very helpful computationally, especially in improving or generating the initial lower bound; see also the results of Anstreicher and Watteyne [4]. Similarly straightforward is the implementation of the lower bound z_F instead of z_{AT} -- according to section 3, we merely remove the last inequality in (ATD). While quantities such as c_p, d_p and e_p are not directly available in the implementation, they can easily be derived using lemma 3.2; hence we were able to implement the lower bound update z_{GV} also. The combination of three direction choices and three lower bound updates yielded nine algorithms. The computation of projections to calculate search directions and update lower bounds requires careful implementation. We used the orthogonal matrix Q from a QR factorization of the scaled matrix $\hat{A}^T$. In addition, after each iteration, the current solution was again projected onto the null space of the scaled matrix $\hat{A}$. Because most of the ill-conditioning is caused by different scales for different rows of $\hat{A}^T$, we ordered the rows by decreasing sizes of the components of the current iterate. All algorithms were coded in FORTRAN using double precision.

For each method we made eight runs, in each of which ten random dense problems of the same characteristics and size were solved. The first four runs involved nondegenerate problems without null (zero in all feasible solutions) or unbounded (in the optimal solution set) variables, of sizes 50x100, 100x200, 150x300 and 200x400 (in each case this is the dimension of $\tilde{A}$ in (LP) : A had an extra column). These were generated by model 1 of [25]. The second set of four runs were generated by model 2 of [25] and had a quarter of the variables null and a quarter unbounded; the dimensions were as in the first four runs, with no primal or dual degeneracy (except for the small amount forced by the null or unbounded variables).

In all problems, the initial lower bound was taken to be -10^{10} when z_{GV} was used; otherwise it was taken to be $-\infty$. When $z = -\infty$, $\hat{u} = \hat{c} - z\hat{d}$ is replaced by $\hat{d}$ in (DFSP) - (DFSP$'''$). The termination criterion was as in [25]; we stopped when the maximum of .95 times the relative error in the constraints, ten times the value of the artificial variable, and the relative error in the objective function value was below a tolerance $\in$, or (unsuccessfully) if the maximum was the first term and exceeded $\in$ or if the algorithm could make no further progress in reducing a suitable potential function. We chose $\in = 10^{-4}$ throughout, although this was almost never achieved for the model 2 problems, and we judged these satisfactorily solved if they achieved a tolerance of $2x10^{-2}$, as in [25]. The results differ slightly from those given in [25] for the same algorithm, because of the change in $\in$.

First we consider model 1 problems. Only one of the forty problems was successfully solved using de Ghellinck and Vial's lower bound update z_{GV} with any of the three direction choices when the monotonicity constraint was imposed; on all but this one problem the initial lower bound was never updated. The reason for this failure is that de Ghellinck and Vial's lower bound update is inconsistent with the addition of the monotonicity constraint, so that the direction produced is not guaranteed to reduce the potential function; in fact, this is how the algorithms terminated, with an indication of impossibility of progress. We therefore modified the methods when z_{GV} was used so that the monotonicity constraint was relaxed -- the resulting directions are denoted $\bar{\hat{g}}_{GV}$, etc. In this case, the algorithms with z_{GV} solved all the problems. The results are given in table 1 (we do not give the unsatisfactory results of z_{GV} with monotonicity). When using the lower bound update z_{GV}, several of the problems show a sequence of iterates with norms increasing rapidly, leading to considerable inaccuracies; the numerical stability of the implementation allows accuracy to be regained, but slow progress is made because the norms of the iterates have to be decreased to that of the optimal solution. All the other methods solved all problems without numerical difficulties. There is a slight advantage (in iteration count) to the directions $\hat{g}_{GV}$ and $\hat{g}_T$ over $\hat{g}_A$. The main difference between the former directions is not in the number of iterations, but in the solutions obtained - in eleven of the forty problems using z_{AT} (ten using z_F) feasibility was attained when $\hat{g}_T$ was employed. Finally, there is a very slight improvement when Fraley's lower bound update z_F replaces z_{AT}. In table 1, for all runs on the same problem set with the same lower bound update, but with different direction choices, identical statistics reflect identical runs, except for the "17.5's" in the first column and the "37.9's" in the third.

The results for model 2 problems are given in table 2. Here we give the average number of iterations until termination and the number of problems (out of 10) solved satisfactorily. Again, de Ghellinck and Vial's lower bound update performs very poorly when the monotonicity constraint is imposed, so the results in table 2 are for the case where this constraint is relaxed. Even in this case, none of the forty problems was successfully solved, but at least some of the time (in 15 of the 40 problems) lower bound updates were performed. Indeed, problems with unbounded variables have the property that there is a nonzero solution x to $Ax = 0$, $(c - zd)^T x = 0$, $x \geq 0$ for any z, and this proves that $q(z) \not< e$. Hence, theoretically, no updates will ever be made. (A similar statement can be made for the other lower bound updates, an observation of Anstreicher; see the discussion in [25]. However, in practice, with suitable tolerances lower bounds are updated; in our runs, in 39 of the 40 problems.) For these particular problems, the Anstreicher-Todd and Fraley lower bound updates perform identically. Finally, the Anstreicher and de Ghellinck-Vial directions happen to give identical results on this problem set, and the results are very similar to those with the Todd direction; on only one out of the forty problems does the latter yield a feasible solution. In table 2, for all runs on the same set of problems with the same lower bound update, but with different direction choices, identical statistics reflect identical runs.

These computational results support our earlier claims. The directions $\hat{g}_T$ and $\hat{g}_{GV}$ are comparable in terms of iteration counts, and slightly superior to $\hat{g}_A$, while $\hat{g}_T$ attains feasibility where the other directions cannot. The lower bound updates z_{AT} and z_F are comparable, with a very slight advantage to the latter; the update z_{GV} performs very poorly. Overall, the combination of $\hat{g}_T$ and z_F is recommended. As in [25], the major difficulty is in obtaining the first lower bound when the problems have null or unbounded variables. This is consistent with the numerical results of Fraley and Vial [6,7] on the NETLIB problems.

		50x100	100x200	150x300	200x400
$\bar{\hat{g}}_{GV}$	z_{GV}	32.5	37.5	37.9	43.3
$\hat{g}_{GV}$	z_{AT}	17.5	20.4	22.5	23.0
$\hat{g}_{GV}$	z_F	17.5	20.3	22.4	23.0
$\bar{\hat{g}}_A$	z_{GV}	32.5	37.5	37.9	43.3
$\hat{g}_A$	z_{AT}	17.5	23.4	26.7	26.9
$\hat{g}_A$	z_F	17.5	23.3	26.6	26.9
$\bar{\hat{g}}_T$	z_{GV}	32.3	37.1	37.9	42.9
$\hat{g}_T$	z_{AT}	17.7	20.2	22.4	23.3
$\hat{g}_T$	z_F	17.5	20.1	22.5	23.3

Table 1. Average number of iterations for model 1 problems

		50x100	100x200	150x300	200x400
$\bar{\hat{g}}_{GV}$	z_{GV}	32.0/0/7	31.1/0/2	30.5/0/0	32.9/0/6
$\hat{g}_{GV}$	z_{AT}	31.0/9	31.4/6	32.2/3	32.5/4
$\hat{g}_{GV}$	z_F	31.0/9	31.4/6	32.2/3	32.5/4
$\bar{\hat{g}}_A$	z_{GV}	32.0/0/7	31.1/0/2	30.5/0/0	32.9/0/6
$\hat{g}_A$	z_{AT}	31.0/9	31.4/6	32.2/3	32.5/4
$\hat{g}_A$	z_F	31.0/9	31.4/6	32.2/3	32.5/4
$\bar{\hat{g}}_T$	z_{GV}	32.0/0/7	31.1/0/2	30.5/0/0	32.9/0/6
$\hat{g}_T$	z_{AT}	30.9/9	31.4/5	32.0/2	32.5/6
$\hat{g}_T$	z_F	30.9/9	31.4/5	32.0/2	32.5/6

Table 2. Average number of iterations/number of problems satisfactorily solved for model 2 problems

(For the methods using z_{GV}, the last entry is the number of problems for which a lower bound update was made.)

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Title: On Combined Phase 1-Phase 2 Projective Methods for Linear Programming

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Abstract: We compare the projective methods for linear programming due to de Ghellinck and Vial, Anstreicher, Todd and Fraley. These algorithms have the feature that they approach feasibility and optimality simultaneously, rather than requiring an initial feasible point. We compare the directions used in these methods and the lower bound updates employed. In many cases the directions coincide and two of the lower bound updates give the same result. It appears that Todd's direction and Fraley's lower bound update have slight advantages, and this is borne out in limited computational testing.