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THE AFFINE–SCALING DIRECTION FOR LINEAR PROGRAMMING IS A LIMIT OF PROJECTIVE–SCALING DIRECTIONS

by

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Abstract. There are two ways to convert a standard-form linear programming problem to a form suitable for application of a projective-scaling interior point algorithm based on Karmarkar's method. One of these adds a dummy variable which is identically one. We show that, as the number of dummy variables (all identically one) added tends to infinity, the resulting direction in the original variables tends to the direction chosen by Dikin's affine-scaling algorithm.

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1. Introduction

Consider the standard form linear programming problem

$$(P) \qquad \begin{aligned} \min \quad & c^T x \\ & A\, x = b \\ & x \geq 0, \end{aligned}$$

where A is $m \times n$ of rank m, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let us assume that $b \neq 0$, and that a lower bound z_0 on the optimal value z_* is known. If we add p extra dummy variables to (P), we obtain the equivalent problem

$$(EVP) \qquad \begin{aligned} \min \quad & c^T x \\ & A\, x \quad = b \\ & \qquad w = e \\ & x \geq 0, \quad w \geq 0, \end{aligned}$$

where $w \in \mathbb{R}^p$ and e is the vector of ones in $\mathbb{R}^p$. We are concerned in this paper with the equivalence, or lack of it, of algorithms applied to (P) and (EVP) at the corresponding feasible points.

It is easy to see that the simplex method performs equivalently on (P) and (EVP). Starting at corresponding basic feasible solutions, and employing the same pivot rules, corresponding sequences of basic feasible solutions will be generated.

Now consider interior point methods. We will show in section 2 that Dikin's affine-scaling method [4,5] (see also [2,17]) generates corresponding sequences of iterates for (P) and for (EVP). There are two ways to convert a standard-form problem to a form suitable for application of a projective-scaling algorithm based on Karmarkar's method [10]. The first, due to Gonzaga [7] (see also [8]) gives a problem of the same dimension; the other, due independently to Anstreicher [1], de Ghellinck and Vial [3], Gay [6], Jensen and Steger [14] and Ye and Kojima [18], introduces an extra variable and yields a problem of dimension one higher. In section 3, we show that the method resulting from the latter transformation on (P) is equivalent to the method resulting from Gonzaga's transformation on (EVP) with a single dummy variable ($p = 1$).

Moreover, we show that Gonzaga's method applied to (EVP) with p dummy variables (or the other method applied to (EVP) with $p-1$ dummy variables) generates a direction for the original variables that tends as $p \to \infty$ to that generated by the affine-scaling algorithm. Hence the projective algorithms are not invariant under the addition of nonnegatively-constrained dummy variables.

We should note that, if the extra variable added in the algorithms of [1,3,6,14,18] is not constrained to be nonnegative, and the resulting problem is treated by the method of Mitchell and Todd [13] for problems with free variables, then the resulting iterates correspond exactly to those of Gonzaga's algorithm [7]. Similarly, the addition of further dummy variables that are also free does not affect the iterates generated by the algorithm of [13]. Here we assure that the line search approximately minimizes a potential function of the form of (18) below, where n is the number of nonnegatively-constrained variables and the barrier term only includes such variables. The proof of these statements is fairly straightforward (although the linear algebra involved is cumbersome) and will be omitted.

Section 4 presents conclusions. In particular, we describe a connection, suggested by Jim Renegar, between the question of the existence of an $O(L)$ iteration bound for the projective-scaling algorithm and whether the affine-scaling algorithm (in its pure form) can be proved polynomial.

We note that the results presented here remain valid with appropriate rewording if the constraints $w = e$ are replaced by $Fw = F\tilde{w}$ for any nonsingular F and strictly positive $\tilde{w}$.

2. The affine-scaling algorithm.

Let $x^k > 0$ be feasible in (P). The affine-scaling algorithm generates an improved feasible solution by scaling x^k to the vector e of ones in $\mathbb{R}^n$ and making a step in the projected steepest descent direction for the transformed objective function in the transformed space.

Let $X := X_k := \mathrm{diag}(x^k)$ be the diagonal matrix with diagonal entries the components of x^k. The scaled version of (P) is then

$$(\hat{P}) \qquad \begin{aligned} \min \ & \hat{c}^T \hat{x} \\ & \hat{A}\hat{x} = b \\ & \hat{x} \geq 0 \end{aligned}$$

in terms of the scaled variables $\hat{x} := X^{-1}x$, where

$$\hat{A} := AX \text{ and } \hat{c} := Xc. \tag{1}$$

The feasible solution $x = x^k$ of (P) corresponds to the feasible solution $\hat{x} = e$ of $(\hat{P})$.

Dikin's [4,5] affine-scaling algorithm takes a step from $\hat{x} = e$ in the direction of the negative projected gradient. Let P_M denote the projection onto the null space of M, for any matrix M; if M has full row rank,

$$P_M = I - M^T(MM^T)^{-1}M. \tag{2}$$

Then the affine-scaling direction in the transformed space is

$$\hat{d}_{AFF} = -P_{\hat{A}}\hat{c} = -P_{AX}Xc,$$

and in the original space corresponds to moving from x^k in the direction

$$d_{AFF} = -X P_{AX} Xc. \tag{3}$$

Now suppose this algorithm is applied to (EVP), which we rewrite as

$$(\bar{P}) \qquad \begin{aligned} \min\ & \bar{c}^T \bar{x} \\ & \bar{A}\ \bar{x} = \bar{b} \\ & \bar{x} \geq 0 \end{aligned}$$

with

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b \\ e \end{bmatrix} \quad \text{and} \quad \bar{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}. \tag{4}$$

If the current solution is $\bar{x}^k = \left[x^{k^T}, e^T\right]^T$, then we find

$$\hat{\bar{A}} = \begin{bmatrix} \hat{A} & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad \hat{\bar{c}} = \begin{bmatrix} \hat{c} \\ 0 \end{bmatrix} \tag{5}$$

where $\hat{\bar{A}} := \bar{A}\,\bar{X}$, $\hat{\bar{c}} := \bar{X}\,\bar{c}$, and $\bar{X} := \bar{X}_k := \mathrm{diag}(\bar{x}^k)$. Now

$$P_{\hat{\bar{A}}} = I - \bar{X}\,\bar{A}^T(\bar{A}\,\bar{X}^2\,\bar{A}^T)\bar{A}\,\bar{X}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \hat{A}^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}\hat{A}^T & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_{\hat{A}} & 0 \\ 0 & 0 \end{bmatrix}; \qquad (6)$$

it follows that the direction generated in the original space is

$$\bar{d}_{AFF} = -\bar{X}\,P_{\bar{A}\,\bar{X}}\,\bar{X}\,\bar{c} = -\begin{pmatrix} XP_{AX}\,Xc \\ 0 \end{pmatrix} = \begin{pmatrix} d_{AFF} \\ 0 \end{pmatrix}. \qquad (7)$$

The standard step size in the affine-scaling algorithm is to move a fixed proportion (say .9) of the way to the boundary. If $d_{AFF} = 0$, it is easy to see that all feasible points are optimal. If $d_{AFF} \geq 0$, $d_{AFF} \neq 0$, then (P) is unbounded. Otherwise, the rule above defines the next iterate x^{k+1}. Similar comments apply to $(\bar{P})$.

From (7) we conclude

Proposition 1. If started at corresponding strictly positive feasible solutions, the affine-scaling algorithm applied to (P) and to (EVP) generates corresponding sequences of iterates.

3. Projective-scaling algorithms.

The variants of Karmarkar's method for dealing with problems in standard form with unknown optimal value are most easily described in terms of a problem

$$(\dot{P}) \qquad \begin{aligned} \min\ & \dot{c}^T \dot{x} \\ & \dot{A}\ \dot{x} = 0 \\ & \dot{g}^T \dot{x} = 1 \\ & \dot{x} \geq 0 \end{aligned}$$

with subspace constraints $\dot{A}\,\dot{x} = 0$ and a normalizing constraint $\dot{g}^T\dot{x} = 1$.

To reformulate (P) into this form, Gonzaga [7,8] chooses some nonzero b_i, lets $\dot{g} = a_i/b_i$ where a_i^T is the ith row of A, and then lets the rows of $\dot{A}$ be of the form $a_j^T - (b_j/b_i)a_i^T$ for $j \neq i$. Thus $\dot{c} = c$ and $\dot{x} = x$. He shows that the resulting algorithm is independent of the choice of i.

A different method is chosen by Anstreicher [1], de Ghellinck and Vial [3], Gay [6], Jensen and Steger [14] and Ye and Kojima [18]. They rewrite (P) as

$$(\check{P}) \qquad \begin{aligned} \min\ & (c^T,0)\binom{x}{\omega} \\ & (A,-b)\binom{x}{\omega} = 0 \\ & (0^T,1)\binom{x}{\omega} = 1 \\ & \binom{x}{\omega} \geq 0, \end{aligned}$$

which is clearly of form $(\dot{P})$, using an extra dummy variable ω.

Suppose Gonzaga's transformation is applied to (EVP) with $p = 1$, i.e. a single extra constraint $\omega = 1$. By choosing this row to give the normalizing constraint, we easily see that the resulting problem $(\dot{P})$ is exactly $(\check{P})$ above.

Given a feasible solution $\dot{x}^k > 0$ to $(\dot{P})$, the methods of [1,3,6,7, 8,14,18] all proceed similarly (de Ghellinck and Vial's method [3] is more general, since it does not require feasibility). They assume known a lower bound z_k on the optimal value of $(\dot{P})$. Now define $\dot{X} := \dot{X}_k := \mathrm{diag}(\dot{x}^k)$ and

$$\hat{A} := \dot{A}\,\dot{X},\ \hat{c} := \dot{X}\,\dot{c},\ \text{and}\ \ \hat{g} := \dot{X}\dot{g}. \tag{8}$$

In terms of the scaled variables $\hat{x} = \dot{X}^{-1}\dot{x}$, we have the equivalent problem

$$(\hat{P}) \qquad \begin{aligned} \min\ & \hat{c}^T\hat{x} \\ & \hat{A}\ \hat{x} = 0 \\ & \hat{g}^T\hat{x} = 1 \\ & \hat{x} \geq 0. \end{aligned}$$

Clearly, it is equivalent to minimize $\hat{c}(z)^T\hat{x}$, where $\hat{c}(z) = \hat{c} - z\,\hat{g}$ for any z. The methods move from $\hat{x} = e$ (corresponding to $\dot{x} = \dot{x}^k$) in the direction given by the negative projection of $\hat{c}(z)$, where z is a possibly updated lower bound. Here the projection is onto the null space of $\hat{A}$, ignoring the normalizing constraint. Then a radial (or conical [7]) projection of the resulting point gives one that is feasible in $(\hat{P})$, and scaling back gives $\dot{x}^{k+1}$ feasible in $(\dot{P})$.

Since the details are somewhat complicated, we outline the results, referring to [1,3,6,7,8,14,18] for motivation and further elaboration. If

$$P_{\hat{A}}\,\hat{c}(z_k) = P_{\hat{A}}\,\hat{c} - z_k\,P_{\hat{A}}\,\hat{g} \tag{9}$$

has a nonpositive component, then the lower bound is unchanged: we set $z_{k+1} = z_k$. Otherwise,

$$z_{k+1} = \max\{z: P_{\hat{A}} \hat{c}(z) \geq 0\}. \tag{10}$$

In this case, if we set

$$\dot{y}^{k+1} = (\hat{A}\,\hat{A}^T)^{-1}\,\hat{A}\,\hat{c}\,(z_{k+1}), \tag{11}$$

then $(\dot{y}^{k+1}, z_{k+1})$ is feasible in the dual $(\dot{D})$ of $(\dot{P})$, with value z_{k+1}, certifying z_{k+1} as a lower bound.

Having obtained $z := z_{k+1}$, we set

$$\hat{x}_+ = e - \alpha P_{\hat{A}}\,\hat{c}(z) \tag{12}$$

for some $\alpha > 0$ so that $\hat{x}_+ > 0$, radially project to $\hat{x}_+/\hat{\dot{g}}^T \hat{x}_+$, and rescale to get $\dot{x}^{k+1} = \dot{X}\,\hat{x}_+/\hat{g}^T\,\hat{x}_+$. The choice of α is made to minimize approximately Karmarkar's potential function [10]

$$\dot{f}(\dot{x};z) := \dot{n}\,\ell n\,(\dot{c} - z\dot{g})^T\dot{x} - \sum_j \ell n\,\dot{x}_j, \tag{13}$$

where $\dot{n}$ is the number of $\dot{x}$ variables.

Suppose Gonzaga's method is applied to the problem $(\dot{P})$ resulting from (P) by his transformation, without increasing the dimension. Note that

$$e - P_{AX}\, e = XA^T(AX^2A^T)^{-1}AXe$$
$$= XA^T(AX^2A^T)^{-1}b. \tag{14}$$

Thus, Gonzaga's Lemma 3.1 [8] shows that in the original variables, we search from x^k in the direction

$$d^G_{PRO} := -XP_{AX}\, Xc + \upsilon_G\, XP_{AX}\, e, \tag{15}$$

where

$$\upsilon_G := \upsilon_G(z_{k+1}) := \frac{c^TX^2A^T(AX^2A^T)^{-1}b - z_{k+1}}{b^T(AX^2A^T)^{-1}b}. \tag{16}$$

If $z_{k+1} > z_k$, then the corresponding y is

$$y_G := y_G(z_{k+1}) := (AX^2A^T)^{-1}AX^2c - \upsilon_G(AX^2A^T)^{-1}b. \tag{17}$$

Moreover, in finding $x^{k+1} = x^k + \alpha d^G_{PRO}$, we approximately minimize

$$f_G(x;z) := n\, \ell n\, (c^Tx - z) - \Sigma\, \ell n\, x_j. \tag{18}$$

Now suppose the methods of [1,3,6,14,18] are applied to (P). Then it can be shown (see, e.g., [11,15]) that x^{k+1} is obtained by searching in the direction

$$d^O_{PRO} := -XP_{AX}\, Xc + \upsilon_O\, XP_{AX}e, \tag{19}$$

where

$$\nu_O := \nu_O(z_{k+1}) := \frac{c^TX^2A^T(AX^2A^T)^{-1}b - z_{k+1}}{b^T(AX^2A^T)^{-1}b + 1}. \tag{20}$$

If $z_{k+1} > z_k$, the corresponding y is $y_O = y_O(z_{k+1})$, defined as in (17) with ν_O instead of ν_G (here "O" stands for "others"). Finally, $x^{k+1} = x^k + \alpha d^O_{PRO}$ is chosen to minimize approximately

$$f_O(x;z) := (n+1)\ \ell n\ (c^Tx - z) - \Sigma\ \ell n\ x_j. \tag{21}$$

Several observations can be made here. Both projective-scaling methods generate directions that are linear combinations of the affine-scaling direction d_{AFF} of (3) and the "centering" direction $XP_{AX}e$, and differ only in the combination chosen. See Gonzaga [8] and Mitchell and Todd [12]. A similar remark holds for the dual solutions y. Finally, different potential functions are chosen by the two methods; Gonzaga's puts a weight n on the objective function part, while the others put weight $n + 1$. Hence, even if they used the same bound, the methods would generate different directions; even if they generated the same direction, minimizing the potential function would give different step sizes.

However, suppose Gonzaga's method is applied to the problem $(\bar{P})$ with $p = 1$, i.e., one dummy variable is added. Then the resulting $(\dot{P})$ exactly corresonds to (P). Moreover, the appropriate potential function is now

$$\bar{f}_G(\bar{x};z) = \bar{n}\ \ell n(\bar{c}^T\bar{x}-z) - \sum_{j=1}^{\bar{n}} \ell n\ \bar{x}_j$$

$$= (n+1)\ \ell n(c^Tx-z) - \sum_{j=1}^{n} \ell n\ x_j \qquad (22)$$

where $\bar{x}^T = (x^T,1)$. Hence we have

Proposition 2. Gonzaga's standard-form projective-scaling algorithm applied to (EVP) with $p = 1$ generates a sequence of points corresponding exactly to that generated by the standard-form projective-scaling algorithm of Anstreicher, Gay, Jensen and Steger, and Ye and Kojima applied to (P), assuming corresponding line search methods are used. The same is true for de Ghellinck and Vial's method, given that the initial point is feasible.

This result can also be seen more formally by substituting $\bar{A}$, $\bar{b}$ and $\bar{c}$ into (13)-(18). Note in particular that

$$\bar{b}^T(\bar{A}\ \bar{X}^2\ \bar{A}^T)^{-1}\bar{b} = (b^T,1) \begin{bmatrix} AX^2A^T & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix}$$

$$= b^T(AX^2A^T)^{-1}b + 1. \qquad (23)$$

An interesting observation can be made about the way the two methods update their lower bounds, assuming now that Gonzaga's method is applied directly to (P). Then Gonzaga sets z_{k+1} to the maximum z so that $y_G(z)$ is feasible in

$$
(\mathrm{D}) \qquad \begin{aligned} \max \quad & b^T y \\ & A^T y \le c \\ & y \ge 0; \end{aligned}
$$

this corresponds to the maximum z so that $P_{\hat{A}} \hat{c}(z)$ is nonnegative. Note that

$$
b^T y_G(z) = z, \qquad (24)
$$

so that z_{k+1} is the value of the feasible dual solution $y^{k+1} = y_G(z_{k+1})$. However, the other method sets z_{k+1} to the maximum z so that $P_{\hat{A}} \hat{c}(z)$ is nonnegative, where now $\dot{A}$ and $\dot{c}$ are as in $(\check{P})$. This corresponds to the maximum z so that

$$
A^T y_O(z) \le c \quad \text{and} \quad b^T y_O(z) \ge z. \qquad (25)
$$

Note that $b^T y_O(z)$ may be different from z, so that z_{k+1} may be less than the value of the feasible dual solution $y_O(z_{k+1})$.

The present development (and proposition 2) give some explanation. The dual of $(\overline{P})$ (with p = 1) is

$$
(\overline{\mathrm{D}}) \qquad \begin{aligned} \max \quad & b^T y + \eta \\ & A^T y \qquad \le c \\ & \qquad \eta \le 0 \end{aligned}
$$

and Gonzaga's method applied to $(\overline{P})$ will generate a dual solution

$\overline{y}_G(z) = \begin{bmatrix} \tilde{y}_G(z) \\ \eta_G(z) \end{bmatrix}$ and set z_{k+1} to the maximum z such that $\overline{y}_G(z)$ is feasible in $(\overline{D})$. Then it is quite possible that $\eta_G(z) < 0$, so that $\overline{b}^T\overline{y}_G(z) < b^T\tilde{y}_G(z)$.

Several authors (de Ghellinck and Vial [3], Todd [15] and Ye and Kojima [18]) have proposed improved lower bounds. In the notation of the previous paragraph, these correspond to choosing the maximum z' such that $\tilde{y}_G(z')$ satisfies $A^Ty \leq c$, and then setting $z_{k+1} = b^T\tilde{y}_G(z')$, i.e. to choosing $\eta = 0$ instead of $\eta = \eta_G(z)$. Since $\{\tilde{y}_G(z') = y_0(z')\} = \{y_G(z)\}$ (although the parameterizations differ), this improved bound is exactly that produced by Gonzaga's method applied to (P).

Finally, we compute the limiting direction of the projective-scaling methods applied to (EVP) as the number p of dummy variables tends to infinity. We confine ourselves to Gonzaga's method, as the argument above shows that it will coincide with the other method with p-1 dummy variables.

Let

$$\overline{\upsilon}_G := \overline{\upsilon}_G(z_{k+1}) = \frac{\overline{c}^T\overline{X}^2\overline{A}^T(\overline{A}\overline{X}^2\overline{A}^T)^{-1}\overline{b} - z_{k+1}}{\overline{b}^T(\overline{A}\overline{X}^2\overline{A}^T)^{-1}\overline{b}}$$

<u>Lemma 3</u>. For any $z_0 \leq z \leq z_*$, $\overline{\upsilon}_G(z) \to 0$ as $p \to \infty$.

<u>Proof</u>. A computation like that in (23) shows that the denominator in $\overline{\upsilon}_G$ is $b^T(AX^2A^T)^{-1}b + p$. Now

$$\overline{c}^T\,\overline{X}^2\,\overline{A}^T(\overline{A}\,\overline{X}^2\,\overline{A}^T)^{-1}\,\overline{b} = (c^T,0)\begin{bmatrix} X^2A^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (AX^2A^T) & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} b \\ e \end{bmatrix}$$

$$= c^TX^2A^T(AX^2A^T)^{-1}b$$

is independent of p, while z is bounded. Hence the conclusion follows

We now easily obtain

Proposition 4. If Gonzaga's method is applied to (EVP) with p dummy variables, the component of the resulting direction $\bar{d}^G_{PRO}$ in the original variables converges to the affine-scaling direction d_{AFF} as $p \to \infty$.

Proof. As in (7), we find

$$-\bar{X} P_{\bar{A}\bar{X}} \bar{X} \bar{c} = \begin{bmatrix} -X P_{AX} X c \\ 0 \end{bmatrix} = \begin{bmatrix} d_{AFF} \\ 0 \end{bmatrix}$$

and

$$\bar{X} P_{\bar{A}\bar{X}} e = \begin{bmatrix} X P_{AX} e \\ 0 \end{bmatrix} .$$

Hence the x-component of $\bar{d}^G_{PRO}$ is

$$d_{AFF} + \bar{\nu}_G XP_{AX} e$$

which converges to d_{AFF} as $p \to \infty$ by lemma 3.

It follows immediately that the projective-scaling methods are not invariant under the addition of dummy variables.

4. Conclusions.

We have seen that, while the simplex method and the affine-scaling interior-point method are invariant under the addition of dummy variables, the projective-scaling interior-point methods are not. In a sense, the latter are driven by potential functions, which strive to keep all variables bounded away from zero. The addition of dummy variables distorts the definition of such a potential function, since as far as the algorithm is concerned, all these new variables might approach zero and thus require barrier terms to prevent this. In turn, the extra barrier terms force an increase in the weight assigned to the objective function compared to the original barrier terms.

Proposition 4 shows that, at any given feasible point, as the number of dummy variables tends to infinity, the direction generated by the projective-scaling methods in the original variables converges to that generated by the affine-scaling algorithm. Now suppose the feasible region of (P) is bounded and all feasible solutions are nondegenerate. Then, as shown by Kallio [9] and Todd [15], $(AX^2A^T)^{-1}$ is uniformly bounded for feasible x. In this case, the convergence of the x-component of $\bar{d}^G_{PRO}$ to d_{AFF} as $p \to \infty$ is uniform in the current feasible solution, since clearly all other quantities remain bounded. In addition, d_{AFF} is a uniformly continuous function of the current solution. Hence, assuming step sizes are chosen in a consistent way (e.g., going a fixed proportion of the way to the boundary), the result in the original variables of taking <u>any</u> <u>fixed</u> number of steps in the projective-scaling methods applied to (EVP) tends to the result of taking the same number of steps in the affine-scaling method applied to (P) as $p \to \infty$.

Jim Renegar (private communication) has pointed out an intriguing consequence of this fact. The excellent computational experience with both affine- and projective-scaling methods has led some to conclude that the number of steps required to attain a given precision depends only on the precision required, and not on the dimension. Thus, while Karmarkar [10], proved on $O(nL)$ iteration bound, and various path-following methods have an $O(\sqrt{n}\,L)$ bound (see, e.g., [16]), perhaps a bound of $O(L)$ for the original algorithm and the projective-scaling variants of section 3 can be established. Here L denotes the input length of the problem, assumed to have integer data. If L appears only because of the precision required, and if the step sizes chosen converged appropriately, such a bound would imply a similar bound on the number of steps required for the affine-scaling method because of the statement at the end of the previous paragraph. However, a polynomial time bound for the latter is believed unlikely, partly based on the limiting results of Megiddo and Shub [11]; this suggests that some dependence on n (perhaps only logarithmic) is necessary in the projective-scaling methods. Unfortunately, obtaining such lower bounds in interior-point methods appears difficult.

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