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ABSTRACT COMPLEMENTARY PIVOT THEORY

Michael J. Todd

A Dissertation Presented to the Faculty of the
Graduate School of Yale University
in Candidacy for the Degree of
Doctor of Philosophy

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ABSTRACT

This thesis introduces combinatorial systems possessing a generalization of the pivot step of linear programming and complementary pivot theory. These systems provide a natural setting for the study of complementary pivot algorithms. Our aim is to provide a theoretical basis for complementary pivot theory, in the same way as the study of polytopes provides a basis for the theory of linear programming. In particular, we find bounds of the maximum diameter of these combinatorial systems. The diameters of these systems are closely related to the number of iterations required in complementary pivot algorithms, just as the diameters of polytopes are closely related to the number of iterations required for the simplex algorithm applied to a bounded non-degenerate linear program. We show that systems of comparable size to those arising from polytopes in n-space with k facets have maximum diameter of the order of

\[ \frac{5(k-n)^2}{54} \]
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INTRODUCTION

One of the major techniques applied to the solution of managerial and economic problems has been that of linear programming algorithms. In recent years, several writers have found that a new class of algorithms, known as complementary pivot algorithms, can be used to solve a much wider class of problems in economics, mathematical programming and game theory. For example, algorithms of this type can be used to find approximate fixed points of continuous mappings on a simplex [20, 28], compute approximate economic equilibria [14, 15, 29], find a core vector in a balanced game [27], find a Nash equilibrium point of a bimatrix game [21, 22], and solve quadratic programming problems [6, 10].

These algorithms are based on a radically new approach, incorporating a novel proof of convergence using combinatorial reasoning which does not rely on the monotonicity arguments fundamental to the convergence proof of optimization algorithms. In fact, the problems solved by these algorithms usually have no objective function to be optimized.

The common feature of the well known simplex algorithm for linear programming and the complementary pivot algorithms is the so-called pivot step. In the case of a non-degenerate bounded linear program, this step entails entering a column into a set of columns which forms a feasible basis; this column forces a unique column to leave that set, and a new
feasible basis results. In the case of Scarf's primitive assertion [12] or Kuhn's simplicial subdivisions [20] we find a different kind of pivot step --from a set of vectors which forms a primitive set, a vector is dropped; and a unique replacement can be found. This can be called a dual pivot step.

In linear programming the pivot step is just one property of the simplex algorithm. In fact, all properties of linear programming can be thought of as properties of polyhedra, which are the intersections of a finite number of half-spaces in real n-spaces. Polyhedra, and more particularly bounded polyhedra or polytopes, have been studied by mathematicians from the ancient Greeks to the present day. Linear programming gave a new impetus to this study. The present state of knowledge can be found in the survey papers of Klee [10] and Grünbaum [12].

In complementary pivot algorithms, on the other hand, the pivot step is the only property necessary to obtain a solution. Algorithms of this type have been applied to systems which seem to have little in common besides this pivot step.

The aim of this dissertation is primarily to study from a theoretical standpoint all systems possessing a generalization of this pivot operation.

Let us indicate how our defining axioms generalize the pivot step. We confine ourselves throughout to the case corresponding to a bounded non-degenerate linear program or a simple polytope. An n-dimensional polytope is simple if each of its vertices lies on just n of its bounding hyperplanes or facets.

The vertices of the feasible region of a bounded non-degenerate linear program correspond exactly to the feasible bases of the linear program.
We may describe each feasible basis by the set of indices of its basic variables—each such set will have the same number of elements, say \( n \). Moreover, if we introduce any non-basic variable into the basis, exactly one of the basic variables will become non-basic. This corresponds to moving along an edge of the polyhedral feasible region from one vertex to a neighboring one. Correspondingly, if we introduce any new index into a set of indices corresponding to a feasible basis, just one of these indices will drop out, to give a new set corresponding to a feasible basis. Equivalently, every set of \( n+1 \) indices contains 0 or 2 sets of \( n \)-indices corresponding to a feasible basis.

We will generalize this property by defining a semi-primoid of order \( n \) on a set \( S \) as a collection of \( n \)-element subsets of \( S \), called vertices, such that any \( n+1 \)-element subset of \( S \) contains an even number of vertices of the semi-primoid.

Dually, we have the notion of semi-duoid, a system which possesses a dual pivot step. Consider a simple polytope in \( n \) dimensions, with \( k \) bounding facets. Each vertex may be described by the set of \( n \) of these \( k \) facets on which it lies. If we take any \( n-1 \) of the facets, their intersection will either be empty or will be an edge of the polytope. This edge will be incident to two vertices of the polytope. Correspondingly, any set of \( n-1 \) facets will be contained (as a set) in either 0 or 2 of the \( n \)-sets corresponding to the vertices.

We will generalize this by defining a semi-duoid of order \( n \) on a set \( S \) as a collection of \( n \)-element subsets of \( S \), called vertices, such that any \( n-1 \) element subset of \( S \) is contained in an even number of vertices of the semi-duoid.
Although complementary pivot algorithms can be applied to any semi-primoid and semi-duoid, in almost all cases the semi-primoid and semi-duoid are minimal collections satisfying these defining properties. In this case we will call them primoids and duoids. Thus the class of duoids does not include such semi-duoids as those arising from two disjoint polytopes.

To justify the use of such general structures, we must show that the abstraction allows us to prove theorems which are worth the price we pay for this generalization. In our case, the abstraction allows us to use matroid theory or combinatorial topology to prove an important characterization result, which states that all semi-duoids or semi-primoids can be built up from very simple building blocks. Without this abstraction, it is certainly impossible to use matroid theory or combinatorial topology, and it is far from clear how the characterization result may be obtained.

It may be useful here to give an intuitive example showing how a complementary pivot algorithm works. The full description of the algorithm must be deferred until Chapter 2, but this small example shows some common properties of all complementary pivot algorithms.

Imagine a simple polytope in 3-dimensional space bounded by 6 facets. On this polytope are two timid bugs who are constrained to the vertices and edges of the polytope. Initially, bug I and bug II are on two opposite vertices, i.e., vertices which do not lie together on any one plane, or facet, of the polytope. Thus the vertices lie on 3 planes each, and these sets of planes are complementary.

Bug I sets out to investigate his world by crawling along any edge, leaving say facet 1, and reaching a new facet, say facet 2. Bug II's vertex lies on facet 2, and being timid, bug II crawls away down the only
edge which leaves facet 2. If he reaches facet 1, the bugs are once again
or: opposite vertices, and the operation stops. Otherwise, he reaches a
new facet, say facet 3, on which lies bug I's vertex. Now bug I crawls
away. The process continued until the bugs are once again on opposite
vertices. Let us illustrate this process:

In this example, five steps are necessary; they are indicated on
the diagram by arrows on the appropriate edges.

We note the following properties:

1) The two bugs crawl alternately (complementary pivots).

2) The bugs start and finish at complementary (or opposite) vertices.

3) At each step except the first and last, bug I is on 3 facets
   and bug II on 3 facets; these two sets share one member (almost
   complementary vertices).
4) The process terminates whenever bug I returns to facet I, or bug II first visits facet I; at each stage the only facet containing neither of the bugs' vertices is facet I.

5) Each crawl is uniquely determined by the position of the two bugs and a knowledge of which bug should crawl next.

We now describe briefly the contents of each chapter.

In Chapter 1 we provide formal definitions and illustrate them with many examples.

Chapter 2 is devoted to complementary pivot theory and establishes that the systems we have defined do in fact provide a natural setting for the study of complementary pivot theory. We present a generalized complementary pivot problem which includes as special cases almost all problems which have been solved by complementary pivot algorithms. The exceptional problems which do not fit into this framework are those involving systems with many vertices from which no pivot step can be performed. However, the special structure of these problems guarantees that no such pivot step will ever be required during the course of the algorithm. To provide a general theory, we must ignore these problems with special structure, and restrict ourselves to those problems with a complete pivot structure.

We prove the existence of a solution to the generalized problem and give an algorithm to find such a solution.

Chapter 3 investigates the combinatorial structure of semi-primoids, semi-duoids, primoids, and duoids. We show that the primoids and duoids of a given size form the circuits and co-circuits of a binary matroid. As noted earlier, our level of abstraction is necessary to construct this matroid. By means of matroid theory we obtain a major result providing
a characterization of semi-duoids (or, dually, semi-primoids) which shows that all semi-duoids can be formed from very simple "building blocks."
The rest of the chapter is devoted to tests to determine whether a semi-duoid is in fact a duoid. These tests are mainly used to check examples in later chapters, but they can also be used to list all duoids of a certain size.

Chapter 4 compares our duoids and semi-duoids with combinatorial structures of other kinds. Since the most important examples of semi-duoids are those arising from simple polytopes, we first derive a number of properties possessed by these semi-duoids. The first of these properties is obviously that these semi-duoids must be duoids. Using results from Chapters 2 and 3, we justify our focus on duoids rather than on semi-duoids. In the course of deriving these properties, we compare our duoids with the abstract polytopes of Adler and Dantzig [1, 2, 3].

In the second half of Chapter 4 we investigate another similar combinatorial structure, arising from combinatorial topology. We show how this structure, introduced by Crapo and Rota [8], resembles our duoids and investigate fully the relationships of our semi-duoids and duoids to combinatorial topology. In the course of this study we see that topological reasoning provides an alternative proof to the characterization of semi-duoids obtained in Chapter 3.

Chapter 5 is a short technical section which introduces three operations under which duoids are closed. Each of them has a simple geometrical interpretation. The operations of product and wedge are based on those of Klee and Walkup [19]; the sum operation was introduced by Adler [3]. These operations are useful for providing examples in the later chapters.
In Chapter 6 we investigate the range of Euler characteristics of duoids. The Euler characteristic of a duoid gives another method of demonstrating that the duoid does not represent a polytope. Moreover, the range of Euler characteristics shows that duoids arising from polytopes are an insignificant proportion of all duoids. For dimensions larger than 3, any integer can be realized as the Euler characteristic of some duoid. In dimension 3, all integers less than two can be realized; the proof of this result leads us to a comparison of 3-dimensional duoids and triangulated 2-manifolds.

In Chapter 7 we obtain lower bounds on the maximum diameter of duoids, proper duoids, and normal duoids. In linear programming much interest is focused on the maximum diameter of polytopes with a given dimension and number of facets. The same measure is of interest concerning duoids in complementary pivot algorithms. In both cases the maximum diameter will provide bounds on the minimum number of iterations required in the worst possible case.

Hirsch (see [9], pp. 160 and 168) conjectured the maximum diameter of convex polytopes in \( n \) dimensions with \( k \) facets to be at most \( k-n \). This conjecture has been confirmed for \( n \leq 3 \) [17] and \( k-n \leq 5 \) [19]. This last result has been extended to abstract polytopes [2].

We present a class of duoids for which the generalized Hirsch conjecture is violated. Some researchers suspect that the Hirsch conjecture is false, and Klee is working on a counter-example at the level \( k-n = 6, 7 \) or 8. However, our results stand as the first negative results obtained in this area.
We will show that the maximum diameter of duoids of dimension $n$ on a $k$-set is at least \[
\left[ \frac{5}{6} \left( k - n + 5 \right) \left( \frac{k - n - 3}{9} \right) - 5 \right] \] if $n \geq 6$. This is of the order $\frac{5}{54} (k-n)^2$; thus the discrepancy with the Hirsch bound is large.

Moreover, we show that duoids which possess some of the "regularity" properties of simple polytopes introduced in Chapter 4 attain a diameter as large as that above.
CHAPTER 1

PRELIMINARIES

1.1. **Notation**

A ground set $S$ of finite cardinality $k$ is given throughout almost all the thesis. Elements of $S$ will be called facets or hyperplanes and denoted by $e$, $f$, $g$, etc.; in examples we often take $S$ to be the set of the first $k$ integers. Subsets of $S$ will be denoted by capitals $P$, $D$, $G$ and $T$ will generally be of cardinality $n$, and usually called vertices; $U$ will generally have cardinality $n-1$ ($U$ for undersized) and $V$ generally $n+1$ ($V$ for oversized). The cardinality of $H$ will be denoted by $|H|$. Collections of subsets of $S$ will be denoted by Greek letters, $\pi$, $\delta$, $\gamma$, $\sigma$, etc., or by a set surrounded by brackets, $<V>$ or $>U<$. When we give examples of such collections, we will often omit the braces for each set; thus, $\{123, 345\}$ is shorthand for $\{[1,2,3], \{3,4,5\}\}$. The usual set-theoretic symbols $\cup$, $\cap$, $\setminus$ (set subtraction) will be used. We will write $D \cup f$ for $D \cup \{f\}$ and $P/e$ for $P/\{e\}$ when no confusion will result. The symbol $+$ will be reserved almost entirely for symmetric difference of sets; it should be clear when the normal number-theoretic meaning is intended. Recall that the symmetric difference of two sets is just the set of elements in exactly one. Hence the symmetric difference of two disjoint sets is just
their union; when we want to indicate that the sets are disjoint we use
the symbol $\oplus$. It is easily seen that symmetric difference is associa-
tive and commutative; thus, we may define the symmetric difference of a
number of sets without confusion. The symmetric difference is simply the
set of elements in an odd number of these sets; we will use the $\Sigma$ nota-
tion for this symmetric difference.

1.2. Definitions

By means of the following definition, we can avoid using cardinality
repeatedly.

Definition 1.2.1. $\sigma_j \triangleq \{T \subseteq S \mid \lvert T \rvert = j\}$

If $\sigma \subseteq \gamma \subseteq \tau_n$, we call $\sigma$ a vertex of $\gamma$.

Now we may define semi-primoids. We give two equivalent axioms.
The first is a generalization of the fact that each edge of a polytope
contains two vertices of the polytope, while the second is a generalization
of the well-known pivot step in linear programming.

Definition 1.2.2. $\pi \subseteq \sigma_n$ is a semi-primoid of order $n$ on $S$ if

P1) If $V \in \sigma_{n+1}$, then $V$ contains an even number of vertices of $\pi$;

or equivalently, if

P2) If $P \in \pi$, $e \in S/P$, then there is an odd number of facets (i.e.

elements of $S$) $f \in P$ such that $P' = P \cup e/f \in \pi$.

We say $P'$ is obtained from $P$ by pivoting in $e$.

If the even number in (P1) is 0 or 2 for all $V \in \sigma_{n+1}$, or equi-

valently if the odd number in (P2) is 1 for all $P \in \pi$, $e \in S/P$, then

$\pi$ is called proper. If $\pi$ is a minimal non-empty semi-primoid, then
\( \pi \) is called a **primoid**, \( k-n \) is the **dimension** of \( \pi \) (recall that \( k = |S| \)); if \( \pi \) represents some polytope \( P \) (see 1.3.1) then this is also the dimension of \( P \).

The equivalence of (P1) and (P2) is easily seen.

(P1) \( \implies \) (P2) Let \( V = P \cup e \).

(P2) \( \implies \) (P1) If \( V \) contains no vertices of \( \pi \) then (P1) holds. If it contains at least one, say \( G \), take \( P = G \) and \( e \in V/G \) in (P2) and (P1) follows.

**Definition 1.2.3.** \( \delta \subseteq \sigma_n \) is a **semi-duoid of order** \( n \) on \( S \) if

1. If \( U \in \sigma_{n-1} \), \( U \) is contained in an even number of vertices of \( \delta \); or equivalently, if
2. If \( D \in \delta \), \( e \in D \), then there is an odd number of facets (i.e., elements of \( S \)) \( f \in S/D \) such that \( D' = D \cup f/e \in \delta \).

We say that \( D' \) is obtained from \( D \) by **pivoting out** \( e \).

Proper semi-duoids and duoids are defined in the obvious manner, analogously to the case of semi-primoids. Note that if \( \delta \) is a (semi-) duoid on a set \( S \), then it is also a (semi-) duoid on any superset of \( S \). \( n \) is the dimension of \( \delta \); if \( \delta \) represents some polytope \( P \) (see 1.3.1*), then \( n \) is also the dimension of \( P \).

We note that proper duoids are exactly the pseudomanifolds familiar in topology and used by Kuhn [20] in his statement of the complementary pivot algorithm. The other close resemblance is with the abstract polytopes of Adler and Dantzig [1, 2, 3]; these similarities will be studied in Chapter 4.
1.3. Examples

1.3.1. A simple d-dimensional polytope is one whose every vertex is on precisely d of its d-1-faces, or facets. The vertices of such a polytope, each described by the set of facets on which it does not lie, form a proper primoid on the set of all bounding facets.

1.3.2(a) Let the feasible region of a bounded non-degenerate linear program be \( \{x| Ax = b, x \geq 0\} \) where \( A \) is \( m \times n \). Let \( S \) be \( \{1, 2, \ldots, n\} \); and for each feasible basis \( x_{j_1}, \ldots, x_{j_m} \) let \( \{j_1, j_2, \ldots, j_m\} \) be a vertex of \( \pi \). Then \( \pi \) is a proper primoid of order \( m \) on \( S \).

(b) Drop the bounded assumption, but assume that \( A \) has the special form \( [-I, M] \), where \( M > 0 \) is \( m \times n-m \), \( I \) is \( m \times m \), and \( b > 0 \).
As before, let \( S = \{1, 2, \ldots, n\} \), and for each feasible basis \( x_{j_1}, \ldots, x_{j_m} \), let \( \{j_1, j_2, \ldots, j_m\} \) be a vertex of \( \pi \). Then \( \pi \cup \{1, 2, \ldots, m\} \) is a proper primoid on \( S \).

The reader is urged to think of primoids in terms of 1.3.1 or 1.3.2(a); it is for this reason that elements of \( S \) are called facets, and \( n \)-sets vertices, to encourage the geometric visualization. \( n+1 \)-sets (or \( n-1 \)-sets in the case of a duoid) can then be naturally associated with the edges of the polytope. Usually we use the numbers 1, 2, \ldots, \( k \) to refer to the \( k \) facets of the simple polytope; obviously the resulting primoid depends on the particular correspondence of the numbers with the facets, or, loosely speaking, on the labelling. Sometimes to provide pathological examples, we give the same label to two different facets. If the two facets are sufficiently distant, a primoid will result.
It is usually helpful to think of a primoid or a duoid as a set of vertices, each of which is also an n-set, rather than as a family of subsets of S.

Other examples depend on a partition \( S_1, S_2, \ldots, S_m \) of S.

1.3.3. \( \pi_1 = \{P|P \cap S_i = 1 \text{ for all } i\} \) is a proper primoid of order m on S.

If \( j_1, j_2, \ldots, j_m \) are given odd numbers,

1.3.4. \( \pi_2 = \{P|P \cap S_i = j_i \text{ for all } i\} \) is a semi-primoid of order \( \sum_{i=1}^{m} j_i \) on S.

1.3.5. \( \pi_3 = \{P|\sum_{i=1}^{m} |P \cap S_i| = n \text{ and each summand odd}\} \) is a semi-primoid of order n on S.

1.3.6. Given \( U \in \sigma_{n-1} \), \( \pi_4 = \{P|U \subseteq P \in \sigma_n\} \) will be called a simplicial primoid and denoted by \( \uparrow U \downarrow \).

The reason for this nomenclature is simply that such a primoid corresponds (in the sense of 1.3.1) to a simplex. In fact, if we take the simplex to be bounded in \( \mathbb{R}^{k-n} \) by \( x_1 \geq 0, x_2 \geq 0, \ldots, x_{k-n} \geq 0 \), and \( \Sigma x_i \leq 1 \), let these k-n+1 facets be given distinct labels in S/U. Let \( x_1 \geq -1, \ldots, x_i \geq -n+1 \) be redundant facets with labels in U. Then the vertices of the simplex are \( (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1) \) and \( (0, 0, \ldots, 0) \), and each of these lies off all the redundant facets, and off just one of the bounding facets. Thus, referring to 1.3.1, each vertex is described by an n-set containing U.
Simplicial primoids (and duoids, see 1.3.9) are fundamental. We shall see later that they are the "building blocks" for all primoids and duoids (3.8 and following).

A transversal of a family of sets \( \{S_i\}_{i=1}^{m} \) is a system of distinct representatives from these sets, i.e., a set \( E = \{e_1, e_2, \ldots, e_m\} \) with \( e_i \neq e_j \) if \( i \neq j \), and \( e_i \in S_i \) for \( i = 1, 2, \ldots, m \). (For an account of transversal theory, see [25].) Then \( \pi_1, \pi_2 \) and \( \pi_4 \) above can also be considered as the set of all transversals of a certain family of sets; e.g., for \( \pi_2 \) let the family consist of \( j_i \) copies of \( S_i \) for each \( i \).

It is easy to see that if \( \delta = [D\mid S/D \in \pi] \) then \( \delta \) is a (proper) \([\text{semi-}]\) duoid of order \( n \) on \( S \) if and only if \( \pi \) is a (proper)[\( \text{semi-} \)] primoid of order \( k-n \) on \( S \). If \( \delta \) is obtained in this way from \( \pi \), it is denoted by \( \xi_{\pi} \); similarly, \( \pi \) is denoted by \( \xi_{\delta} \).

This fundamental result gives us a duality between (semi-) primoids and (semi-) duoids. Corresponding to every theorem about primoids (duoids), there will be a dual theorem about duoids (primoids), obtained by this complementation. It will be referred to by the same number with an asterisk affixed. We will sometimes quote such a dual result even if it has not been explicitly stated, if its content is immediately obvious from the "primal" result.

The same reasoning applies to examples. For instance, we have:

1.3.1*. The vertices of a simple polytope, each described by the set of facets on which it does lie, form a proper duoid on the set of all facets.
1.3.2*. (a) Consider the linear program introduced in 1.3.2(a). For each feasible basic solution, let \( x_{i_1}, x_{i_2}, \ldots, x_{i_{n-m}} \) be the non-basic variables, and let \( \{i_1, i_2, \ldots, i_{n-m}\} \) be a vertex of \( S \). Then \( S \) is a proper duoid.

1.3.3*. \( S = \{D \mid D \cap S_1 = |S_1| - 1 \text{ for all } i \} \) is a semi-duoid of order \( |S| - m \) on \( S \).

For other examples of semi-duoids and duoids, note the following:

1.3.7. If we are given a simplex (usually the standard simplex in \( \mathbb{R}^n \), the portion of the flat \( \sum_{i=1}^{n} x_i = 1 \) which is in the positive orthant) and a grid of points \( \pi^{n+1}, \ldots, \pi^k \) on the simplex satisfying the non-degeneracy condition that no two of them have any coordinate equal, then we form a proper duoid as follows. See [14, 15, 27-29].

Let the sides of the simplex be denoted by \( \pi^1, \pi^2, \ldots, \pi^n \), where \( \pi^i \) is the side on which \( x_i \) is zero. Then the \( n-m \) vectors \( \pi^1, \pi^2, \ldots, \pi^{n-m} \), along with the \( m \) sides \( \pi^i_1, \pi^i_2, \ldots, \pi^i_m \), form a primitive set if there is no vector from \( \pi^{n+1}, \ldots, \pi^k \) which lies in the interior of the simplex defined by \( x_{i_1} \geq 0, \ldots, x_{i_m} \geq 0 \) and \( x_i \geq \min(\pi^1_i, \pi^2_i, \ldots, \pi^{n-m}_i) \) for \( i \neq i_1, \ldots, i_m \).

If \( S' \) is the collection of primitive sets, \( S_1', S_2', \ldots, S_n' \), then \( S_1 = S' \cup \{\pi^1, \pi^2, \ldots, \pi^n\} \) is a proper duoid. We have in effect closed the simplex by pasting it onto itself.
1.3.8. If $\delta_2$ is the collection of n-simplices in a triangulation of an n-manifold, then $\delta_2$ is a proper duoid.

1.3.9. If $V \in \sigma_{n+1}$ we call $\delta_3 = \{D | D \subseteq V, D \in \sigma_n\}$ a simplicial duoid and denote it $\langle V \rangle$.

The following concrete examples may help to show how a primoid or duoid is obtained from its corresponding polytope. The primoid corresponding to the 2-polytope below, with its hyperplanes (lines) labelled as shown:

```
   3
   |
2   4
   |
   5
```

is $\{1,2,3\}$

(see example 1.3.1); we abbreviate it by 123

```
P
```

is $\{234, 345, 451, 512\}$.

The duoid corresponding to the 3-polytope below

```
 3 5
```

Q is described by $\{1,3,4\}$,
the hyperplanes on which it does lie.

```
 2
```

is $\{134, 145, 153, 234, 245, 253\}$. Note that in both cases, the dimension of the primoid or duoid equals that of the polytope.

We note the following implications (and analogously for duoids, etc.):
To show that no other implications exist, let $S = \{1, 2, \ldots, 6\}$; then,

\[1.3.10. \quad \pi_1 = \{125, 126, 134, 136, 234, 235, 245, 246, 345, 356\} \] shows that arrows (i) cannot be reversed, and

\[1.3.11. \quad \pi_2 = \{1234, 1235, 1236, 1456, 2456, 3456\} \] shows that arrows (ii) cannot be reversed.
CHAPTER 2

THE GENERALIZED COMPLEMENTARY PIVOT ALGORITHM

In this chapter we introduce a generalized complementary pivot problem, stated in terms of semi-primoids and semi-duoids, which includes as special cases almost all known problems solvable by complementary pivot algorithms. We prove the existence of a solution to this general problem and give an algorithm to solve it. Moreover, this algorithm is a generalization of those complementary pivot algorithms already used to solve particular problems.

We will begin by giving a brief history of the field. The first complementary pivot algorithm was devised by Lemke and Howson [21] to find a Nash equilibrium point for a bimatrix game. Previous algorithms were known for this problem, but none had the elegant simplicity or the creative genius of theirs. They used a new kind of convergence proof which until then had been the prerogative of combinatorialists. They invoked no monotonicity arguments—solutions generated prior to termination of the algorithm gave no clue to the final solution. Each individual step was a simple pivot step which had been familiar in linear programming for over a decade; but the manner in which these steps were fitted together was entirely novel. The proof of convergence was based implicitly on Euler's famous solution to the Königsberg bridge problem—its itself a creative leap forward which
stimulated the birth of graph theory (see [5] Chapter 3). The basic idea of Lemke and Howson found surprising applications in wide-ranging fields. Scarf [27] used the idea to constructively find the core of a balanced game. Kuhn [20] and Scarf [28] employed similar reasoning to find fixed points of continuous mappings on a simplex. Hansen [14] then found a method to make Scarf's and Kuhn's algorithms computationally feasible. A paper [15] by Hansen and Scarf listed six applications of their algorithm, including non-linear programming and the computation of equilibrium prices in an activity-analysis model of an economy. Meanwhile in the mathematical programming field, Cottle and Dantzig [6] with Lemke [21] worked on the so-called linear complementarity problem (find \( w, z \) satisfying \( w = Mz + q, \ w \geq 0, \ z \geq 0, \ w^Tz = 0 \) and showed it to be the "fundamental" problem of linear programming, quadratic programming, and bimatrix games. Many other authors have dealt with extensions of this theory; Eaves' paper [10] and its bibliography show the current state of the research.

A more complete description of some of these applications will follow our statement of the generalized complementary pivot problem.

The chapter is organized as follows. First we introduce the general problem and show how it applies to the particular problems mentioned above. Then we introduce the graph on which the proof of convergence is based. A simple graph theory result shows the existence of a solution to the generalized complementary pivot problem. We next give two graph theoretic algorithms which form the basis for the generalized complementary pivot algorithm. With the aid of a modified graph, constructed to resolve non-degeneracy, we present the generalized complementary pivot algorithm.
Finally we exhibit a possible application for which this level of generalization is necessary.

Before we introduce the generalized complementary pivot problem, recall that in Chapter 1, as well as examples of semi-primoids and semi-duoids, we also have examples of semi-primoids minus one vertex (1.3.2(b)) and semi-duoids minus one vertex (1.3.7). This missing vertex is crucial for providing a starting point for the algorithm. We now state

2.1. The generalized complementary pivot problem: We are given three things—a starting point \( G \in \sigma_n \), and two sets of vertices \( \gamma_1, \gamma_2 \subseteq \sigma_n \), where either

a) \( G \in \gamma_1/\gamma_2 \), \( \gamma_1 = \pi \) is a semi-primoid, and \( \gamma_2 \cup \{G\} = \delta \) is a semi-duoid; or

b) \( G \in \gamma_2/\gamma_1 \), \( \gamma_1 \cup \{G\} = \pi \) is a semi-primoid, and \( \gamma_2 = \delta \) is a semi-duoid.

The problem is to find a vertex \( H \) common to \( \gamma_1 \) and \( \gamma_2 \).

We will now give some examples of 2.1, to show that it does indeed include almost all known applications. In fact the applications of complementary pivot theory which are not examples of 2.1 are those which have vertices from which pivot steps cannot be made. These vertices are never reached in the course of the algorithm because of the special structure of these particular problems. As examples of 2.1, consider the following:

1) Scarf's problem [15, 27, 28, 29] includes as special cases finding fixed points, economic equilibria, etc. We are given a fine grid of points on the standard \( n \)-simplex, say \( \pi^{n+1}, \ldots, \pi^k, \pi^1, \pi^2, \ldots, \pi^n \) represent the \( n \) facets of the simplex. With each \( \pi^j \) is associated
an n-vector; for \(1 \leq j \leq n\), the associated vector is the \(j^{th}\) unit vector, while for \(j > n\), the vector is arbitrary. These vectors form a matrix \([I,A]\) whose \(j^{th}\) column is the vector associated with \(\pi^j\).

We also choose a right hand side vector \(b > 0\), and we assume that the feasible region \(\{x|[I,A]x = b, x \geq 0\}\) is a simple polytope. Scarf's problem is to find a primitive set \(\pi^1, \ldots, \pi^n\) such that \(x_j^1, \ldots, x_j^n\) form a basis of \([I,A]x = b, x \geq 0\).

To see that this is an example of 2.1, let \(\gamma_1\) be generated by the feasible bases of this system as in 1.3.2(a), and let \(\gamma_2\) be the collection of index sets of the primitive sets as in 1.3.7.

\[G = \{1, 2, \ldots, n\} \in \gamma_1/\gamma_2.\] Then \(\gamma_1\) is a semi-primoid and \(\gamma_2 \cup \{G\}\) a semi-duoid. 2.1(a) holds, and we see that an \(H \in \gamma_1 \cap \gamma_2\) solves Scarf's problem.

As an example of this problem, assume we wish to find a fixed point of a continuous mapping from the standard simplex into itself. Then associate with \(\pi^j\) the \(i^{th}\) unit vector, where \(i\) is the smallest index such that the \(i^{th}\) coordinate of \(f(\pi^j)\) is greater than that of \(\pi^j\).

Let \(b = (1, 1, \ldots, 1)^T\). Then a basis of the system is just a complete set of unit vectors. By solving Scarf's problem, we will find a primitive set, which is a set of very close vectors, such that for each \(i\), there is a vector in the set whose \(i^{th}\) coordinate increases under the mapping. Any vector in this primitive set is thus close to its image; a limiting argument gives an existence proof, and any solution gives an approximate fixed point.

2) Lemke-Howson's problem is to find a Nash equilibrium point for a bimatrix game [22]. Let player I(II) have pure strategies \(1, 2, \ldots, n\).
(1, 2, ..., n), and let the payoff to player I (II) when I plays $i$ and II plays $j$ be $a_{ij} (b_{ij})$. Assume without loss of generality that $A = \{a_{ij}\} > 0$ and $B = \{b_{ij}\} < 0$, since the addition of any constant to the payoffs of either player does not affect the optimal strategies.

Lemke and Howson showed that to find an equilibrium point it was sufficient to find solutions to $\begin{bmatrix} I, A \end{bmatrix} x = e, \ x \geq 0 \text{ and } \begin{bmatrix} -B^T, -I \end{bmatrix} y = e', \ y \geq 0$, with $x^T y = 0$. (Here the first $I$ is $n \times n$, $x$ and $y$ are $(n+m) \times 1$, $e$ is a vector of $n$ 1's, the second $I$ is $m \times m$, and $e'$ is a vector of $m$ 1's.) Assume that the two systems are non-degenerate. Then by comparing dimensions, we see that both $x$ and $y$ must be basic solutions.

Let $\gamma_1$ be formed from the first $(x)$ system as in 1.3.2(a).

Let $\gamma_3$ be formed from the second $(y)$ system as in 1.3.2(b). Then $\gamma_3 \cup \{n+1, ..., n+m\}$ is a semi-primoid. Let $\gamma_2 = \mathcal{G}_3$ (see remarks after 1.3.6). Then $\gamma_2 \cup \{1, 2, ..., n\}$ is a semi-duoid. If we take $G = \{1, 2, ..., n\} \cap \gamma_1 / \gamma_2$, 2.1(a) holds. A vertex $\mathcal{H}$ common to $\gamma_1$ and $\gamma_2$ yields complementary solutions $x$ and $y$, and thus solves Lemke-Howson's problem.

3) The linear complementarity problem is to find $w, z$ satisfying $w = Mz + q, w \geq 0, z \geq 0, w^T z = 0$. (Here $M$ is $n \times n, w, z, q$ are $n \times 1$.) In [10], a full characterization of matrices $M$ which can be processed by this type of algorithm is given, where "processed" means the algorithm converges or shows that no solution exists. Much special structure of this particular form is used (to deal with secondary rays, etc.). To show how this relates to our generalized problem we take the special case $M > 0, q < 0$. Then the linear relation can be written
as \([-I,M]\left(\begin{array}{c} w \\ z \end{array}\right) = -q ; \ w, z \geq 0\). Let \(\gamma_1\) be formed from the feasible bases of this system as in 1.3.2(b). Let \(\gamma_2 = \{D \mid D \cap \{i, i+n\} = 1 \text{ for } i = 1, 2, \ldots, n\}\). It is immediately clear that \(\gamma_2\) is a semi-duoid; it is also an instance of 1.3.3*. Let \(G = \{1, 2, \ldots, n\}\). Then 2.1(b) holds. A solution to 2.1 is a vertex \(H\) in \(\gamma_1\), i.e., a feasible basis of the linear system, also in \(\gamma_2\), i.e., satisfying the orthogonality condition \(w^Tz = 0\).

4) The generalized linear complementarity problem of Cottle and Dantzig [7] is to find a solution \(w, z\) to \(w = Mz + q, w, z \geq 0\)

\[
\begin{pmatrix}
\vdots \\
w^1 \\
\vdots \\
w^m
\end{pmatrix}, M \text{ is } n \times m, q = \begin{pmatrix}
\vdots \\
q^1 \\
\vdots \\
q^m
\end{pmatrix}, \text{ and } w^j \text{ and } q^j \text{ are}
\]

\(p_j \times 1, \sum_{j=1}^{m} p_j = n\) which also satisfies the generalized complementarity condition: \(z \prod_{i=1}^{m} w_j^i = 0 \text{ for } j = 1, 2, \ldots, m\). Assume that \(M > 0\), \(q < 0\) for simplicity. Let \(\gamma_1\) be generated by the feasible bases of \([-I,M]\left(\begin{array}{c} w \\ z \end{array}\right) = -q ; \ w, z \geq 0\), and let \(\gamma_3\) be generated by the partition \(S_1, S_2, \ldots, S_m\) of \(S = \{1, 2, \ldots, n+m\}\) where \(S_j = \{\sum_{i=1}^{j-1} p_i + 1, \ldots, \sum_{i=1}^{j-1} p_i + p_j, n+j\}\) as in 1.3.4. Let \(\gamma_2 = \gamma_3\), and let \(G = \{1, 2, \ldots, n\}\).

Then \(G \in \gamma_2 / \gamma_1\), \(\gamma_1 \cup \{G\}\) is a semi-primoid and \(\gamma_2\) is a semi-duoid. 2.1(b) holds and a solution of 2.1 will yield a feasible basis (because \(H \in \gamma_1\)) which also satisfies the generalized complementarity condition (because \(H \in \gamma_2\)).

Now we introduce the fundamental graph which provides an existence
theorem for a solution to 2.1, and is the basis of the generalized complementary pivot algorithm.

**Definition 2.2.** Given a semi-primoid \( \pi \), and a semi-duoid \( \delta \), both of order \( n \) on \( S \), and a facet \( e \in S \), then \( G(\pi, \delta, e) \) is the following graph:

The nodes are pairs \( (P, D) \in \pi \times \delta \) such that \( P \subseteq D \cup e \). It can easily be seen that the nodes consist of pairs \( (P, D) \) with \( P = D \) and pairs \( (P, D) \) with \( P \cap D \in \pi - 1 \), \( P/D = \{e\} \). The adjacencies are described as follows. \( (P, D) \) is adjacent to \( (P', D') \) iff any of 1)-4) holds.

1) \( P \neq D \), \( P' = P \), \( D' \) is obtained from \( D \) by pivoting out \( D/P \).
2) \( P \neq D \), \( D' = D \), \( P' \) is obtained from \( P \) by pivoting in \( D/P \).
3) \( P = D \), \( e \in P \), \( P' = P \), \( D' \) is obtained from \( D \) by pivoting out \( e \).
4) \( P = D \), \( e \notin P \), \( D' = D \), \( P' \) is obtained from \( P \) by pivoting in \( e \).

We can easily check that this is symmetric. The important fact about \( G(\pi, \delta, e) \) concerns the parity of the degrees of its nodes. Let \( (P, D) \) be any node of \( G(\pi, \delta, e) \).

**Case 1.** \( P \neq D \). Then \( (P, D) \) is adjacent to an odd number of nodes via

1) (since \( \delta \) is a semi-duoid, using (D2)) and an odd number of nodes via
2) (since \( \pi \) is a semi-primoid, using (P2)). For the former set of nodes, the first member of the pair is unchanged, while the second changes--this is a dual pivot. For the latter set of nodes, the situation is reversed--this is a primal pivot. We see that all these nodes are distinct. So \( (P, D) \) is adjacent to an even number of nodes in total.
Case 2. \( P = D \). Then \((P,D)\) is adjacent to an odd number of nodes via 3) (if \( e \in P \)) or an odd number of nodes via 4) (if \( e \notin P \)) but not both. In all, \((P,D)\) is adjacent to an odd number of nodes.

This characterization of the nodes of \( G(\pi, \delta, e) \) is fundamental. The nodes corresponding to shared vertices of \( \pi \) and \( \delta \) (i.e., nodes \((G,G)\) where \( G \in \pi \cap \delta \)) have odd degree, while all others have even degree. This fact immediately gives us our first result:

**Theorem 2.3.** If \( \pi \) is a semi-primoid, \( \delta \) a semi-duoid, both of order \( n \) on \( S \), then \(|\pi \cap \delta|\) is even. (\( \pi \) and \( \delta \) share an even number of vertices.)

**Proof.** By the first result of graph theory (Theorem 1.2 of [5]), the sum of the degrees of the nodes of a graph is equal to twice the number of edges of the graph, i.e., an even number. Thus the number of nodes of odd degree in a graph is even. Considering \( G(\pi, \delta, e) \) for any \( e \in S \) gives us the result immediately, in the light of the characterization above.

Theorem 2.3 guarantees the existence of a solution to 2.1. Let \( \pi \) and \( \delta \) be defined as in 2.1(a) or (b). Then \( \pi \) and \( \delta \) share an even number of vertices. One of these is \( G \). Any other, say \( H \), must be in \( \gamma_1 \) and \( \gamma_2 \).

We note in passing two corollaries.

**Corollary 2.4.** If \( \pi \) is a semi-primoid of odd dimension, \(|\pi|\) is even.
Proof. Consider $\sigma_n$. If $U \in \sigma_{n-1}$, there are an even number of vertices in $\sigma_n$ containing $U$ (obtained by adjoining to $U$ any of the even number of facets in $S/U$). Thus $\sigma_n$ is itself a semi-duoid, and so $|\pi| = |\pi \cap \sigma_n|$ is even.

Corollary 2.4*. If $\delta$ is a semi-duoid of odd dimension, $|\delta|$ is even.

Proof. Recall the remarks made after 1.3.6.

Before we go on to the algorithms, let us observe an interesting interpretation of 2.3. The definition of a semi-primoid is that it is a subset of $\sigma_n$, and that each $n+1$-set $V$, which we can think of as an edge, contains an even number of vertices of the semi-primoid. Using our notation for simplicial duoids (see 1.3.9), we see that $\pi$ is a semi-primoid iff $\pi \subseteq \sigma_n$ and

(P3) For every $V \in \sigma_{n+1}$, $|\pi \cap \langle V \rangle|$ is even; or

(P3') For every simplicial duoid $\delta$, $|\pi \cap \delta|$ is even.

The same analysis can be carried out for semi-duoids—$\delta$ is a semi-duoid iff $\delta \subseteq \sigma_n$ and

(D3) For every $U \in \sigma_{n-1}$, $|\delta \cap \langle U \rangle|$ is even; or

(D3') For every simplicial primoid $\pi$, $|\delta \cap \pi|$ is even.

Simplicial primoids and duoids can be thought of as "small" or "local" systems in the sense that each vertex is adjacent to every other. The "local" property of semi-primoids is that they share an even number of vertices with each "local" semi-duoid (or simplicial duoid). 2.3 can thus
be interpreted as showing that this local property extends to the corresponding global property— if \( \pi \) has the local property, it is a semi-primoid; and by 2.3, for any semi-duoid \( \delta \), \(|\pi \cap \delta|\) is even, i.e., it has the corresponding global property.

Now we find an algorithm to solve problem 2.1. This algorithm is based on traversing the edges of a graph without repetition, starting and finishing at two different odd nodes (nodes of odd degree). Such a walk will be called an Euler trail. We will follow the notation of [5]. A walk from \( v_0 \) to \( v_m \) in a graph \( G \) is a sequence \((v_0, e_0, v_1, e_1, \ldots, e_{m-1}, v_m)\) of alternating nodes and edges of \( G \), such that for \( 1 \leq i \leq m \), \( e_{i-1} \) is the edge \( \{v_{i-1}, v_i\} \). This walk is a trail if all the edges \( e_i \) are distinct; it is a path if all the nodes \( v_i \) (and so all the edges \( e_i \)) are distinct.

The algorithms we consider are based on reasoning similar to that used in Euler's solution to the Königsberg bridge problem. (See Chapter 3 of [5]). To show that we will always reach an odd node, consider the following algorithm.

**Algorithm 1.** Given an odd node \( v \) of a graph \( G = (V, E) \), find another odd node of \( G \).

**Step 1.** Set \( v_0 = v \), the given odd node. Set \( E_0 = E \), and \( i = 0 \).

**Step 2.** Pick any edge \( e_i \), not a loop, in \( E_i \) and incident to \( v_i \). Call its other endpoint \( v_{i+1} \). Stop if \( v_{i+1} \neq v_0 \) is odd. Otherwise, let \( E_{i+1} = E_i / e_i \).

**Step 3.** Increase \( i \) by one; return to step 2.
Proof of finite convergence to a solution: Since the $E_i$ are decreasing, the algorithm must terminate with a solution if all the steps can be carried out. We will show that, at each step before termination, $v_i$ has odd degree in $(V, E_i)$. This guarantees the existence of an edge $e_i$ of the required form.

For $i = 0$, $v_0$ has odd degree in $(V, E_0) = G$ by assumption.

For $i > 0$, let $F_i = E/E_i = \{e_0', e_1', \ldots, e_{i-1}'\}$. Then the edges in $F_i$ are the edges of a trail from $v_0$ to $v_i$.

Case 1. $v_i = v_0$. Then the trail finishes at its starting point. So we return to $v_i$ as many times as we leave it. Hence $v_i$ has even degree in $(V, F_i)$, and odd degree in $G = (V, E)$. So $v_i$ has odd degree in $(V, E/F_i) = (V, F_i)$.

Case 2. $v_i \neq v_0$. Then the trail does not finish at its starting point. Thus we leave $v_i$ one less time than we reach it. Hence $v_i$ has odd degree in $(V, F_i)$, and even degree in $G = (V, E)$; otherwise, the algorithm would stop at $v_i$. So $v_i$ has odd degree in $(V, E/F_i) = (V, E_i)$.

Algorithm 1 has been shown to converge, giving an Euler trail from $v$ to another odd node $v'$.

The computational difficulty with this algorithm is that each edge $e_i$ must be recorded to ensure that it is not repeated. A much simpler algorithm is available if all nodes of the graph are known to have degree less than three.
Algorithm 2. Given an odd node \( v \) of a graph \( G = (V,E) \) whose every node has degree less than three, find another odd node of \( G \).

Step 1. Set \( v_0 = v \), the given odd node, and \( i = 0 \).

Step 2. Pick any edge \( e_i \), not equal to \( e_{i-1} \) if \( i > 0 \), incident to \( v_i \). Call its other endpoint \( v_{i+1} \). Stop if \( v_{i+1} \neq v_0 \) is odd.

Step 3. Increase \( i \) by one; return to Step 2.

Note that in Algorithm 2, we only "remember" each edge for one more iteration; thus the storage necessary to avoid cycling is minimal.

Proof of finite convergence to a solution: We need to show only that

1) \( e_i \) can always be found in Step 2.
2) \( e_i \) is not equal to \( e_j \) for \( j < i \).

Then proof of convergence will follow from the argument used for Algorithm 1.

Assume 1) and 2) are true for \( 0, 1, 2, \ldots, i-1 \). We will show that they hold for \( i \). \( F_i = \{e_0, e_1, \ldots, e_{i-1}\} \) is a trail from \( v_0 \) to \( v_i \). The same argument used in Algorithm 1 will show that an edge \( e_i \) in \( E/F_i \) can be found incident to \( v_i \). This satisfies the requirements of Step 2. Thus 1) is proved for \( i \).

To prove 2), let us assume to the contrary that \( e_i = e_j \) for some \( j < i \). We distinguish two cases.

Case 1. \( v_j = v_i \), \( v_{j+1} = v_{i+1} \) (i.e., \( e_i \) and \( e_j \) are traversed in the same direction). Then \( e_{j-1} \) is distinct from \( e_j \) and is incident to \( v_j \), and \( e_{i-1} \) is distinct from \( e_i \) and is incident to \( v_i \). By our induction hypothesis, \( e_{i-1} \neq e_{j-1} \). So \( v_j \) is incident to at least
three distinct edges, \( e_{i-1}, e_j, \) and \( e_i = e_j \), contrary to our hypothesis that all nodes have degree at most two.

**Case 2.** \( v_j = v_{i+1}, v_{j+1} = v_i \) (i.e., \( e_i \) and \( e_j \) are traversed in opposite directions). Then \( e_{j+1} \) is distinct from \( e_j \) and is incident to \( v_{j+1} \), and \( e_{i-1} \) is distinct from \( e_i \) and is incident to \( v_i \).

If \( e_{i-1} \neq e_{j+1} \), we obtain the same contradiction as in Case 1. So \( e_{i-1} = e_{j+1} \), and by our induction hypothesis \( i-1 = j+1 \). But then \( e_{j+1} \) joins \( v_{j+1} = v_i \) and \( v_{j+2} = v_i \), so \( e_{j+1} \) is a loop. Then \( v_i \) has degree at least three (two from \( e_{j+1} \) and one from \( e_j \)) contrary to hypothesis.

1) and 2) hold trivially for \( i = 0 \). Thus 1) and 2) and hence convergence of Algorithm 2, follow by induction.

Note that in this case the Euler trail is in fact an Euler path.

No nodes can be repeated, for this would imply a degree greater than two.

We now have the machinery to solve the generalized complementary pivot problem 2.1. By considering the definition of \( G(\pi, \delta, e) \) (2.2), where \( \pi \) and \( \delta \) are as in 2.1(a) or (b), we see that each node of this graph has degree at most two if \( \pi \) and \( \delta \) are proper. By starting at the "artificial" node (\( G, G \)) of \( G(\pi, \delta, e) \) and using Algorithm 2, we can find another odd node, which will be a solution (\( H, H \)) to 2.1. If either \( \pi \) or \( \delta \) is not proper, then nodes of \( G(\pi, \delta, e) \) may have degree at least three, and Algorithm 2 can no longer be used. We could use Algorithm 1 to solve this problem, but as noted earlier we then need a large storage capacity to avoid cycling. We also increase the time for each iteration, since we must search through our list of excluded edges at each Step 2.
Even refinements of Algorithm 1 (combinations of Algorithms 1 and 2 which record only edges which might be repeated) require a search through such a list.

Instead we will modify the graph $G(\pi, \delta, e)$ in such a way as to reduce the degree of each node to at most two. Basically we are resolving the degeneracy of non-proper semi-primoids and semi-duoids. This modification will not affect $G(\pi, \delta, e)$ when $\pi$ and $\delta$ are proper.

By considering the rules of adjacency (1)-(4) of 2.2, we see that a node $(P, D)$ of $G(\pi, \delta, e)$ has degree greater than two only if pivoting $D/P$ into $P$ can give more than one $P'$, or pivoting $D/P$ out of $D$ can give more than one $D'$. We call these pivots non-unique. Let us concentrate on the primal pivot step.

Assume we have a node $(P_1, D)$ and pivoting $D/P_1$ into $P_1$ can give us $P_2$, $P_3$, ..., $P_{2m}$ $(m > 1)$. Considering just the part of $G(\pi, \delta, e)$ involving the nodes $(P_1, D)$, $(P_2, D)$, ..., $(P_{2m}, D)$, we see that it is the complete graph on $2m$ nodes, $K_{2m}$ (every node is adjacent to every other). Our modified graph $G'(\pi, \delta, e)$ will replace this $K_{2m}$ with a graph only having $m$ edges, say connecting $(P_1, D)$ and $(P_2, D)$, $(P_3, D)$ and $(P_4, D)$, ..., $(P_{2m-1}, D)$ and $(P_{2m}, D)$. If $m = 3$

![Diagram](image)

becomes

(see example 2.7)

To define $G'$ formally, we need some added notation.

We assume that we have a total order on $S$. Consider an edge
V \in \sigma_{n+1} \text{ of a semi-primoid } \pi \text{ which contains } 2m \text{ vertices of } \pi. \text{ Each of these vertices is obtained by dropping one facet from } V; \text{ let these facets be } e_1, e_2, \ldots, e_{2m}, \text{ with } e_1 < e_2 < \ldots < e_{2m}. \text{ Then we define } P_i(V, \pi) \text{ as } V/e_i.

Dually, if an edge } U \in \sigma_{n-1} \text{ of a semi-duoid } \delta \text{ is contained in } 2m \text{ vertices of } \delta, \text{ each obtained by adding a facet to } U, \text{ let these facets be } f_1, f_2, \ldots, f_{2m}, \text{ with } f_1 < f_2 < \ldots < f_{2m}. \text{ Then we define } D_i(U, \delta) \text{ as } U \cup f_i.

Now we may define } G'(\pi, \delta, e).

**Definition 2.5.** The nodes of } G'(\pi, \delta, e) \text{ are just the nodes of } G(\pi, \delta, e). \text{ The node } (P, D) \text{ is adjacent to } (P', D') \text{ iff any of (1)-(4) holds.}

1. \( P \neq D, \ P' = P, \ D = D_{2j}(P \cap D, \delta) \text{ and } D' = D_{2j-1}(P \cap D, \delta) \text{ for some } j. \)

2. \( P \neq D, \ D' = D, \ P = P_{2j}(P \cup D, \pi) \text{ and } P' = P_{2j-1}(P \cup D, \pi) \text{ for some } j. \)

3. \( P = D, \ e \in P, \ P' = P, \ D = D_{2j}(D/e, \delta) \text{ and } D' = D_{2j-1}(D/e, \delta) \text{ for some } j. \)

4. \( P = D, \ e \notin P, \ D' = D, \ P = P_{2j}(P \cup e, \pi) \text{ and } P' = P_{2j-1}(P \cup e, \pi) \text{ for some } j. \)

It is immediate that this is symmetric.

**Example 2.7** gives an illustration of } G'(\pi, \delta, e).
As before, we are concerned with the degrees of the nodes of 
$G'(\pi, \delta, e)$. Let $(P, D)$ be any such node.

**Case 1.** $P \neq D$. Then $D = D_i(P \cup D, \delta)$ for just one $i$. So $(P, D)$ is adjacent to just one node via (1). Similarly, $P = P_i(P \cup D, \pi)$ for just one $i$. So $(P, D)$ is adjacent to just one node via (2). These two nodes are distinct; one is obtained by a primal pivot, the other by a dual pivot. Thus $(P, D)$ has degree 2.

**Case 2.** $P = D$. If $e \subseteq P$, then $D = D_i(D/e, \delta)$ for just one $i$, so $(P, D)$ is adjacent to just one node via (3). If $e \notin P$, then $P = P_i(P \cup e, \pi)$ for just one $i$, so $(P, D)$ is adjacent to just one node via (4). In either case, $(P, D)$ has degree 1.

Thus as with $G(\pi, \delta, e)$, the only odd nodes of $G'(\pi, \delta, e)$ are shared vertices of $\pi$ and $\delta$. However, in $G'(\pi, \delta, e)$, even if $\pi$ or $\delta$ is not proper, every node has degree at most two. Thus we may use Algorithm 2 to solve the problem.

Now we solve the generalized complementary pivot problem 2.1 by means of:

2.6. **The generalized complementary pivot algorithm**

**Step 1.** Form $\pi$ and $\delta$ as in (a) or (b) of 2.1. Then pick any $e \in S$, and construct $G'(\pi, \delta, e)$. This construction is merely a convenience for describing the algorithm. In practice, nodes and edges of $G'$ are generated only when they are needed.

**Step 2.** Starting at the "artificial" node $(C, G)$, use Algorithm 2 to find another odd node of $G'(\pi, \delta, e)$. This will have the form $(H, H)$,
H ≠ G, and so will give a vertex H common to both \( \gamma_1 \) and \( \gamma_2 \). This solves problem 2.1.

Let us illustrate this algorithm with an example.

**Example 2.7.** Let \( S = \{1,2,3,4,5,6\} \), with the natural ordering. Let \( \pi = \{125,126,134,136,234,235,245,246,345,356\} \), and let \( \delta = \{124,126,135,136,146,156,245,256,345,346\} \).

If we pick \( e = 1 \), \( G(\pi, \delta, e) = G'(\pi, \delta, e) \) is the following:

\[
\begin{array}{cccccccc}
136,136 & 136,346 & 134,346 & 134,345 & 345,345 \\
126,126 & 126,256 & 125,256 & 125,245 & 245,245
\end{array}
\]

Assuming \( v_0 = (126,126) \) is the starting node, and using algorithm 2, the steps will proceed as follows.

1) Find an edge incident to \( v_0 = (125,126) \). By the rules of adjacency, the only adjacent vertex is obtained by pivoting out 1 from D = 126. The pivot is unique, and we obtain \( v_1 = (126,256) \).

2) Find an edge incident to \( v_1 \), not \( e_0 \). By the rules of adjacency, we must pivot 5 into \( P = 126 \). This pivot is unique, and we obtain \( v_2 = (125,256) \).

3) Find an edge incident to \( v_2 \), not \( e_1 \). We must pivot 6 out of D = 256. This pivot is unique and we obtain \( v_3 = (125,245) \).

4) Find an edge incident to \( v_3 \), not \( e_2 \). We must pivot 4 into \( P = 125 \). This pivot is unique and gives us \( v_4 = (245,245) \). Since both coordinates are equal, we stop. 245 is the \( H \in \gamma_1 \cap \gamma_2 \) we were looking for.
Now let us pick $e = 2$, to show why $G(\pi, \delta, e)$ is not always sufficient. In the following diagram all edges, solid or dotted, are edges of $G(\pi, \delta, e)$. The edges of $G'(\pi, \delta, e)$ are the solid edges.

Note that this example illustrates the resolving of non-uniqueness in both primal pivots (on the right) and dual pivots (on the left).

Assume we start at $v_0 = (136, 136)$.

1) Find an edge $e_0$ incident to $v_0$. We must pivot 2 into $P = 136$. This pivot is unique, giving us $v_1 = (126, 126)$.

2) Find an edge $e_1$, not $e_0$, incident to $v_1$. We must pivot 3 out of $D = 136$. Here we have a choice. The edge 16 of $\delta$ is contained in the four vertices $126, 136, 146$, and 156 of $\delta$. Since $2 < 3 < 4 < 5$, $D_1(16, \delta) = 126$, $D_2(16, \delta) = 136$, $D_3(16, \delta) = 146$, and $D_4(16, \delta) = 156$. Since we now have $D = 136 = D_2(16, \delta)$, we see that $j = 1$ in (1) of the rules of adjacency in 2.5, so the adjacent node $v_2 = (P, D_1(16, \delta))$ or $(126, 126)$. Since both coordinates are equal, we stop. 126 is the $H \in \gamma_1 \cap \gamma_2$ we were looking for.

During the course of applying the algorithm to this example, note that we did not need to construct the full graph $G'$ except for exposition purposes. We merely generated nodes and edges as required.

For further illustration let us consider what would happen if we
chose the order on $S$ to be $1 < 2 < 4 < 3 < 5 < 6$. Then at step 2, we would have $D_1(16, 5) = 126$, $D_2(16, 5) = 146$, $D_3(16, 5) = 136$, and $D_4(16, 5) = 156$. In this case, the two edges of $G'$ in the left hand $K_4$ would be those running northwest-southwest. We obtain as our next node $(126, D_4(16, 5))$ or $(126, 156)$. The algorithm would then proceed along the bottom string of nodes to $(235, 345)$. At that stage, pivoting 4 into $P = 235$ would not be a unique pivot. So we would have $P_1(2345, π) = 345$, $P_2(2345, π) = 235$, $P_3(2345, π) = 245$, and $P_4(2345, π) = 234$ (recall the new ordering of $S$). Thus since $P = P_2(2345, π)$, our next node would be $(P_1(2345, π), 345) = (345, 345)$, and we would stop with the solution $H = 345$.

In this case, seven steps in the graph would be required, or four iterations, where an iteration is a primal pivot and a dual pivot. Note that an application of algorithm 1 to $G(π, δ, ε)$ might require fourteen steps in the graph, or seven iterations (here an iteration is hard to define, as one primal pivot may follow another). Also, added storage is required.

We remark that if at each non-unique pivot it is clear exactly which vertices can be obtained, then the non-degeneracy routine to obtain the $D_i(U, δ)'s$ or $P_i(V, π)'s$ and make the appropriate pivot should not be computationally burdensome. Certainly it will be much faster than recording edges to exclude cycling.

The generalized complementary pivot algorithm has the following properties:
a) The algorithm starts at a shared vertex of \( \pi \) and \( \delta \), the "artificial" vertex \((G,G)\), and finishes at a shared vertex, the solution to the problem.

b) At each intermediate step, we have a primal vertex and a dual vertex which share \( n-1 \) facets. The special facet \( e \) is always in the first and never in the second.

c) The algorithm terminates when \( e \) leaves the primal vertex, or enters the dual vertex in a pivot step.

d) Primal and dual pivot steps alternate in the course of the algorithm.

e) Each pivot is uniquely given if we know the primal vertex, the dual vertex, and whether a primal or dual pivot is necessary. Thus almost no storage is needed.

All of the examples of complementary pivot theory considered so far have been proper and could be solved by using just \( G(\pi, \delta, e) \). A possible non-proper application is the following:

Given a bipartite graph \( G = (V,E) \), the spanning trees of \( G \) form a semi-prinoid on \( E \). For if an edge is added to a spanning tree, it forms a unique cycle, with an even number of edges. Any one of these edges can be dropped to obtain a spanning tree. This verifies (P1).

So the cotrees (edge complements of the trees) form a semi-duoid.

Now let us associate with each edge of \( G \) an \( n \)-vector, where \( n = |E| - |V| + 1 \), in such a way that to the edges of one cotree correspond the \( n \) negative unit vectors, while every other edge corresponds to a positive vector. Let \( b > 0 \). Then by letting \( \gamma_1 \) be generated by
feasible bases of \([-I,A]\) \(x = b, \ x \geq 0\), and \(\gamma^2\) be generated by the cotrees, we can find a cotree which corresponds to a feasible basis.

I can suggest no use for this application; worthwhile algorithms are at present confined to the proper case. The usefulness of admitting non-proper primoids and duoids will become apparent in the next chapter, where we characterize all semi-primoids and semi-duoids in a simple way, using the simplicial primoids and duoids as building blocks. We will see that admitting all primoids and duoids allows us to use matroid theory to investigate their structure.
CHAPTER 3

THE UNDERLYING MATROIDS AND CHARACTERIZATIONS OF DUOIDS, ETC.

Now we investigate the combinatorial structure of primoids and duoids. The characterizations for semi-primoids and semi-duoids can be achieved by topological reasoning, as we shall see in Chapter 4, but we prefer to use matroid theory, since our major concern is primoids and duoids. We will show that the primoids and duoids of order $n$ on $S$ form the circuits and co-circuits of a binary matroid. We use the existence theorem of Chapter 2 (2.3) as a major stepping stone to prove this result.

Then we concentrate on semi-duoids and duoids (dual results of course hold for semi-primoids and primoids) and characterize them, using the matroids constructed. Semi-duoids are easy to characterize (3.9) but it is hard to do the same for duoids without using minimality. The last four theorems address this problem, and give a number of tests to determine whether a given semi-duoid is in fact a duoid. These last four theorems may be omitted on the first reading; they are used only to verify that examples introduced in the later chapters are in fact duoids.

Our first result is important because it gives a non-geometric operation on semi-primoids and has many very useful corollaries.
3.2

\textbf{Lemma 3.1.} If $\pi_1$, $\pi_2$ are semi-primoids of order $n$ on $S$, then so is $\pi_1 \oplus \pi_2$.

Recall that $\pi_1 \oplus \pi_2$ is the symmetric difference of $\pi_1$ and $\pi_2$, i.e., $(\pi_1 \setminus \pi_2) \cup (\pi_2 \setminus \pi_1)$.

\textbf{Proof:} $\pi_1 \oplus \pi_2 \subseteq \pi_1 \cup \pi_2 \subseteq \sigma_n$. Now let $V \in \sigma_{n+1}$. Then $V$ contains, say, $2q$ vertices of $\pi_1$; say, $2r$ vertices of $\pi_2$; and, say $s$ vertices of $\pi_1 \cap \pi_2$. (Let $q$, $r$, and $s$ be integers.) So $V$ contains $(2q - s) + (2r - s) = 2(q + r - s)$ vertices of $\pi_1 \oplus \pi_2$; this establishes (PI).

The dual result about semi-duoids holds: if $\delta_1$ and $\delta_2$ are semi-duoids, so is $\delta_1 \oplus \delta_2$. This is 3.1*.

\textbf{Corollary 3.2.} Every semi-primoid is a union of disjoint primoids.

\textbf{Proof:} Choose if possible the semi-primoid $\pi$ with the fewest vertices for which the corollary does not hold. Then $\pi$ is not a primoid, so it contains one, say $\pi_1$. Then by 3.1, $\pi \oplus \pi_1 = \pi / \pi_1$ is a semi-primoid and has fewer vertices than $\pi$. So $\pi / \pi_1$ is a union of disjoint union of primoids $\pi_2$, $\pi_3$, ..., $\pi_m$. Then $\pi$ is a union of disjoint primoids $\pi_1$, $\pi_2$, ..., $\pi_m$.

\textbf{Corollary 3.3.} The semi-primoids of order $n$ on $S$ form a group under $+$ whose every element has order 2.

\textbf{Proof.} We have shown in 3.1 that they are closed under $\oplus$. But $\emptyset$ is a unit, and $\pi \oplus \pi = \emptyset$ shows that $\pi$ is its own inverse. In Chapter 4, we will see that this is a well known group in combinatorial topology.
Our third corollary introduces the matroids which we will use later to characterize semi-ducoids.

Matroids can be defined in terms of several different primitive notions. We will use circuits as our primitive notions. Then a matroid is a pair \((E, \mathcal{C})\) where \(\mathcal{C}\) is a collection of subsets of \(E\), called circuits, such that

1. No circuit properly contains another.
2. If \(C_1\) and \(C_2\) are circuits, \(a \in C_1 \cap C_2\), \(b \in C_1 / C_2\), then there is a circuit \(C_3\) such that \(b \in C_3 \subseteq C_1 \cup C_2 / \{a\}\).

For our purposes, little need be known about matroids. Matroids were introduced by Whitney [31] as abstract linear dependence structures. It may be helpful to think of the elements of the matroid as vectors. Each subset of vectors is either linearly dependent or not. Then the circuits of the matroid correspond to minimal dependent sets. In terms of the primitive notion of circuit, an independent set is one which contains no circuit; a base is a maximal independent set. To each matroid there corresponds a unique dual matroid whose bases are the complements of the bases of the original matroid. Circuits of the dual matroid are called co-circuits of the original. Theorems about matroids will be merely quoted where necessary. Minty [24] is a sufficient reference for most of the development. Tutte [30] is a more complete reference.

**Corollary 3.4.** The primoids of order \(n\) on \(S\) form the circuits of a matroid on \(\sigma_n\).
Proof: (C1) is immediately satisfied by the definition of primoids as minimal semi-primoids.

To prove (C2), let \( \pi_1, \pi_2 \) be primoids, \( P_1 \in \pi_1 \cap \pi_2 \), \( P_2 \in \pi_1/\pi_2 \). Then \( \pi_1 + \pi_2 \) is a semi-primoid (3.1) which is a union of disjoint primoids (3.2), one of which, say \( \pi_3 \), contains \( P_2 \). Then \( P_2 \in \pi_3 \subseteq \pi_1 \cup \pi_2/P_1 \) as required.

We have now constructed two matroids—with primoids as circuits as in 3.4; and with duoids as circuits as in 3.4*. These will be denoted by \( M_{nk} \) and \( M^*_{nk} \) respectively. This notation will be justified when we prove them dual to each other.

First, however, we want to stress that \( M_{nk} \) and \( M^*_{nk} \) are matroids not on \( S \), but on \( \sigma_n \). Each individual primoid or duoid is a family of subsets, but these subsets do not in general form the circuits, bases, dependent or independent sets of any matroid on \( S \). The higher level of abstraction is necessary to investigate the structure. As mentioned in Chapter 1, it is convenient in most cases to imagine the vertices as elements of the set \( \sigma_n \), not as subsets of \( S \). Thus when we write \( \delta = \{12, 23, 13\} \), each vertex is a set \((12 \text{ is really } \{1, 2\})\); but it is less confusing to think of 12 as just a member of a set \( \sigma_2 \).

Let us recall again that \( M_{nk} \) (\( M^*_{nk} \)) has as circuits all primoids (duoids) of a certain order, each considered rather as a set of vertices than as a family of subsets of \( S \).

Now we return to the relationship between these two matroids. We will prove them dual to each other, but perhaps even more significant is the fact that they are both binary, a property which allows the simple
characterization of semi-duoids (or semi-primoids) seen in 3.9. Binary matroids are, in the simplest definition, those for which every circuit intersects each co-circuit in an even number of members. It was a result of noticing the similarity between this property and theorem 2.3 which first indicated to the author the possible relation with matroid theory.

**Theorem 3.5.** $M_{nk}$ and $M_{nk}^*$ are dual matroids, and both are binary.

Since a matroid is determined by its circuits or co-circuits, $M_{nk}$ and $M_{nk}^*$ are dual if the circuits of the latter, duoids, are the co-circuits of the former.

Minty’s theorem 5.1 [24] characterizes the co-circuits of a matroid as minimal non-empty sets which do not intersect any circuit in just one element.

Thus $\gamma \subseteq \sigma_n$ is a co-circuit of $M_{nk}$ iff

1) $|\pi \cap \gamma| \neq 1$ for all primoids $\pi$.

2) $\gamma$ is minimal non-empty with this property.

Before proving the theorem, we need a preliminary result:

**Lemma 3.6.** If $G, H \in \gamma$, then there is a primoid $\pi$ such that $\pi \cap \gamma = \{G, H\}$ (where $\gamma$ satisfies (1) and (2) above).

**Proof of the Lemma:** Define $G \sim H$ if such a primoid exists, or if $G = H$. Trivially, the relation $\sim$ is reflexive and symmetric.

Now let $\pi_1$, $\pi_2$ be primoids intersecting $\gamma$ in just $\{G, H\}$, $\{H, I\}$ respectively ($G, H, I$ distinct). Then $\pi_1 + \pi_2$ is a semi-primoid intersecting $\gamma$ in just $\{G, I\}$ (3.1). Pick a primoid $\pi_3$ contained in
$\pi_1 + \pi_2$ and containing $G$ (3.2). Then by property (1) of $\gamma$, \( \pi_3 \) contains $I$. So $G \sim I$. We have therefore that $\sim$ is an equivalence relation. To prove the lemma, we must show that the only equivalence class is $\gamma$ itself.

Assume to the contrary that $\gamma_1$ is an equivalence class with $\emptyset \subset \gamma_1 \subset \gamma$. Then we will show that $\gamma/\gamma_1$ satisfies (1), so that $\gamma$ does not satisfy (2). Once again we proceed by contradiction.

Assume $\gamma/\gamma_1$ does not satisfy (1). Then there is a primoid $\pi$ intersecting $\gamma$ in \( \{G_0, G_1, \ldots, G_m\} \) where $G_0 \in \gamma/\gamma_1$ and $G_i \in \gamma_1$ for $i = 1, 2, \ldots, m$.

Then, because $\gamma_1$ is an equivalence class, we have primoids $\pi_1, \pi_2, \ldots, \pi_{[m/2]}$ such that $\pi_1 \cap \gamma = \{G_{2i-1}, G_{2i}\}$. Then $\pi + \sum_{i=1}^{[m/2]} \pi_i$ is a semi-primoid by 3.1. We distinguish two classes.

**Case 1.** $m$ is even. Then $\pi + \sum_{i=1}^{m/2} \pi_i$ intersects $\gamma$ in just $G_0$, and contains a primoid with the same property (3.2). This contradicts (1).

**Case 2.** $m$ is odd. Then $\pi + \sum_{i=1}^{m/2-1/2} \pi_i$ intersects $\gamma$ in just $\{G_0, G_m\}$. We choose a primoid contained in this and containing $G_0$.

By (1) it must also contain $G_m$. So $G_0 \sim G_m$, contrary to hypothesis.

We have established that $\gamma$ is the only equivalence class, thus proving the lemma.

**Proof of the Theorem:** We first show that if $\gamma$ satisfies (1) and (2), it is a semi-duoid. Let us assume to the contrary, i.e. that there is a $U \in \sigma_{n-1}$ such that the vertices of $\gamma$ containing $U$ are odd in number,
say $G_0, G_1, \ldots, G_{2m}$. We will show a contradiction.

By the lemma, for $i = 1, 2, \ldots, 2m$, there is a primoid $\pi_i$ which intersects $\gamma$ in just $\{G_0, G_i\}$. Also let $\pi_0 = \emptyset$.

Consider $\pi = \sum_{i=0}^{2m} \pi_i$. This is a semi-primoid by 3.1. Moreover, for $i > 0$, $G_i$ is in just $\pi_0$ and $\pi_i$, so not in $\pi$. But $G_0$ is in all the $\pi_i$'s, so in $\pi$. By 3.2 there is a primoid $\pi'$ contained in $\pi$ which intersects $\gamma$ in just $G_0$. This contradicts (1). So $\gamma$ is a semi-duoid.

Moreover, if $\gamma$ contains a duoid properly, this duoid satisfies (1); so $\gamma$ does not satisfy (2). Hence $\gamma$ must itself be a duoid. Now it is easy to see that all duoids satisfy (1) and (2). So $M^{\pm}_{nk}$ and $M^{\pm}_{nk}$ are dual. The theorem is completed when we note that our theorem 2.3, the existence theorem for complementary pivot algorithms, states that a circuit and co-circuit intersect in an even number of vertices. By Minty's theorem 7.2 [24], this assures us that both matroids are binary.

3.6 can now be rewritten as follows:

**Lemma 3.6.** If $\delta$ is a duoid, $D_1, D_2 \in \delta$, then there is a primoid $\pi$ such that $\pi \cap \delta = \{D_1, D_2\}$.

Note also that (1) and (2) provide a characterization of duoids; however, this characterization is no advance on the definition of duoids.

In order to use the binary nature of these matroids to characterize semi-duoids, we need to have a base for the matroid and a set of fundamental circuits. Since a base is a maximal independent set, adding any extra
vertex will add a circuit; moreover, a simple application of (C2) shows that this circuit is unique. This circuit is called the fundamental circuit associated with the added vertex and the base. For each base, we have a set of fundamental circuits, one for each element not in the base. Luckily, some of the bases for our matroids have a very simple form, and so do the associated fundamental circuits. Let $\beta^e$ denote $\{G \in \sigma | e \in G\}$ and $\beta_e = \sigma_u / \beta^e$.

**Lemma 3.7.** For any $e \in S$, $\beta^e$ is a base of $M^*_n$ and for any $G \in \beta_e$, $<G \cup e>$ is the fundamental circuit associated with $G$ and $\beta^e$.

**Proof:** First we show that $\beta^e$ contains no duoid $\delta$. If it did, let $D \in \delta$; then we should be able to pivot out $e$ from $D$. But this would give a vertex of $\delta$ outside $\beta^e$. Now we must show $\beta^e$ is maximal. But for any $G \in \beta_e$, $<G \cup e>$ is a duoid contained in $\beta^e \cup \{G\}$. This also gives us the fundamental circuits.

Dually, for any $e \in S$, $\beta_e$ is a base of $M^n_k$, and for any $G \in \beta^e$, $>G/e<$ is the fundamental circuit associated with $G$ and $\beta^e$ in the matroid $M^n_k$.

We use this lemma to give us our first major result.

**Theorem 3.8.** Let $\delta$ be any duoid; then $\delta = \sum_{D \in \delta \cap \beta_e} <D \cup e>$.

**Proof:** Minty's theorem 7.3 [24] states that for a binary matroid, given any circuit $C$ and base $B$, then $C = \sum_{a \in C/B} FC(a,B)$, where $FC(a,B)$ is the unique circuit contained in $B \cup a$ (fundamental circuit associated...
with a and B). 3.8 follows immediately from the base and fundamental circuits characterized above.

Now we can easily characterize semi-duoids.

**Corollary 3.9.** \( \delta \) is a semi-duoid iff \( \delta = \Sigma_{D \in \delta \cap \beta} <D \cup e> \).

Dually, we can characterize semi-primoids.

**Corollary 3.9*.** \( \pi \) is a semi-primoid iff \( \pi = \Sigma_{P \cap \delta \cap \beta} >P/e< \).

**Proof of 3.9*.** If \( \delta \) is a semi-duoid, then it splits into disjoint duoids (3.2*), each of which can be expressed in a similar form from Theorem 3.8; taking the union, we obtain the above. Conversely, if \( \delta \) satisfies the relation above, it is a semi-duoid by 3.1*.

**Corollary 3.10.** If \( \delta = \Sigma_{i=1}^{m} <D_i \cup e> \) is a semi-duoid, then \( \delta \) is a union of disjoint duoids as follows:

\[
\delta = \Sigma_{I_1} <D_i \cup e> + \Sigma_{I_2} <D_i \cup e> \oplus \ldots \oplus \Sigma_{I_l} <D_i \cup e>
\]

where \( I_1, I_2, \ldots, I_l \) is a partition of \( \{1, 2, \ldots, m\} \).

**Proof.** Assume all the \( D_i \)'s are distinct (if not, we can drop any identical pair). For each \( i \), \( D_i \) is the only vertex of \( <D_i \cup e> \) not containing \( e \). So \( D_i \in \delta \) for each \( i \). Let \( \delta \) be decomposed into disjoint duoids, say \( \gamma_1, \gamma_2, \ldots, \gamma_l \).

This decomposition partitions the \( D_i \)'s. Let \( D_i \in \gamma_j \) for \( i \in I_j \). Since the \( D_i \)'s are the only vertices of \( \delta \) not containing \( e \),
\( \gamma_j = \sum_{D_i \subseteq U \ e} \) by theorem 3.8. This establishes the corollary.

**Corollary 3.11.** For any fixed \( e \), \( \delta \) is a semi-duoid iff for all \( U \in \sigma_{n-1} \) with \( e \notin U \), \( U \) is contained in an even number of vertices of \( \delta \).

**Proof:** We need to show that if \( U' \in \sigma_{n-1} \), \( e \in U' \), then \( U' \) is contained in an even number of vertices of \( \delta \), i.e., \(|\delta \cap U'|\) is even. But \( |U'| = \sum_{P/e} \geq \sum_{P/e < U'} \) by 3.8; so \( \delta \cap U' = \sum_{P/e > U'} \geq \sum_{P/e < U'} \). But each of the sets in the symmetric difference has even cardinality by hypothesis, so their symmetric difference will have also.

Corollary 3.11 allows us to examine only a fraction of all the possible edges to determine whether a given subset of \( \sigma_n \) is a semi-duoid; but it is often most convenient to use corollary 3.9, which allows us to write a duoid or semi-duoid in the following simple way.

List each vertex in \( \delta \cap \beta_e \) in a column. Draw a line to the right of this column. Then on the right of this line, and in the same row as \( D \in \delta \cap \beta_e \), we write the remaining vertices of \( <D \cup e> \). Finally, we draw a line through every vertex to the right of the line which appears an even number of times. The remaining vertices are the vertices of \( \delta \); to the left of the line are those which do not contain \( e \), and to the right those that do.

For example, the duoid corresponding to the triangular prism at the end of 1.3 can be expressed as follows. We take \( e = 4 \); then the vertices of \( \delta \cap \beta_e \), i.e., the vertices of \( \delta \) which do not contain 4,
are \{135\} and \{235\}. Then the simplicial duoids \(<D \cup e>\) for \(D\) in \\
\(\delta \cap \beta_e\) are \(<1345>\) and \(<2345>\), or \{135, 134, 145, 345\} and \\
\{235, 234, 245, 345\}. Note that the symmetric difference of these is in \\
fact the duoid \(\delta\). We can write this in shorthand as follows:

\[
\begin{array}{c|cccc}
135 & 134 & 145 & 345 \\
235 & 234 & 245 & 345 \\
\end{array}
\]

We now turn to the question of how to identify or recognize duoids \\
without recourse to the minimality criterion in the definition. A number \\
of conditions will be presented in the next four theorems.

The first two give only sufficient conditions. They are necessary \\
in the case of proper semi-duoids.

We define initially two graphs, \(G(\delta)\) and \(G'(\delta)\). \(G(\delta)\) will \\
be used again in Chapter 7; it can be called the associated graph of the \\
duoid or semi-duoid.)

The nodes of each are just the vertices of \(\delta\). In \(G(\delta)\), \(D_1\) \\
and \(D_2\) are adjacent iff \(D_1 \cap D_2 \in \sigma_{n-1}\). In \(G'(\delta)\), \(D_1\) and \(D_2\) are \\
adjacent iff \(D_1 \cap D_2 \in \sigma_{n-1}\) and moreover \(D_1 \cap D_2 \subseteq D \in \delta\) only if \\
\(D = D_1\) or \(D_2\). Note that edges in \(G(\delta)\) correspond to pivots in \(\delta\); \\
while edges in \(G'(\delta)\) correspond to unique pivots in \(\delta\).

**Theorem 3.12.** If \(\delta\) is a semi-duoid, then all the vertices of any component \\
of \(G'(\delta)\) must lie together in one of the duoids in \(\delta\). In particular, \\
if \(G'(\delta)\) is connected, \(\delta\) is a duoid. The converse is not true in general.

However, if \(\delta\) is a proper semi-duoid, then \(G'(\delta)\) is connected iff \(\delta\) \\
is a duoid. Finally, if \(\delta\) is a duoid, \(G(\delta)\) is connected.
Proof: To prove the first part, assume $\delta$ is a semi-duoid, $D_1$ and $D_2$ connected in $G'(\delta)$, but $\delta = \delta_1 \oplus \delta_2$ where $\delta_1$, $\delta_2$ are semi-duoids with $D_i \in \delta_i$, $i = 1, 2$. We seek a contradiction. $D_1$ is connected by a path in $G'(\delta)$ to $D_2$. Let this path be $D_1 = G_0, G_1, \ldots, G_p = D_2$.

Let $m$ be the smallest integer such that $G_m \in \delta_2$. Then $G_{m-1} \in \delta_1$.

But $U = G_{m-1} \cap G_m \in \sigma_{n-1}$ and $U \subseteq D \in \delta$ only for $D = G_{m-1}$ or $G_m$. So by considering $U$, we see $\delta_1$ is not a semi-duoid. This shows the first part of the theorem.

To show that the converse is not true in general, let $S = \{a, b, \ldots, l\}$ and let $\delta = \{\{abcde\} > \{abce\} + \{abcd\} > \{acdf\} > \{acdg\} > \{abdgh\} > \{abcdh\} > \{abchi\} > \{abci\} > \{acdi\} > \{acdij\} > \{abcdj\} > \{abdjk\} > \{abdk\} > \{abcl\} > \{abcd\}$.

It is straightforward to see that $G'(\delta)$ is disconnected (the vertices in $\{abcd\}$ are disconnected from the rest), but $\delta$ is a duoid. By Corollary 3.10, if it splits into duoids, the simplicial duoids are partitioned. But if $\{abcde\}$ is in $\delta_1$, say, so is $\{abce\}$ (for let $D = \{abc\} \in \delta$ so not in $\delta_1$), $\{abcd\}$ (take $D = \{abcf\}$) .... and $\{abdk\}$. The only possible partition is thus the first 13 simplicial duoids and the last one. But $\{abcd\}$ is in the last one, but not in $\delta$. So $\delta$ must be a duoid.

Next note that if $\delta$ is proper $G(\delta)$ and $G'(\delta)$ coincide, for if $D_1 \cap D_2 \in \sigma_{n-1}$, $D_1 \cap D_2$ can only be in $D_1$ or $D_2$ by definition. So it remains to be proved that if $\delta$ is a duoid, $G(\delta)$ is connected.

Let $D_1, D_2 \in \delta$; by Lemma 3.6 there is a primoid $\pi$ such that $\pi \cap \delta = \{D_1, D_2\}$, and by Chapter 2 we have a path in $G(\pi, \delta, e)$ from
\((D_1', D_1')\) to \((D_2', D_2')\). Considering the duoid vertices in this path, we have a sequence \(D_1 = G_0, G_1, \ldots, G_\ell = D_2\) where \(G_{i-1} \cap G_i \in \sigma_{n-1}\) for \(i = 1, 2, \ldots, \ell\), i.e., \(G_{i-1}\) and \(G_i\) are adjacent in \(G(\delta)\). Thus \(G(\delta)\) is connected.

Theorem 3.12 requires us to examine a graph \(G'(\delta)\) with \(|\delta|\) vertices. Another sufficient condition will be given in the next theorem, which examines a smaller graph, but does not give us such a strong test.

For any fixed \(e \in S\), we define two more graphs, \(G(\delta,e)\) and \(G'(\delta,e)\). The nodes of each are just the vertices of \(\delta \cap \beta_e\). Two such vertices are adjacent in \(G(\delta,e)\) iff they are adjacent in \(G(\delta)\), and adjacent in \(G'(\delta,e)\) iff they are adjacent in \(G'(\delta)\). Note that, by Theorem 3.8, \(D_1\) and \(D_2\) are adjacent in \(G'(\delta,e)\) iff \(D_1 \cap D_2 \in \sigma_{n-1}\) and \(D_1 \cap D_2 \subseteq D \in \delta \cap \beta_e\) only if \(D = D_1\) or \(D_2\).

**Theorem 3.13.** For any fixed \(e \in S\), Theorem 3.12 holds if we replace \(G(\delta)\) by \(G(\delta,e)\) and \(G'(\delta)\) by \(G'(\delta,e)\) throughout. However, it is possible that \(G'(\delta)\) is connected, but that \(G'(\delta,e)\) is not.

**Proof:** The first part will hold if it can be shown that if \(D_1\), \(D_2\) lie in the same component of \(G'(\delta,e)\) then they lie in the same component of \(G'(\delta)\). But this is obvious from the definitions of adjacency. If \(G'(\delta,e)\) is connected, all the vertices of \(\delta \cap \beta_e\) must lie in one duoid in \(\delta\); but by Corollary 3.10, this must be \(\delta\) itself. The same example as used in the proof of 3.12 shows that the converse is not true. If \(\delta\) is proper then \(G(\delta,e)\) and \(G'(\delta,e)\) coincide as before. So it remains to be shown that if \(\delta\) is a duoid, \(G(\delta,e)\) is connected. The same argument used in Theorem 3.12 can be used here, since all the vertices of \(\delta\).
in the path in $G(\pi, \delta, e)$ will not contain the facet $e$, so will be nodes of $G(\delta, e)$.

Now we give an example of a duoid $\delta$ with $G'(\delta)$ connected but $G'(\delta, e)$ disconnected. Let $S = \{a, b, c, \ldots, h\}$ and let $\delta = \langle\text{abcdef}\rangle + \langle\text{abcdef}\rangle + \langle\text{acdfg}\rangle + \langle\text{abcdg}\rangle + \langle\text{abdh}\rangle$. It is readily checked that $G'((\delta)$ is connected, but in $G'(\delta, a)$, the vertex $\{bcdh\}$ is isolated.

Theorems 3.12 and 3.13 do not completely identify duoids unless the given semi-duoid is proper. They also require us to check that a given $\delta$ is a semi-duoid--this can be done as the graph is constructed. Note that if we know $\delta$ to be a semi-duoid, we need not construct the entire graph; we only find enough edges to show it connected.

The following two theorems will settle the question of identifying duoids, at the cost of additional computation.

The tests require us to determine whether a given set of column vectors is independent or not; this can be done by Gaussian elimination or more sophisticated methods.

We say that a matrix is just-dependent if the set of its columns is a minimal dependent set in $GF(2)$. Since the only coefficients in $GF(2)$ are 0 and 1, we see that a set of columns is dependent iff it contains a subset which sums to zero mod 2. Thus a matrix is just-dependent iff the sum of its columns is zero, but the sum of no proper subset of its columns is zero.

We can verify that a matrix $M$ is just-dependent by
1) the sum of all the columns of $M$ is zero.

2) if the first column of $M$ is deleted, the remaining columns of $M$ are independent.

Obviously, if $M$ is just-dependent, (1) and (2) hold. Conversely, if (1) holds and $M$ is not just-dependent, then there is a proper subset of its columns which sums to zero. Then, using (1), the complementary set of columns also sums to zero. One of these two sets of columns does not include the first column, and so (2) does not hold.

We form a matrix $M = M(S,e,n)$ as follows. The columns of $M$ correspond to all vertices in $\sigma_n^e$. The rows of $M$ correspond to all vertices in $\beta^e$. The entries are 0 or 1 with the entry $m_{HG}$ in the row corresponding to $H$ and the column corresponding to $G$ being 1 iff $H \subseteq G \cup e$. We denote by $M(\delta)$ the submatrix of $M$ consisting of all the columns of $M$ corresponding to vertices of $\delta$.

**Theorem 3.14.** $\delta$ is a duoid of order $n$ on $S$ iff $M(\delta)$ is just-dependent.

**Proof:** First we show that the sum of the columns of $M(\gamma)$ is zero iff $\gamma$ is a semi-duoid. The sum of the columns of $M(\gamma)$ is zero $\iff$ for every $H \in \beta^e$, the number of vertices of $\gamma$ containing $H/e$ is even $\iff$ for every $U \in \sigma_{n-1}$ with $e \notin U$, the number of vertices of $\gamma$ containing $U$ is even $\iff$ $\gamma$ is a semi-duoid, by 3.11.

Now assume $M(\delta)$ is just-dependent. Then by the above, $\delta$ is a semi-duoid. But if $\delta$ contained a duoid $\delta'$, the columns of $M(\delta')$ would sum to zero, so $M(\delta)$ would not be just-dependent. Thus $\delta$ is a duoid.
Conversely, if $\delta$ is a duoid, then the columns of $M(\delta)$ sum to zero. If $M(\delta)$ were not just-dependent, there would be a proper subset of its columns, say those of $M(\delta')$, which would sum to zero. But then $\delta'$ would be a non-empty semi-duoid strictly contained in $\delta$. So $M(\delta)$ is just-dependent.

In fact, those familiar with Tutte [30] will recognize $M(S,e,n)$ as just the standard representative matrix of $M^*_{nk}$ with respect to the cobase (complement of a base; dendroid in [30]) $\beta_e$. See Sections 5.2 and 5.3 of [30].

Theorem 3.14 requires us to examine a matrix with $|\delta|$ columns. Analogously to the way 3.13 is obtained from 3.12, we may simplify matters by considering only vertices of $\delta \cap \beta_e$.

We construct a matrix $M'(S,e,n)$ as follows. The columns of $M'$ correspond to all vertices in $\beta_e$. The rows of $M'$ correspond to all vertices in $\beta^e$. The entries are all 0 or 1 with $m'_{HG} = 1$ iff $H \subseteq G \cup e$. We denote by $M'(\delta)$ the submatrix of $M'$ consisting of all the columns of $M'$ corresponding to vertices in $\delta \cap \beta_e$. We form $Q(\delta)$ from $M'(\delta)$ by

$$q_{DG} = \begin{cases} m_{DG}^t & \text{if } \sum_{H \subseteq G \cup e} m_{DH}^t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.15. For any $\delta$ and any $e \in S$, $\delta$ is a semi-duoid iff

$$D \in \delta \cap \beta^e \iff \sum_{G \subseteq \delta \cap \beta^e} m_{DG}^t = 1.$$  A semi-duoid $\delta$ is a duoid iff $Q(\delta)$ is just-dependent.
\[\begin{align*}
\text{Proof: } \delta \text{ is semi-duoid } & \iff \delta = V (H \cup e) \iff D \in \delta \text{ iff } D \text{ is contained in an odd number of such } \langle H \cup e \rangle \iff D \in \delta \cap \beta_e \text{ iff } D \text{ is contained in an odd number of such } \langle H \cup e \rangle \iff D \in \delta \cap \beta_e \text{ iff } \sum_{G \in \delta \cap \beta_e} \gamma^i_D G = 1.
\end{align*}\]

This proves the first part.

For the second part, first we note that the sum of the columns of \(Q(5)\) is always zero. In fact, if \(\sum_{H \in \delta \cap \beta_e} \gamma^i_D H = 0\), then \(\sum_{G \in \delta \cap \beta_e} \gamma^i_D G = 0\). If on the other hand, \(\sum_{H \in \delta \cap \beta_e} \gamma^i_D H = 1\), then by the definition of \(Q(5)\), the row of \(Q\) corresponding to \(D\) is a row of zeroes.

We will show the second part by proving the following string of propositions equivalent:

a) \(\delta\) is not a duoid.

b) There is a non-null semi-duoid \(\delta'\) strictly contained in \(\delta\).

c) There is a proper subset \(\gamma\) of \(\delta \cap \beta_e\) with \(G \not\in \delta \implies G \not\in \langle H \cup e \rangle\) for all \(G \in \beta_e\).

d) There is a proper subset \(\gamma\) of \(\delta \cap \beta_e\) with \(\sum_{H \in \gamma \cap \beta_e} \gamma^i_D H = 0\) for all \(G \in \beta_e\).

e) \(Q(5)\) is not just-dependent.

(a) \(\iff\) (b) and (d) \(\iff\) (e) are immediate from the definitions.

(b) \(\implies\) (c). Let \(\gamma = \delta' \cap \beta_e\). Then by 3.9, \(\emptyset \subset \gamma \subset \delta \cap \beta_e\).

Also, \(G \notin \delta \implies G \notin \delta' \implies G \notin \langle H \cup e \rangle\). Thus (c) follows.

e) \(\implies\) (b). Let \(\delta' = \sum_{H \in \langle H \cup e \rangle} \gamma^i_D H\). Then \(\delta'\) is a non-null semi-duoid.
Moreover, \( \delta' \subseteq \delta \) because \( G \in \delta^e / \delta \implies G \notin \delta' \), and \( \delta' \neq \delta \) because
\[
\gamma \subseteq \delta \cap \beta_e.
\]

(c) \( \implies \) (d). If \( G \in \delta \cap \beta^e \), then \( \Sigma m_{GH}^i = 1 \), so \( \Sigma q_{GH}^\gamma = \Sigma 0 = 0 \).

If on the other hand, \( G \in \delta^e / \delta \), then \( G \notin \delta \), so \( \Sigma m_{GH}^i = 0 \). By
\[
\Sigma q_{GH}^\gamma = 0.
\]

Thus (d) is verified.

(d) \( \implies \) (c). If \( G \notin \delta \), then \( \Sigma m_{GH}^i = 0 \). Then \( \Sigma m_{GH}^i = 0 \).

This completes the proof of the theorem.

Let us illustrate these four tests by an example.

**Example 3.16.** Let \( \delta = \{124, 126, 135, 136, 140, 156, 245, 256, 345, 346\} \).

We will show that \( \delta \) is a duoid using the four tests of theorems 3.12-3.15.

a) 3.12

\[
G'(\delta):
\]
\( G(\delta) \) has in addition the dotted edges.

Since \( G'(\delta) \) is connected, \( \delta \) is a duoid.

b) 3.13

\[ G'(\delta,1) = G(\delta,1) : 256 \rightarrow 245 \rightarrow 345 \rightarrow 346 \]

Since \( G'(\delta,1) \) is connected, \( \delta \) is a duoid (note the simple form of \( G'(\delta,1) \)).

c) 3.14. Take \( e = 1 \).

We show the matrix over \( GF(2) \), \( M(\delta) \) (zeroes are omitted):

\[
\begin{bmatrix}
124 & 126 & 135 & 136 & 146 & 156 & 245 & 256 & 345 & 346 \\
123 & & & & & & & & & \\
124 & 1 & & & & & & & & \\
125 & & 1 & 1 & & & & & & \\
126 & 1 & & & & 1 & & & & \\
134 & & & & 1 & 1 & & & & \\
135 & & 1 & & & & 1 & & & \\
136 & 1 & & & & & & 1 & & \\
145 & & 1 & & & & & & 1 & \\
146 & 1 & & & & & & & & 1 \\
156 & & & & & & & & 1 & \\
\end{bmatrix}
\]

It can be shown by Gaussian elimination that this matrix is just-

dependent; so \( \delta \) is a duoid.
d) 3.15. Take $e = 1$.

Finally we form $M'(e)$ and $Q$:

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CHAPTER 4

RELATIONSHIPS OF DUOIDS TO PREVIOUSLY STUDIED COMBINATORIAL STRUCTURES

In this chapter we will see how our semi-duooids and duooids relate to various other combinatorial structures--representations of simple polytopes (1.3.1*), the abstract polytopes of Adler and Dantzig [1,2,3] and the circuits and co-circuits of some matroids introduced by Crapo and Rota [8].

Simple polytopes are the primary examples of primoids and duooids. However, the results of the last chapter were obtained at the expense of considerable abstraction of one basic property of polytopes. An ideal program would be to develop a sequence of tests which would determine in a finite number of steps whether a given semi-duoid corresponded to a simple polytope. Unfortunately, this is not possible if \( n > 3 \), but we will give a number of useful tests.

Some obvious properties of polytopes present themselves. Firstly, the semi-duoid must be a duoid. It was partly for this reason that duooids were considered more important than semi-duooids. In fact, all useful examples of semi-duooids are also duooids. This is why the tests at the end of the last chapter were refined to reasonable computational efficiency. In later chapters we will generally use only the first two of these tests. But the last is the most powerful, and could be used in conjunction with
3.9 to find all duoids of order \( n \) on a \( k \)-set. From this, at least in theory, it is possible to find all combinatorial types of simple polytopes of dimension \( n \) with at most \( k \) facets; some redundant duoids will of course be introduced, which we can perhaps discard for other reasons.

Another reason for the focus on duoids comes from complementary pivot theory. A complementary pivot algorithm searches for an object with two properties—corresponding to a vertex belonging to the semi-primoid and to the semi-duoid. 3.2 and 3.2* show us that every semi-primoid or semi-duoid is a disjoint union of primoids or duoids. Thus it seems plausible that in actual applications, a semi-primoid corresponds to the disjunction of a number of properties, whereas a primoid corresponds to a single property. A single property will arise more commonly in applications.

A more compelling reason is apparent in the proper case. 3.2* and 3.12 imply that a proper semi-duoid is a unique union of disjoint duoids, given by the components of the associated graph \( G(\delta) \). In a complementary pivot algorithm, the dual vertices encountered must all lie in just one of these components. Thus, in effect, the algorithm is just applied to the primoid and duoid, contained in the semi-primoid and semi-duoid respectively, which contain the starting vertex. Thus the additional generality of semi-primoids and semi-duoids gains nothing in this case.

A second obvious property of semi-duoids which are generated by simple polytopes is that they are proper. We will now present some other properties of such semi-duoids, which will lead us to the comparisons of duoids with the abstract polytopes of Adler and Dantzig [1, 2, 3]. To
do this, we will investigate the facial structure of duoids.

We recall that a facet is an element of the ground set \( S \), so named because it corresponds to a facet of a simple polytope. A face of a polytope is the intersection of the polytope with a subset of its facets. It is also the affine hull of all vertices of the polytope which lie on all of the subset of its facets. Correspondingly, we can define a face of a duoid.

**Definition 4.1.** For any \( U \in \sigma_{n-d} \), \( 0 \leq d \leq n \), we define the face \( \delta(U) \) generated by \( U \) by \( \delta(U) = \{ D/U | U \subseteq D \in \delta \} \). For convenience, we write \( \delta(e) \) for \( \delta(\{ e \}) \).

**Lemma 4.2**

a) If \( \delta \) is a semi-duoid of order \( n \) on \( S \), then \( \delta(U) \) is a semi-duoid of order \( d \) on \( S/U \).

b) If \( \delta = \delta_1 + \delta_2 \) and all are semi-duoids, then \( \delta(U) = \delta_1(U) + \delta_2(U) \).

In particular if \( \delta(U) \) is a duoid, then all the vertices of \( \delta \) containing \( U \) lie in one duoid contained in \( \delta \).

**Proof.**

a) Let \( \sigma_m^k = \{ W \subseteq S/U | |W| = m \} \). Then \( \delta(U) \subseteq \sigma_d^k \). Now let \( W \in \sigma_{d-1}^k \).

Then \( W \cup U \in \sigma_{n-1}^k \). Moreover, \( W \cup U \subseteq D \in \delta \iff W \subseteq D/U \in \delta(U) \).

There are an even number of such \( D \), so an even number of such \( D/U \).

b) \( \delta(U) = \{ D/U | U \subseteq D \in \delta_1 + \delta_2 \} = \{ D/U | U \subseteq D \in \delta_1 \} \cup \{ D/U | U \subseteq D \in \delta_2 \}/(D/U | U \subseteq D \in \delta_1 \cap \delta_2) = \delta_1(U) + \delta_2(U) \). If \( \delta(U) \) is a duoid, then one of \( \delta_1(U) \), \( \delta_2(U) \) must be empty if \( \delta_1 \), \( \delta_2 \) are disjoint. The result follows.
We now translate some facial properties of polytopes into properties of duoids. Obviously, every face of a polytope is itself a polytope. In particular, every n-1-dimensional face of an n-dimensional polytope is a polytope.

**Definition 4.3.** A facet e is said to be normal in δ (or, if there is no confusion, just normal) if δ(e) is a duoid or empty.

**Definition 4.4.** δ is said to be normal if all its facets are normal.

We use this intermediate property of normality because from any duoid it is easy to generate normal duoids which share some of their properties, such as Euler characteristic or large diameter. We will use this in later chapters to obtain results not only about duoids, but also about the more restricted class of normal duoids. Unfortunately, it is difficult to generate the even more regular class of duoids introduced below. Later we will show that these correspond exactly to abstract polytopes.

**Definition 4.5.** δ is said to be supernormal iff all its faces are duoids or empty. This implies in particular that its 1-dimensional faces, or edges, are duoids; therefore, δ is proper.

Let us now define the abstract polytopes of Adler and Dantzig [1, 2, 3]. In our terms, δ is an abstract polytope iff it satisfies the following three conditions:

AP1) δ ⊆ σ_n.

AP2) If U ∈ σ_{n-1}, U is contained in 0 or 2 vertices of δ.

A path of vertices in δ is a sequence D_0, D_1, ..., D_m of vertices.
of \( \delta \) such that \( |D_{i+1} \cap D_i| = m-1 \) for \( i = 1, 2, \ldots, m \).

AP3) if \( D_1, D_2 \in \delta \) then there is a path in \( \delta \) from \( D_1 \) to \( D_2 \) whose every vertex contains \( D_1 \cap D_2 \).

Let \( \delta \) be an abstract polytope. From AP1 and AP2, \( \delta \) is a proper semi-duoid. But by AP3, \( G(\delta) \) is connected; thus by our Theorem 3.12, \( \delta \) is a duoid.

However, not all proper duoids are abstract polytopes. For example, let us take a dodecahedron and label all of its facets differently, except two opposite ones which share a label. We then get a proper duoid which is not an abstract polytope, because the face with the duplicated label is not connected. Abstract polytopes are "more strongly connected" than duoids.

**Lemma 4.6.** Supernormal duoids correspond exactly to abstract polytopes.

**Proof.**

a) Given a supernormal duoid \( \delta \), then, since it is proper, it satisfies axioms (AP1) and (AP2). Now let \( D_1, D_2 \in \delta \). Consider \( \delta(D_1 \cap D_2) \).

This is a duoid, and so by Lemma 3.6 there is a prismo \( \pi \) intersecting it in just \( D_1/D_2 \), \( D_2/D_1 \). By the algorithms of Chapter 2, there is a path from \( D_1/D_2 \) to \( D_2/D_1 \) in \( \delta(D_1 \cap D_2) \). Adjoining the symbols of \( D_1 \cap D_2 \) to this path, we obtain a path in \( \delta \) all of whose vertices contain \( D_1 \cap D_2 \). This proves (AP3).

b) Given an abstract polytope \( \delta \), we have already seen that it is a duoid. Moreover, by Adler’s remarks [2] each face of \( \delta \) is an abstract polytope, and hence a duoid. So \( \delta \) is supernormal.
Another useful result is the following:

Lemma 4.7.

a) All 2-dimensional duoids are representations of simple polygons.
b) If a 3-dimensional duoid is normal, it is proper and hence supernormal.

Proof.

a) Let $\delta$ be a 2-dimensional duoid, and let $D_0 = \{e_0', e_1\} \in \delta$. Pivoting out $e_0'$, we obtain, say $D_1 = \{e_1, e_2\} \in \delta$, where a choice is made arbitrarily if necessary. At the $m$th step, we will have $D_{m-1} = \{e_{m-1}', e_m\}$, pivoting out $e_{m-1}'$, we obtain $D_m = \{e_m', e_{m+1}\}$. Since $S$ is finite, there must be repetitions in $(e_0', e_1', e_2', \ldots)$; let the first such be $e_\ell' = e_j$ ($j < \ell$).

Then $\delta' = \{D_j, D_{j+1}, \ldots, D_{\ell-1}\}$ forms a semi-duoid contained in $\delta$ — thus it must be $\delta$ itself. But $\delta'$ is the representation of a simple $\ell$-gon with sides labelled $e_j', e_{j+1}', \ldots, e_{\ell-1}'$ in that order.

b) Assume the edge $\{e,f\}$ is contained in more than two vertices of a normal three-dimensional duoid $\delta$. Then the edge $\{f\}$ is contained in more than two vertices of the 2-dimensional duoid $\delta(e)$, which is a duoid because $\delta$ is normal. This statement contradicts (a); therefore, $\delta$ is proper.

Now we exhibit an abstract polytope which does not correspond to a simple polytope.

Example 4.8. Let $S = \{1,2,3,4,5,6\}$. Let $\delta = \{123,126,134,145,156,235,245,246,346,356\}$. That this cannot be generated by a polytope in 3-space can be most easily seen by noting that
if so, the polytope would have 10 vertices, 15 edges, and 6 facets, thus violating Euler's well-known relation.

The boundary of this duoid can, however, be represented on the projective plane. Let this be drawn as a disc, with diametrically opposed points of its boundary identified. Then the duoid above can be represented as follows:

\[\text{P is the point 126} \]
\[\text{Q is the point 235}\]

The duoid is also associated with the standard triangulation of the projective plane:
A third way of viewing this duoid is as that generated by a dodecahedron when opposite facets have the same label. That $\delta$ is a supernormal duoid is easily checked—in fact, its 2-faces are just pentagons.

For the three-dimensional case, a well-known theorem of Steinitz (see e.g. [23]) can be interpreted to show that Euler's relation is sufficient for a proper duoid to be representable in 3-space.

The higher dimensional case is not known and likely to be very difficult. It is easy to verify by a listing of all duoids that if $k \leq 5$ or $n \leq 2$, then every duoid is proper and representable. (See Appendix.)

The following picture shows our classification of duoids and semi-duoids:

```
        duoids
         /
    supernormal /    
       I       III    V     VII
     /   
   II    
  /
IV     VI     VIII
```

where the category I corresponds to duoids which represent simple polytopes.

Lemma 4.7 shows that if $n = 2$, the categories II, III, IV, V and VI are empty, and that if $n = 3$, III and IV are empty.

If $n \geq 4$ and $k \geq n+6$, then each category is non-empty. We
must quote some results from Chapter 5 to show this.

I contains, for example, the simplicial duoids.

II contains, for example, triangulations of n-1-manifolds other
than the sphere, such as projective n-1-space. However, not all duoids
in this category are of this type—for instance a prism erected on the
duoid of Example 4.8 (see 5.1).

III contains normal proper duoids which are not supernormal; the
existence of these is shown in Lemma 5.2.4.

IV contains normal duoids which are not proper—see 5.2.4.

V contains proper duoids which are not normal—for example $\delta_1$
in the proof of 5.2.4; in higher dimensions take prisms.

VI contains duoids which are neither normal nor proper—see example
1.3.10* for $n = 3$; in higher dimensions, take prisms.

VII contains proper semi-duoids which are not duoids—see Example
1.3.11* for $n = 4$; in higher dimensions, take prisms.

VIII contains semi-duoids which are not duoids, not proper, and
not normal; take $\sigma_n$ if $k-n$ is odd, or $\beta_e$ if $k-n$ is even.

The appendix seems to indicate that when $k$ and $n$ become even
moderately large ($n \geq 3$, $k \geq 6$) then category VI comprises by far the
largest part of all duoids. Even if we restrict ourselves to supernormal
duoids or abstract polytopes, the examples given for category II will
probably be far more extensive than the duoids in category I. Thus either
of these two abstractions—abstract polytopes or all duoids—implies a
large number of redundant objects if used to study simple polytopes, the
latter considerably more than the former. On the other hand, no result
similar to 3.9 is available for abstract polytopes; the test for a duoid to be supernormal is likely to be much harder than our Theorem 3.12. Thus it would be much harder to find all abstract polytopes of a given dimension than all duoids.

We now turn to the relationship between the matroids we have constructed, and others on the set $\sigma_n$ introduced by H.H. Crapo and G.-C. Rota [33].

First there is the matroid mentioned briefly at the end of Chapter 3 of [33]. We will denote this matroid by C-R I. It has since been discovered that in fact C-R I is not a matroid if $n > 2$. We will show that, if $n = 2$, then C-R I is exactly $M_{2k}^k$. Thus the matroids we have constructed provide a possible generalization of the case $n = 2$ sought by Crapo and Rota.

To define C-R I as in [8], we would need to introduce other, cryptomorphic versions of a matroid. As remarked earlier, matroids can be defined in terms of their circuits, co-circuits, bases, independent sets or dependent sets. Other definitions can be found in terms of a rank function, a closure operator, or the collection of closed sets. All of these definitions are equivalent, but it is sometimes hard to translate a definition in terms of one concept into that in terms of another. Rather than introducing more definitions, we will make the transformation from Crapo and Rota's definition of C-R I in terms of its closed sets to one in terms of its co-circuits. In our terminology, the co-circuits of C-R I are sets $\gamma \subseteq \sigma_n$ (for $n = 2$) which satisfy

1) If $\gamma$ is a simplicial duoid, $|\gamma \cap \delta| \neq 1$
2) \( \gamma \) is minimal non-empty with this property.

In the presence of (1), we will show (2) to be equivalent to (2'):

2') \( G^+(\gamma) \) is connected, where \( G^+(\cdot) \) is the dual notion to \( G^-'(\cdot) \) introduced in Chapter 3. To be precise, the nodes of \( G^+(\gamma) \) are the vertices of \( \gamma \). \( G_1 \) and \( G_2 \in \gamma \) are adjacent in \( G^+(\gamma) \) iff \( |G_1 \cup G_2| = n+1 = 3 \) and \( G \subseteq G_1 \cup G_2 \), \( G \in \gamma \)
only if \( G = G_1 \) or \( G_2 \).

(1) and (2) imply (2'), for if \( G^+(\gamma) \) is not connected, the vertices of any of its components satisfy (1), so (2) is violated. Conversely, (1) and (2') imply (2), for if \( \emptyset \subset \gamma_1 \subset \gamma \), then there is an edge in \( G^+(\gamma) \) joining a vertex of \( \gamma_1 \) say \( G_1 \) and a vertex of \( \gamma/\gamma_1 \), say \( G_2 \). Then by considering the simplicial duoid \( \langle G_1 \cup G_2 \rangle \) we see that \( \gamma_1 \) does not satisfy (1).

Trivially, every primoid of order 2 is proper, for there are only 3 2-sets contained in any 3-set. Thus the equivalence above and 3.12* show us that every primoid of order 2 on \( S \) is a co-circuit of C-R I.

To show the converse holds, we must prove that (1) and (2) imply 3) If \( \delta \) is a simplicial duoid, \( |\gamma \cap \delta| \neq 3 \).

To every \( \gamma \) satisfying (1), there corresponds an equivalence relation on \( S \), defined by \( e \sim f \) iff \( e = f \) or \( \{e, f\} \in \gamma \). Trivially, this relation is symmetric and reflexive, and if \( e \sim f \), \( f \sim g \), then by considering the simplicial duoid \( \langle e, f, g \rangle \) we must have \( e \sim g \); so the relation is transitive. Conversely, to every equivalence relation on \( S \), there corresponds a set \( \gamma \subseteq \sigma_n \) given by \( \{e, f\} \in \gamma \) iff \( e \neq f \).

That this \( \gamma \) satisfies (1) follows from transitivity. If \( \gamma \) satisfies
(1) and (2), the corresponding relation \( \sim \) is maximal non-trivial, i.e.
there is no equivalence relation \( \sim' \neq \sim \) with \( e \sim f \implies e \sim' f \) except
the universal relation "everything equivalent to everything." So \( \sim \)
has exactly two equivalence classes. Thus, for every three elements of
\( S, e, f \) and \( g \), either \( e \sim f \) or \( f \sim g \) or \( e \sim g \). So
\[ |\gamma \cap \langle e, f, g \rangle| \neq 3 \]
and (3) holds. We have proved the following:

**Lemma 4.2.** If \( n = 2 \), C-R I is exactly \( M_{2k}^* \). The co-circuits of either
are the primoids of order 2 on a set \( S \), are all proper, and have the
form of 1.3.3 with \( m = 2 \). They correspond to the product of two simplices,
whose dimensions add to \( k-2 \).

The proof of the last part of this lemma follows from the character-
ization of the equivalence relations corresponding to satisfying (1)
and (2). These have just two equivalence classes, say \( S_1 \) and \( S_2 \).
Then the corresponding \( \gamma = \{ G | G \cap S_i | = 1, i = 1, 2 \} \). This corresponds
to the product of two simplices of dimensions \( |S_1| - 1 \) and \( |S_2| - 1 \),
see for example [19].

The lemma shows that the closed sets of \( M_{2k}^* \) are just those subsets
of \( \sigma_2 \) which do not contain exactly two of the three 2-subsets of any
3-set (see [8]). Unfortunately, if \( n > 2 \), it is difficult to give any
of the cryptomorphic versions of \( M_{nk} \) or \( M_{nk}^* \) in a simple form. Note
that 3.7 identifies only some of the bases of these matroids.

Now we turn to a comparison between our matroids and those introduced
by Crapo and Rota in Chapter 6 of [8]. These matroids are based on combi-
natorial topology; we shall find a very close connection between topological
concepts and our combinatorial constructions.
We will let \( S = \{1, 2, \ldots, k\} \). Then the matroid on the \( n \)-sets of \( S \) defined in Chapter 6 of [8] will be denoted by C-R II. We must defer its description until we have introduced some more terminology.

We will merely use the topological concepts necessary for the development, and we will present them in the simplest way possible. For a complete reference on algebraic and combinatorial topology, see [11]. The following notation allows us to sidestep the explanation of orientation.

We denote by \( \tau_m \) the set of all \( m \)-tuples \((t_1, t_2, \ldots, t_m)\) with \( 1 \leq t_1 < t_2 < \ldots < t_m \leq k \). If \( T \in \sigma_m \), \( \tau \) is the corresponding \( m \)-tuple in \( \tau_m \).

An \( m \)-chain, \( \emptyset \), is a function from \( \tau_m \) into a coefficient group. For the moment, we take this group to be the additive group of the integers, \( \mathbb{Z} \). An \( m \)-chain, \( \emptyset \), on \( \gamma \) satisfies \( \emptyset(T) = 0 \) if \( T \not\in \gamma \). Each \( m \)-chain \( \emptyset \) generates an \( m \)-l-chain, its boundary, denoted by \( \partial_m \emptyset \), given by

\[
\partial_m \emptyset(U) = \sum_{T \in \sigma_m} \emptyset(T) - \sum_{T \in \sigma_m} \emptyset(T)
\]

\( T/U \) in even place in \( \tau \), \( T/U \) in odd place in \( \tau \)

For example, if \( \emptyset \) is a 3-chain taking \((123)\) and \((234)\) into 1, all other ordered 3-sets into 0, then \( \partial_3 \emptyset \) is a 2-chain taking \((12)\) into \(-1\), \((13)\) into 1, \((23)\) into \(-2\), \((34)\) into \(-1\), \((24)\) into 1; and all other 2-sets into 0. \( \emptyset \) is an \( n \)-cycle on \( \gamma \) if it is a chain on \( \gamma \) and \( \partial_n \emptyset = 0 \).

For example, if \( \gamma = \{12, 13, 23\} \) and \( \emptyset(12) = \emptyset(23) = 1 \), \( \emptyset(13) = -1 \), then \( \emptyset \) is a 2-cycle on \( \gamma \).

Now we may define C-R II.
\( \gamma \) is a circuit of the C-R II matroid if there is just one n-cycle and its multiples on \( \gamma \), and \( \gamma \) is minimal with this property.

It is easy to see that this n-cycle must have the property that \( \emptyset (T) \neq 0 \) if \( T \) is in \( \gamma \), for otherwise \( \gamma \) is not minimal.

We construct a chain on a simplicial duoid \( <V> \) by letting

\[
\emptyset (G) = \begin{cases} 
(-1)^i & \text{if } G \subseteq V \text{ and } V/G \text{ is in the } i^{th} \text{ place in } V \\
0 & \text{otherwise.}
\end{cases}
\]

It is easily seen that \( \emptyset \) is a cycle on \( <V> \), and moreover that \( \emptyset \) and its multiples are the only cycles on \( <V> \). Thus every simplicial duoid is a C-R II circuit.

Now consider \( \beta^e \) for any \( e \). If there were a non-zero cycle \( \tau \) on \( P^e \), then let \( \emptyset (G) \neq 0 \). Then \( \partial_\gamma \emptyset (G/e) = \pm \emptyset (G) \neq 0 \). So there is no such cycle, and \( \beta^e \) is C-R II independent. Since the simplicial duoids are C-R II circuits, \( \beta^e \) is maximal independent, and thus a C-R II base. Thus if the C-R II matroid were binary, it would coincide with our dual matroid. That it does not is seen by Example 4.8. If this had a cycle \( \emptyset \), then let \( \emptyset (235) = i \neq 0 \). Then \( \emptyset (245) = -i \) (taking \( \gamma = (25) \); \( \emptyset (246) = i \); \( \emptyset (346) = -i \) and \( \emptyset (356) = i \) (each time by \( \partial_3 \emptyset = 0 \)). But then \( \partial_3 \emptyset (35) = -i - i = -2i \neq 0 \).

The following two theorems show the relationships between the circuits of C-R II and duoids.

**Theorem 4.10.** Every C-R II circuit contains a duoid.

**Proof.** Let the circuit be \( \gamma \), and its cycle \( \emptyset \). We assume that \( \emptyset \) takes at least one n-set into an odd integer; otherwise \( \emptyset /2 \) is a cycle.
Let $\delta = \{ T \in \gamma | \delta(T) \text{ is odd} \}$. $\delta \subseteq \gamma$.

Reducing equation (3) modulo 2, and remembering that $\chi_n^0 = 0$,

we have

$$0 = \chi_n^0(U) = \sum_{T \in \delta} \chi_T^0 + \sum_{T \in \delta} \chi_T^0 \quad \text{for each } U \in \sigma_{n-1}.$$  

$T/U$ in even place in $\mathcal{T}$, $T/U$ in odd place in $\mathcal{T}$

This establishes (D1); so $\delta$ is a semi-duoid, and therefore contains a duoid.

**Theorem 4.11.** If $\delta$ is a duoid, $\delta$ has at most one cycle and its multiples.

**Proof.** Assume the duoid $\delta$ has two cycles, $\psi_1$ and $\psi_2$. $\psi_1$ and $\psi_2$ must be non-zero on $\delta$; otherwise, the support of $\delta_1$, say, is strictly contained in $\delta$, and contains a duoid by 4.10.

Let $D \in \delta$, and let $\emptyset \equiv \delta_2(D) - \delta_1(D)$. Then $\emptyset$ is a cycle on $\delta/D$. Hence $\emptyset \equiv 0$, and $\delta_1$ and $\delta_2$ are dependent. Thus $\delta$ has at most one cycle and its multiples.

The fact that there is no cycle on the duoid of Example 4.8 is not surprising when we recall that it corresponds to a triangulation of the projective plane, a non-orientable surface. The existence of a cycle would give an orientation to the surface (see the section on simple n-circuits at the end of Chapter 3 of [11]).

There is no concept of orientability for duoids; Example 6.8 is a duoid that can be considered as a degenerate triangulation of either an orientable surface, the torus, or a non-orientable surface, the Klein
bottle. This fact accounts for the difference between $M_{nk}^*$ and C-R II. We can resolve this difference by choosing the coefficient group for the chains to be $Z_2$, the additive group of residues modulo 2. In this case, orientability cannot be determined from the cycle structure.

If we make this change to C-R II, it coincides with $M_{nk}^*$. For, by the argument of 4.10, slightly modified, every C-R II circuit contains a duoid. On the other hand, if $\delta$ is a duoid, then the $n$-chain taking $D \in \delta$ into 1 and $D \notin \delta$ into 0 is a cycle by (D2); thus every duoid contains a C-R II circuit.

This relation will be clarified by a more extensive study of the $n$-cycles over $Z_2$. We may consider such an $n$-chain to be a function from $\sigma_n$ into $Z_2$. The introduction of $\tau_n$ is no longer necessary, since in $Z_2$, $-1 = +1$. The support of an $n$-chain, $\emptyset$, is $\{C \in \sigma_n | \emptyset(C) = 1\}$. In Tutte [30], this support is denoted by $||\emptyset||$; we will use instead $\gamma_{\emptyset}$. Conversely, $\gamma$ will denote the indicator function of $\gamma$. There is obviously a 1-1 correspondence between $n$-chains and subsets of $\sigma_n$. This will be used repeatedly. We denote the set of all $n$-chains $2^{\sigma_n}$.

There are two special functions on $n$-chains; the boundary operator $\delta_n$ and the coboundary operator $\delta_n^*$. These yield respectively an $(n-1)$-chain and an $(n+1)$-chain.

\[ \delta_n : 2^{\sigma_n} \rightarrow 2^{\sigma_{n-1}} \] is defined by: \((\delta_n \emptyset)(U) = \sum_{C \subseteq U} \emptyset(C)\) for all $U \in \sigma_{n-1}$, where all arithmetic here and for the rest of this chapter is modulo 2. It can easily be checked that this definition agrees with that introduced earlier when the coefficient group is $Z_2$.

Similarly, \(\delta_n^* : 2^{\sigma_n} \rightarrow 2^{\sigma_{n+1}}\) is defined by: \((\delta_n^* \emptyset)(V) = \sum_{C \subseteq V} \emptyset(C)\) for all $V \in \sigma_{n+1}$.  

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The subscripts will often be omitted. We use $\partial_n \gamma$ to denote $\partial_{n+1} \partial_n \gamma$, and similarly $\partial^* \gamma$ for $\partial_{n+1} \partial^* \gamma$.

We call $\emptyset$ a cycle (cocycle) iff $\partial \emptyset = 0$ $(\partial^* \emptyset = 0)$. Correspondingly, we call $\gamma$ a cycle (cocycle) iff $\partial \gamma$ is.

We call $\emptyset$ a boundary (coboundary) iff $\exists \emptyset$ s.t. $\emptyset = \partial \emptyset$ $(\emptyset = \partial^* \emptyset)$. Correspondingly, we call $\gamma$ a boundary (coboundary) iff $\partial \gamma$ is.

Now we show the relationship between these terms and semi-primoids and semi-duoids. First we note that $\emptyset \gamma_1 \gamma_2 = \emptyset \gamma_1 + \emptyset \gamma_2$ where the first + denotes symmetric difference, the second addition in $\mathbb{Z}_2$. This follows immediately from the definitions.

Now $\gamma$ is a cycle $\iff \emptyset \gamma$ is a cycle $\iff \emptyset \gamma = 0 \iff$ for all $\emptyset \in \mathcal{G}_{n-1}$, $\sum \emptyset \gamma (G) - \emptyset \iff$ for all $\emptyset \in \mathcal{G}_{n-1}$, $\emptyset \in G \in \gamma$ for an even number of $G$ $\iff \gamma$ is a semi-duoid.

Similarly we can show that $\gamma$ is a cocycle iff it is a semi-primoid.

Also, $\gamma$ is a boundary $\iff \emptyset \gamma$ is a boundary $\iff \exists \emptyset$ s.t. $\partial \emptyset = \emptyset \gamma$$\iff \exists \emptyset$ s.t. $\emptyset \gamma (G) = \emptyset \sum \gamma (V) \iff \exists \emptyset$ s.t. $\gamma = \emptyset \sum V \iff \gamma$ is the symmetric difference of simplicial duoids.

Similarly, we can show that $\gamma$ is a coboundary iff it is the symmetric difference of simplicial primoids.

Thus by 3.1 and 3.1*, boundaries are cycles and coboundaries are cocycles. Thus $\partial \partial_n \gamma = 0$, $\partial^* \partial_n \gamma = 0$. Furthermore, by 3.9 and 3.9*, the converses are true; cycles are boundaries and cocycles are coboundaries.

3.9 and 3.9* can also be proved using homology and cohomology theory, as we indicated in Chapter 3. To see this, we must be more precise in
our definitions of chains, cycles and boundaries. A simplicial complex $\mathcal{C}$ is a subset of $\sigma_k \cup \sigma_{k-1} \cup \ldots \cup \sigma_0 = 2^S$ such that if $T \in \mathcal{C}$, $U \subseteq T$, then $U \in \mathcal{C}$. An $n$-chain of $\mathcal{C}$ is a function of $\mathcal{C} \cap \sigma_n$ into the coefficient group $\mathbb{Z}_2$. An $n$-cycle of $\mathcal{C}$ is an $n$-chain of $\mathcal{C}$ whose boundary is zero. An $n$-boundary of $\mathcal{C}$ is an $n$-chain of $\mathcal{C}$ which is the boundary of an $(n+1)$-chain of $\mathcal{C}$. What we have been calling the boundaries are in fact the boundaries of $2^S$. The $n$-cycles of $\mathcal{C}$ and the $n$-boundaries of $\mathcal{C}$ form two groups on $2^\sigma_n$, denoted by $\mathbb{Z}_n(\mathcal{C})$ and $\mathbb{B}_n(\mathcal{C})$. It is a simple result of homology theory that $\mathbb{B}_n(\mathcal{C})$ is a normal subgroup of $\mathbb{Z}_n(\mathcal{C})$. The $n^{th}$ homology group of $\mathcal{C}$, $\mathbb{H}_n(\mathcal{C})$ is the factor group $\mathbb{Z}_n(\mathcal{C})/\mathbb{B}_n(\mathcal{C})$.

A well-known result of homology theory is that the homology groups of $2^S$, which can be thought of as all simplices of a $(k-1)$-simplex, described by their sets of vertices, are all trivial. This immediately implies $\mathbb{Z}_n(2^S) = \mathbb{B}_n(2^S)$, which when the coefficient group is $\mathbb{Z}_2$ is exactly 3,9. The corresponding cohomology result implies 3,9*. Conversely, we can use 3,9 and 3,9* to give a new proof that $\mathbb{H}_n(2^S)$ and $\mathbb{H}^n(2^S)$ are trivial if the coefficient group is $\mathbb{Z}_2$.

We have the following mappings (or, rather, homomorphisms):

\[ \begin{array}{ccc}
\cdots & \sigma_{n+1} & \sigma_n & \sigma_{n-1} & \cdots \\
\delta_{n+1} & \delta_n & \delta_{n-1}
\end{array} \]

Let us now consider the composite mappings of $2^\sigma_n$ into itself:

$\delta \delta^* = \delta_{n+1} \circ \delta_n$ and $\delta^* \delta = \delta_{n-1} \circ \delta_n$. We will confine ourselves to the
case $k$ odd to avoid complications. This involves no real loss of
generality, for each semi-primoid or semi-duoid can be considered on a ground
set $S$ of odd cardinality by trivial modifications.

**Lemma 4.12**

a) $\pi$ is a semi-primoid iff $\pi = \partial^* \partial_\pi$

b) $\delta$ is a semi-duoid iff $\delta = \partial \partial^* \delta$

**Proof.** We prove a); b) follows by duality. Firstly, if $\pi = \partial^* \partial_\pi$, $\pi$
is a coboundary and hence a semi-primoid. So assume $\pi$ is a semi-primoid,
and consider $\partial^* \partial_\pi$.

$$\partial^* \partial_\pi(P) = \sum_{U \subseteq P} \partial_\pi(U) = \sum_{U \subseteq P} \sum_{G \subseteq U} \partial_\pi(G).$$

Noting that there are $n$ $(n-1)$-sets contained in $P$,

$$\partial^* \partial_\pi(P) = n\partial_\pi(P) + \sum_{G \subseteq P} \partial_\pi(G) = n\partial_\pi(P) + \sum_{G \subseteq P} \partial_\pi(G).$$

If $P \notin \pi$, then $\partial_\pi(P) = 0$, and each of the sums in square brackets
is zero by (P1). So $\partial^* \partial_\pi(P) = 0$.

Now assume $P \in \pi$. Then $\partial_\pi(P) = 1$, and each of the sums in
the square brackets is 1 by (P2); so $\partial^* \partial_\pi(P) = n+k-n=k=1 \mod 2$.
Hence $\partial^* \partial_\pi = \partial_\pi$ or $\partial \partial^* \pi = \pi$.

A corollary to this lemma is the existence of a 1-1 correspondence
between semi-primoids of order $n$ and semi-duoids of order $n-1$, given
by the inverse mappings $\partial_n$ and $\partial^{*}_{n-1}$.

We conclude with an interesting but singularly useless theorem.
Theorem 4.13. If $\gamma \subseteq \sigma_n$, then $\gamma = \partial^* \partial \gamma + \partial \partial^* \gamma$ and this is the only way $\gamma$ can be expressed as the symmetric difference of a semi-primoid and a semi-duoid.

Proof. $\partial^* (\gamma + \partial \partial^* \gamma) = \partial^* \gamma + \partial \partial^* (\partial^* \gamma) = \partial^* \gamma + \partial^* \gamma = \emptyset$ since $\partial^* \gamma$ is a coboundary, hence a semi-primoid. So $\gamma + \partial \partial^* \gamma$ is a cocycle, hence a semi-primoid. Similarly, $\gamma + \partial^* \partial \gamma$ is a semi-duoid. Thus $\gamma$ can be expressed as the symmetric difference of a semi-primoid and a semi-duoid.

Now let $\gamma = \pi_1 + \delta_1 = \pi_2 + \delta_2$ where $\pi_1$, $\pi_2$ are semi-primoids and $\delta_1$, $\delta_2$ are semi-duoids. Then $\tau = \pi_1 + \pi_2 = \delta_1 + \delta_2$ is a semi-primoid and a semi-duoid. So $\partial^* \partial \tau = \tau$ but $\partial \tau = \emptyset$; thus $\tau = \emptyset$ and $\pi_1 = \pi_2$, $\delta_1 = \delta_2$. Since the decomposition is unique, $\gamma = \partial^* \partial \gamma + \partial \partial^* \gamma$. This can be obtained immediately by $\gamma + \partial \partial^* \gamma - \partial^* \partial (\gamma + \partial \partial^* \gamma) = \partial \partial^* \gamma + \partial^* (\partial \partial^*) \partial \partial^* \gamma = \partial^* \partial \gamma + \emptyset = \partial^* \partial \gamma$. 

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CHAPTER 5

OPERATIONS ON DUOIDS

We have already mentioned the most useful operation on duoids—symmetric difference. This has the drawback that from a pair of duoids, the result may not be a duoid; the operation is closed on semi-duoids, but not on duoids. The operations we consider here are all closed on duoids. They also have a simple geometric interpretation. In other words, the operations are analogues of operations closed on polytopes.

The product and the wedge are based on operations used by Klee and Walkup [19]; the sum operation was introduced by Adler [3].

These operations will be used in the following two chapters to construct duoids with certain properties.

Note that all these operations can be performed on primoids dually.

5.1. Products of Duoids

Definition 5.1.1. If $\delta_i$ is a semi-duoid of order $n_i$ on $S_i$, $i = 1, 2$ and $S_1 \cap S_2 = \emptyset$, then we define the product $\delta_1 \otimes \delta_2$ by $\delta_1 \otimes \delta_2 = \{D_1 \cup D_2 | D_1 \in \delta_1, i = 1, 2\}$.

Lemma 5.1.2. If $\delta_1$, $\delta_2$ are (semi-) duoids, then so is $\delta_1 \otimes \delta_2$ of order $n_1 + n_2$ on $S_1 \cup S_2 = S$; if $\delta_1$, $\delta_2$ are proper, so is $\delta_1 \otimes \delta_2$;
\( \delta_1 \oplus \delta_2 \) is normal (supernormal) iff both \( \delta_1 \) and \( \delta_2 \) are.

**Proof:** From the definition it is immediate that \( \delta_1 \oplus \delta_2 \subseteq \sigma_{n_1+n_2} \). Now let \( U \subseteq \sigma_{n_1+n_2-1} \), and let \( U_i = U \cap S_i, \ i = 1, 2 \).

**Case 1.** \( |U_1| = n_1 \), \( |U_2| = n_2 - 1 \). Then if \( U_1 \nsubseteq \delta_1 \), there are no vertices of \( \delta_1 \oplus \delta_2 \) containing \( U \). If \( U_1 \subseteq \delta_1 \), then \( U \subseteq D_1 \cup D_2 \in \delta_1 \oplus \delta_2 \iff U_2 \subseteq D_2 \in \delta_2 \) and \( D_1 = U_1 \). There are an even number of such \( D_2 \), thus an even number of such \( D_1 \cup D_2 \).

**Case 2.** \( |U_1| = n_1 - 1 \), \( |U_2| = n_2 \). Proceed as in Case 1.

**Case 3.** Neither or the above. Then either \( |U_1| > n_1 \) or \( |U_2| > n_2 \). Thus there are no vertices of \( \delta_1 \oplus \delta_2 \) containing \( U \).

Hence \( \delta_1 \oplus \delta_2 \) is a semi-duoid of order \( n_1 + n_2 \) on \( S_1 \cup S_2 = S \). Moreover, we see that if \( \delta_1 \) and \( \delta_2 \) are proper, so is \( \delta_1 \oplus \delta_2 \).

Let \( D_i \in \delta_i, \ i = 1, 2 \), and \( D_i^t \in \delta_i^t, \ i = 1, 2 \). Then \( (\delta_1 \oplus \delta_2)(D_2) = \delta_1 \) is a face of \( \delta_1 \oplus \delta_2 \) which is a duoid. Thus by Lemma 4.2, all vertices containing \( D_2 \) lie in the same duoid contained in \( \delta_1 \oplus \delta_2 \). Repeating the argument for \( (\delta_1 \oplus \delta_2)(D_i^t) \), we see that \( D_1^t \cup D_2^t \) is in the same duoid as \( D_1^t \cup D_2 \), which is in the same duoid as \( D_1 \cup D_2 \). But the \( D_1 \), \( D_1^t \) were arbitrary; thus \( \delta_1 \oplus \delta_2 \) is a duoid.

Now \( (\delta_1 \oplus \delta_2)(U) = \delta_1(U \cap S_1) \oplus \delta_2(U \cap S_2) \) for any \( U \subseteq S \). By taking \( U \) first as all one-element subsets of \( S \), then as all subsets of \( S \), we see that \( \delta_1 \oplus \delta_2 \) is normal iff \( \delta_1 \) and \( \delta_2 \) are, and supernormal iff \( \delta_1 \) and \( \delta_2 \) are.
Example 5.1.3. If \( \delta_2 = \langle [t, b] \rangle \) (for top, bottom) the 1-dimensional duoid, then \( \delta_1 \otimes \delta_2 \) just corresponds to a prism erected on \( \delta_1 \). Let
\[
\delta_1 = \{[1,2], [2,3], [3,4], [4,1]\}
\]
the 2-dimensional square. (In fact, with some relabelling we could consider \( \delta_1 = \delta_2 \otimes \delta_2 \).) Then
\[
\delta_1 \otimes \delta_2 = \{12t, 23t, 34t, 41t, 12b, 23b, 34b, 41b\}
\]
which is just the cube:

![Diagram of a cube](image)

We already have some examples of the product for semi-primoids; let
\[
\sigma^1_n = \{T \subseteq S_1 \mid |T| = n\}
\]
Then \( \pi_1 \) of 1.3.3 = \( \sigma^1_1 \otimes \sigma^2_1 \otimes \cdots \otimes \sigma^m_1 \) and \( \pi_2 \) of 1.3.4 = \( \sigma^1_j \otimes \sigma^2_i \otimes \cdots \otimes \sigma^m_i \), each of the constituents being semi-primoids as remarked in corollary 2.4*.

5.2. Wedges of Duoids

Definition 5.2.1. Given a semi-duoid \( \delta \) of order \( n \) on \( S \) and a facet \( e \) of \( S \), two symbols \( t \) and \( b \) not in \( S \), then the wedge of \( \delta \) on \( e \) is defined by \( W(\delta, e) \triangleq (\delta \otimes \langle [t, b] \rangle) + (\delta(e) \otimes \langle [t, b, e] \rangle) \). Note that the definition depends on \( t \) and \( b \), but this will not be confusing.

Lemma 5.2.2. If \( \delta \) is a (semi-) duoid, then \( W(\delta, e) \) is a (semi-) duoid of order \( n+1 \) on \( S' = S \cup [t, b]_/e \). It is proper if \( \delta \) is proper.
and b are normal facets in $W(\delta, e)$. A normal facet of $\delta$ remains normal in $W(\delta, e)$.

**Proof:** By Lemma 5.1.2, $W(\delta, e)$ is a semi-duoid if $\delta$ is. Furthermore, it is of order $n+1$, and we note that every vertex containing $e$ is eradicated by the symmetric difference. Note also that every vertex contains either $t$ or $b$ or both. Let $\delta$ be a duoid, and consider $(W(\delta, e))(t)$, the face generated by $t$. From the definition it is easily seen that $(W(\delta, e))(t) = \delta + (\delta(e) \otimes \langle b, e \rangle)$, and hence it is isomorphic to $\delta$ with $e$ replaced with $b$. Thus it is a duoid. So by Lemma 4.2, all vertices containing $t$ lie in the same duoid contained in $W(\delta, e)$; the same is true for $b$. Moreover, let $D$ be a vertex of $W(\delta, e)$ on $t$ but not $b$, then the $n$-set $D/t$ is contained in just $D$ and $n$-set $b/t$. Thus these must lie together in one of the duoids of $W(\delta, e)$. Putting these facts together, we see that $W(\delta, e)$ must itself be a duoid. We have also seen above that $t$ and $b$ are normal facets in $W(\delta, e)$. Moreover, if $f \in S/e$ $(W(\delta, e))(f) = W(\delta(f), e)$; thus a normal facet of $\delta$ remains normal in $W(\delta, e)$. Now assume $\delta$ is proper and let $U \subseteq S'$, $|U| = n$.

**Case 1.** $t \in U$. Then the vertices of $W(\delta, e)$ containing $U$ correspond to those of $W(\delta(e))(t)$ containing $U/t$. But $W(\delta(e))(t)$ is isomorphic to $\delta$ with relabeling. So these vertices are 0 or 2 in number.

**Case 2.** $b \in U$. Proceed as in Case 1.

**Case 3.** $t \notin U$, $b \notin U$. Since every vertex of $W(\delta, e)$ contains either $t$ or $b$, the only ones which can contain $U$ are $UUt$, $UUb$. So again there are at most 2.
Hence \( W(\delta, e) \) is proper if \( \delta \) is.

**Example 5.2.3.** Let \( \delta = \{[1, 2], [2, 3], [3, 4], [4, 5], [5, 1]\} \). Then \( W(\delta, 5) = \{12t, 23t, 34t, 4tb, 1tb, 12b, 23b, 34b\} \).

![Diagram of \( \delta \) and \( W(\delta, 5) \)](image)

We can use this construction to show that Lemma 4.7 is the strongest possible, in the following sense.

**Lemma 5.2.4.**

(a) For \( n \geq 4 \), there is a normal proper duoid of dimension \( n \) which is not supernormal.

(b) For \( n \geq 4 \), there is a normal duoid of dimension \( n \) which is not proper.

**Proof:** a) Let \( \delta_1 = \{123, 124, 134, 156, 157, 167, 235, 346, 247, 356, 467, 257\} \). It can be checked that \( \delta_1 \) is a proper duoid and that each of its facets except 1 is normal. Then \( W(\delta_1, 1) \) will be a normal proper duoid of dimension 4, but its \( \{t, b\} \) face will be \( \delta_1(1) \), which is not a duoid. So \( W(\delta_1, 1) \) is not supernormal. For higher dimensions, erect prisms on \( W(\delta_1, 1) \) as in 5.1.3.
b) Let $\delta_2 = \{124, 126, 135, 136, 146, 156, 245, 256, 345, 346\}$. Let $\gamma_1 = \delta_2 \otimes [7, 8]$. Let $\delta_3 = \{81, 19, 9e, 6f, 66\}$. Let $\gamma_2 = [2, 3] \otimes W_{45}(\delta_3, g)$ where the subscript 45 means that we use 4 and 5 instead of t and b. Then let $\gamma = \gamma_1 + \gamma_2$. We can check that $\gamma$ is in fact a duoid by Theorem 3.13. It is not proper, because $U = \{167\}$ is contained in 1267, 1367, 1467, and 1567, all vertices of $\gamma$. So we must show $\gamma$ to be normal. Notice that if we switch the labels 2 and 3 and also the labels 4 and 5, then all the duoids, in particular $\gamma$ remain fixed. So $\gamma(2)$ is just a relabelling of $\gamma(3)$; the same is true for $\gamma(4)$ and $\gamma(5)$. We can also switch 2 and 3, 1 and 6, and 9 and f; thus $\gamma(1)$ is a relabelling of $\gamma(6)$ and $\gamma(9)$ a relabelling of $\gamma(f)$. It is fairly simple to check the six remaining cases and verify that $\gamma$ is normal. For higher dimensions erect prisms on $\gamma$.

5.3. Sums of Duoids

Definition 5.3.1. Let $\delta_1$, $\delta_2$ be semi-duoids of order $n$ on $S_1$, $S_2$. Relabel if necessary, obtaining $\delta'_1$, $\delta'_2$, $S'_1$, $S'_2$, to ensure that $|S'_1 \cap S'_2| = n$, and $S'_1 \cap S'_2$ is a vertex of both $\delta'_1$ and $\delta'_2$. Then $\text{Sum}(\delta_1, \delta_2) = \delta'_1 + \delta'_2$. Once again the labelling is arbitrary, and the vertices of $\delta'_1$ and $\delta'_2$ chosen to coincide are also arbitrary.

Lemma 5.3.2. If $\delta_1$, $\delta_2$ are (semi-) duoids, then $\text{Sum}(\delta_1, \delta_2)$ is a (semi-) duoid of order $n$ on $S = S'_1 \cup S'_2$; it is proper if $\delta_1$ and $\delta_2$ are proper, and normal if $\delta_1$ and $\delta_2$ are normal.
Proof: Let $D_1, D'_1 \in \delta_1/\delta_2'$. Since $\delta_1'$ is a duoid on $S_1'$ it is also a duoid on $S$. Let $\pi$ be a proid on $S$ with $\pi \cap \delta_1' = \{D_1, D_1'\}$.

Consider $G(\pi, \text{sum}(\delta_1, \delta_2), e)$ where $e \in S_1' \cap S_2'$. The component of this graph that contains $(D_1, D_1)$ can contain no $(D_2, D_2)$ for $D_2 \in \delta_2'$ since this component contains only $(P, D)'$'s with $e \notin D$ except for its odd nodes. So it contains as odd nodes just $(D_1, D_1)$ and $(D_1', D_1')$.

Thus any duoid contained in $\text{sum}(\delta_1, \delta_2)$ must contain $D_1$ and $D_1'$ or neither. The same holds for any $D_2, D_2' \in \delta_1/\delta_2'$. Thus the only way in which $\text{sum}(\delta_1, \delta_2)$ could split into semi-duoids is if these were $\delta_1'/\{S_1' \cap S_2'\}$ and $\delta_2'/\{S_1' \cap S_2'\}$; but these cannot be semi-duoids because they are strictly contained in $\delta_1'$, $\delta_2'$. Thus $\text{sum}(\delta_1, \delta_2)$ is a duoid.

Now assume $\delta_1$ and $\delta_2$ are proper, then so are $\delta_1'$, $\delta_2'$. Let $U \in \sigma_{n-1}$. Let $U_1 = U \cap S_1'$, $U_2 = U \cap S_2'$.

Case 1. $U = U_1$, $U_2 = \emptyset$. Then the vertices of $\text{sum}(\delta_1, \delta_2)$ containing $U$ are just those of $\delta_1'$ containing $U$; and there are 0 or 2 of these.

Case 2. $U_2 = U$, $U_1 = \emptyset$. Proceed as in case 1.

Case 3. $U = U_1 = U_2$. Then $U \subseteq S_1' \cap S_2'$. $U$ is contained in just $D_1$, say, and $S_1' \cap S_2'$ of $\delta_1'$, $i = 1, 2$. So $U$ is contained in just $D_1$, $D_2$ of $\text{sum}(\delta_1, \delta_2)$.

Case 4. $U_1 \subseteq U$, $U_2 \subseteq U$. Since all vertices of $\text{sum}(\delta_1, \delta_2)$ are contained in either $S_1'$ or $S_2'$, there are no vertices of $\text{sum}(\delta_1, \delta_2)$ containing $U$, which has facets of $S_1'/S_2'$ and $S_2'/S_1'$.

So $\text{sum}(\delta_1, \delta_2)$ is proper.

Now let $\delta_1$, $\delta_2$ be normal. Let $e \in S$. 

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Case 1. \( e \in S_1^1/S_2^1 \); then \((\text{Sum}(\delta_1, \delta_2))(e) = \delta_1^!(e)\) and this is a duoid.

Case 2. \( e \in S_2^1/S_1^1 \). Proceed as in Case 1.

Case 3. \( e \in S_1^1 \cap S_2^1 \). \((\text{Sum}(\delta_1, \delta_2))(e) = \delta_1^!(e) + \delta_2^!(e)\). The same argument that was used to show that \(\text{Sum}(\delta_1, \delta_2)\) was a duoid will show that \((\text{Sum}(\delta_1, \delta_2))(e)\) is a duoid. Hence \(\text{Sum}(\delta_1, \delta_2)\) is normal.

Example 5.3.3. Let \( \delta_1 = W(8, 5) \) as in 5.2.3 and let \( \delta_2 = \langle 1, 2, 3, 4 \rangle \).

We will consider two different Sums, to show the choices involved.

a) Let \( \delta_1^! = \delta_1, \ \delta_2^! = \langle 4, t, b, 5 \rangle \). These satisfy the hypotheses of definition 5.3.1, and \(\text{Sum}(\delta_1, \delta_2) = \{12t, 23t, 34t, 45t, 5tb, 1tb, 12b, 23b, 34b, 45b\}\); geometrically,

\[
\text{Sum}(\delta_1, \delta_2) \text{ is the duoid obtained by selecting } P_1 \text{ and } P_2, \text{ and joining the edges as shown to obtain:}
\]
b) Let $\delta_1' = \delta_1$, $\delta_2' = \langle 1, 2, t, 5 \rangle$. Then $\text{Sum}(\delta_1', \delta_2') = \{125, 15t, 25t, 23t, 34t, 4tb, 1tb, 12b, 23b, 34b\}$. It can be seen that these two sums are essentially different—i.e., they are not just relabellings of each other. However, our use of Sums does not depend on the choice—$\text{Sum}(\delta_1', \delta_2')$ can be chosen to be any Sum so obtained.
Note that the sum of a duoid $\delta$ and a simplicial duoid corresponds to truncating one vertex of the duoid $\delta$. 
CHAPTER 6

THE RANGE OF EULER CHARACTERISTICS OF DUOIDS

AND SOME TOPOLOGICAL PROPERTIES OF DUOIDS OF DIMENSION 3

In Chapter 4 we showed the relationship between our duoids and abstract polytopes. In [1, 3] Adler studied the range of Euler characteristics of the latter. We will comment on his results and obtain corresponding results for duoids, normal duoids, and proper duoids. In the three-dimensional case we will be led to study 2-manifolds.

In Chapter 4 we defined a face of a duoid; now we are interested in the number of faces of a given dimension.

Definition 6.1. Let $\delta$ be a semi-duoid of order $n$ on $S$. For $0 \leq d \leq n$, define $f_d(\delta) = \sum_{\sigma \in S_{n-d}} [\exists D \in \delta \text{ s.t. } U \subseteq D]$. In particular, $f_0(\delta) = |\delta|$, the number of vertices, and $f_{n-1}(\delta) = k$, if we exclude superfluous facets from $S$. From the definition, $f_n(\delta) = 1$; we may consider that this represents the one face of dimension $n$, i.e., $\delta$ itself. In terms of the $f_d(\delta)$, we may now define the (modified) Euler characteristic as follows.

Definition 6.2. $X^*(\delta) = \sum_{d=0}^{n} (-1)^d f_d(\delta)$.

Note that, for a polytope, the normal definition is
\[ X(P) \equiv \sum_{d=0}^{n-1} (-1)^d f_d(P) = 1 - (-1)^n \] be the Euler-Poincaré relation. Using

\[ X^* \] instead enables us to replace this with \[ X^*(P) = 1 \], a much simpler relation. This definition of \( X^* \) corresponds to Adler's; it has other advantages in terms of simpler results also.

Example 4.8 has already shown us that \( X^*(\delta) \) is not necessarily 1, even for normal proper duoids. This example is the same as that used by Adler [3].

As remarked in that paper, 3-dimensional abstract polytopes correspond 1-1 to triangulated 2-manifolds. It is also easy to see that triangulated n-1-manifolds give rise to n-dimensional abstract polytopes in higher dimensions also. This is because the link of any face will be a triangulated manifold, and the result will follow by induction.

This will enable us to settle Adler's remaining question. In fact, \( S^2 \times S^2 \) is a 4-dimensional manifold with Euler characteristic 3; it can be triangulated, to give a 5-dimensional abstract polytope, also with Euler characteristic 3. From this all Euler characteristics can be obtained by an argument similar to that in [3] or in our Theorem 6.6.

**Lemma 6.3.** \( X^*(\delta_1 \otimes \delta_2) = X^*(\delta_1)X^*(\delta_2) \).

**Proof.** It is easily seen from the definitions that \( U \) generates a \( d \)-face of \( \delta_1 \otimes \delta_2 \) iff \( U \cap S_1 \) generates a \( d_1 \)-face of \( \delta_1 \) with \( d_1 + d_2 = d \).

Thus \( f_d(\delta_1 \otimes \delta_2) = \sum_{j=0}^{d} f_j(\delta_1)f_{d-j}(\delta_2) \).
\[ x^*(\delta_1 \boxplus \delta_2) = \sum_{d=0}^{n_1+n_2} (-1)^d f_d(\delta_1 \boxplus \delta_2) \]

\[ = \sum_{d=0}^{n_1+n_2} (-1)^d \sum_{d_1+d_2=d} f_{d_1}(\delta_1)f_{d_2}(\delta_2) \]

\[ = \left( \sum_{d_1=0}^{n_1} (-1)^{d_1} f_{d_1}(\delta_1) \right) \left( \sum_{d_2=0}^{n_2} (-1)^{d_2} f_{d_2}(\delta_2) \right) \]

\[ = x^*(\delta_1)x^*(\delta_2). \]

**Lemma 6.4.** \( x^*(W(\delta,e)) = x^*(\delta) \).

**Proof.** For \( 0 \leq d \leq n+1 \), we consider the \( d \)-faces of \( W(\delta,e) \) generated by \( U \)'s \( \in \sigma_{n+1-d} \) in four categories.

**Case 1.** \( t \in U, \ b \notin U \). Then \( U \) generates a \( d \)-face of \( W(\delta,e) \) iff \( U/t \), with \( e \notin U/t \) generates a \( d \)-face of \( \delta \).

**Case 2.** \( t \notin U, \ b \in U \). Then \( U \) generates a \( d \)-face of \( W(\delta,e) \) iff \( U/b \), with \( e \notin U/b \) generates a \( d \)-face of \( \delta \).

**Case 3.** \( t, b \in U \). Then \( U \) generates a \( d \)-face of \( W(\delta,e) \) iff \( U/[t,b] \) generates a \( d \)-face of \( \delta(e) \).

**Case 4.** \( t, b \notin U \). Then \( U \) generates a \( d \)-face of \( W(\delta,e) \) iff \( U \cup t \) generates a \( d-1 \)-face of \( W(\delta,e) \) iff \( U \), with \( e \notin U \), generates a \( d-1 \)-face of \( \delta \).

Condensing, we obtain, for \( 0 \leq d \leq n+1 \):
\[ f_d(W(\delta, e)) = [f_d(\delta) - f_d(\delta(e))] + [f_d(\delta) - f_d(\delta(e))] + [f_d(\delta(e))] \\
+ [f_{d-1}(\delta) - f_{d-1}(\delta(e))] \\
= 2f_d(\delta) - f_d(\delta(e)) + f_{d-1}(\delta) - f_{d-1}(\delta(e)). \]

Thus \( X^*(W(\delta, e)) = \sum_{d=0}^{n+1} [2f_d(\delta) - f_d(\delta(e)) + f_{d-1}(\delta) - f_{d-1}(\delta(e))](-1)^d \)
\[ = 2X^*(\delta) - X^*(\delta(e)) - X^*(\delta) + X^*(\delta(e)) \]
\[ = X^*(\delta). \]

**Lemma 6.5.** \( X^*(\text{Sum}(\delta_1, \delta_2)) = X^*(\delta_1) + X^*(\delta_2) - 1. \)

**Proof.** Let \( U \subseteq \sigma_{n-d} \) for \( 0 \leq d \leq n \).

**Case 1.** \( U \cap (S_2^* \setminus S_1^*) \neq \emptyset \). Then \( U \) does not generate a face of \( \delta_2^* \) and generates a face of \( \text{Sum}(\delta_1, \delta_2) \) iff it generates a face of \( \delta_1^* \).

**Case 2.** \( U \cap (S_2^* \setminus S_1^*) \neq \emptyset \). Then \( U \) does not generate a face of \( \delta_1^* \) and generates a face of \( \text{Sum}(\delta_1, \delta_2) \) iff it generates a face of \( \delta_2^* \).

**Case 3.** \( U \subseteq S_1^* \cap S_2^* \). Then let \( U' \subseteq S_1^* \cap S_2^* \) contain \( U \) and \( |U'| = n-1 \).

\( U' \) is contained in a vertex of \( \delta_1^* \) besides \( S_1^* \cap S_2^* \), so \( U \) generates a face of \( \text{Sum}(\delta_1, \delta_2) \). There are \( \binom{n}{n-d} \) of such faces, and each is a face of \( \delta_1^* \) and of \( \delta_2^* \). Hence \( f_d(\text{Sum}(\delta_1, \delta_2)) = f_d(\delta_1) + f_d(\delta_2) - \binom{n}{n-d} \)

Also \( f_0(\text{Sum}(\delta_1, \delta_2)) = f_0(\delta_1) + f_0(\delta_2) - 2 - \binom{n}{n-0} - 1 \).
So \( X^*(\text{Sum}(\delta_1, \delta_2)) \) = \( \sum_{d=0}^{n} (-1)^d \left( f_d(\delta_1) + f_d(\delta_2) - \binom{n}{n-d} \right) - 1 \)

\[ = X^*(\delta_1) + X^*(\delta_2) + (1-1)^n - 1 \]

\[ = X^*(\delta_1) + X^*(\delta_2) - 1 . \]

Theorem 6.6.

a) All 2-dimensional duoids have Euler characteristic 1.

b) All 3-dimensional duoids have Euler characteristic \( \leq 1 \); and for any integer \( m \leq 1 \), there is a normal proper 3-duoid with Euler characteristic \( m \).

c) All 4-dimensional normal duoids have Euler characteristic \( \leq 1 \); for any integers \( k \leq 1 \) and \( m \), there is a normal proper 4-dimensional duoid with Euler characteristic \( k \) and a proper 4-dimensional duoid with Euler characteristic \( m \).

d) For \( n \geq 5 \), and \( m \) any integer, there is a normal proper duoid with Euler characteristic \( m \).

Proof. The first parts of b) and c) are difficult and will be deferred to lemmas; it is there that we investigate the topological nature of 3-dimensional duoids.

Part a) follows from Lemma 4.10. For higher dimensions, more notation will be useful. Let \( 1\delta = \delta \), and let \( m\delta \) denote \( \text{Sum}((m-1)\delta, \delta) \).

Let \( \delta^1 = \delta \) and let \( \delta^R \) denote \( \delta^{n-1} \otimes \delta^* \) where \( \delta^* \) is a relabelling of \( \delta \) so that the required ground sets are disjoint.
Let $\tau_n$ denote $\langle e_0, e_1, \ldots, e_n \rangle$, let $\delta_1$ be as in Example 4.8, and let $\delta_2 = \langle \{1, 2, 3, 4, 5\} \rangle + \langle \{2, 3, 4, 5, 6\} \rangle + \langle \{3, 4, 5, 6, 7\} \rangle + \langle \{4, 5, 6, 7, 8\} \rangle + \langle \{5, 6, 7, 8, 9\} \rangle + \langle \{6, 7, 8, 9, 1\} \rangle$. We can check that $\delta_2$ is a duoid by considering $G'(\delta_2, 5)$ and using 3.13.

Second part of b). $X^*(\tau_3) = 1$, $X^*(\delta_1) = 0$, and by repeated use of 6.5, $X^*(m\delta_1) = 1 - m$ for all $m \geq 1$.

Second part of c) $X^*(\tau_4) = 1$, $X^*(m\delta_1 \otimes \tau_1) = 1 - m$ by 6.3 for all $m \geq 1$.

Furthermore, all of these are normal and proper by 5.1.2 and 5.3.2.

Third part of c). It can be verified that $X^*(\delta_2) = 2$. Hence $X^*(m\delta_2) = 1 + m$ by 6.5 for all $m \geq 1$. Since $\delta_2$ is proper, so is $m\delta_2$ for all $m \geq 1$.

Part d) $X^*(\tau_5) = 1$; $X^*(m\delta_1 \otimes \tau_1^2) = 1 - m$; and $X^*(W(\delta_2, 1)) = 2$ by 6.4. Letting $\delta_3$ denote $W(\delta_2, 1)$, we first see that $\delta_3$ is normal, since every facet of $\delta_2$ except 1 is. Then all the duoids above are normal and proper, and $X^*(m\delta_3) = 1 + m$.

For dimension $n > 5$, we use $\tau_n$, $m\delta_1 \otimes \tau_1^{n-3}$, and $\delta_3 \otimes \tau_1^{n-5}$ for Euler characteristic 1, 1 - m and 1 + m respectively.

We must now settle the remaining parts of b) and c).

**Lemma 6.7.** If $\delta$ is a 3-dimensional duoid, then $X^*(\delta) \leq 1$.

**Proof.** We first prove the lemma in the case when $\delta$ is proper. The argument for the general case is similar, but relies more on topological results.

From our proper 3-dimensional duoid $\delta$, we proceed as follows to obtain a normal duoid $\delta'$. For each facet $e$ of $\delta$, define $g(e)$ to
be the smallest number of duoids whose disjoint union is $\delta(e)$. (So if $e$ is normal in $\delta$, $g(e) = 1$.) Then we define the abnormality of $\delta$, 
\[ \text{Ab}(\delta) \triangleq \sum_{e \in S} (g(e) - 1). \]

If, say, $g(e) > 1$, let $\delta_1(e)$ be one of the $g(e)$ duoids contained in $\delta(e)$, and let $f \in S$. We obtain $\delta'$ from $\delta$ by relabelling $e$ by $f$ on $\delta_1(e)$; more formally, $\delta' = \delta + \delta_1(e) \otimes \langle e, f \rangle$. It can be easily seen that $\delta'$ is a duoid ($G(\delta')$ is still connected). In addition $\text{Ab}(\delta') < \text{Ab}(\delta)$, since $g'(e) = 1$, $g'(e) < g(e)$ and $g'(e') \leq g(e')$ for $e' \neq e$ where $g'$ refers to $\delta'$. We can also verify that $f_0(\delta') = f_0(\delta)$, $f_1(\delta') = f_1(\delta)$, $f_2(\delta') = 1 + f_2(\delta)$, $f_3(\delta') = f_3(\delta)$; so $X^*(\delta') > X^*(\delta)$. Continuing, we must arrive at a normal duoid $\delta^*(\text{Ab}(\delta^*) - \delta)$ with $X^*(\delta^*) \geq X^*(\delta)$.

Now $\delta^*$ is a duoid on some set $S^*$ containing $S$. Let us put points in general position in $E^6$ in 1-1 correspondence with $S^*$—in fact, let $P_e$ correspond to $e \in S^*$, let $\Delta_D$ be the closed triangle whose vertices are $\{P_e | e \in D\}$, and let $M^2 = \cup_{D \subseteq \delta^*} \Delta_D$. The operations shown thus far are illustrated in Example 6.8.

We now show that $M^2$ is a 2-manifold. First note that since the points are in general position in $E^6$, no triangles can intersect except at shared vertices or edges between them. Thus if $Q$ is interior to one of the triangles, it has a neighborhood homeomorphic to a 2-disc. If $Q$ is on the line joining $P_e$ and $P_f$, then $Q$ is in two such triangles because $\delta^*$ is proper, so once again $Q$ has a neighborhood homeomorphic to a 2-disc. Finally if $Q$ is $P_e$, then since $\delta^*$ is normal $\delta^*(e)$ is duoid, hence a simple polygon. Thus the triangles containing $P_e$ occur
in a single circuit round $P_e$; again $Q$ has a neighborhood homeomorphic
to a 2-disc. Also $M^2$ is connected because $G(s^k)$ is.

Now $X^k(M^2) = X^k(s^k)$ because the vertices, edges, and facets of $s^k$
correspond 1-1 to regions, edges, and vertices of the triangulation
of $M^2$. Thus by a well-known result, $X^k(M^2) \leq 1$, so $X^k(s) \leq 1$.

We have also shown that proper 3-dimensional duoids correspond 1-1
with triangulated 2-manifolds, with possibly some vertices identified.

Having proved that $X^k(s) \leq 1$ for $s$ proper and 3-dimensional,
we may go to the general case where $s$ is possibly improper. By the pre-
ceding argument, we may obtain a normal duoid $s^*$ from $s$ by splitting
its non-normal facets. This normal duoid $s^*$ corresponds to a triangu-
lation $T$ of a 2-manifold $M^2$. To obtain a representation of $s$, we
must do some surgery on $M^2$. Since vertices of $s$ correspond exactly
to vertices of $s^*$, we need only four operations:

1) Identify two vertices of $T$.
2) Identify two edges of $T$ with separate end-points.
3) Identify two edges of $T$ sharing one end-point.
4) Identify two edges of $T$ with common end-points.

We will thus proceed to construct surfaces $M^2 = a_0, a_1, \ldots, a_m$
where each surface is obtained from the last by one of the operations above,
and there is a triangulation of $a_m$ corresponding exactly to $s$, so
that $X^k(s) = X^k(a_m)$. Now we need some relation between the Euler charac-
teristics of the $a$'s.

Denote by $\gamma(a)$ the connectivity of $a$; then $\gamma(a)$ is the number
of generators of the fundamental group of $a$, or more simply, the number
of independent cycles which do not separate $a$. Then by a well-known
result in topology, $X^*(M^2) + \psi(M^2) = 1$, see, for example, [4].

To illustrate, the sphere has Euler characteristic 1 and no non-separating cycles; the torus has Euler characteristic -1 and two non-separating cycles:

![Diagram](image)

We wish to show that $X^*(a_m) + \psi(a_m) = 1$; then $X^*(\delta) = X^*(a_m) \leq 1$.

We will show that $X^* + \psi$ is unchanged by all the operations listed above; the result will then follow.

First note that each $a$ is pathwise connected, since $M^2$ is; this follows from Theorem 3.12 applied to $\delta^*$.  

**Operation 1.** $X^*$ is reduced by 1 since there is one less vertex, the same number of edges, and the same number of triangles. But the two vertices were previously connected by a path, and under the identification this path becomes a new independent cycle; so $\psi$ is augmented by 1.

**Operation 2.** $X^*$ is reduced by 1 since there are two less vertices, one less edge, and the same number of triangles. But if we take one vertex of each edge and connect them by a path, this path becomes a new independent cycle; so $\psi$ is augmented by 1.
Operation 3. $X^\ast$ is unaltered since there is one less vertex and one less edge, and the number of triangles is unchanged. It is easy to see that $\psi$ is also unaltered.

Operation 4. $X^\ast$ is increased by 1, since the number of edges is reduced by one while the number of vertices and triangles is unaltered. We must show that $\psi$ is reduced by 1; this will be true if the old cycle consisting of the two edges to be identified was a non-separating cycle. But if this cycle did separate, then consider all the triangles on one side of it. These will form a semi-duoid, since the boundary is just a repeated edge whose vertices have the same labels in $\delta$. By relabelling this semi-duoid, we obtain a non-null semi-duoid strictly contained in $\delta$; but this violates the hypothesis that $\delta$ is a duoid.

Thus $X^\ast(a_m) + \psi(a_m) = X^\ast(M^2) + \psi(M^2) = 1$, and the lemma is proved, hence proving 6.6(b).

Before proceeding to the 4-dimensional case, we note that the manifold obtained from $\delta$ is not unique; in fact, we can construct either an orientable or a non-orientable manifold from the following duoid:

Example 6.8. Let $\delta = \{124, 125, 127, 128, 134, 135, 137, 138, 246, 256, 279, 289, 345, 378, 458, 467, 478, 589, 589, 679\}$

i) $\delta$ is a duoid. It can be checked that $\delta$ is a semi-duoid; and it is a duoid because $G^\ast(\delta, 1)$ is connected.

ii) $\delta(1) = \{24, 25, 27, 28, 34, 35, 37, 38\}$ is not a duoid. It can be checked that all other facets of $\delta$ are normal. $\delta(1)$ can be split into two duoids in two ways:
a) \( \delta(1) = \{24, 27, 34, 37\} \oplus \{25, 28, 35, 38\} \)

b) \( \delta(1) = \{24, 28, 34, 38\} \oplus \{25, 27, 35, 37\} \)

Let us form \( \delta^*_a \) and \( \delta^*_b \) by taking the two decompositions; let the new facet in each case be 1'.

Then \( \delta^*_a \) and \( \delta^*_b \) correspond to triangulations of a) the torus, b) the Klein bottle (the surfaces are represented by rectangles with identified edges):

---

[Diagrams showing triangulations of a torus and a Klein bottle]
The first part of Theorem 6.5(c) now follows from:

**Lemma 5.9.** If $\delta$ is a normal 4-dimensional duroid, then $X^{k}(\delta) \leq 1$.

**Proof.** Since $\delta$ is normal, each facet is a 3-dimensional duroid; we will count the faces of $\delta$ by counting the faces of these facets. Note that each vertex of $\delta$ will appear in 4 facets of $\delta$; each edge in 3 facets of $\delta$; each plane in 2 facets of $\delta$ and each facet in one facet of $\delta$.

Thus

$$X^{k}(\delta) = f_{0}(\delta) - f_{1}(\delta) + f_{2}(\delta) - f_{3}(\delta) + 1$$

$$= \frac{1}{4} \sum_{e \in S} f_{0}(\delta(e)) - \frac{1}{3} \sum_{e \in S} f_{1}(\delta(e)) + \frac{1}{2} \sum_{e \in S} f_{2}(\delta(e)) - \sum_{e \in S} f_{3}(\delta(e)) + 1$$

$$= \sum_{e \in S} \left( \frac{1}{4} f_{0}(\delta(e)) - \frac{1}{3} f_{1}(\delta(e)) + \frac{1}{2} f_{2}(\delta(e)) - f_{3}(\delta(e)) \right) + 1$$

$$= \sum_{e \in S} \left( \frac{1}{2} X^{k}(\delta(e)) - \frac{1}{4} f_{0}(\delta(e)) + \frac{1}{5} f_{1}(\delta(e)) - \frac{1}{2} f_{3}(\delta(e)) \right) + 1.$$

Now since each $\delta(e)$ is a duroid, $X^{k}(\delta(e)) \leq 1$ by 6.7 and $f_{3}(\delta(e)) = 1$. Also, each vertex of $\delta(e)$ is on three edges of $\delta(e)$; and each edge contains at least two vertices of $\delta(e)$. Thus,

$$3f_{0}(\delta(e)) \geq 2f_{1}(\delta(e)),$$

and

$$\frac{1}{5} f_{1}(\delta(e)) - \frac{1}{4} f_{0}(\delta(e)) \leq 0.$$ Combining these results, we obtain $X^{k}(\delta) \leq 1$. 

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CHAPTER 7

LOWER BOUNDS ON THE MAXIMUM DIAMETERS OF DUOIDS

This chapter will demonstrate the importance of diameters of duoids in complementary pivot theory. Indeed, these diameters provide bounds on the number of iterations required in complementary pivot algorithms in the same way that diameters of simple polytopes give bounds on the number of iterations required in non-degenerate bounded linear programs. Our main result gives lower bounds for the maximum diameter of all duoids of order $n$ on a $k$-set, by exhibiting duoids which do in fact attain these lower bounds.

The distance between two vertices of a polytope $P$ is the length of any shortest path connecting them, measured by the number of edges traversed. The diameter of $P$ is the largest such distance. Correspondingly, the distance between two vertices of a duoid $\delta$ is the length of any shortest path in $G(\delta)$ (see p. 311) connecting them; and the diameter of $\delta$, $\Lambda(\delta)$, is the largest such distance. Also, $\Lambda_d(n,k)$ will represent the maximum diameter of all duoids of order $n$ on a set of $S$ of cardinality $k$. The superscripts $P$, $N$, and $NP$ correspond to the restrictions that the duoid must be proper, normal, and normal and proper respectively. Trivially, we have $\Lambda_d(n,k) \geq \begin{cases} \Lambda^P_d(n,k) \\ \Lambda^N_d(n,k) \end{cases} \geq \Lambda^{NP}_d(n,k)$.
are interested in these restricted measures mainly because almost all examples of complementary pivot algorithms are based on prismoids and duoids which are in fact normal and proper.

The diameter of a polytope is of great interest in the study of the number of iterations required in linear programming. Any variant of the well known simplex algorithm proceeds from vertex to vertex of the polyhedral feasible region. The cases when this polyhedron is unbounded or not simple can be reduced to the case of a simple polytope by adding a bounding artificial constraint or by perturbing the constraints. Thus the diameter of the resulting simple polytope is the number of iterations necessary when the starting vertex and the optimal vertex are at maximum distance and when the pivots are chosen to minimize their number. Of course these pivots may not be those indicated by the operational rules of the algorithm; in fact, Klee [18] has an example where the standard rules take one through every vertex of the polytope.

Of course, neither the diameter nor the number of vertices of a polytope is very important on its own. What is important is the expected number of iterations for a linear programming problem of a certain size, say one with k-n equality constraints on k non-negative variables. (This gives a feasible region which is isomorphic to an n-dimensional polytope with k facets.) Vast computational experience indicates that the expected number of iterations for such problems is of the order of 2(k-n) or 3(k-n). These values are much closer to the conjectured maximum diameter of all polytopes of this size (see below) than to the maximum number of vertices. We hope that complementary pivot algorithms
will exhibit similar behavior. The only evidence for this optimism is that
in Scarf's limited experience in using his algorithm, he has observed that
the primitive sets encountered usually lie approximately on a straight
line from the starting position to the solution obtained. Obviously in
this case the number of iterations is close to the shortest distance between
these points. In the worst possible case, when the solution is maximally
distant from the starting position, the number of iterations will be close
to the diameter of the duoid generated by the primitive sets.

The maximum diameter of all n-dimensional polytopes with k facets
is denoted by \( A_b(n,k) \). Hirsch conjectured that \( A_b(n,k) \leq k-n \) (see
[9], pp. 160 and 168). This conjecture has been confirmed for the case
\( n \leq 3 \) by Klee [17]; and for the case \( k-n < 6 \) by Klee and Walkup [19].
Adler and Dantzig [2] extended this last result to abstract polytopes.
The conjecture has also been confirmed for all \( n,k \) for the very special
cases when the polytope arises from a shortest route problem [26] or a
Léontief substitution system [12]. No negative results have been obtained.

Our results show for the first time a generalization of polytopes
for which the Hirsch conjecture fails. Indeed, we construct normal and
proper duoids which violate the generalized Hirsch conjecture by an order
of magnitude. A further justification for considering the restricted
measures \( \Lambda^P_d(n,k) \) and \( \Lambda^NP_d(n,k) \) is that we thereby provide stronger
evidence that the Hirsch conjecture may be false. Unfortunately, we have
been unable to construct supernormal duoids, or abstract polytopes, which
violate the Hirsch conjecture.

The general results of Klee, Walkup, Adler and Dantzig are in fact
very limited. If \( k-n < .6 \), an n-dimensional polytope with k facets
is just an $n$-simplex with a few vertices truncated. Our results are much more general, although they only give lower bounds on $\Delta_d(n,k)$.

We now show that $\max\{\Delta_d(n,k), \Delta_d(k-n,k)\}$ gives a lower bound on the number of iterations required in a complementary pivot algorithm applied to the "worst" prizoid and duoid of order $n$ on a $k$-set. Here "worst" means that the prizoid and duoid achieve the maximum diameters for their sizes and share only two maximally distant (in both $\pi$ and $\delta$) vertices. Also, we use iteration to mean a primal pivot step and a dual pivot step; in some cases (for example, the linear complementarity problem) one type of pivot step is much easier to perform than the other. Thus the basic step is a primal pivot step followed by a dual pivot step, or vice versa.

It is immediately obvious from Chapter 2 that the sequence of primal vertices forms a path in $G^\pi(\pi)$ (the definition of this graph is analogous to that of $G(\delta)$; it is isomorphic to $G(\overline{\delta} \pi)$, and the sequence of dual vertices forms a path in $G(\delta)$.

If we start and finish at maximally distant vertices for both $\pi$ and $\delta$ (the worst case), then the number of iterations required is at least $\max\{\Delta^*(\pi), \Delta(\delta)\}$. ($\Delta^*(\pi)$ is defined in the obvious manner: $\Delta^*(\pi) = \Delta(\overline{\delta} \pi)$.) If, moreover, $\pi$ and $\delta$ achieve the maximum diameter for their sizes, $\Delta(\delta) = \Delta_d(n,k)$ and $\Delta^*(\pi) = \Delta(\overline{\delta} \pi) = \Delta_d(k-n,k)$ . In this case the number of iterations required is at least $\max\{\Delta_d(n,k), \Delta_d(k-n,k)\}$.

When $\pi$ and $\delta$ are both proper, we can obtain an upper bound on the number of iterations. In this case it is easy to see that the number of nodes of $G(\pi,\delta,e)$ is at most $2 \min\{|\pi|, |\delta|\}$. Since no node is
Theorem 7.1.

\[ a) \quad \Delta_d^P(3,k) \geq \left[ \frac{2}{3} \left( k' \left[ \frac{k'}{54} \right] - k' - 2 \right) \right] \]

\[ b) \quad \Delta_d^P(4,k) \geq \left[ \frac{3}{4} \left( k' \left[ \frac{k'}{14} \right] - k' - 3 \right) \right] \]

\[ c) \quad \Delta_d^P(5,k) \geq \left[ \frac{4}{5} \left( k' \left[ \frac{k'}{8} \right] - k' - 4 \right) \right] \]

\[ d) \quad \Delta_d^P(n,k) \geq \left[ \frac{5}{6} \left( (k-n+6) \left[ \frac{k-n-3}{9} \right] - 5 \right) \right] \text{ if } n \geq 6 \]

\[ e) \quad \Delta_d^{NP}(n,k) \geq \Delta_d^{P}\left( n - k + \left[ \frac{k}{2} \right], \left[ \frac{k}{2} \right] \right) \text{ if } n - k + \left[ \frac{k}{2} \right] > 1. \]

We now introduce the special class of duoids of large diameter.

We restrict ourselves initially to the case \( n \geq 3 \), and \( k \) prime.

Our ground set \( S \) is again a set of \( k \) elements, but for ease of exposition it is convenient to take \( S \) to be the cyclic group on \( k \) elements, \( C_k \). It will often be convenient, at least mentally, to have a pictorial representation of \( S \) or various subsets of it. Thus if \( S = C_k = \{c^0, c^1, \ldots, c^{k-1}\} \), we represent \( c^h \) by \( \exp \left( \frac{2\pi i h}{k} \right) \), i.e. our points are evenly set around the unit circle in the complex plane, with \( c^0 \) represented by the point 1.

The gap between two such points \( c^j \) and \( c^h \) will be the points contained in the smaller arc joining them in the picture. This gap is adjacent to its two endpoints \( c^j \) and \( c^h \); its size is

\[ \min\{|h-j|, k-|h-j|\} . \]

As an example, let us look at the pictorial representation of \( C_5^5 \):
The gap between $c^4$ and $c^1$ is 
$\{c^4, c^0, c^1\}$. It is adjacent to 
$c^4$ and $c^1$. Its size is 2.

The constructed duoids will also depend on an $r$-tuple $p = (p_1, \ldots, p_r)$ 
of integers all $\leq k-1$.

We now define $\delta_{nk}(p)$ as follows:

**Definition 7.2.**

$$
\delta_{nk}(p) = \sum_{i=0}^{k-n} <c^i p_1, (i+1)p_1, \ldots, (i+n)p_1> 
+ \sum_{j=1}^{n-1} <c^j p_1, \ldots, c^0 p_2, p_2, c, \ldots, c^0 p_2> 
+ \sum_{i=0}^{k-n} <c^i p_2, (i+1)p_2, \ldots, (i+n)p_2> 
+ \sum_{j=1}^{n-1} <c^j p_2, \ldots, c^0 p_3, p_3, c, \ldots, c^0 p_3> + \ldots 
+ \sum_{i=0}^{k-n} <c^i p_r, (i+1)p_r, \ldots, (i+n)p_r> .
$$

$\delta_{nk}(p)$ is a semi-duoid since it is the symmetric difference of 
simplicial duoids—these are called its constituent simplicial duoids.
(c.s.d.'s) and are generated by its constituent simplicial sets (c.s.s.'s). These are arranged in blocks as shown. The top block is called the p₁-block, the next the p₁-p₂-block, and so on.

Let us restrict p as follows (more restrictions will be placed on it later): all p_i distinct, and all p_i < \frac{k}{m+3}. Then, by looking at the c.s.s.'s, we note that they are a set of evenly spaced points on the circle (unless the c.s.s. is in the p₁-p₁₊₁ block, in which case the step size changes from p_i to p_i₊₁ at the point Q). Consecutive gaps are of the size p_i (p_i < \frac{k}{m+3}), and there are n+1 points, so somewhere around the circle there will be a single recognizable gap of size \frac{3k}{m+3}. By considering the following gap, we will be able to recognize the c.s.s.'s. Thus all the c.s.s.'s are distinct.

It may be helpful to refer to the example at the end of this chapter, 7.10, to see the way in which the c.s.d.'s fit together. This example does not satisfy the conditions already placed on the pₙ's, nor those to be placed on them later. However, it serves our purpose as a small counterexample to the generalized Hirsch conjecture, and the structure of the duoid does illustrate that of the whole class of duoids of the form δₙk(p).

We now define the height of a c.s.d. or equivalently of the corresponding c.s.s. The formal definition is rather inelegant, but it corresponds exactly to listing vertically all c.s.d.'s as given in the definition, starting with the p₁-block, and writing each summand in the order in which the sum (or rather, symmetric difference) is made. The lowest height is 1.
Definition 7.3.

\[ h(<c^{(i+1)}p_k, ..., c^{(i+n)}p_k>) = (r-k)k + k - n - i + 1 \]

\[ h(<c^{-(n-j)}p_k, ..., c^{-(n+j)}p_k>) = (r-k)k - j + 1. \]

Two heights are neighboring if they differ by 1. Notice that height is well defined when \( p \) satisfies the conditions mentioned above.

Theorem 7.4

If \( p \) satisfies

1) all \( p_i \) distinct and \( < \frac{k}{n+3} \)

2) \( p_1 < p_2 ... < p_m \) and \( 2p_k > p_{k+1} \) for \( k = 1, 2, ..., m-1 \)

3) if \( n \leq 4 \), all \( p_k \) odd

4) if \( n = 3 \), the sets \( \{p_k\}, \{2p_k\}, \{3p_k\}, \{p_k + p_{k+1}\}, \{2p_k + p_{k+1}\} \) and \( \{p_k + 2p_{k+1}\} \) are all disjoint;

then every vertex of a c.s.d. lies in either just one, or in two of neighboring heights. Furthermore, \( \delta_{nk}(p) \) is proper.

Note that a vertex of \( \delta_{nk}(p) \) is just a c.s.s. minus one point.

We will find it convenient in this proof and in the description of the adjacencies in \( \delta_{nk}(p) \) later to classify vertices by type.

Definition 7.5. A vertex of a c.s.d. has type \( h \) iff it is obtained from a c.s.s. by omitting the \( h^{th} \) symbol of the c.s.s., in the natural order in which they appear in definition 7.2.
Proof* of 7.4. For the first part, we will show that a vertex of type 1 is equal to the vertex of type n+1 one lower, but that otherwise all vertices are distinct. To prove this, we take any vertex of a c.s.d.\( D = \{ c_1, c_2, \ldots, c_n \} \) say, where \( 0 \leq i_1 < i_2 < \cdots < i_n \leq k-1 \), and consider the \( n \) gaps \( b_1 = i_2 - i_1, b_2 = i_3 - i_2, \ldots, b_{n-1} = i_n - i_{n-1} \), and \( b_n = i_1 + k - i_n \). Subscripts of the \( b_j \)'s and \( i_j \)'s will be considered modulo \( n \), thus \( b_{n+1} = b_1 \), etc.

Since \( D \) is a c.s.s. minus one point (we can picture \( D \) as a set of spokes in a wheel) and since all the \( p_i \)'s are all \( < \frac{k}{n+3} \), just one of the \( b_j \)'s is \( > \frac{3k}{n+3} \). The other \( b_j \)'s are all \( < \frac{2k}{n+3} \).

We will show that each vertex, except those of type 1 or \( n+1 \), has a different set of \( b_j \)'s, and \( i_j \)'s. Let \( b_0 \) be the "large" \( b \).

Let \( D \) be a type 1 vertex in the \( p_h \)-block. Then if \( i = 1 \) or \( n+1 \), we have \( b_m = k - (n-1)p_h \), \( b_{m+1} = b_{m+2} = \cdots = b_{m+n-1} = p_h \).

If \( 1 < i < n+1 \), we have \( b_m = k - np_i \), \( b_{m+1} = b_{m+2} = \cdots = b_{m+i-2} = p_h \), \( b_{m+i-1} = 2p_h \) and \( b_{m-i} = \cdots = b_{m+n-1} = p_h \).

Now let \( D \) be the type 1 vertex in the \( j \)th summand of the \( p_h \)-block (see definition 7.2).

If \( i = 1 \), we have \( b_m = -(n-j-1)p_h - j p_{h+1} \) (mod k), \( b_{m+1} = \cdots = b_{m+n-1} = p_{h+1} \).

If \( 1 < i \leq n-j \), we have \( b_m = -(n-j)p_h - j p_{h+1} \) (mod k), \( b_{m+1} = \cdots = b_{m+i-2} = p_h \), \( b_{m+i-1} = 2p_h \), \( b_{m+i} = \cdots = b_{m+n-j-1} = p_h \), and \( b_{m+n-j} = \cdots = b_{m+n-1} = p_{h+1} \).

If \( i = n-j+1 \) (i.e. \( c^0 \) is missing from the vertex) we have

---

*This proof may be omitted on a first reading.
Let \( b_m = -(n-j)p_h - jp_{h+1} \pmod{k} \), \( b_{m+1} = \cdots = b_{m+n-1-j} = p_h \), \( b_{m+n-j} = p_{h+1} \), \( b_{m+n-j+1} = \cdots = b_{m+n-1} = p_{h+1} \).

If \( n-j+1 < i < m+1 \), we have \( b_m = -(n-j)p_h - jp_{h+1} \pmod{k} \), \( b_{m+1} = \cdots = b_{m+n-j} = p_h \), \( b_{m+n-j+1} = \cdots = b_{m+i-1} = p_{h+1} \), \( b_{m+i} = 2p_{h+1} \), and \( b_{m+i+1} = \cdots = b_{m+n-1} = p_{h+1} \).

Finally, if \( i = m+1 \), we have \( b_m = -(n-j)p_h - (j-1)p_{h+1} \pmod{k} \), \( b_{m+1} = \cdots = b_{m+n-j} = p_h \), and \( b_{m+n-j+1} = \cdots = b_{m+n-1} = p_{h+1} \).

Now we show that all vertices are distinct, except types 1 and \( n+1 \) of neighboring heights.

**Case 1.** \( n = 3 \). Then \( b_{m+1} \), the size of the gap after the single "large" gap, is of the form \( p_h \), \( 2p_h \) or \( p_h + p_{h+1} \) for some \( h \); and by using the hypothesis (5iv), we can express \( b_{m+1} \) uniquely in one of these forms.

Now the argument is straightforward; basically, when we have identified one of the gaps, we can find "where the vertex is" (and in just one way). This argument will hold for the higher dimensional cases also.

If \( b_{m+1} = p_h \), then the vertex can only be in the \( p_{h-1} - p_h \) block, the \( p_h \)-block, or the \( p_h - p_{h+1} \) block. We consider \( b_{m+1} \), the point following the large gap, and calculate \( \frac{i_{m+1}}{p_h} \pmod{k} = i \), say.

If \( i = 0 \), then the vertex is in the 0th summand \((i = 0)\) of the \( p_h \)-block, and possibly the \((n-1)\)\(^{st}\) summand of the \( p_{h-1} - p_h \) block.

If \( 0 < i \leq k-n \), the vertex is in the \( i \)\(^{th}\) summand, and possibly the \((i-1)\)\(^{st}\), of the \( p_h \)-block.

If \( i = k-n+1 \), the vertex is in the 1st summand of the \( p_h - p_{h+1} \)
of the form \( p_h, 2p_h \) or \( p_h + p_{h+1} \). But at most one is of the latter two kinds. Since \( p_h < 2p_{h-1} \) (ii), the smallest of these three b's is a \( p_h \). Hence, working backwards is necessary, we can express \( b_{m+1} \) uniquely as \( p_h, 2p_h \) or \( p_h + p_{h+1} \). The rest of the proof is contained in Case 1.

This completes the proof of the first part.

To prove the second part, we will use the same reasoning. Note that if an edge (a \((n-1)\)-set) is contained in any vertex of \( \delta_{nk}(P) \), then it is a c.s.s. minus two points. Once again we find the associated b's, the sizes of the gaps in the edge; there are now \( n-1 \) of them. As before, one of the b's is larger than \( \frac{3k}{n+3} \). All other b's will be smaller than this (even if the two missing points are next to each other). So we can distinguish, say, \( b_m \), the large one. Subscripts of the b's and i's will now be modulo \((n-1)\).

As in the first part, we will try to identify \( b_{m+1} \) uniquely as a \( p_h \), a \( 2p_h \), a \( 3p_h \), a \( p_h + p_{h+1} \), a \( 2p_h + p_{h+1} \) or a \( p_h + 2p_{h+1} \)--these are all possible gaps in an edge, by an analysis similar to that for vertices.

If we have determined which of these \( b_{m+1} \) must be, then we know \( h \). Once again, let \( i = \frac{m+1}{p_h} \) (mod \( k \)). Then if \( i \leq k-n \), the edge can be contained only in a vertex in the \( i^{th} \) summand of the \( p_h \)-block, and possibly the two c.s.d.'s immediately higher (since the edge is a c.s.s. minus two points). If \( i > k-n \), then the edge can be contained only in a vertex in the \((i-k+n)^{th}\) summand of the \( p_h - p_{h+1} \) block and possibly
the two c.s.d.'s immediately higher. Within this range of three c.s.d.'s of consecutive heights, the edge can only be contained in two vertices, as can be easily checked. In fact, the symmetric difference of three such c.s.d.'s corresponds to a simple polytope, an n-simplex truncated, and then truncated again at one of the vertices created by the first truncation. Reference to example 7.10 will clarify this. So it remains to show how to determine \( b_{m+1} \) uniquely as one of the expressions shown in the last paragraph.

**Case 1.** \( n = 3 \). Then all these expressions are distinguishable, by hypothesis (iv). Thus \( b_{m+1} \) can be identified.

**Case 2.** \( n = 4 \). Consider \( b_{m+1} \) and \( b_{m+2} \). As in the proof of the first part, if \( m \neq n-2 \), \( b_{m+1} \) must be a \( p_h \), a \( 2p_h \), or a \( 3p_h \). If \( m = n+2 \), \( b_{m+2} \) must have one of these forms. Since no \( p_i \) is even (iii), the only confusion possible is when, say, \( b_{m+1} = p_i = 3p_j \). (The proof for \( b_{m+2} \) is similar.) We will show that no confusion can arise.

We will use the loose expression "if \( b_{m+1} \) were 'really' \( p_i \)" as shorthand for "if the edge were contained in a vertex in the \( p_i \)-block or the \( p_i-p_{i+1} \) block, so that \( b_{m+1} \) could be naturally expressed as \( p_i \)."

We distinguish three cases:

(i) Neither \( i_{m+1} \) nor \( i_{m+2} \) is 0. Then if \( b_{m+1} \) were "really" \( p_i \), \( b_{m+2} \) would be \( p_i \), \( 2p_i \), \( 3p_i \), \( p_i + p_{i+1} \), \( 2p_i + p_{i+1} \), or \( p_i + 2p_{i+1} \). In all cases, \( b_{m+2} \geq b_{m+1} \). If \( b_{m+1} \) were "really" \( 3p_j \), then \( b_{m+2} \) would be \( p_j < b_{m+1} \). So no confusion can arise in this case, and we may express \( b_{m+1} \) uniquely.
(ii) If $i_{m+1} = 0$, then if $b_{m+1}$ were "really" $p_i$, $b_{m+2}$ would be $p_i$, $2p_i$ or $3p_i$, and $b_{m+2} \geq b_{m+1}$. If $b_{m+1}$ were "really" $3p_j$, then $b_{m+2}$ would be $p_j < b_{m+1}$. No confusion is possible.

(iii) If $i_{m+2} = 0$, then if $b_{m+1}$ were "really" $p_i$, $b_{m+2}$ would be $p_{i+1}$, $2p_{i+1}$ or $3p_{i+1}$, and $b_{m+2} > b_{m+1}$. If $b_{m+1}$ were "really" $3p_j$, then $b_{m+2}$ would be $p_{j+1} < b_{m+1}$. Once again we may determine the proper expression for $b_{m+1}$ uniquely.

**Case 3.** $n = 5$. We have at least 4 points in the edge, and so at least 3 small ($< \frac{3k}{n+3}$) gaps $b_{m+1}$, $b_{m+2}$, and $b_{m+3}$. Since the edge is a c.s.s. minus two points, one of these b's is unchanged from its size in the c.s.s., and equal to $p_h$, say. Moreover, $p_h$ is smaller than all the other gaps which might occur in the same edge, i.e. $2p_h$, $3p_h$, $2p_{h+1}$, $3p_{h+1}$, $2p_{h-1}$, $3p_{h-1}$, $p_{h-1} + p_h$, $p_{h-1} + 2p_h$, $2p_{h-1} + p_h$, $p_h + p_{h+1}$, $2p_h + p_{h+1}$, and $p_h + 2p_{h+1}$.

So the smallest of $b_{m+1}$, $b_{m+2}$, $b_{m+3}$ is equal to $p_h$. Working backwards to $b_{m+1}$ if necessary, we can express $b_{m+1}$ uniquely as one of the possible forms.

The theorem is now complete. By showing how to identify vertices and edges as arising from just one or possibly two blocks of the duoid, we have also shown that the vertices appear only once in the description of the duoid. The vertices that appear more than once appear just twice in c.s.d.'s of neighboring heights, and cancel each other out in the symmetric difference.

Also, we have seen that edges are essentially "local" in that a vertex is only adjacent to others of fairly close heights. This is what
accounts for the large diameter of this class of duoids—we cannot jump
a large distance. In fact, any $p_i$-block of the duoid, considered as a
duoid itself, is a representation of a simple polytope, obtained as
follows. Take an $n$-dimensional simplex and truncate one of its vertices
(this vertex corresponds to the type 1 vertex of maximum height in the
block). This truncation gives $n$ new vertices (corresponding to the
vertices of types $1, 2, \ldots, n$ of height one lower in the block).
Truncate one of these vertices (the one corresponding to the type 1 vertex).
The process continues, and we finally obtain a simple polytope isomorphic
to that used by Klee in [ ]. The $p_i$-block is obtained from this polytope
by labelling the facets all differently, except that the facet created by
the last truncation and the facet opposite the first vertex truncated
share the label $c^0$. We can see this most clearly by considering example
7.10.

To describe the adjacencies of $\delta_{nk}(p)$ (i.e. the adjacencies in
$G(\delta_{nk}(p))$), we use our classification of vertices by type; from 7.4 we
have that the c.s.d. of minimum height has vertices of types $1, 2, \ldots, n$;
that of maximum height has vertices of type $2, 3, \ldots, n+1$; and all
others just have those of type $2, 3, \ldots, n$.

**Theorem 7.6.** Under the hypotheses of 7.4, the adjacencies of the vertices
of $\delta_{nk}(p)$ can be described as follows:

The type 1 vertex is adjacent to

1) type 1 vertex of same height $i = 2, 3, \ldots, n$

2) type 2 vertex one higher
The type $n+1$ vertex is adjacent to

i) type $i$ vertex of same height $i = 2, 3, \ldots, n$

ii) type $n$ vertex one lower

For $2 \leq i \leq n$: A type $i$ vertex is adjacent to

i) type $j$ vertex same height for $i \neq j = 2, 3, \ldots, n$

ii) type $i+1$ vertex one higher, or if none, type 2 vertex two higher, or if none, type $n+1$ vertex same height

iii) type $i-1$ vertex one lower, or if none, type $n$ vertex two lower, or if none, type 1 vertex same height.

Proof: Consider 5 heights of c.s.d. and their associated vertices, which all have the following form:

$$<[a_1, a_2, \ldots, a_{n+1}]>$$
$$<[a_2, a_3, \ldots, a_{n+2}]>$$
$$<[a_3, a_4, \ldots, a_{n+3}]>$$
$$<[a_4, a_5, \ldots, a_{n+4}]>$$
$$<[a_5, a_6, \ldots, a_{n+5}]>$$

We will consider the vertices of this c.s.d.

The type $i$ vertex of the middle c.s.d. for $3 \leq i \leq n-1$ is

$$\{a_3, \ldots, a_{i+1}, a_{i+3}, \ldots, a_{n+3}\}.$$ 

It can be seen that this is adjacent to all vertices of the same height, to

$$\{a_2, a_3, \ldots, a_{i+1}, a_{i+3}, \ldots, a_{n+2}\}.$$
the type \( i+1 \) vertex one higher, and to
\[
\{a_4, a_5, \ldots, a_{i+1}, a_{i+3}, \ldots, a_{n+4}\}
\]
the type \( i-1 \) vertex one lower. If \( i = 2 \), the latter is not a vertex of \( \delta \); we replace it with
\[
\{a_5, a_6, \ldots, a_{n+3}, a_{n+5}\}
\]
the type \( n \) vertex two lower. If \( i = n \), the former is not a vertex of \( \delta \); we replace it with
\[
\{a_1, a_3, \ldots, a_{n+1}\}
\]
the type \( 2 \) vertex two higher. The special cases at either end of the height scale are easy to check.

**Corollary 7.7.** If \( p \) satisfies the conditions of 7.4, then these are the only adjacencies, and \( \delta_{nk}(p) \) is a duoid.

**Proof:** In this case, \( \delta_{nk}(p) \) is proper, so each vertex is adjacent to exactly \( n \) other vertices. But theorem 7.6 exhibits \( n \) adjacent vertices, so these are the only adjacencies.

Secondly, since \( \delta_{nk}(p) \) is proper, by theorem 3.12, we need only check that \( G(\delta_{nk}(p)) \) is connected. But this is trivial from the adjacencies mentioned above.

**Corollary 7.8.** If \( p \) satisfies the conditions of 7.4, the shortest path between the type 1 vertex of minimum height and the type \( n+1 \) vertex of maximum height in \( \delta_{nk}(p) \) gains height \( n \) every \( n-1 \) steps.
Proof: Obviously the shortest path joining these two vertices must be the "steepest" such path. The path type 1 of height 1 → type 2 of height 2 → ... → type n of height n → type 2 of height n+2 gains height 1 in every step but the last, which gains 2; this path gains an average of height n every n-1 steps. (The cases of extreme height are very slightly different.) But by corollary 7.7 the only adjacencies are those stated in the theorem. So the only way to improve on an average steepness of 1 is to incorporate the step type n → type 2. On the other hand, any time we stay at the same height to get to this type n, we have a height gain of 0 adjacent to our height gain of 2, so an average of just 1. Thus the procedure mentioned is optimal.

We are now in a position to show some duoids of large diameter.

For n ≥ 5, let p = (2, 3, ..., m) where m is the largest integer smaller than \( \frac{k}{n+3} \). Then the total height of \( \delta_{nk}(p) \) is \( k - n + 1 + (m-2)k \).

So its diameter is at least \( \left[ (k-n+1) + (m-2)k \right] \left( \frac{n-1}{n} \right) \) or

\[
\left( \frac{n-1}{n} \right) (k - n + 1 + k \left( \left[ \frac{k}{n+3} \right] - 2 \right))
\]

where \([a]\) denotes the largest integer not larger than \(a\). For n = 4, let p = (3, 5, 7, ..., m) where m is the largest odd number less than \( \frac{k}{n+3} = \frac{k}{7} \). Then the total height of \( \delta_{4k}(p) \) is \( k - n + 1 + k \left( \frac{m-1}{2} - 1 \right) \); so its diameter is at least

\[
\left[ \frac{3}{4} k - 3 + k \left( \frac{m-1}{2} - 1 \right) \right] \quad \text{or} \quad \left[ \frac{3}{4} \left( k - 3 + k \left( \left[ \frac{k}{14} \right] - 2 \right) \right) \right].
\]

For n = 3, let p = (11, 13, 29, 31, ..., m) where m is the largest number of the form \( 18n + 11 \) or \( 18n + 13 \) less than \( \frac{k}{6} \). It can be checked that \( p \) satisfies the conditions of theorem 7.4. Then the total height of \( \delta_{3k}(p) \).
is \( k - 2 + k \left( \left\lfloor \frac{m+7}{18} \right\rfloor + \left\lfloor \frac{m+5}{18} \right\rfloor - 1 \right) \). Thus its diameter is at least

\[
\left[ \frac{2}{3} \left( k - 2 + k \left( \left\lfloor \frac{m}{9} \right\rfloor - 2 \right) \right) \right] \text{ (simplifying) or } \left[ \frac{2}{3} \left( k - 2 + k \left( \left\lfloor \frac{m}{9} \right\rfloor - 2 \right) \right) \right].
\]

We now indicate how to adjust this construction for the case when \( k \) is not a prime. This adjustment will be valid when \( n \) is sufficiently large to allow us to make all identification of vertices and edges. For example, if \( n \geq 6 \), then there will be two genuine gaps (see Theorem 7.4) of which at most one is destroyed by the construction. The problem that arises is that the sequence \( c^0, p_2, c^1, p_3, c^2, \ldots \) may return to \( c^0 \) before having covered all of \( S \). Let \( j \) be the smallest number such that

\[
j_{p_2} = c^0. \text{ If } j = k, \text{ there is no problem. If } j < k, \text{ then we "fudge" our arithmetic by letting } c^{j+1}_{p_2} = c^1, \quad c^{1+j}_{p_2} = c^{1+1}_{p_2} \text{ and so on.}
\]

When we return to \( c^1 \) (at \( 2j \)), we make the same "fudge"--we let

\[
c^{2j}_{p_2} = c^2 \text{ and so on. The same analysis will give us similar bounds.}
\]

We now show how to obtain normal duoids of large diameter. Note that every facet of \( \delta_{nk}(p) \) is abnormal. To see this, note that each facet appears in each one of the \( p_i \)-blocks. However, no edge connects any two vertices of two distinct \( p_i \)-blocks. Thus, each facet gives rise to a disconnected face, which cannot be a duoid (3.12). However, let

\[
\delta_i = W(\delta_{nk}(p), c^0), \text{ and in general } \delta_{i+1} = W(\delta_i, c^i) \text{ for } i = 1, 2, \ldots, k-1.
\]
Then \( \delta_k \) will be a duoid of dimension \( n+k \) and \( 2k \) facets, by induction on \( i \), using Lemma 5.2.2. Moreover, \( \delta_1 \) has only \( k-1 \) abnormal facets \( (c_1, c_2, \ldots, c_{k-1}) \), \( \delta_2 \) only \( k-2 \), and \( \delta_k \) is a normal duoid. Since \( \delta_{nk} \) is proper, \( \delta_k \) is also. We now show that \( \Delta(\delta_k) \geq \Delta(\delta_{nk}(p)) \) by means of the following technical lemma, based on a similar result of Klee and Walkup [19].

**Lemma 7.9.** a) \( \Delta(W(\delta,e)) \geq \Delta(\delta) \)

b) \( \Delta_d(n+1, k+1) \geq \Delta_d(n, k) \)

c) \( \Delta_d(n, k+1) \geq \Delta_d(n, k) \).

Parts b) and c) also hold when the superscripts \( P \), \( N \), or \( NP \) are added throughout.

**Proof:** Let two maximally distant vectors of \( \delta \) be \( D_1 \) and \( D_2 \). Let

\[
D'_i = \begin{cases} 
D_i \cup t & \text{if } e \notin D_i \\
D_i \cup \{t, b\}/e & \text{if } e \in D_i 
\end{cases} \quad \text{for } i = 1, 2.
\]

Then \( D'_1 \) and \( D'_2 \) are vertices of \( W(\delta, e) \). We now show that their distance is at least the diameter of \( \delta \). Let \( D'_1 = G'_0, G'_1, \ldots, G'_m = D'_2 \) be any path in \( W(\delta, e) \) connecting them. Let

\[
G_i = \begin{cases} 
G'_i/t & \text{if } t \notin G'_i, b \notin G'_i \\
G'_i/b & \text{if } t \notin G'_i, b \in G'_i \quad \text{for } i = 1, 2, \ldots, m \\
G'_i \cup e/\{t, b\} & \text{if } t, b \in G'_i
\end{cases}
\]

Then \( G_0 = D_1 \), \( G_m = D_2 \) and for \( i = 1, 2, \ldots, m, G_i \) and \( G_{i-1} \) are either equal or adjacent. So we have a path of length at most \( m \) joining \( D_1 \) and \( D_2 \). This proves part (a).
Part (b) follows immediately from (a) if \( \Delta(\delta) = \Delta_d(n, k) \).

For part (c) let \( \Delta(\delta) = \Delta_d(n, k) \) as before. Let \( \tau_n \) denote a simplicial duoid, and let \( \delta' = \text{Sum}(\delta, \tau_n) \). Then \( \delta' \) corresponds to a truncation of a single vertex of \( \delta \). An argument similar to that for part (b) shows that \( \Delta(\delta') \geq \Delta(\delta) \). Then (c) follows immediately.

Part (a) of this lemma shows by an inductive argument that the \( \delta^k \) which was introduced just before the lemma has diameter at least that of the \( \delta_{nk}(p) \) which generated it.

Proof of Theorem 7.1. We will refer continually to the duoids of large diameter introduced on page 7.19.

(a) follows from lemma 7.9(c) and \( \delta_{3k}(p) \) introduced on page 7.19, simplifying.

(b) and (c) similarly follow from 7.9(c) and \( \delta_{4k}(p) \) and \( \delta_{5k}(p) \).

(d) follows from repeated use of 7.9(b) and \( \delta_{6, k-n+6}(p) \) on simplifying the resulting expression for the diameter.

This bound is generally better than that given by \( \delta_{nk}(p) \); complications arise because of the integer part function, but if this is ignored, we can show by straightforward calculus that \( \Delta(\delta_{6, k-n+6}(p)) > \Delta(\delta_{nk}(p)) \). We are also using the required modifications if \( k-n+6 \) is not a prime.

(e) follows from the construction of normal proper duoids given at the bottom of page 7.20, starting with a duoid \( \delta \) of order \( n-k+\left\lceil\frac{k}{2}\right\rceil \) on a set \( S \) of cardinality \( \left\lceil\frac{k}{2}\right\rceil \) with maximum diameter.

The smallest example of a duoid violating the generalized Hirsch conjecture known to the author is \( \delta_{3, 19}(1, 4) \). That this is of the required
form does not follow from the theorems above, since \( p = (1,4) \) does not satisfy the conditions of 7.4. But these restrictions are not necessary, as this example shows. It is worthwhile to exhibit this duoid as a simple illustration of the adjacencies involved.

The duoid will be presented in a way showing clearly the height and type of each vertex. To be concise, we will relabel, letting \( a, b, c, \ldots, s \) represent \( c, c_1, \ldots, c_{18} \).

Then \( \delta_{3,19}(1,4) \) has the following form:

**Example 7.10**

<table>
<thead>
<tr>
<th>Height</th>
<th>Constituent Simplicial Set</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>abcd</td>
<td>( \delta_{3} )</td>
</tr>
<tr>
<td>35</td>
<td>hiode</td>
<td>( \delta_{4} )</td>
</tr>
<tr>
<td>34</td>
<td>cdef</td>
<td>( \delta_{5} )</td>
</tr>
<tr>
<td>33</td>
<td>defg</td>
<td>( \delta_{6} )</td>
</tr>
<tr>
<td>32</td>
<td>efgh</td>
<td>( \delta_{7} )</td>
</tr>
<tr>
<td>31</td>
<td>fghi</td>
<td>( \delta_{8} )</td>
</tr>
<tr>
<td>30</td>
<td>ghij</td>
<td>( \delta_{9} )</td>
</tr>
<tr>
<td>29</td>
<td>hijk</td>
<td>( \delta_{10} )</td>
</tr>
<tr>
<td>28</td>
<td>ijkl</td>
<td>( \delta_{11} )</td>
</tr>
<tr>
<td>27</td>
<td>jklm</td>
<td>( \delta_{12} )</td>
</tr>
<tr>
<td>26</td>
<td>klmn</td>
<td>( \delta_{13} )</td>
</tr>
<tr>
<td>25</td>
<td>lmno</td>
<td>( \delta_{14} )</td>
</tr>
<tr>
<td>24</td>
<td>mnop</td>
<td>( \delta_{15} )</td>
</tr>
<tr>
<td>23</td>
<td>nopq</td>
<td>( \delta_{16} )</td>
</tr>
<tr>
<td>22</td>
<td>opqr</td>
<td>( \delta_{17} )</td>
</tr>
<tr>
<td>21</td>
<td>pqr</td>
<td>( \delta_{18} )</td>
</tr>
<tr>
<td>20</td>
<td>qrs</td>
<td>( \delta_{19} )</td>
</tr>
</tbody>
</table>

\( P_1 \)-block
<table>
<thead>
<tr>
<th>Height</th>
<th>Constituent Simplicial Set</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
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<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>19</td>
<td>rsae, saei</td>
<td>sae, rae, rse, rsa</td>
</tr>
<tr>
<td>18</td>
<td>aem, eim</td>
<td>aei, sei, sai, aae</td>
</tr>
<tr>
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<td>imq</td>
<td>aim, aim, aem, aei</td>
</tr>
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<td>16</td>
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<td>qbf, mbf, mqb, mbf</td>
</tr>
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<td>bfj, qfj, qbj, qbf</td>
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<tr>
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<td>jnr, fnr, fjr, jfr</td>
</tr>
<tr>
<td>10</td>
<td>jnr, jrc</td>
<td>rcg, nrc, jnc, jnr</td>
</tr>
<tr>
<td>9</td>
<td>nrg</td>
<td>reg, nrg, nrc, arc</td>
</tr>
<tr>
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<td>rgk</td>
<td>egk, rgk, rek, xeg</td>
</tr>
<tr>
<td>7</td>
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<td>sko, cgo, cgk, xsk</td>
</tr>
<tr>
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<td>kso, gks, sks, gks</td>
</tr>
<tr>
<td>5</td>
<td>kosd</td>
<td>osd, kod, kso, los</td>
</tr>
<tr>
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<td>sdh, osd, osi, osd</td>
</tr>
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<td>dh, dhl, shl, sdl</td>
</tr>
<tr>
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<td>dhlp</td>
<td>hl, dp, dhp, dhp</td>
</tr>
<tr>
<td>1</td>
<td>hlpa</td>
<td>lpa, hpa, hla, hlp</td>
</tr>
</tbody>
</table>

From this chart we can see very clearly the type and height of eachvertex; the fact that there are only type 2 and type 3 vertices for all heights except 1 and 36; and the shortest path which has maximum length, the diameter. This last is:

1pa → dlp → sdl → ksd → gks → rgk → nrg → fnr → bfn → mbf → imb →
aim → sai → qsa → pqs → npq → mnp → kmn → jkm → hjk → ghj → egh →
deg → bde → abd → abc

of length 25 > 19 - 3, the bound given by the Hirsch conjecture. We remark that this path alternates between vertices of type 2 and those of type 3, except at its ends, and that it gains height 1 and 2 alternatively except at its ends.
Note also that this duoid has 74 vertices. The Upper Bound Theorem for polytopes (see [25]) states that an n-dimensional polytope with k facets has at most \( \binom{k - \left\lceil \frac{n+1}{2} \right\rceil}{k-n} + \binom{k - \left\lceil \frac{n+2}{2} \right\rceil}{k-n} \) vertices. For \( k = 19 \), \( n = 3 \), this number is \( \binom{17}{16} + \binom{17}{16} = 34 \). Thus proper duoids do not satisfy the theorem, and the duoids we have constructed give lower bounds on the maximum number of vertices of duoids. The normal duoid generated from this example as at the bottom of page 7 does not violate the theorem; it is an open question whether normal and proper duoids satisfy the theorem. We will not pursue this line; we will just remark that obviously these results relate to the worst possible behavior of complementary pivot algorithms.

It can also be verified that this duoid is Hamiltonian, i.e. there is a path passing through every vertex of the duoid exactly once. This path consists of three disjoint "spirals," joined at the top and bottom. In fact, every \( \delta_{nk}(p) \) satisfying the conditions of 7.4 is Hamiltonian. We will not prove this incidental result.

In conclusion, we have shown a class of duoids which indicates that the maximum diameter of duoids increases at least as fast as the square of \( k-n \) times a constant. This follows from part (d) of 7.1. Furthermore, if \( k < 2n - 12 \), duoids of this large diameter can be both normal and proper (7.1(e)); in any case, they can be proper.

As topic for further study, we would be interested in knowing how sharp this bound is—can we find a class of duoids whose diameters increase as the cube of \( k-n \) ? Another important subject is supernormal duoids or
abstract polytopes. What is their maximum diameter? Do they violate the Hirsch conjecture? I would think that an operation on proper--or normal and proper--duoids, which would yield supernormal duoids, would be a great step in this direction. Unfortunately, to construct such an operation seems to be very difficult; the operation of iterated wedging, as performed to obtain normal duoids on page 7.20, does not "normalize" any of the smaller dimensional faces of a duoid.

The results of Adler and Dantzig for supernormal duoids with \( k-n \leq 5 \) [2] are much more restricted than ours for proper duoids. Polytopes with only at most five facets more than dimensions are mainly just simplexxes with a few vertices truncated. Our results are much more general, and seem to me very worthwhile for the study of both polytopes and the more general systems which arise in complementary pivot theory.
We will list, apart from permutations, all primoids and duoids with $n \leq 3$, $k \leq 6$, indicating whether they arise from polytopes. If the corresponding matroid $M_{nk}$ is cographic (i.e. if its circuits correspond to the polygons of a graph), we show the associated graph. This will give a compact description of the structure of the matroid. In each case, $S = \{1, 2, \ldots, k\}$.

<table>
<thead>
<tr>
<th>Primoids</th>
<th>Duoids</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $n = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>i) $k = 1$ &amp; Just one -- {1} or {1!}</td>
<td>None</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (point or 0-simplex)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii) $k = 2$ &amp; Just one -- {1, 2}</td>
<td>Just one -- {1, 2}</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (line-segment or 1-simplex)</td>
<td>(1-simplex)</td>
<td></td>
</tr>
<tr>
<td>iii) $k = 3$ &amp; Just one -- {1, 2, 3}</td>
<td>3 permutations of {1, 2}</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (triangle or 2-simplex)</td>
<td>(1-simplex)</td>
<td></td>
</tr>
<tr>
<td>iv) $k = 4$ &amp; Just one -- {1, 2, 3, 4}</td>
<td>6 permutations of {1, 2}</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (tetrahedron or 3-simplex)</td>
<td>(1-simplex)</td>
<td></td>
</tr>
<tr>
<td>v) $k = 5$ &amp; Just one -- {1, 2, 3, 4, 5}</td>
<td>10 permutations of {1, 2}</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (4-simplex)</td>
<td>(1-simplex)</td>
<td></td>
</tr>
<tr>
<td>vi) $k = 6$ &amp; Just one -- {1, 2, 3, 4, 5, 6}</td>
<td>15 permutations of {1, 2}</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>&amp; (5-simplex)</td>
<td>(1-simplex)</td>
<td></td>
</tr>
<tr>
<td>Circuits</td>
<td>Co-circuits</td>
<td>Graph</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td>-------</td>
</tr>
<tr>
<td><strong>b) n = 2</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>i) k = 2</strong></td>
<td>Just one--{1,2} or {1,2} (0-simplex)</td>
<td>None</td>
</tr>
<tr>
<td><strong>ii) k = 3</strong></td>
<td>The circuits (co-circuits) are the sets of vertices which are the complements in S of the vertices in the co-circuits (circuits) of the case ( n=1, \ k=3 ). The associated graph is:</td>
<td></td>
</tr>
<tr>
<td><strong>iii) k = 4</strong></td>
<td>4 permutations of ( {1} ) or ( {2} ) (2-simplex)</td>
<td>4 permutations of ( {1,2,3} ) or ( {123} ) (2-simplex)</td>
</tr>
<tr>
<td></td>
<td>3 permutations of ( {1} + {2} ) (square)</td>
<td>3 permutations of ( {123} + {234} ) (square)</td>
</tr>
<tr>
<td><strong>iv) k = 5</strong></td>
<td>5 permutations of ( {1} ) (3-simplex)</td>
<td>10 permutations of ( {123} ) (2-simplex)</td>
</tr>
<tr>
<td></td>
<td>10 permutations of ( {1} + {2} ) (triangular prism)</td>
<td>15 permutations of ( {12} \oplus {34} ) (square)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12 permutations of {12,23,34,45,51} (pentagon)</td>
</tr>
<tr>
<td><strong>v) k = 6</strong></td>
<td>6 permutations of ( {1} ) (4-simplex)</td>
<td>20 permutations of ( {123} ) (2-simplex)</td>
</tr>
<tr>
<td></td>
<td>15 permutations of ( {1} + {2} ) (prism on tetrahedron)</td>
<td>45 permutations of ( {12} \oplus {34} ) (square)</td>
</tr>
<tr>
<td></td>
<td>20 permutations of ( {1} + {2} + {3} ) (a triangle times a triangle)</td>
<td>72 permutations of {12,23,34,45,51} (pentagon)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60 permutations of {12,23,34,45,56,61} (hexagon)</td>
</tr>
</tbody>
</table>

Not cographic (it is graphic, with associated graph \( K_5 \))
c) \( n = 3 \)

1) \( k = 3 \)  
Just one--\( \{123\} \)  
(0-simplex)

2) \( k = 4 \)  
"Complement" (in the sense of the statement in case \( n = 2, k = 3 \) of the case \( n = 1, k = 4 \).

3) \( k = 5 \)  
"Complement" of the case \( n = 2, k = 5 \).

4) \( k = 6 \)  
Primoids are just the "complements" of duoids. We will list just duoids.  
15 permutations of \( <1234> \) (3-simplices)  
60 permutations of \( <1234> + <2345> \) (triangular prisms)  
120 permutations of \( <1234> : <2345> : <3456> \) (pentagonal wedges)  
15 permutations of \( <12> x <34> x <56> \) (cubes)  
*360 permutations of \( <1234> + <1345> + <1456> + <1562> \)  
(These are triangular prisms, truncated at two non-adjacent vertices by facets with the same label, as shown below.)

![Diagram](image)

12 permutations of Example 4.1. (These can be thought of as dodecahedra with opposite facets having the same label.)
Of all these duoids, only those of the asterisked type are abnormal or improper—but this type contributes over half the duoids (and similarly primoids).
BIBLIOGRAPHY


