Mathematical Proofs.

**EC.1. Proof of Table 1.**

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<th>Table EC.1</th>
<th>Example Terminal Payoff Functions and Their Properties</th>
</tr>
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<td></td>
<td>0-1 Terminal Payoff</td>
</tr>
<tr>
<td>$h_{0x}(s)$</td>
<td>$m_{0x} [1 - p_x(s)]$</td>
</tr>
<tr>
<td>$h_{1x}(s)$</td>
<td>$m_{1x} \cdot p_x(s)$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>${ x : p_x(S_{n,x}) \geq \frac{m_{0x}}{m_{0x} + m_{1x}} }$</td>
</tr>
</tbody>
</table>

Payoff Condition 1 holds because function $g$ defined by $g(t) \mapsto \max \{ m_{0x}(1 - t), m_{1x} \}$ is a convex function and that $E_0 [p_{x_1}(S_{1,x_1})] = E_0 [E_1 [1_{\{x_1 \in \mathbb{B}\}}]] = E_0 [1_{\{x_1 \in \mathbb{B}\}}] = p_{x_1}(S_{0,x_1})$. Hence by Jensen’s inequality, we have $h_{x_1}(S_{0,x_1}) = g(p_{x_1}(S_{0,x_1})) = g(E_0 [p_{x_1}(S_{1,x_1})]) \leq E_0 [g(p_{x_1}(S_{1,x_1}))] = E_0 [h_{x_1}(S_{1,x_1})]$. Since for any $x$ and $s \in \Lambda$,

$$h_x(s) = \max \{ h_{0x}(s), h_{1x}(s) \} = h_{0x}(s) \cdot 1_{\{s \notin B(x)\}} + h_{1x}(s) \cdot 1_{\{s \in B(x)\}} \leq m_{0x} \cdot 1_{\{s \notin B(x)\}} + m_{1x} \cdot 1_{\{s \in B(x)\}},$$

hence for any $n$,

$$E [h_x(S_{n,x}) \mid S_{0,x} = s, x_1 = \cdots = x_n = x] - h_x(s) \leq m_{0x} \cdot 1_{\{s \notin B(x)\}} + m_{1x} \cdot 1_{\{s \in B(x)\}} - h_x(s).$$

Thus Payoff Condition 2 holds with $H_x$ specified in the table.
EC.1.2. Linear Terminal Payoff

$h_{0x}(s) = \mathbb{E}[r_0(x; \theta_x) | \eta_x \sim \mathcal{D}(s)] = m_{0x}[d_x - \mu_x(s)]$. Similarly $h_{1x}(s) = m_{1x}[\mu_x(s) - d_x]$. $B_n$ follows directly from definition (1). Hence $h_x(s) = m_{0x}[\mu(s) - d_x]^{-} + m_{1x}[\mu(s) - d_x]^{+}$.

Payoff Condition 1 holds because $\mathbb{E}_0[\mu_{1x}] = \mathbb{E}_0[\mathbb{E}_1(\theta_{x_1})] = \mathbb{E}_0[\theta_{x_1}] = \mu_{0x1}$. Hence by Jensen’s inequality $(\mathbb{E}_x)^{+} \leq \mathbb{E}(X^{+})$ and $(\mathbb{E}_x)^{-} \leq \mathbb{E}(X^{-})$, we have

\[
    h_{x_1}(S_{0,x_1}) = m_{0x} (\mu_{0,x_1} - d_x)^{-} + m_{1x} (\mu_{0,x_1} - d_x)^{+} \\
    \leq \mathbb{E}_0 \left[ m_{0x} (\mu_{1,x_1} - d_x)^{-} + m_{1x} (\mu_{1,x_1} - d_x)^{+} \right] = \mathbb{E}_0 [h_{x_1}(S_{1,x_1})].
\]

Now for any $n$ and $x$, by Jensen’s inequality,

\[
    h_x(S_{n,x}) = m_{0x} (\mu_{nx} - d_x)^{-} + m_{1x} (\mu_{nx} - d_x)^{+} \leq \mathbb{E}_n \left[ m_{0x} (\theta_x - d_x)^{-} + m_{1x} (\theta_x - d_x)^{+} \right].
\]

Hence

\[
    \mathbb{E} [h_x(S_{n,x}) | S_{0,x} = s, x_1 = \cdots = x_n = x] \\
    \leq \mathbb{E} \left\{ \mathbb{E}_n \left[ m_{0x} (\theta_x - d_x)^{-} + m_{1x} (\theta_x - d_x)^{+} \right] | S_{0,x} = s, x_1 = \cdots = x_n = x \right\} \\
    = \mathbb{E} \left[ m_{0x} (\theta_x - d_x)^{-} + m_{1x} (\theta_x - d_x)^{+} | \eta_x \sim \mathcal{D}(s) \right].
\]

Thus Payoff Condition 2 holds with $H_x$ specified in the table.

EC.2. Proof of Proposition 1.

**Proposition 1.**

\[
    V(s) = \sup_{\pi} \mathbb{E}^{\pi} \left[ \sum_{n=1}^{\tau} \alpha^{n-1} \mathcal{R}_{x_n}(S_{n-1,x_n}) \bigg| S_0 = s \right], \tag{EC.1}
\]

where the reward functions $\mathcal{R}_x : \Lambda \rightarrow \mathbb{R}$ for $x = 1, \ldots, k$ are defined by

\[
    \mathcal{R}_x(s) = \mathbb{E}[h_x(S_{1,x}) | S_{0,x} = s, x_1 = x] - h_x(s) - c_x. \tag{EC.2}
\]

**Proof of Proposition 1.** By the tower property of conditional expectation,

\[
    \mathbb{E}^{\pi} [r(B_{\tau \wedge T}; \theta, d)] = \mathbb{E}^{\pi} [\mathbb{E}^{\pi}_{\tau \wedge T} [r(B_{\tau \wedge T}; \theta, d)]] = \mathbb{E}^{\pi} \left[ \sum_{x=1}^{k} h_x(S_{\tau \wedge T,x}) \right].
\]

We can therefore write (5) as follows,

\[
    V(s) = \sup_{\pi} \mathbb{E}^{\pi} \left[ \sum_{x=1}^{k} h_x(S_{\tau \wedge T,x}) - \sum_{n=1}^{\tau \wedge T} c_{x_n} \bigg| S_0 = s \right].
\]
We restructure this into a sequence of single period rewards. For a fixed policy \( \pi \) and hence a stopping rule \( \tau \), since \( T \) is geometrically distributed with parameter \( 1 - \alpha \) and independent of the sampling filtration,

\[
\mathbb{E}^\pi \left[ \sum_{x=1}^k h_x (S_{\tau \land T, x}) - \sum_{n=1}^{\tau \land T} c_{x_n} \right]
= \mathbb{E}^\pi \left[ \sum_{t=1}^\tau \left[ (1 - \alpha) \alpha^{t-1} \left( \sum_{x=1}^k h_x (S_{t, x}) - \sum_{n=1}^t c_{x_n} \right) \right] + \alpha^\tau \left( \sum_{x=1}^k h_x (S_{\tau, x}) - \sum_{n=1}^\tau c_{x_n} \right) \right]
= \mathbb{E}^\pi \left[ \sum_{x=1}^k \sum_{t=1}^\tau \left[ (1 - \alpha) \alpha^{t-1} h_x (S_{t, x}) + \alpha^t h_x (S_{\tau, x}) \right] - \sum_{t=1}^\tau (1 - \alpha) \alpha^{t-1} \sum_{n=1}^t c_{x_n} \right]
= \mathbb{E}^\pi \left[ \sum_{x=1}^k h_x (S_{0, x}) + \sum_{t=1}^\tau \alpha^{t-1} [h_x (S_{t, x}) - h_x (S_{t-1, x})] - (1 - \alpha) \sum_{n=1}^\tau c_{x_n} \right]
= \mathbb{E}^\pi \left[ \sum_{x=1}^k h_x (S_{0, x}) + \sum_{t=1}^\tau \alpha^{t-1} [-c_{x_t} + h_{x_t} (S_{t, x_t}) - h_{x_t} (S_{t-1, x_t})] \right]
= \mathbb{E}^\pi \left[ \sum_{x=1}^k h_x (S_{0, x}) + \sum_{t=1}^\tau \alpha^{t-1} [\pi t \land S_{t-1:x_t}] - h_{x_t} (S_{t-1, x_t}) \right].
\]

The second to last equation follows from simple computation and the fact that at each time \( n = 1, 2, \ldots, \tau \), for all non-selected alternatives \( x \neq x_n \), we have \( S_{n, x} = S_{n-1, x} \).

Define for all \( n \geq 1 \)

\[
R_n = -c_{x_n} + h_{x_n} (S_{n, x_n}) - h_{x_n} (S_{n-1, x_n}).
\]

We then have

\[
V(s) = R_0(s) + \sup_{\pi} \mathbb{E}^\pi \left[ \sum_{n=1}^{\tau} \alpha^{n-1} R_n \right| S_0 = s].
\]

We see from (EC.4) that a problem with stopping rule \( \tau \) that provides a fixed initial reward \( R_0(s) \) and a discounted single period reward \( \alpha^{n-1} R_n \) at each time \( n \geq 1 \) is equivalent to the original problem. Since \( R_0 \) does not affect the optimal policy, we may subtract it from the value function and instead think of \( V \) as the optimal expected incremental reward over \( R_0 \). This provides the equivalent problem,

\[
V(s) = \sup_{\pi} \mathbb{E}^\pi \left[ \sum_{n=1}^{\tau} \alpha^{n-1} R_n \right| S_0 = s],
\]

where we have redefined \( V \) to correspond to this equivalent problem in a slight abuse of notation. A policy \( \pi \) that attains the supremum in (EC.5) also attains the supremum in (EC.4) and (5).

For later work, it is convenient to make one additional transformation in which we replace the random variables \( R_n \) in (EC.5) with deterministic reward functions of the alternatives’ states. From (EC.5) and the tower property, we know

\[
V(s) = \sup_{\pi} \mathbb{E}^\pi \left[ \sum_{n=1}^{\tau} \alpha^{n-1} \mathbb{E}^\pi [R_n | S_{n-1}] \right| S_0 = s].
\]
Now \( \mathcal{R}_x(s) = \mathbb{E}[R_1 | S_{0,x} = s, x_1 = x] \) by (EC.2). Hence

\[
\mathbb{E}^x[R_n | S_{n-1}] = \mathcal{R}_{x_n}(S_{n-1,x_n}),
\]

and then (EC.1) follows.

**EC.3. Proof of Proposition 2.**

**Proposition 2.** Under Payoff Condition 1, \( \nu_x \geq -c_x \). Under Payoff Conditions 1 and 2, \(-c_x \leq \nu_x \leq -c_x + H_x\).

**Proof of Proposition 2.** Under Payoff Condition 1, \( \nu_x \geq -c_x \) follows directly from \( R_x \geq -c_x \).

Now under both conditions, we take the following expectations with respect to the sub-problem with a single alternative \( x \). It then follows from (EC.3) and (EC.6) that

\[
\mathcal{R}_x(S_{n,x}) = \mathbb{E}[R_{n+1} | S_n].
\]

Since \( \mathcal{R}_x + c_x \geq 0 \) and \( 0 < \alpha < 1 \), we write

\[
\nu_x(s) + c_x = \max_{\tau > 0} \mathbb{E} \left[ \sum_{n=1}^{\tau} \alpha^{n-1} \left[ \mathcal{R}_x(S_{n-1,x}) + c_x \right] \big| S_{0,x} = s \right] \\
\leq \max_{\tau > 0} \mathbb{E} \left[ \sum_{n=1}^{\tau} \left[ \mathcal{R}_x(S_{n-1,x}) + c_x \right] \big| S_{0,x} = s \right] = \mathbb{E} \left[ \sum_{n=0}^{\infty} \mathcal{R}_x(S_{n,x}) + c_x \big| S_{0,x} = s \right] \\
= \sum_{n=0}^{\infty} \mathbb{E} \left[ \mathcal{R}_x(S_{n,x}) + c_x \big| S_{0,x} = s \right] \\
= \sum_{n=1}^{\infty} \mathbb{E} \left[ R_n + c_x \big| S_{0,x} = s \right] \\
= \sum_{n=1}^{\infty} \mathbb{E} \left[ h_x(S_{n,x}) - h_x(S_{n-1,x}) \big| S_{0,x} = s \right] \\
= \sum_{n=1}^{\infty} \left\{ \mathbb{E} \left[ h_x(S_{n,x}) \big| S_{0,x} = s \right] - \mathbb{E} \left[ h_x(S_{n-1,x}) \big| S_{0,x} = s \right] \right\} \\
= \mathbb{E} \left[ h_x(S_{1,x}) \big| S_{0,x} = s \right] - h_x(s) + \mathbb{E} \left[ h_x(S_{2,x}) \big| S_{0,x} = s \right] - \mathbb{E} \left[ h_x(S_{1,x}) \big| S_{0,x} = s \right] + \cdots \\
= \lim_{n \to \infty} \mathbb{E} \left[ h_x(S_{n,x}) \big| S_{0,x} = s \right] - h_x(s) \leq H_x(s),
\]

Here, the third line uses the Monotone Convergence Theorem; the fourth line uses (EC.7) and the tower property; and the fifth line uses (EC.3). The limit in the last equality exists since \( \{h_x(S_{n,x})\}_{n \geq 0} \) is a sub-martingale by Payoff Condition 1 and hence \( \{\mathbb{E} \left[ h_x(S_{n,x}) \big| S_{0,x} = s \right] \}_{n} \) is an increasing sequence; and the last inequality follows from Payoff Condition 2.
EC.4. Proof of Proposition 3.

PROPOSITION 3. \( V_x \geq 0 \). Under Payoff Condition 2, \( V_x \leq H_x \).

Proof of Proposition 3. We receive a zero reward if we take \( \tau_x = 0 \). Thus \( V_x \geq 0 \). By (7), we can write (9) as \( V_x(s) = \sup_{\tau_x} \{ -c_x \tau_x + \mathbb{E} [ h_x(S_{\tau_x,x}) \mid S_{0,x} = s, x_1 = \cdots = x_{\tau_x} = x] - h_x(s) \} \). Since \( \tau_x \geq 0 \), we know that \( V_x(s) \leq H_x(s) \) under Payoff Condition 2.

EC.5. Proof of Proposition 4

PROPOSITION 4 Under Payoff Condition 3, \( \tau^*_x \) has a deterministic upper bound \( N_x \), where

\[
N_x := \min \left\{ n \geq 0 : \left[ \sup_{s \in PS(x;n')} \hat{H}_x(s) \leq c_x, \forall n' \geq n \right] \right\}. \tag{EC.8}
\]

Proof of Proposition 4. Since \( c_x > 0 \), we know that \( N_x \) exists under Payoff Condition 3. Applying Payoff Condition 3 to (7) gives \( \mathcal{R}_x \leq -c_x + \hat{H}_x \). Hence for all \( n \geq N_x \) and \( s \in PS(x;n) \), \( \mathcal{R}_x(s) \leq 0 \). Now in the sub-problem with single alternative \( x \) and initial state \( S_{0,x} = s \in PS(x;N_x) \), we know that \( S_{t,x} \in PS(x;N_x + t) \) for all \( t \geq 0 \), and hence \( \mathcal{R}_x(S_{t,x}) \leq 0 \) for all \( t \geq 0 \). It follows that

\[
V_x(s) = \sup_{\tau_x} \mathbb{E} \left[ \sum_{t=0}^{\tau_x-1} \mathcal{R}_x(S_{t,x}) \mid S_{0,x} = s \right] = 0, \quad \forall s \in PS(x;N_x)
\]

Thus \( \tau^*_x \leq N_x \).

EC.6. Proof of Theorem 1

THEOREM 1. The value function is given by, \( V(s) = \sum_{n=1}^{k} V_x(s_x) \). Furthermore, any policy with sampling decisions \( (x^*_n)_{n \geq 1} \) and stopping time \( \tau^* \) satisfying the following conditions is optimal:

\[
x^*_{n+1} \in \{ x : S_n,x \in C_x \}, \quad \forall n \geq 0; \quad \tau^* = \inf \left\{ n \geq 0 : S_{n,x} \notin C_x, \forall x \right\}. \tag{EC.9}
\]

Proof of Theorem 1. For any arbitrary policy \( \pi \) with stopping time \( \tau \), we denote the number of times we sample from each alternative \( x \) by \( m_x \). Then \( \tau = \sum_{x=1}^{k} m_x \). Denote the collection of times when we sample from \( x \), \( \{ 1 \leq n \leq \tau : x_n = x \} \), by \( \{ n^*_i \}_{1 \leq i \leq m_x} \).

Since the reward for each period only depends on the alternative being sampled during that period, and the states of all the other alternatives remain frozen, we know that the order of the sequence of sampling decisions does not affect the expected total reward. Hence the original problem can be naturally decomposed into \( k \) sub-problems as follows.

\[
\mathbb{E}_\pi \left[ \sum_{n=1}^{\tau} \mathcal{R}_{x_n}(S_{n-1,x_n}) \mid S_0 = s \right] = \sum_{x=1}^{k} \left\{ \mathbb{E}_\pi \left[ \sum_{i=1}^{m_x} \mathcal{R}_x(S^*_n - 1, x) \mid S_{0,x} = s_x \right] \right\}
\]

\[
= \sum_{x=1}^{k} \left\{ \mathbb{E}_\pi \left[ \sum_{n=1}^{m_x} \mathcal{R}_x(S_{n-1,x}) \mid S_{0,x} = s_x, x_1 = \cdots = x_{m_x} = x \right] \right\} \leq \sum_{x=1}^{k} V_x(s_x), \tag{EC.10}
\]
where the last inequality follows from (9). Thus

\[ V(s) = \sup_{\pi} \mathbb{E}^\pi \left[ \sum_{n=1}^{\tau} \mathcal{R}_{x_n}(S_{n-1,x_n}) \middle| S_0 = s \right] \leq \sum_{x=1}^{k} V_x(s_x). \]

On the other hand, if we adopt a policy satisfying \( x_{n+1} \in \{ x : S_{n,x} \in \mathbb{C}_x \} \) for all \( n \geq 0 \) and \( \tau = \inf\{ n \geq 0 : S_{n,x} \notin \mathbb{C}_x, \forall x \} \), then for each \( x \), \( m_x \) is exactly the number of samples from alternative \( x \) needed for the state of \( x \) to leave \( \mathbb{C}_x \) for the first time. Hence in each decomposed sub-problem with single alternative \( x \), \( m_x \) is an optimal solution equivalent to \( \tau_x^* \). As a result,

\[ \mathbb{E}^\pi \left[ \sum_{n=1}^{m_x} \mathcal{R}_x(S_{n-1,x}) \middle| S_{0,x} = s_x, x_1 = \cdots = x_{m_x} = x \right] = V_x(s_x), \]

the inequality in (EC.10) becomes equality, and \( V(s) = \sum_{x=1}^{k} V_x(s_x) \). This also shows that any policy satisfying the conditions (EC.9) is optimal.

**EC.7. Proof of Proposition 5.**

**Proposition 5.** Suppose that each \( \tau_x^* \) has a deterministic upper bound \( N_x \). Then the optimal stopping rule \( \tau^* \), as characterized in Theorem 1, has a deterministic upper bound \( \sum_{x=1}^{k} N_x \).

**Proof of Proposition 5.** We apply ideas similar to those in the proof of Theorem 1. First, \( \tau^* = \sum_{x=1}^{k} m_x \). Under the optimal policy, since \( m_x \) is the number of samples from alternative \( x \) needed for the state of \( x \) to leave \( \mathbb{C}_x \) for the first time, its distribution is the same as the distribution of \( \tau_x \) in the decomposed sub-problem with single alternative \( x \). It follows that \( m_x \leq N_x \), for each \( x \). Thus \( \tau^* \leq \sum_{x=1}^{k} N_x \).

**EC.8. Remark 1.**

Consider the distribution of \( S_i \) given \( S_0 = (a, b) \) and \( x_1 = x \). Since \( \mathbb{P}_0\{y_1 = 1\} = \mathbb{E}_0[y_1] = \mathbb{E}_0[\mathbb{E}_0[y_1 | \theta]] = \mathbb{E}_0[\theta_{x_1}] = \mu_{0x_1} \), we immediately have the following expressions:

\[ \mathbb{P}\{S_1 = (a + e_x, b) \mid S_0 = (a, b), x_1 = x\} = \mathbb{P}\{y_1 = 1 \mid S_0 = (a, b), x_1 = x\} = a_x / (a_x + b_x), \]

\[ \mathbb{P}\{S_1 = (a, b + e_x) \mid S_0 = (a, b), x_1 = x\} = \mathbb{P}\{y_1 = 0 \mid S_0 = (a, b), x_1 = x\} = b_x / (a_x + b_x). \]

**EC.9. Proof of Table 2.**

**EC.9.1. Preparatory Material**

Use Stirling’s approximation, for large \( a \) and \( b \),

\[ B(a,b) \sim \sqrt{2\pi} \frac{a^{a-\frac{1}{2}}b^{b-\frac{1}{2}}}{(a+b)^{a+b-1}}. \]

More generally, we have the following lemma.
Table EC.2  Example Terminal Payoff Functions with Bernoulli Sampling

<table>
<thead>
<tr>
<th>Payoff Condition 3</th>
<th>0-1 Terminal Payoff</th>
<th>Linear Terminal Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{H}_x(a,b)$</td>
<td>$\frac{m_0 x + m_{1x}}{2\sqrt{2\pi(a+b)}}$</td>
<td>$\frac{\max{m_0, m_{1x}} + m_0}{4(a+b+1)}$</td>
</tr>
<tr>
<td>$N_x$</td>
<td>$\left(\left\lfloor \frac{(m_0 x + m_{1x})^2}{8\pi c_x^2} \right\rfloor - 2 \right)^+$</td>
<td>$\left(\left\lfloor \frac{(m_0 x + m_{1x} + m_0 x)}{4c_x} \right\rfloor - 3 \right)^+$</td>
</tr>
</tbody>
</table>

**Lemma EC.1.** For $a, b \geq 1$,
\[
B(a, b) = \sqrt{2\pi} a^{a-b-\frac{1}{2}} (a+b)^{a-b}\frac{(a+b) - a}{2}.
\]

**Proof of Lemma EC.1.** By Stirling’s asymptotic series (see, e.g., Abramowitz and Stegun (1964) and Sloane (2007)), we write
\[
\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi} e^z, \quad \text{with} \quad \frac{1}{12z+1} < \lambda_z < \frac{1}{12z}.
\]
Hence
\[
B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} = \sqrt{2\pi} \exp\{\lambda_a + \lambda_b - \lambda_{a+b}\} \frac{a^{a-b-\frac{1}{2}} (a+b)^{a-b}\frac{(a+b) - a}{2}}{a+b}.
\]
The result then follows since for $a, b \geq 1$,
\[
\lambda_a + \lambda_b - \lambda_{a+b} > \frac{1}{12a+1} + \frac{1}{12b+1} - \frac{1}{12(a+b)} = \frac{(12a + \frac{1}{2})^2 + (12b + \frac{1}{2})^2 + 144ab - \frac{3}{2}}{12(a+b)(12a+1)(12b+1)} > 0.
\]

**Lemma EC.2.** For $a, b \geq 1$ and $d_x \in (0, 1)$,
\[
\frac{d_x^a(1-d_x)^b}{B(a, b)} \leq \frac{1}{2} \sqrt{\frac{a+b}{2\pi}}.
\]

**Proof of Lemma EC.2.** Denote $t = a+b$ and $\mu = \frac{a}{a+b}$. Then by Lemma EC.1,
\[
\frac{d_x^a(1-d_x)^b}{B(a, b)} \leq \frac{d_x^a(1-d_x)^b}{\sqrt{2\pi(\mu)\mu^{\mu-\frac{1}{2}}(1-\mu)\mu^{1-\mu-\frac{1}{2}}}} = \sqrt{\frac{\mu(1-\mu)t}{2\pi}} \left[ \left( \frac{d_x}{\mu} \right)^\mu \left( \frac{1-d_x}{1-\mu} \right)^{1-\mu} \right]^t.
\]
Define function $g(\cdot)$ on $(0, 1)$ by
\[
g(u) = \left( \frac{d_x}{u} \right)^u \left( \frac{1-d_x}{1-u} \right)^{1-u}.
\]
Then since $\frac{d}{du} [\log g(u)] = \log \frac{d_x(1-u)}{u(1-d_x)}$, it is easy to see that $g(\cdot)$ is unimodal on $(0, 1)$ with peak at $u = d_x$. Finally, since $\mu \in (0, 1)$, we know $\sqrt{\mu(1-\mu)} \leq \frac{1}{2}$, and hence
\[
\frac{d_x^a(1-d_x)^b}{B(a, b)} \leq \frac{1}{2} \sqrt{\frac{t}{2\pi}} [g(\mu)]^t \leq \frac{1}{2} \sqrt{\frac{t}{2\pi}} [g(d_x)]^t = \frac{1}{2} \sqrt{\frac{t}{2\pi}} = \frac{1}{2} \sqrt{\frac{a+b}{2\pi}}.
\]
EC.9.2. 0-1 Terminal Payoff

We first state the following property of the regularized incomplete beta function $I_d(\cdot, \cdot)$,

$$I_d(a+1, b) = I_d(a, b) - \frac{d^a(1-d)^b}{aB(a, b)}; \quad I_d(a, b+1) = I_d(a, b) + \frac{d^a(1-d)^b}{bB(a, b)}.$$

Hence by Remark 1, if

$$I_{dx}(a, b) \geq \frac{m_{1x}}{m_{0x} + m_{1x}} + \frac{d^x(1-d)^b}{aB(a, b)},$$

then

$$E[h_x(S_{1,x}) \mid S_{0,x} = (a, b), x_1 = x] - h_x(a, b)$$

$$= \frac{a}{a+b} h_x(a+1, b) + \frac{b}{a+b} h_x(a, b+1) - h_x(a, b)$$

$$= m_{0x} \left[ \frac{a}{a+b} I_{dx}(a+1, b) + \frac{b}{a+b} I_{dx}(a, b+1) - I_{dx}(a, b) \right]$$

$$= m_{0x} \left[ \frac{a}{a+b} \cdot \frac{d^x(1-d)^b}{aB(a, b)} + \frac{b}{a+b} \cdot \frac{d^x(1-d)^b}{bB(a, b)} \right] = 0.$$

Similarly, if

$$I_{dx}(a, b) \leq \frac{m_{1x}}{m_{0x} + m_{1x}} - \frac{d^x(1-d)^b}{bB(a, b)},$$

we have $E[h_x(S_{1,x}) \mid S_{0,x} = (a, b), x_1 = x] - h_x(a, b) = 0$.

Now, if

$$\frac{m_{1x}}{m_{0x} + m_{1x}} \leq I_{dx}(a, b) < \frac{m_{1x}}{m_{0x} + m_{1x}} + \frac{d^x(1-d)^b}{aB(a, b)},$$

then

$$E[h_x(S_{1,x}) \mid S_{0,x} = (a, b), x_1 = x] - h_x(a, b)$$

$$= \frac{a}{a+b} m_{1x} \left[ 1 - I_{dx}(a+1, b) \right] + m_{0x} \left[ \frac{b}{a+b} I_{dx}(a, b+1) - I_{dx}(a, b) \right]$$

$$= \frac{a}{a+b} m_{1x} - \left( m_{0x} + m_{1x} \right) I_{dx}(a, b) + \frac{m_{0x} + m_{1x}}{a+b} \cdot \frac{d^x(1-d)^b}{B(a, b)}$$

$$\leq \frac{m_{0x} + m_{1x}}{a+b} \cdot \frac{d^x(1-d)^b}{B(a, b)}.$$

Similarly, if

$$\frac{m_{1x}}{m_{0x} + m_{1x}} - \frac{d^x(1-d)^b}{bB(a, b)} < I_{dx}(a, b) \leq \frac{m_{1x}}{m_{0x} + m_{1x}},$$

we still have

$$E[h_x(S_{1,x}) \mid S_{0,x} = (a, b), x_1 = x] - h_x(a, b) \leq \frac{m_{0x} + m_{1x}}{a+b} \cdot \frac{d^x(1-d)^b}{B(a, b)}.$$  \hspace{1cm} (EC.11)

Thus (EC.11) holds for all $(a, b) \in \Lambda = [1, +\infty) \times [1, +\infty)$. Now applying Lemma EC.2, we know

$$E[h_x(S_{1,x}) \mid S_{0,x} = (a, b), x_1 = x] - h_x(a, b) \leq \frac{m_{0x} + m_{1x}}{2\sqrt{2\pi(a+b)}} := \bar{H}_x(a, b).$$
Since in the sub-problem with single alternative \( x, a_{n,x} + b_{n,x} = a_{0,x} + b_{0,x} + n \), and hence \( PS(x;n) = \{(a,b) \in \Lambda : a + b \geq 2 + n \} \). It follows that

\[
\lim_{n \to \infty} \frac{\sup_{(a,b) \in PS(x;n)} \bar{H}_x(a,b)}{2\sqrt{2\pi(2 + n)}} = \lim_{n \to \infty} \frac{m_{0x} + m_{1x}}{2\sqrt{2\pi(2 + n)}} = 0.
\]

Thus Payoff Condition 3 holds and

\[
N_x = \min \left\{ n \geq 0 : \frac{m_{0x} + m_{1x}}{2\sqrt{2\pi(2 + n)}} \leq c_x \right\} = \left( \frac{(m_{0x} + m_{1x})^2}{8\pi c_x^2} - 2 \right)^+.
\]

**EC.9.3. Linear Terminal Payoff**

First suppose \( a \geq b \), then \( \frac{a}{a+b} \geq \frac{a+1}{a+b+1} \). Since for all \( x, y \in \mathbb{R}, x^+ - y^+ \leq |x-y| \), we have

\[
\mathbb{E}[h_x(S_{1,x}) \mid S_{0,x} = (a,b), x_1 = x] - h_x(a,b) = \frac{a}{a+b} h_x(a+1,b) + \frac{b}{a+b} h_x(a,b+1) - h_x(a,b)
\]

\[
= \frac{a}{a+b} \left[ m_{0x} \left( d_x - \frac{a+1}{a+b+1} \right)^+ + m_{1x} \left( \frac{a+1}{a+b+1} - d_x \right)^+ \right]
+ \frac{b}{a+b} \left[ m_{0x} \left( d_x - \frac{a}{a+b} \right)^+ + m_{1x} \left( \frac{a}{a+b} - d_x \right)^+ \right]
- \left[ m_{0x} \left( d_x - \frac{a}{a+b} \right)^+ + m_{1x} \left( \frac{a}{a+b} - d_x \right)^+ \right]
\]

\[
= \frac{a}{a+b} \left[ \left( d_x - \frac{a+1}{a+b+1} \right)^+ - \left( d_x - \frac{a}{a+b} \right)^+ \right] + m_{1x} \left[ \left( \frac{a+1}{a+b+1} - d_x \right)^+ - \left( \frac{a}{a+b} - d_x \right)^+ \right]
+ \frac{b}{a+b} \left[ m_{0x} \left( d_x - \frac{a}{a+b} \right)^+ - \left( d_x - \frac{a}{a+b} \right)^+ \right] + m_{1x} \left[ \left( \frac{a}{a+b} - d_x \right)^+ - \left( \frac{a}{a+b} - d_x \right)^+ \right]
\]

\[
\leq \frac{a \cdot m_{0x}}{a+b} \left| \frac{a+1}{a+b+1} - \frac{a}{a+b} \right| + \frac{b \cdot m_{0x}}{a+b} \left| \frac{a}{a+b+1} - \frac{a}{a+b} \right| + \left( \frac{a}{a+b} - d_x \right)^+ + \left( \frac{a}{a+b} - d_x \right)^+
\]

\[
= \frac{2ab \cdot m_{0x}}{(a+b)^2(a+b+1)}.
\]

Similarly if \( a < b \), then

\[
\mathbb{E}[h_x(S_{1,x}) \mid S_{0,x} = (a,b), x_1 = x] - h_x(a,b) \leq \frac{ab(m_{0x} + m_{1x})}{(a+b)^2(a+b+1)}.
\]

In both cases, we have

\[
\mathbb{E}[h_x(S_{1,x}) \mid S_{0,x} = (a,b), x_1 = x] - h_x(a,b) \leq \frac{ab \max\{m_{0x},m_{1x}\} + m_{0x}}{(a+b)^2(a+b+1)} \leq \frac{\max\{m_{0x},m_{1x}\} + m_{0x}}{4(a+b+1)} := \bar{H}_x(a,b).
\]

Since \( PS(x;n) = \{(a,b) \in \Lambda : a + b \geq 2 + n \} \), we have

\[
\lim_{n \to \infty} \sup_{(a,b) \in PS(x;n)} \bar{H}_x(a,b) = \lim_{n \to \infty} \left[ \frac{\max\{m_{0x},m_{1x}\} + m_{0x}}{4(3+n)} \right] = 0.
\]
Thus Payoff Condition 3 holds and
\[ N_x = \min \left\{ n \geq 0 : \frac{\max\{m_{0x}, m_{1x}\} + m_{0x}}{4(3 + n)} \leq c_x \right\} = \left( \frac{\max\{m_{0x}, m_{1x}\} + m_{0x}}{4c_x} - 3 \right)^+. \]

**EC.10. Remark 2.**

From here we let \( \Phi \) and \( \varphi \) denote the standard normal cdf and pdf respectively. Define the following events for a standard normal random variable \( Z \):
\[
A^i = \{ \mu + \tilde{\sigma}_x(\beta) Z \in \left[ \mu_x^i(\beta + \beta_x^i) - \delta/2, \mu_x^i(\beta + \beta_x^i) + \delta/2 \right] \} \quad \text{for all } i,
\]
\[
A = \left\{ \mu + \tilde{\sigma}_x(\beta) Z \notin \left[ \mu_x(\beta + \beta_x^i), \varphi_x(\beta + \beta_x^i) \right] \right\}.
\]

Then
\[
\mathbb{E} [V_x (\mu + \tilde{\sigma}_x(\beta) Z, \beta + \beta_x^i)]
\approx \sum_i \left\{ \mathbb{E} [V_x (\mu + \tilde{\sigma}_x(\beta) Z, \beta + \beta_x^i) \left| A^i \right] \cdot \mathbb{P}(A^i) \right\} + \mathbb{E} [V_x (\mu + \tilde{\sigma}_x(\beta) Z, \beta + \beta_x^i) \left| A \right] \cdot \mathbb{P}(A)
\approx \sum_i \left\{ V_x (\mu_x^i(\beta + \beta_x^i), \beta + \beta_x^i) \cdot \mathbb{P}(A^i) \right\} + 0 \cdot \mathbb{P}(A)
= \sum_i \left\{ V_x (\mu_x^i(\beta + \beta_x^i), \beta + \beta_x^i) \cdot \left[ \Phi \left( \frac{\mu_x^i(\beta + \beta_x^i) + \delta/2 - \mu}{\tilde{\sigma}_x(\beta)} \right) - \Phi \left( \frac{\mu_x^i(\beta + \beta_x^i) + \delta/2 - \mu}{\tilde{\sigma}_x(\beta)} \right) \right] \right\}.
\]

**EC.11. Remark 3.**

We only need to give explicit expressions for \( \mathbb{E} [h_x (\mu + \tilde{\sigma}_x(\beta) Z, \beta + \beta_x^i)] \).

**EC.11.1. 0-1 Terminal Payoff**

Let \( \rho = d_x - \mu \) and \( \xi = \tilde{\sigma}_x^{-1}(\beta) \left[ \rho - \Phi^{-1} \left( \frac{m_{1x}}{m_{0x} + m_{1x}} \right) / \sqrt{\beta + \beta_x^i} \right] \), then
\[
\mathbb{E} [h_x (\mu + \tilde{\sigma}_x(\beta) Z, \beta + \beta_x^i)]
= \mathbb{E} \left[ \max \left\{ m_{0x} \cdot \Phi \left( \sqrt{\beta + \beta_x^i} (d_x - \mu - \tilde{\sigma}_x(\beta) Z) \right), m_{1x} \left[ 1 - \Phi \left( \sqrt{\beta + \beta_x^i} (d_x - \mu - \tilde{\sigma}_x(\beta) Z) \right) \right] \right\} \right]
= m_{0x} \int_0^\xi \Phi \left( \sqrt{\beta + \beta_x^i} (\rho - \tilde{\sigma}_x(\beta) z) \right) \varphi(z) dz + m_{1x} \int_\xi^{+\infty} \left[ 1 - \Phi \left( \sqrt{\beta + \beta_x^i} (\rho - \tilde{\sigma}_x(\beta) z) \right) \right] \varphi(z) dz
= m_{0x} \cdot X + m_{1x} \cdot Y,
\]
where \( X \) and \( Y \) are defined to be the first and second integral respectively in the second line. Let \( Z_1 \) and \( Z_2 \) be two independent standard normal random variables. Then
\[
X = \mathbb{P} \left[ Z_1 \leq \sqrt{\beta + \beta_x^i} (\rho - \tilde{\sigma}_x(\beta) Z_2) , Z_2 \leq \xi \right] = \mathbb{P} \left[ (Z_2, Z_3) \leq (\xi, 0) \right],
\]
where \( Z_3 := Z_1 - \sqrt{\beta + \beta_x^i} (\rho - \tilde{\sigma}_x(\beta) Z_2) \sim N \left( -\rho \sqrt{\beta + \beta_x^i} , 1 + \beta_x^i / \beta \right) \) and \( \text{Cov}(Z_2, Z_3) = \sqrt{\beta_x^i / \beta} \). It follows that \( X \) can be evaluated from the cdf of a bivariate normal distribution. A similar argument can be applied to \( Y \).
EC.11.2. Linear Terminal Payoff

\[ R_x(\mu, \beta) + c_x = \mathbb{E}[h_x(\mu + \hat{\sigma}_x(\beta) Z, \beta + \beta^*_x)] - h_x(\mu, \beta) \]

\[ = m_{0x} \left\{ \mathbb{E} \left[ (d_x - \mu - \hat{\sigma}_x(\beta) Z)^+ \right] - (d_x - \mu)^+ \right\} + m_{1x} \left\{ \mathbb{E} \left[ (\mu + \hat{\sigma}_x(\beta) Z - d_x)^+ \right] - (\mu - d_x)^+ \right\} \]

\[ = m_{0x} \cdot \Delta \left( d_x - \mu, \hat{\sigma}_x^2(\beta) \right) + m_{1x} \cdot \Delta \left( \mu - d_x, \hat{\sigma}_x^2(\beta) \right), \]

which can be computed using Lemma EC.4.

EC.12. Proof of Table 3

<table>
<thead>
<tr>
<th>Payoff Condition 3</th>
<th>0-1 Terminal Payoff</th>
<th>Linear Terminal Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ H_x(\mu, \beta) ]</td>
<td>( \max{m_{0x}, m_{1x}} \left( \sqrt{1 + \beta_x^2/\beta} - 1 \right) A_3 + \pi^{-1} \sqrt{\beta_x^2/\beta} )</td>
<td>( \frac{m_{0x} + m_{1x}}{\sqrt{2\pi\beta}} )</td>
</tr>
<tr>
<td>[ N_x ]</td>
<td>( \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1} )</td>
<td>( \frac{(m_{0x} + m_{1x})^2}{2\pi c_\beta^2} )</td>
</tr>
<tr>
<td>Special Condition 1</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Special Condition 2</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

where

\[ A_1 = A_4^2 + 2A_4 + A_5, \quad A_2 = 2(A_4 + A_5)/\pi, \quad A_3 = \pi^{-2} - A_4^2, \]

\[ A_4 = 1 + 1/\sqrt{2\pi e}, \quad A_5 = c_x/\max\{m_{0x}, m_{1x}\}. \]

EC.12.1. Preparatory Material

LEMMA EC.3. Let \( u_0 := \Phi^{-1} \left( \frac{m_{0x}}{m_{0x} + m_{1x}} \right) \). Define functions \( g_x(\cdot), m_x(\cdot), s_x(\cdot), t_x(\cdot), s(\cdot) \) at \((-\infty, +\infty)\) and \( g(\cdot) \) at \((-\infty, +\infty)^2\) by

\[
\begin{align*}
g_x(u) &= \max \left\{ m_{0x} \Phi(u), m_{1x} \left[ 1 - \Phi(u) \right] \right\}, \\
m_x(u) &= m_{0x} \mathbb{1}_{\{u > u_0\}} + m_{1x} \mathbb{1}_{\{u < u_0\}}, \\
s_x(u) &= \begin{cases} 
\sup_{v} \left[ m_{0x} \Phi(u) - m_{1x} \Phi(v) \right] & \text{if } u \geq u_0, \\
\sup_{v} \left[ m_{0x} \Phi(u) - m_{1x} \Phi(v) \right] & \text{if } u < u_0,
\end{cases} \\
t_x(u) &= \begin{cases} 
m_{0x} \sup_{v \geq u} \left[ \Phi(v) - \Phi(u) \right] & \text{if } u \geq u_0, \\
m_{1x} \sup_{v \leq u} \left[ \Phi(v) - \Phi(u) \right] & \text{if } u < u_0,
\end{cases} \\
s(u) &= \max \left\{ s_x(u), t_x(u) \right\}, \\
g(u, v) &= g_x(u) + |v - u| s(u).
\end{align*}
\]

Then for all \( u \) and \( v \),
1. $0 \leq s(u) \leq \max\{m_{0x}, m_{1x}\}/\sqrt{2\pi}$.
2. $|u|s(u) \leq \max\{m_{0x}, m_{1x}\}(1 + 1/\sqrt{2\pi e})$.
3. $g_x(v) \leq g(u, v)$.

Proof of Lemma EC.3.  

- If $u \geq u_0$, then $g_x(u) = m_{0x}\Phi(u)$ and $m_x(u) = m_{0x}$. Also, if $m_{0x}\Phi(u) \geq m_{1x}$, then $s_x(u) = 0$; otherwise there exists a maximum $v^* < u$ satisfying the first order condition, i.e.,

$$s_x(u) = \frac{m_{1x}\Phi(-v^*) - m_{0x}\Phi(u)}{u - v^*} = m_{1x}\varphi(v^*). \tag{EC.13}$$

If $u \geq 0$, then $t_x(u) = m_{0x}\varphi(u)$; otherwise there exists some $v^{**} > u$ such that

$$t_x(u) = m_{0x}\Phi(v^{**}) - \Phi(u) = m_{0x}\varphi(v^{**}). \tag{EC.14}$$

- Similarly if $u < u_0$, then $g_x(u) = m_{1x}\Phi(-u)$ and $m_x(u) = m_{1x}$. If $m_{1x}\Phi(-u) \geq m_{0x}$, then $s_x(u) = 0$; otherwise there exists some $v^* > u$ such that

$$s_x(u) = \frac{m_{0x}\Phi(v^*) - m_{1x}\Phi(-u)}{v^* - u} = m_{0x}\varphi(v^*).$$

If $u \leq 0$, then $t_x(u) = m_{1x}\varphi(u)$; otherwise there exists $v^{**} < u$ such that

$$t_x(u) = m_{1x}\Phi(v^{**}) - \Phi(u) = m_{1x}\varphi(v^{**}).$$

It follows that $s(u) \leq \max\{m_{0x}, m_{1x}\}\varphi(0) = \max\{m_{0x}, m_{1x}\}/\sqrt{2\pi}$. Hence the first inequality holds.

For the second inequality, we notice that function $u \mapsto |u|\varphi(u)$ is maximized at $u = \pm 1$ with maximum value $\varphi(1) = 1/\sqrt{2\pi e}$. Assume $u \geq u_0$. We first show $|u|s_x(u) \leq \max\{m_{0x}, m_{1x}\}(1 + 1/\sqrt{2\pi e})$. This result holds immediately when $m_{0x}\Phi(u) > m_{1x}$, i.e., $s_x(u) = 0$. Otherwise by (EC.13),

$$|u|s_x(u) \leq m_{1x}\Phi(-v^*) - m_{0x}\Phi(u) + m_{1x}|v^*|\varphi(v^*) \leq m_{1x}(1 + 1/\sqrt{2\pi e}) \leq \max\{m_{0x}, m_{1x}\}(1 + 1/\sqrt{2\pi e}).$$

We then show $|u|t_x(u) \leq \max\{m_{0x}, m_{1x}\}(1 + 1/\sqrt{2\pi e})$. This result holds immediately when $u \geq 0$. Otherwise by (EC.14),

$$|u|t_x(u) = m_{0x}\Phi(v^{**}) - m_{0x}\Phi(u) + m_{0x}|v^{**}|\varphi(v^{**}) \leq m_{0x}(1 + 1/\sqrt{2\pi e}) \leq \max\{m_{0x}, m_{1x}\}(1 + 1/\sqrt{2\pi e}).$$

The case when $u < u_0$ is similar.

We now show the last inequality. WLOG, suppose $m_{0x} \leq m_{1x}$. Then $u_0 \geq 0$.

For $u \geq u_0$, we have $g(u, v) = m_{0x}\Phi(u) + |v - u| \max\{s_x(u), m_{0x}\varphi(u)\}$.

1. For $v \geq u$, by the Mean Value Theorem, $\Phi(v) = \Phi(u) + (v - u) \cdot \varphi(w)$, where $w \in (u, v)$ and $\varphi(w) < \varphi(u)$. Hence $g_x(v) = m_{0x}\Phi(v) < m_{0x}[\Phi(u) + |v - u|\varphi(u)] \leq g(u, v)$.
2. For $u_0 \leq v < u$, $g_x(v) = m_{0x} \Phi(v) < m_{0x} \Phi(u) < g(u, v)$.

3. For $v < u_0$, $g_x(v) = m_{1x} \Phi(-v) = m_{0x} \Phi(u) + (u - v) \left[\frac{m_{1x} \Phi(-v) - m_{0x} \Phi(u)}{u - v}\right] \leq m_{0x} \Phi(u) + |v - u| s_x(u) \leq g(u, v)$.

For $0 < u < u_0$ (if $u_0 > 0$), $g(u, v) = m_{1x} \Phi(-u) + |v - u| \max\{s_x(u), t_x(u)\}$.

1. For $v \leq u$, $g_x(v) = m_{1x} \Phi(-v) = m_{1x} \left[\Phi(-u) + (u - v) \left(\frac{\Phi(-v) - \Phi(-u)}{u - v}\right)\right] \leq m_{1x} \Phi(-u) + |v - u| t_x(u) \leq g(u, v)$.

2. For $u < v \leq u_0$, $g_x(v) = m_{1x} \Phi(-v) < m_{1x} \Phi(-u) = g_x(u) \leq g(u, v)$.

3. For $v > u_0$, $g_x(v) = m_{0x} \Phi(v) = m_{1x} \Phi(-u) + (v - u) \left[\frac{m_{0x} \Phi(v) - m_{1x} \Phi(-u)}{v - u}\right] \leq m_{1x} \Phi(-u) + |v - u| s_x(u) \leq g(u, v)$.

For $u \leq 0$, we can show $g_x(\cdot) \leq g(u, \cdot)$ similar to the case when $u \geq u_0$.

**Lemma EC.4.** Define $\Delta: (-\infty, +\infty) \times (0, +\infty) \to [0, +\infty)$ by

$$\Delta(\mu, \beta) := [E(X^+) - (EX)^+]|X \sim N(\mu, 1/\beta)].$$

Then

$$\Delta(\mu, \beta) = \begin{cases} 
\mu \Phi(\mu \sqrt{\beta}) + \sqrt{\beta}^{-1} \varphi(\mu \sqrt{\beta}), & \text{if } \mu \leq 0; \\
\mu \left[\Phi(\mu \sqrt{\beta}) - 1\right] + \sqrt{\beta}^{-1} \varphi(\mu \sqrt{\beta}), & \text{if } \mu > 0.
\end{cases}$$

For any fixed $\beta > 0$, $\Delta(\cdot, \beta)$ has the following properties:

- it is strictly increasing on $(-\infty, 0]$ and strictly decreasing on $[0, +\infty)$;
- it reaches its maximum at $\mu = 0$, with maximum value $1/\sqrt{2\pi\beta}$;
- it converges to 0 as $\mu \to +\infty$ and $\mu \to -\infty$.

**Proof of Lemma EC.4** For any fixed $\beta > 0$ and $\mu \in \mathbb{R}$, suppose $X \sim N(\mu, 1/\beta)$, then by Clark (1961),

$$E(X^+) = \mu \Phi(\mu \sqrt{\beta}) + \sqrt{\beta}^{-1} \varphi(\mu \sqrt{\beta})$$

where $\Phi$ and $\varphi$ are the standard normal cdf and pdf respectively. Since $\Phi'(x) = \varphi(x)$, $\varphi'(x) = -x \varphi(x)$, it follows that

$$[E(X^+)]'_\mu = \Phi(\mu \sqrt{\beta}) > 0,$$

which indicates that $E(X^+)$ is a strictly increasing function of $\mu$. Similarly,

$$E(X^-) = E(X^+) - \mu = \mu \left[\Phi(\mu \sqrt{\beta}) - 1\right] + \sqrt{\beta}^{-1} \varphi(\mu \sqrt{\beta})$$

$$[E(X^-)]'_\mu = \Phi(\mu \sqrt{\beta}) - 1 < 0,$$

hence $E(X^-)$ is a strictly decreasing function of $\mu$.

Now consider the objective function

$$\Delta(\mu, \beta) = E(X^+) - \left[ E(X^+) - E(X^-) \right]^+ = \begin{cases} 
E(X^+) = E(X^-), & \text{if } \mu = 0; \\
E(X^+), & \text{if } E(X^+) < E(X^-), \text{ i.e., } \mu < 0; \\
E(X^-), & \text{if } E(X^+) > E(X^-), \text{ i.e., } \mu > 0.
\end{cases}$$
It follows that $\Delta(\cdot, \beta)$ is strictly increasing on $(-\infty, 0]$, strictly decreasing on $[0, +\infty)$, and maximized at $\mu = 0$, where

$$
\Delta(0, \beta) = \left[ E(X^+) | X \sim \mathcal{N}(0, 1/\beta) \right] = 1/\sqrt{\beta} \cdot \left[ E(X^+) | X \sim \mathcal{N}(0, 1) \right] = 1/\sqrt{2\pi\beta}.
$$

Finally, since for a standard normal r.v. $Z \in L_1$, we have

$$
0 = \lim_{x \to +\infty} xP(Z > x) = \lim_{x \to -\infty} xP(Z \leq x) = \lim_{x \to -\infty} x(1 - \Phi(x)) = \lim_{x \to -\infty} x\Phi(x).
$$

Thus by (EC.15) and (EC.16),

$$
\lim_{\mu \to -\infty} \Delta(\mu, \beta) = \lim_{\mu \to -\infty} [E(X^+) | X \sim \mathcal{N}(\mu, 1/\beta)] = \sqrt{\beta}^{-1} \left[ \lim_{x \to -\infty} x\Phi(x) + \varphi(-\infty) \right] = 0;
$$

$$
\lim_{\mu \to +\infty} \Delta(\mu, \beta) = \lim_{\mu \to +\infty} [E(X^-) | X \sim \mathcal{N}(\mu, 1/\beta)] = \sqrt{\beta}^{-1} \left[ \lim_{x \to +\infty} x(\Phi(x) - 1) + \varphi(+\infty) \right] = 0.
$$

**EC.12.2. 0-1 Terminal Payoff**

**EC.12.2.1. Payoff Condition 3** Denote $\rho = d_x - \mu$. Applying Lemma EC.3, we have

$$
E[h_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^2)] - h_x(\mu, \beta) = E\left[ g\left( \sqrt{\beta + \beta_x^2}(\rho - \tilde{\sigma}_x(\beta)Z) \right) - g\left( \sqrt{\beta}\rho \right) \right] \\
\leq E\left[ g\left( \sqrt{\beta}\rho, \sqrt{\beta + \beta_x^2}(\rho - \tilde{\sigma}_x(\beta)Z) \right) - g\left( \sqrt{\beta}\rho \right) \right] = E\left[ \left( \sqrt{1 + \beta_x^2/\beta} - 1 \right) \sqrt{\beta}\rho \right. \\
\left. + \sqrt{\beta_x^2/\beta} \cdot E[Z] \cdot s \left( \sqrt{\beta}\rho \right) \right] \\
\leq \max\{m_{0x}, m_{1x}\} \left[ \left( \sqrt{1 + \beta_x^2/\beta} - 1 \right) \left( 1 + 1/\sqrt{2\pi e} \right) + \pi^{-1}\sqrt{\beta_x^2/\beta} \right] = \tilde{H}_x(\mu, \beta).
$$

Now

$$
\lim_{n \to \infty} \sup_{(\mu, \beta) \in PS(x; n)} \tilde{H}_x(\mu, \beta) = \max\{m_{0x}, m_{1x}\} \lim_{n \to \infty} \left[ \left( \sqrt{1 + 1/n - 1} \right) \left( 1 + 1/\sqrt{2\pi e} \right) + 1/\left( \sqrt{n}\pi \right) \right] = 0,
$$

where $PS(x; n) = \{ (\mu, \beta) \in \Lambda : \beta \geq n \cdot \beta_x^2 \}$. Thus Payoff Condition 3 holds. Notice that the expression inside the brackets is a continuous, strictly decreasing function of $n$. Simple algebra then yields

$$
N_x = \min\left\{ n : \max\{m_{0x}, m_{1x}\} \left[ \left( \sqrt{1 + 1/n - 1} \right) \left( 1 + 1/\sqrt{2\pi e} \right) + 1/\left( \sqrt{n}\pi \right) \right] \leq c_x \right\} \\
= \left[ \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \right].
$$

**EC.12.2.2. Special Condition 1** We write

$$
H_x(\mu, \beta) = m_{0x} \cdot 1_{\{p_x(\mu, \beta) \leq m_{0x} \}} + m_{1x} \cdot 1_{\{p_x(\mu, \beta) \geq m_{0x} + m_{1x} \}} \\
- \max\{m_{0x} [1 - p_x(\mu, \beta)], m_{1x} \cdot p_x(\mu, \beta)\}.
$$

(EC.17)

For any fixed $\beta$, $p_x(\mu, \beta) = 1 - \Phi(\sqrt{\beta}(d_x - \mu))$ converges to 1 as $\mu \to +\infty$, and converges to 0 as $\mu \to -\infty$. In both cases, we have $H_x(\mu, \beta) \to 0$ and thus Special Condition 1 holds.
EC.12.2.3. Special Condition 2  Fix $\beta > 0$. Let $u_0 := \Phi^{-1} \left( \frac{m_{1x}}{m_{0x} + m_{1x}} \right)$. Then

$$p_x(\mu, \beta) \leq \frac{m_{0x}}{m_{0x} + m_{1x}} \iff \sqrt{\beta}(d_x - \mu) \geq u_0.$$

Case I  $c_x < 2 \max\{m_{0x}, m_{1x}\}$.

We know that for any $\alpha \in (0, 1)$, $I_\alpha = [-z_\alpha, z_\alpha] := [-\Phi^{-1}(1 - \alpha/2), \Phi^{-1}(1 - \alpha/2)]$ is the 100(1 - $\alpha$)% confidence interval for the standard normal distribution. By Proposition 3 and (EC.17),

$$\begin{align*}
E[V_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x)] &\leq E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x)] \\
&= (1 - \alpha)E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x) \mid Z \in I_\alpha] + \alpha E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x) \mid Z \in I_\alpha^c] \\
&\leq (1 - \alpha)E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x) \mid Z \in I_\alpha] + \alpha \cdot \max\{m_{0x}, m_{1x}\}. \\
\end{align*}$$

(EC.18)

Pick $\alpha = c_x / [2 \max\{m_{0x}, m_{1x}\}]$.

When $\mu \leq d_x - \tilde{\sigma}_x(\beta)z_\alpha - u_0 / \sqrt{\beta + \beta'_x}$, we have $\sqrt{\beta + \beta'_x}(d_x - \mu - \tilde{\sigma}_x(\beta)Z) \geq u_0$ for all $Z \in I_\alpha$, hence

$$\begin{align*}
E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x) \mid Z \in I_\alpha] \\
&= m_{0x} E \left[ 1 - \Phi \left( \sqrt{\beta + \beta'_x}(d_x - \mu - \tilde{\sigma}_x(\beta)Z) \right) \mid Z \in I_\alpha \right] \\
&\leq m_{0x} \left[ 1 - \Phi \left( \sqrt{\beta + \beta'_x}(d_x - \mu - \tilde{\sigma}_x(\beta)z_\alpha) \right) \right],
\end{align*}$$

which goes to 0 as $\mu \to -\infty$.

Moreover, Special Condition 1 indicates that $R_x(\mu, \beta) \to -c_x$ as $\mu \to -\infty$.

(15) and (EC.18) then yields

$$\limsup_{\mu \to +\infty} L_x(\mu, \beta, V_x) \leq \lim_{\mu \to +\infty} R_x(\mu, \beta) + \limsup_{\mu \to +\infty} E[V_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x)] \leq -c_x + c_x / 2 < 0.$$

We can therefore find some $\mu_x(\beta) \leq d_x - \tilde{\sigma}_x(\beta)z_\alpha - u_0 / \sqrt{\beta + \beta'_x}$ such that for all $\mu \leq \mu_x(\beta)$, $L_x(\mu, \beta, V_x) < 0$ and hence $V_x(\mu, \beta) = 0$.

Similarly, we can find some $\overline{\mu}_x(\beta) \geq d_x + \tilde{\sigma}_x(\beta)z_\alpha$ such that $V_x(\mu, \beta) = 0$ for all $\mu \geq \overline{\mu}_x(\beta)$.

Case II  $c_x \geq 2 \max\{m_{0x}, m_{1x}\}$. Then

$$E[V(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x)] \leq E[H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta'_x)] \leq \max\{m_{0x}, m_{1x}\} \leq c_x / 2.$$

It follows that

$$\limsup_{\mu \to +\infty} L_x(\mu, \beta, V_x) \leq -c_x / 2 < 0.$$

Hence we can also find $\overline{\mu}_x(\beta)$ and $\mu_x(\beta)$ satisfying Special Condition 2.
EC.12.3. Linear Terminal payoff

EC.12.3.1. Special Condition 1 Applying Lemma EC.4,

\[
H_x(\mu, \beta) = \mathbb{E} \left[ m_{0x}(d_x - \theta_x)^+ + m_{1x}(\theta_x - d_x)^+ \right| \theta_x \sim \mathcal{N}(\mu, 1/\beta)] - \left[ m_{0x}(d_x - \mu)^+ + m_{1x}(\mu - d_x)^+ \right] = m_{0x} \cdot \Delta(d_x - \mu, \beta) + m_{1x} \cdot \Delta(\mu - d_x, \beta).
\]

(EC.19)

Hence for any fixed \( \beta > 0 \), \( H_x(\mu, \beta) \to 0 \) as \( \mu \to +\infty \) and \( \mu \to -\infty \).

EC.12.3.2. Payoff Condition 3 Notice that

\[
H_x(\cdot, \beta) \leq (m_{0x} + m_{1x})/\sqrt{2\pi \beta} = \tilde{H}_x(\cdot, \beta),
\]

hence the first inequality in Payoff Condition 3 holds. Now in the sub-problem with single alternative \( x, \beta_{nx} = \beta_{0x} + n \cdot \beta_x^* \), hence \( PS(x; n) = \{(\mu, \beta) \in \Lambda : \beta \geq n \cdot \beta_x^* \} \). It follows that

\[
\lim_{n \to \infty} \left[ \sup_{(\mu, \beta) \in PS(x; n)} \tilde{H}_x(\mu, \beta) \right] = \lim_{n \to \infty} \left[ \frac{m_{0x} + m_{1x}}{\sqrt{2\pi n \beta_x^*}} \right] = 0.
\]

Thus the second equation in Payoff Condition 3 holds and

\[
N_x = \min \left\{ n \geq 0 : \frac{m_{0x} + m_{1x}}{\sqrt{2\pi n \beta_x^*}} \leq c_x \right\} = \frac{\left( (m_{0x} + m_{1x})^2 \right)}{2\pi c_x^2 \beta_x^*}.
\]

EC.12.3.3. Special Condition 2 Fix \( \beta > 0 \). Let \( \rho = \mu - d_x \).

Case I \( c_x \sqrt{\beta + \beta_x^*}/(m_{0x} + m_{1x}) < 1 \).

We know that for any \( \alpha \in (0, 1) \), \( I_\alpha = [-z_\alpha, z_\alpha] := [-\Phi^{-1}(1 - \alpha/2), \Phi^{-1}(1 - \alpha/2)] \) is the 100(1 - \( \alpha \))% confidence interval for the standard normal distribution. By Proposition 3 and (EC.19), (EC.20),

\[
\mathbb{E} \left[ V_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right] \leq \mathbb{E} \left[ H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right] = (1 - \alpha)\mathbb{E} \left[ H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right| Z \in I_\alpha] + \alpha\mathbb{E} \left[ H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right| Z \in I'_\alpha]
\]

\[
\leq (1 - \alpha)\mathbb{E} \left[ H_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right| Z \in I_\alpha] + \alpha \tilde{H}_x(\cdot, \beta + \beta_x^*)
\]

\[
= (1 - \alpha)\mathbb{E} \left[ m_{0x} \Delta(\rho - \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) + m_{1x} \Delta(\rho + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right| Z \in I_\alpha] + \alpha \tilde{H}_x(\cdot, \beta + \beta_x^*).
\]

(EC.22)

where \( \tilde{H}_x(\cdot, \beta) = (m_{0x} + m_{1x})/\sqrt{2\pi \beta} \).

Pick \( \alpha = c_x \sqrt{\beta + \beta_x^*}/(m_{0x} + m_{1x}) \). Then \( \alpha \tilde{H}_x(\cdot, \beta + \beta_x^*) = c_x/\sqrt{2\pi} \).

When \( \rho \geq \tilde{\sigma}_x(\beta)z_\alpha \), i.e., \( \mu \geq d_x + \tilde{\sigma}_x(\beta)z_\alpha \), since \( \Delta(\cdot, \beta + \beta_x^*) \) is increasing at \( (-\infty, 0] \) and decreasing at \( [0, +\infty) \), we know

\[
\mathbb{E} \left[ m_{0x} \Delta(\rho - \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) + m_{1x} \Delta(\rho + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^*) \right| Z \in I_\alpha]
\]

\[
\leq m_{0x} \Delta(\rho - \tilde{\sigma}_x(\beta)z_\alpha, \beta + \beta_x^*) + m_{1x} \Delta(\rho - \tilde{\sigma}_x(\beta)z_\alpha, \beta + \beta_x^*),
\]
which goes to 0 as $\rho \to +\infty$, i.e., $\mu \to +\infty$.

Moreover, (EC.12) shows that $R_x(\mu, \beta) \to -c_x$ as $\rho \to +\infty$, i.e., $\mu \to +\infty$.

(15) and (EC.22) then yield

$$\limsup_{\mu \to +\infty} L_x(\mu, \beta, V_x) \leq \lim_{\mu \to +\infty} R_x(\mu, \beta) + \limsup_{\mu \to +\infty} \mathbb{E}[V_x(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta^*_x)] \leq -c_x + c_x/\sqrt{2\pi} < 0.$$

It follows that we can find $\bar{\mu}_x(\beta) \geq d_x + \tilde{\sigma}_x(\beta)z_\alpha$ such that for all $\mu \geq \bar{\mu}_x(\beta)$, $L_x(\mu, \beta, V_x) < 0$ and hence $V_x(\mu, \beta) = 0$.

Similarly, we can find some $\underline{\mu}_x(\beta) \leq d_x - \tilde{\sigma}_x(\beta)z_\alpha$ such that $V_x(\mu, \beta) = 0$ for all $\mu \leq \underline{\mu}_x(\beta)$.

**Case II** $c_x\sqrt{\beta + \beta^*_x}/(m_{0x} + m_{1x}) \geq 1$. Then

$$\mathbb{E}[V(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta^*_x)] \leq \tilde{H}_x(\cdot, \beta + \beta^*_x) = \frac{m_{0x} + m_{1x}}{\sqrt{2\pi(\beta + \beta^*_x)}} \leq c_x/\sqrt{2\pi}.$$

It follows that

$$\limsup_{\mu \to +\infty} L_x(\mu, \beta, V_x) \leq -c_x + c_x/\sqrt{2\pi} < 0.$$

Hence we can also find $\bar{\mu}_x(\beta)$ and $\underline{\mu}_x(\beta)$ satisfying Special Condition 2.

**References**

*See references list in the main paper.*

