Mathematical Proofs.

EC.1. Proof of Table 1.

Table EC.1 Example Terminal Payoff Functions and Their Properties

	0-1 Terminal Payoff	Linear Terminal Payoff
$h_{0x}(s)$	$m_{0x}\left[1-p_x(s)\right]$	$m_{0x}\left[d_x-\mu(s)\right]$
$h_{1x}(s)$	$m_{1x}\cdot p_x(s)$	$m_{1x}\left[\mu(s)-d_x\right]$
B_n	$\left\{x: p_x\left(S_{n,x}\right)\geq \frac{m_{0x}}{m_{0x}+m_{1x}}\right\}$	$\{x:\mu_{nx}\geq d_x\}$
Payoff Condition 1	Yes	Yes
Payoff Condition 2	Yes	Yes
$H_x(s)$	$m_{0x}\cdot\mathbf{1}_{\{s\notin B(x)\}}+m_{1x}\cdot\mathbf{1}_{\{s\in B(x)\}}-h_x(s)$	$A_0(s)-h_x(s)$

where
$$p_x(s):=\mathbb{P}\left\{\theta_x\geq d_x\mid\eta_x\sim\mathcal{D}(s)\right\},\quad \mu(s):=\mathbb{E}\left[\theta_x\mid\eta_x\sim\mathcal{D}(s)\right],\quad \mu_{nx}:=\mathbb{E}_n\left[\theta_x\right]=\mu\left(S_{n,x}\right),$$
$$B(x):=\left\{s: p_x(s)\geq\frac{m_{0x}}{m_{0x}+m_{1x}}\right\},\quad A_0(s):=\mathbb{E}\left[m_{0x}(\theta_x-d_x)^-+m_{1x}(\theta_x-d_x)^+\,\middle|\,\eta_x\sim\mathcal{D}(s)\right].$$

EC.1.1. 0-1 Terminal Payoff

$h_{0x}(s)=\mathbb{E}\left[r_0(x;\theta_x)\mid\eta_x\sim\mathcal{D}(s)\right]=m_{0x}\cdot\mathbb{P}\{x\notin\mathbb{B}\mid\eta_x\sim\mathcal{D}(s)\}=m_{0x}\left[1-p_x(s)\right]$. Similarly $h_{1x}(s)=m_{1x}p_x(s)$. B_n follows directly from definition (1).

Payoff Condition 1 holds because function g defined by $g(t)\mapsto\max\{m_{0x}(1-t),m_{1x}\}$ is a convex function and that $\mathbb{E}_0\left[p_{x_1}\left(S_{1,x_1}\right)\right]=\mathbb{E}_0\left[\mathbb{E}_1\left[\mathbf{1}_{\{x_1\in\mathbb{B}\}}\right]\right]=\mathbb{E}_0\left[\mathbf{1}_{\{x_1\in\mathbb{B}\}}\right]=p_{x_1}\left(S_{0,x_1}\right)$. Hence by Jensen's inequality, we have $h_{x_1}(S_{0,x_1})=g\left(p_{x_1}\left(S_{0,x_1}\right)\right)=g\left(\mathbb{E}_0\left[p_{x_1}\left(S_{1,x_1}\right)\right]\right)\leq\mathbb{E}_0\left[g\left(p_{x_1}\left(S_{1,x_1}\right)\right)\right]=\mathbb{E}_0\left[h_{x_1}\left(S_{1,x_1}\right)\right]$.

Since for any x and $s\in\Lambda$,

$$h_x(s)=\max\{h_{0x}(s),h_{1x}(s)\}=h_{0x}(s)\cdot\mathbf{1}_{\{s\notin B(x)\}}+h_{1x}(s)\cdot\mathbf{1}_{\{s\in B(x)\}}\leq m_{0x}\cdot\mathbf{1}_{\{s\notin B(x)\}}+m_{1x}\cdot\mathbf{1}_{\{s\in B(x)\}},$$

hence for any n,

$$\mathbb{E}\left[h_x(S_{n,x})\mid S_{0,x}=s,x_1=\cdots=x_n=x\right]-h_x(s)\leq m_{0x}\cdot\mathbf{1}_{\{s\notin B(x)\}}+m_{1x}\cdot\mathbf{1}_{\{s\in B(x)\}}-h_x(s).$$

Thus Payoff Condition 2 holds with H_x specified in the table.

EC.1.2. Linear Terminal Payoff

$h_{0x}(s) = \mathbb{E}\left[r_0(x;\theta_x) \mid \eta_x \sim \mathcal{D}(s)\right] = m_{0x}\left[d_x - \mu_x(s)\right]$. Similarly $h_{1x}(s) = m_{1x}\left[\mu_x(s) - d_x\right]$. B_n follows directly from definition (1). Hence $h_x(s) = m_{0x}\left[\mu(s) - d_x\right]^- + m_{1x}\left[\mu(s) - d_x\right]^+$.

Payoff Condition 1 holds because $\mathbb{E}_0\left[\mu_{1x_1}\right] = \mathbb{E}_0\left[\mathbb{E}_1\left(\theta_{x_1}\right)\right] = \mathbb{E}_0\left[\theta_{x_1}\right] = \mu_{0x_1}$. Hence by Jensen's inequality $(\mathbb{E}X)^+ \le \mathbb{E}(X^+)$ and $(\mathbb{E}X)^- \le \mathbb{E}(X^-)$, we have

$$\begin{aligned} h_{x_1}(S_{0,x_1}) &= m_{0x}\left(\mu_{0,x_1} - d_x\right)^- + m_{1x}\left(\mu_{0,x_1} - d_x\right)^+ \\ &\le \mathbb{E}_0\left[m_{0x}\left(\mu_{1,x_1} - d_x\right)^- + m_{1x}\left(\mu_{1,x_1} - d_x\right)^+\right] = \mathbb{E}_0\left[h_{x_1}\left(S_{1,x_1}\right)\right]. \end{aligned}$$

Now for any n and x, by Jensen's inequality,

$$h_x\left(S_{n,x}\right) = m_{0x}\left(\mu_{nx} - d_x\right)^- + m_{1x}\left(\mu_{nx} - d_x\right)^+ \le \mathbb{E}_n\left[m_{0x}\left(\theta_x - d_x\right)^- + m_{1x}\left(\theta_x - d_x\right)^+\right].$$

Hence

$$\begin{aligned} &\mathbb{E}\left[h_x\left(S_{n,x}\right) \mid S_{0,x} = s, x_1 = \cdots = x_n = x\right] \\ &\le \mathbb{E}\left\{\mathbb{E}_n\left[m_{0x}\left(\theta_x - d_x\right)^- + m_{1x}\left(\theta_x - d_x\right)^+\right] \,\middle|\, S_{0,x} = s, x_1 = \cdots = x_n = x\right\} \\ &= \mathbb{E}\left[m_{0x}\left(\theta_x - d_x\right)^- + m_{1x}\left(\theta_x - d_x\right)^+ \,\middle|\, \eta_x \sim \mathcal{D}(s)\right]. \end{aligned}$$

Thus Payoff Condition 2 holds with H_x specified in the table.

EC.2. Proof of Proposition 1.

Proposition 1.

$$V(\mathbf{s}) = \sup_{\pi} \mathbb{E}^{\pi}\left[\sum_{n=1}^{\tau} \alpha^{n-1} \mathcal{R}_{x_n}(S_{n-1,x_n}) \,\middle|\, \mathbf{S}_0 = \mathbf{s}\right], \tag{EC.1}$$

where the reward functions $\mathcal{R}_x : \Lambda \mapsto \mathbb{R}$ for $x = 1, \ldots, k$ are defined by

$$\mathcal{R}_x(s) = \mathbb{E}\left[h_x(S_{1,x}) \mid S_{0,x} = s, x_1 = x\right] - h_x(s) - c_x. \tag{EC.2}$$

Proof of Proposition 1. By the tower property of conditional expectation,

$$\mathbb{E}^{\pi}\left[r\left(B_{\tau\wedge T};\theta,d\right)\right] = \mathbb{E}^{\pi}\left[\mathbb{E}^{\pi}_{\tau\wedge T}\left[r\left(B_{\tau\wedge T};\theta,d\right)\right]\right] = \mathbb{E}^{\pi}\left[\sum_{x=1}^{k} h_x\left(S_{\tau\wedge T,x}\right)\right].$$

We can therefore write (5) as follows,

$$V(\mathbf{s}) = \sup_{\pi} \mathbb{E}^{\pi}\left[\sum_{x=1}^{k} h_x\left(S_{\tau\wedge T,x}\right) - \sum_{n=1}^{\tau\wedge T} c_{x_n} \,\middle|\, \mathbf{S}_0 = \mathbf{s}\right].$$

We restructure this into a sequence of single period rewards. For a fixed policy π and hence a stopping rule τ, since T is geometrically distributed with parameter $1-\alpha$ and independent of the sampling filtration,

$$\begin{aligned}
&\mathbb{E}^{\pi}\left[\sum_{x=1}^{k} h_x\left(S_{\tau\wedge T,x}\right)-\sum_{n=1}^{\tau\wedge T} c_{x_n}\right]\\
&=\mathbb{E}^{\pi}\left[\sum_{t=1}^{\tau}\left[(1-\alpha)\alpha^{t-1}\left(\sum_{x=1}^{k} h_x\left(S_{t,x}\right)-\sum_{n=1}^{t} c_{x_n}\right)\right]+\alpha^{\tau}\left(\sum_{x=1}^{k} h_x\left(S_{\tau,x}\right)-\sum_{n=1}^{\tau} c_{x_n}\right)\right]\\
&=\mathbb{E}^{\pi}\left[\sum_{x=1}^{k}\left[\sum_{t=1}^{\tau}[(1-\alpha)\alpha^{t-1}h_x\left(S_{t,x}\right)]+\alpha^{\tau}h_x\left(S_{\tau,x}\right)\right]-\sum_{t=1}^{\tau}\left[(1-\alpha)\alpha^{t-1}\sum_{n=1}^{t} c_{x_n}\right]-\alpha^{\tau}\sum_{n=1}^{\tau} c_{x_n}\right]\\
&=\mathbb{E}^{\pi}\left[\sum_{x=1}^{k}\left[h_x\left(S_{0,x}\right)+\sum_{t=1}^{\tau}\alpha^{t-1}\left[h_x\left(S_{t,x}\right)-h_x\left(S_{t-1,x}\right)\right]\right]-(1-\alpha)\sum_{n=1}^{\tau}\left[c_{x_n}\sum_{t=n}^{\tau}\alpha^{t-1}\right]-\alpha^{\tau}\sum_{n=1}^{\tau} c_{x_n}\right]\\
&=\mathbb{E}^{\pi}\left[\sum_{x=1}^{k} h_x\left(S_{0,x}\right)+\sum_{t=1}^{\tau}\alpha^{t-1}\left[h_{x_t}\left(S_{t,x_t}\right)-h_{x_t}\left(S_{t-1,x_t}\right)\right]-\sum_{t=1}^{\tau}\alpha^{t-1}c_{x_t}\right]\\
&=\mathbb{E}^{\pi}\left[\sum_{x=1}^{k} h_x\left(S_{0,x}\right)+\sum_{t=1}^{\tau}\alpha^{t-1}\left[-c_{x_t}+h_{x_t}\left(S_{t,x_t}\right)-h_{x_t}\left(S_{t-1,x_t}\right)\right]\right].
\end{aligned}$$

The second to last equation follows from simple computation and the fact that at each time $n=1,2,\ldots,\tau$, for all non-selected alternatives $x\neq x_n$, we have $S_{n,x}=S_{n-1,x}$.

Define for all $n\geq 1$

$$R_n=-c_{x_n}+h_{x_n}\left(S_{n,x_n}\right)-h_{x_n}\left(S_{n-1,x_n}\right). \tag{EC.3}$$

We then have

$$V(\mathbf{s})=R_0(\mathbf{s})+\sup_{\pi}\mathbb{E}^{\pi}\left[\sum_{n=1}^{\tau}\alpha^{n-1}R_n\,\middle|\,\mathbf{S}_0=\mathbf{s}\right]. \tag{EC.4}$$

We see from (EC.4) that a problem with stopping rule τ that provides a fixed initial reward $R_0(\mathbf{s})$ and a discounted single period reward $\alpha^{n-1}R_n$ at each time $n\geq 1$ is equivalent to the original problem. Since R_0 does not affect the optimal policy, we may subtract it from the value function and instead think of V as the optimal expected incremental reward over R_0. This provides the equivalent problem,

$$V(\mathbf{s})=\sup_{\pi}\mathbb{E}^{\pi}\left[\sum_{n=1}^{\tau}\alpha^{n-1}R_n\,\middle|\,\mathbf{S}_0=\mathbf{s}\right], \tag{EC.5}$$

where we have redefined V to correspond to this equivalent problem in a slight abuse of notation. A policy π that attains the supremum in (EC.5) also attains the supremum in (EC.4) and (5).

For later work, it is convenient to make one additional transformation in which we replace the random variables R_n in (EC.5) with deterministic reward functions of the alternatives' states. From (EC.5) and the tower property, we know

$$V(\mathbf{s})=\sup_{\pi}\mathbb{E}^{\pi}\left[\sum_{n=1}^{\tau}\alpha^{n-1}\mathbb{E}^{\pi}[R_n|\mathbf{S}_{n-1}]\,\middle|\,\mathbf{S}_0=\mathbf{s}\right].$$

Now $\mathcal{R}_x(s) = \mathbb{E}[R_1 \mid S_{0,x} = s, x_1 = x]$ by (EC.2). Hence

$$\mathbb{E}^\tau[R_n|\mathbf{S}_{n-1}] = \mathcal{R}_{x_n}(S_{n-1,x_n}), \tag{EC.6}$$

and then (EC.1) follows.

EC.3. Proof of Proposition 2.

PROPOSITION 2. *Under Payoff Condition 1, $\nu_x \geq -c_x$. Under Payoff Conditions 1 and 2, $-c_x \leq \nu_x \leq -c_x + H_x$.*

Proof of Proposition 2. Under Payoff Condition 1, $\nu_x \geq -c_x$ follows directly from $\mathcal{R}_x \geq -c_x$.

Now under both conditions, we take the following expectations with respect to the sub-problem with a single alternative x. It then follows from (EC.3) and (EC.6) that

$$\mathcal{R}_x(S_{n,x}) = \mathbb{E}[R_{n+1}|S_n]. \tag{EC.7}$$

Since $\mathcal{R}_x + c_x \geq 0$ and $0 < \alpha < 1$, we write

$$\begin{aligned}
\nu_x(s) + c_x &= \max_{\tau>0} \mathbb{E}\left[\frac{\sum_{n=1}^\tau \alpha^{n-1}[\mathcal{R}_x(S_{n-1,x}) + c_x]}{\sum_{n=1}^\tau \alpha^{n-1}} \middle| S_{0,x} = s\right] \\
&\leq \max_{\tau>0} \mathbb{E}\left[\sum_{n=1}^\tau [\mathcal{R}_x(S_{n-1,x}) + c_x] \,\middle|\, S_{0,x} = s\right] = \mathbb{E}\left[\sum_{n=0}^\infty [\mathcal{R}_x(S_{n,x}) + c_x] \,\middle|\, S_{0,x} = s\right] \\
&= \sum_{n=0}^\infty \mathbb{E}\left[\mathcal{R}_x(S_{n,x}) + c_x \mid S_{0,x} = s\right] \\
&= \sum_{n=1}^\infty \mathbb{E}\left[R_n + c_x \mid S_{0,x} = s\right] \\
&= \sum_{n=1}^\infty \mathbb{E}\left[h_x(S_{n,x}) - h_x(S_{n-1,x}) \mid S_{0,x} = s\right] \\
&= \sum_{n=1}^\infty \left\{\mathbb{E}\left[h_x(S_{n,x}) \mid S_{0,x} = s\right] - \mathbb{E}\left[h_x(S_{n-1,x}) \mid S_{0,x} = s\right]\right\} \\
&= \mathbb{E}\left[h_x(S_{1,x}) \mid S_{0,x} = s\right] - h_x(s) + \mathbb{E}\left[h_x(S_{2,x}) \mid S_{0,x} = s\right] - \mathbb{E}\left[h_x(S_{1,x}) \mid S_{0,x} = s\right] + \cdots \\
&= \lim_{n\to\infty} \mathbb{E}\left[h_x(S_{n,x}) \mid S_{0,x} = s\right] - h_x(s) \leq H_x(s),
\end{aligned}$$

Here, the third line uses the Monotone Convergence Theorem; the fourth line uses (EC.7) and the tower property; and the fifth line uses (EC.3). The limit in the last equality exists since $\{h_x(S_{n,x})\}_{n\geq 0}$ is a sub-martingale by Payoff Condition 1 and hence $\left\{\mathbb{E}\left[h_x(S_{n,x}) \mid S_{0,x} = s\right]\right\}_n$ is an increasing sequence; and the last inequality follows from Payoff Condition 2.

EC.4. Proof of Proposition 3.

PROPOSITION 3. *$V_x \geq 0$. Under Payoff Condition 2, $V_x \leq H_x$.*

Proof of Proposition 3. We receive a zero reward if we take $\tau_x = 0$. Thus $V_x \geq 0$. By (7), we can write (9) as $V_x(s) = \sup_{\tau_x} \{-c_x\tau_x + \mathbb{E}[h_x(S_{\tau_x,x}) \mid S_{0,x} = s, x_1 = \cdots = x_{\tau_x} = x] - h_x(s)\}$. Since $\tau_x \geq 0$, we know that $V_x(s) \leq H_x(s)$ under Payoff Condition 2.

EC.5. Proof of Proposition 4

PROPOSITION 4 *Under Payoff Condition 3, τ_x^* has a deterministic upper bound N_x, where*

$$N_x := \min\left\{n \geq 0\colon \left[\sup_{s \in PS(x;n')} \widetilde{H}_x(s)\right] \leq c_x,\ \forall n' \geq n\right\}. \tag{EC.8}$$

Proof of Proposition 4 Since $c_x > 0$, we know that N_x exists under Payoff Condition 3. Applying Payoff Condition 3 to (7) gives $\mathcal{R}_x \leq -c_x + \widetilde{H}_x$. Hence for all $n \geq N_x$ and $s \in PS(x;n)$, $\mathcal{R}_x(s) \leq 0$. Now in the sub-problem with single alternative x and initial state $S_{0,x} = s \in PS(x;N_x)$, we know that $S_{t,x} \in PS(x;N_x+t)$ for all $t \geq 0$, and hence $\mathcal{R}_x(S_{t,x}) \leq 0$ for all $t \geq 0$. It follows that

$$V_x(s) = \sup_{\tau_x} \mathbb{E}\left[\sum_{t=0}^{\tau_x - 1} \mathcal{R}_x(S_{t,x}) \,\middle|\, S_{0,x} = s\right] = 0, \quad \forall s \in PS(x;N_x)$$

Thus $\tau_x^* \leq N_x$.

EC.6. Proof of Theorem 1.

THEOREM 1. *The value function is given by, $V(\mathbf{s}) = \sum_{x=1}^k V_x(s_x)$. Furthermore, any policy with sampling decisions $(x_n^*)_{n\geq 1}$ and stopping time τ^* satisfying the following conditions is optimal:*

$$x_{n+1}^* \in \{x : S_{n,x} \in \mathbb{C}_x\},\ \forall n \geq 0; \qquad \tau^* = \inf\{n \geq 0 : S_{n,x} \notin \mathbb{C}_x, \forall x\}. \tag{EC.9}$$

Proof of Theorem 1. For any arbitrary policy π with stopping time τ, we denote the number of times we sample from each alternative x by m_x. Then $\tau = \sum_{x=1}^k m_x$. Denote the collection of times when we sample from x, $\{1 \leq n \leq \tau : x_n = x\}$, by $\{n_i^x\}_{1\leq i\leq m_x}$.

Since the reward for each period only depends on the alternative being sampled during that period, and the states of all the other alternatives remain frozen, we know that the order of the sequence of sampling decisions does not affect the expected total reward. Hence the original problem can be naturally decomposed into k sub-problems as follows.

$$\begin{aligned}
&\mathbb{E}^\pi\left[\sum_{n=1}^{\tau} \mathcal{R}_{x_n}(S_{n-1,x_n}) \,\middle|\, \mathbf{S}_0 = \mathbf{s}\right] = \sum_{x=1}^k \left\{\mathbb{E}^\pi\left[\sum_{i=1}^{m_x} \mathcal{R}_x(S_{n_i^x-1,x}) \,\middle|\, S_{0,x} = s_x\right]\right\} \\
&= \sum_{x=1}^k \left\{\mathbb{E}^\pi\left[\sum_{n=1}^{m_x} \mathcal{R}_x(S_{n-1,x}) \,\middle|\, S_{0,x} = s_x, x_1 = \cdots = x_{m_x} = x\right]\right\} \leq \sum_{x=1}^k V_x(s_x),
\end{aligned} \tag{EC.10}$$

where the last inequality follows from (9). Thus

$$V(\mathbf{s}) = \sup_{\pi} \mathbb{E}^{\pi}\left[\sum_{n=1}^{\tau} \mathcal{R}_{x_n}(S_{n-1,x_n}) \,\middle|\, \mathbf{S}_0 = \mathbf{s}\right] \le \sum_{x=1}^{k} V_x(s_x).$$

On the other hand, if we adopt a policy satisfying $x_{n+1} \in \{x : S_{n,x} \in \mathbb{C}_x\}$ for all $n \ge 0$ and $\tau = \inf\{n \ge 0 : S_{n,x} \notin \mathbb{C}_x, \forall x\}$, then for each x, m_x is exactly the number of samples from alternative x needed for the state of x to leave $\mathbb{C}_x$ for the first time. Hence in each decomposed sub-problem with single alternative x, m_x is an optimal solution equivalent to τ_x^*. As a result,

$$\mathbb{E}^{\pi}\left[\sum_{n=1}^{m_x} \mathcal{R}_x(S_{n-1,x}) \,\middle|\, S_{0,x} = s_x, x_1 = \cdots = x_{m_x} = x\right] = V_x(s_x),$$

the inequality in (EC.10) becomes equality, and $V(\mathbf{s}) = \sum_{x=1}^{k} V_x(s_x)$. This also shows that any policy satisfying the conditions (EC.9) is optimal.

EC.7. Proof of Proposition 5.

Proposition 5. *Suppose that each τ_x^* has a deterministic upper bound N_x. Then the optimal stopping rule τ^*, as characterized in Theorem 1, has a deterministic upper bound $\sum_{x=1}^{k} N_x$.*

Proof of Proposition 5. We apply ideas similar to those in the proof of Theorem 1. First, $\tau^* = \sum_{x=1}^{k} m_x$. Under the optimal policy, since m_x is the number of samples from alternative x needed for the state of x to leave $\mathbb{C}_x$ for the first time, its distribution is the same as the distribution of τ_x in the decomposed sub-problem with single alternative x. It follows that $m_x \le N_x$, for each x. Thus $\tau^* \le \sum_{x=1}^{k} N_x$.

EC.8. Remark 1.

Consider the distribution of $\mathbf{S}_1$ given $\mathbf{S}_0 = (\mathbf{a}, \mathbf{b})$ and $x_1 = x$. Since $\mathbb{P}_0\{y_1 = 1\} = \mathbb{E}_0[y_1] = \mathbb{E}_0\left[\mathbb{E}_0\left[y_1 \mid \theta\right]\right] = \mathbb{E}_0\left[\theta_{x_1}\right] = \mu_{0x_1}$, we immediately have the following expressions:

$$\mathbb{P}\{\mathbf{S}_1 = (\mathbf{a} + \mathbf{e}_x, \mathbf{b}) \mid \mathbf{S}_0 = (\mathbf{a}, \mathbf{b}), x_1 = x\} = \mathbb{P}\{y_1 = 1 \mid \mathbf{S}_0 = (\mathbf{a}, \mathbf{b}), x_1 = x\} = a_x/(a_x + b_x),$$

$$\mathbb{P}\{\mathbf{S}_1 = (\mathbf{a}, \mathbf{b} + \mathbf{e}_x) \mid \mathbf{S}_0 = (\mathbf{a}, \mathbf{b}), x_1 = x\} = \mathbb{P}\{y_1 = 0 \mid \mathbf{S}_0 = (\mathbf{a}, \mathbf{b}), x_1 = x\} = b_x/(a_x + b_x).$$

EC.9. Proof of Table 2.

EC.9.1. Preparatory Material

Use Stirling's approximation, for large a and b,

$$B(a,b) \sim \sqrt{2\pi}\frac{a^{a-\frac{1}{2}} b^{b-\frac{1}{2}}}{(a+b)^{a+b-\frac{1}{2}}}.$$

More generally, we have the following lemma.

Table EC.2 Example Terminal Payoff Functions with Bernoulli Sampling

	0-1 Terminal Payoff	Linear Terminal Payoff
Payoff Condition 3	Yes	Yes
$\widetilde{H}_x(a,b)$	$\frac{m_{0x}+m_{1x}}{2\sqrt{2\pi(a+b)}}$	$\frac{\max\{m_{0x},m_{1x}\}+m_{0x}}{4(a+b+1)}$
N_x	$\left(\lceil\frac{(m_{0x}+m_{1x})^2}{8\pi c_x^2}\rceil-2\right)^+$	$\left(\lceil\frac{\max\{m_{0x},m_{1x}\}+m_{0x}}{4c_x}\rceil-3\right)^+$

Lemma EC.1. *For $a,b\geq 1$,*

$$B(a,b)\geq\sqrt{2\pi}\frac{a^{a-\frac{1}{2}}b^{b-\frac{1}{2}}}{(a+b)^{a+b-\frac{1}{2}}}.$$

Proof of Lemma EC.1. By Stirling's asymptotic series (see, e.g., Abramowitz and Stegun (1964) and Sloane (2007)), we write

$$\Gamma(z)=e^{-z}z^{z-\frac{1}{2}}\sqrt{2\pi}e^{\lambda_z},\quad\text{with }\frac{1}{12z+1}<\lambda_z<\frac{1}{12z}.$$

Hence

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\sqrt{2\pi}\exp\{\lambda_a+\lambda_b-\lambda_{a+b}\}\frac{a^{a-\frac{1}{2}}b^{b-\frac{1}{2}}}{(a+b)^{a+b-\frac{1}{2}}}.$$

The result then follows since for $a,b\geq 1$,

$$\lambda_a+\lambda_b-\lambda_{a+b}>\frac{1}{12a+1}+\frac{1}{12b+1}-\frac{1}{12(a+b)}=\frac{(12a+\frac{1}{2})^2+(12b+\frac{1}{2})^2+144ab-\frac{3}{2}}{12(a+b)(12a+1)(12b+1)}>0.$$

Lemma EC.2. *For $a,b\geq 1$ and $d_x\in(0,1)$,*

$$\frac{d_x^a(1-d_x)^b}{B(a,b)}\leq\frac{1}{2}\sqrt{\frac{a+b}{2\pi}}.$$

Proof of Lemma EC.2. Denote $t=a+b$ and $\mu=\frac{a}{a+b}$. Then by Lemma EC.1,

$$\frac{d_x^a(1-d_x)^b}{B(a,b)}\leq\frac{d_x^{\mu t}(1-d_x)^{(1-\mu)t}t^{t-\frac{1}{2}}}{\sqrt{2\pi}(\mu t)^{\mu t-\frac{1}{2}}[(1-\mu)t]^{(1-\mu)t-\frac{1}{2}}}=\sqrt{\frac{\mu(1-\mu)t}{2\pi}}\left[\left(\frac{d_x}{\mu}\right)^{\mu}\left(\frac{1-d_x}{1-\mu}\right)^{1-\mu}\right]^t.$$

Define function $g(\cdot)$ on $(0,1)$ by

$$g(u)=\left(\frac{d_x}{u}\right)^u\left(\frac{1-d_x}{1-u}\right)^{1-u}.$$

Then since $\frac{d}{du}[\log g(u)]=\log\frac{d_x(1-u)}{u(1-d_x)}$, it is easy to see that $g(\cdot)$ is unimodal on $(0,1)$ with peak at $u=d_x$. Finally, since $\mu\in(0,1)$, we know $\sqrt{\mu(1-\mu)}\leq\frac{1}{2}$, and hence

$$\frac{d_x^a(1-d_x)^b}{B(a,b)}\leq\frac{1}{2}\sqrt{\frac{t}{2\pi}}[g(\mu)]^t\leq\frac{1}{2}\sqrt{\frac{t}{2\pi}}[g(d_x)]^t=\frac{1}{2}\sqrt{\frac{t}{2\pi}}=\frac{1}{2}\sqrt{\frac{a+b}{2\pi}}.$$

EC.9.2. 0-1 Terminal Payoff

We first state the following property of the regularized incomplete beta function $I_d(\cdot,\cdot)$,

$$I_d(a+1,b) = I_d(a,b) - \frac{d^a(1-d)^b}{aB(a,b)}; \quad I_d(a,b+1) = I_d(a,b) + \frac{d^a(1-d)^b}{bB(a,b)}.$$

Hence by Remark 1, if

$$I_{d_x}(a,b) \geq \frac{m_{1x}}{m_{0x}+m_{1x}} + \frac{d_x^a(1-d_x)^b}{aB(a,b)},$$

then

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(S_{1,x}\right) \mid S_{0,x}=(a,b), x_1=x\right] - h_x(a,b) \\
&= \frac{a}{a+b} h_x(a+1,b) + \frac{b}{a+b} h_x(a,b+1) - h_x(a,b) \\
&= m_{0x}\left[\frac{a}{a+b} I_{d_x}(a+1,b) + \frac{b}{a+b} I_{d_x}(a,b+1) - I_{d_x}(a,b)\right] \\
&= m_{0x}\left[-\frac{a}{a+b}\cdot\frac{d_x^a(1-d_x)^b}{aB(a,b)} + \frac{b}{a+b}\cdot\frac{d_x^a(1-d_x)^b}{bB(a,b)}\right] = 0.
\end{aligned}$$

Similarly, if

$$I_{d_x}(a,b) \leq \frac{m_{1x}}{m_{0x}+m_{1x}} - \frac{d_x^a(1-d_x)^b}{bB(a,b)},$$

we have $\mathbb{E}\left[h_x\left(S_{1,x}\right) \mid S_{0,x}=(a,b), x_1=x\right] - h_x(a,b) = 0$.

Now, if

$$\frac{m_{1x}}{m_{0x}+m_{1x}} \leq I_{d_x}(a,b) < \frac{m_{1x}}{m_{0x}+m_{1x}} + \frac{d_x^a(1-d_x)^b}{aB(a,b)},$$

then

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(S_{1,x}\right) \mid S_{0,x}=(a,b), x_1=x\right] - h_x(a,b) \\
&= \frac{a}{a+b} m_{1x}[1 - I_{d_x}(a+1,b)] + m_{0x}\left[\frac{b}{a+b} I_{d_x}(a,b+1) - I_{d_x}(a,b)\right] \\
&= \frac{a}{a+b}\left[m_{1x} - (m_{0x}+m_{1x}) I_{d_x}(a,b)\right] + \frac{m_{0x}+m_{1x}}{a+b}\cdot\frac{d_x^a(1-d_x)^b}{B(a,b)} \\
&\leq \frac{m_{0x}+m_{1x}}{a+b}\cdot\frac{d_x^a(1-d_x)^b}{B(a,b)}.
\end{aligned}$$

Similarly, if

$$\frac{m_{1x}}{m_{0x}+m_{1x}} - \frac{d_x^a(1-d_x)^b}{bB(a,b)} < I_{d_x}(a,b) \leq \frac{m_{1x}}{m_{0x}+m_{1x}},$$

we still have

$$\mathbb{E}\left[h_x\left(S_{1,x}\right) \mid S_{0,x}=(a,b), x_1=x\right] - h_x(a,b) \leq \frac{m_{0x}+m_{1x}}{a+b}\cdot\frac{d_x^a(1-d_x)^b}{B(a,b)}. \tag{EC.11}$$

Thus (EC.11) holds for all $(a,b) \in \Lambda = [1,+\infty) \times [1,+\infty)$. Now applying Lemma EC.2, we know

$$\mathbb{E}\left[h_x\left(S_{1,x}\right) \mid S_{0,x}=(a,b), x_1=x\right] - h_x(a,b) \leq \frac{m_{0x}+m_{1x}}{2\sqrt{2\pi(a+b)}} := \widetilde{H}_x(a,b).$$

Since in the sub-problem with single alternative x, $a_{nx}+b_{nx}=a_{0x}+b_{0x}+n$, and hence $PS(x;n)=\{(a,b)\in\Lambda : a+b\geq 2+n\}$. It follows that

$$\lim_{n\to\infty}\left[\sup_{(a,b)\in PS(x;n)}\widetilde{H}_x(a,b)\right]=\lim_{n\to\infty}\left[\frac{m_{0x}+m_{1x}}{2\sqrt{2\pi(2+n)}}\right]=0.$$

Thus Payoff Condition 3 holds and

$$N_x=\min\left\{n\geq 0:\frac{m_{0x}+m_{1x}}{2\sqrt{2\pi(2+n)}}\leq c_x\right\}=\left(\lceil\frac{(m_{0x}+m_{1x})^2}{8\pi c_x^2}\rceil-2\right)^+.$$

EC.9.3. Linear Terminal Payoff

First suppose $a\geq b$, then $\frac{a}{a+b}\geq\frac{a+1}{a+b+1}$. Since for all $x,y\in\mathbb{R}, x^+-y^+\leq|x-y|$, we have

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(S_{1,x}\right)\mid S_{0,x}=(a,b),x_1=x\right]-h_x(a,b)\\
&=\frac{a}{a+b}h_x(a+1,b)+\frac{b}{a+b}h_x(a,b+1)-h_x(a,b)\\
&=\frac{a}{a+b}\left[m_{0x}\left(d_x-\frac{a+1}{a+b+1}\right)^+ + m_{1x}\left(\frac{a+1}{a+b+1}-d_x\right)^+\right]\\
&\quad+\frac{b}{a+b}\left[m_{0x}\left(d_x-\frac{a}{a+b+1}\right)^+ + m_{1x}\left(\frac{a}{a+b+1}-d_x\right)^+\right]\\
&\quad-\left[m_{0x}\left(d_x-\frac{a}{a+b}\right)^+ + m_{1x}\left(\frac{a}{a+b}-d_x\right)^+\right]\\
&=\frac{a}{a+b}\left\{m_{0x}\left[\left(d_x-\frac{a+1}{a+b+1}\right)^+-\left(d_x-\frac{a}{a+b}\right)^+\right]+m_{1x}\left[\left(\frac{a+1}{a+b+1}-d_x\right)^+-\left(\frac{a}{a+b}-d_x\right)^+\right]\right\}\\
&\quad+\frac{b}{a+b}\left\{m_{0x}\left[\left(d_x-\frac{a}{a+b+1}\right)^+-\left(d_x-\frac{a}{a+b}\right)^+\right]+m_{1x}\left[\left(\frac{a}{a+b+1}-d_x\right)^+-\left(\frac{a}{a+b}-d_x\right)^+\right]\right\}\\
&\leq\frac{a\cdot m_{0x}}{a+b}\left|\frac{a+1}{a+b+1}-\frac{a}{a+b}\right|+0+\frac{b\cdot m_{0x}}{a+b}\left|\frac{a}{a+b+1}-\frac{a}{a+b}\right|+0\\
&=\frac{2ab\cdot m_{0x}}{(a+b)^2(a+b+1)}.
\end{aligned}$$

Similarly if $a<b$, then

$$\mathbb{E}\left[h_x\left(S_{1,x}\right)\mid S_{0,x}=(a,b),x_1=x\right]-h_x(a,b)\leq\frac{ab(m_{0x}+m_{1x})}{(a+b)^2(a+b+1)}.$$

In both cases, we have

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(S_{1,x}\right)\mid S_{0,x}=(a,b),x_1=x\right]-h_x(a,b)\\
&\leq\frac{ab\left[\max\{m_{0x},m_{1x}\}+m_{0x}\right]}{(a+b)^2(a+b+1)}\leq\frac{\max\{m_{0x},m_{1x}\}+m_{0x}}{4(a+b+1)}:=\widetilde{H}_x(a,b).
\end{aligned}$$

Since $PS(x;n)=\{(a,b)\in\Lambda : a+b\geq 2+n\}$, we have

$$\lim_{n\to\infty}\left[\sup_{(a,b)\in PS(x;n)}\widetilde{H}_x(a,b)\right]=\lim_{n\to\infty}\left[\frac{\max\{m_{0x},m_{1x}\}+m_{0x}}{4(3+n)}\right]=0.$$

Thus Payoff Condition 3 holds and

$$N_x = \min\left\{n \geq 0 : \frac{\max\{m_{0x}, m_{1x}\} + m_{0x}}{4(3+n)} \leq c_x\right\} = \left(\lceil \frac{\max\{m_{0x}, m_{1x}\} + m_{0x}}{4c_x} \rceil - 3\right)^+.$$

EC.10. Remark 2.

From here we let Φ and φ denote the standard normal cdf and pdf respectively. Define the following events for a standard normal random variable Z:

$$A^i = \left\{\mu + \tilde{\sigma}_x(\beta)Z \in \left[\mu_x^i(\beta + \beta_x^\epsilon) - \delta/2, \mu_x^i(\beta + \beta_x^\epsilon) + \delta/2\right]\right\} \text{ for all } i,$$
$$A = \left\{\mu + \tilde{\sigma}_x(\beta)Z \notin \left[\underline{\mu}_x(\beta + \beta_x^\epsilon), \overline{\mu}_x(\beta + \beta_x^\epsilon)\right]\right\}.$$

Then

$$\begin{aligned}
&\mathbb{E}\left[V_x\left(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^\epsilon\right)\right] \\
\approx &\sum_i \left\{\mathbb{E}\left[V_x\left(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^\epsilon\right) \mid A^i\right] \cdot \mathbb{P}(A^i)\right\} + \mathbb{E}\left[V_x\left(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^\epsilon\right) \mid A\right] \cdot \mathbb{P}(A) \\
\approx &\sum_i \left\{V_x\left(\mu_x^i(\beta + \beta_x^\epsilon), \beta + \beta_x^\epsilon\right) \cdot \mathbb{P}(A^i)\right\} + 0 \cdot \mathbb{P}(A) \\
= &\sum_i \left\{V_x\left(\mu_x^i(\beta + \beta_x^\epsilon), \beta + \beta_x^\epsilon\right) \cdot \left[\Phi\left(\frac{\mu_x^i(\beta + \beta_x^\epsilon) + \delta/2 - \mu}{\tilde{\sigma}_x(\beta)}\right) - \Phi\left(\frac{\mu_x^i(\beta + \beta_x^\epsilon) + \delta/2 - \mu}{\tilde{\sigma}_x(\beta)}\right)\right]\right\}.
\end{aligned}$$

EC.11. Remark 3.

We only need to give explicit expressions for $\mathbb{E}\left[h_x\left(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^\epsilon\right)\right]$.

EC.11.1. 0-1 Terminal Payoff

Let $\rho = d_x - \mu$ and $\xi = \tilde{\sigma}_x^{-1}(\beta)\left[\rho - \Phi^{-1}\left(\frac{m_{1x}}{m_{0x}+m_{1x}}\right)/\sqrt{\beta + \beta_x^\epsilon}\right]$, then

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(\mu + \tilde{\sigma}_x(\beta)Z, \beta + \beta_x^\epsilon\right)\right] \\
= &\mathbb{E}\left[\max\left\{m_{0x} \cdot \Phi\left(\sqrt{\beta + \beta_x^\epsilon}\left(d_x - \mu - \tilde{\sigma}_x(\beta)Z\right)\right), m_{1x}\left[1 - \Phi\left(\sqrt{\beta + \beta_x^\epsilon}\left(d_x - \mu - \tilde{\sigma}_x(\beta)Z\right)\right)\right]\right\}\right] \\
= &m_{0x}\int_{-\infty}^{\xi} \Phi\left(\sqrt{\beta + \beta_x^\epsilon}\left(\rho - \tilde{\sigma}_x(\beta)z\right)\right)\varphi(z)dz + m_{1x}\int_{\xi}^{+\infty}\left[1 - \Phi\left(\sqrt{\beta + \beta_x^\epsilon}\left(\rho - \tilde{\sigma}_x(\beta)z\right)\right)\right]\varphi(z)dz \\
= &m_{0x} \cdot X + m_{1x} \cdot Y,
\end{aligned}$$

where X and Y are defined to be the first and second integral respectively in the second line. Let Z_1 and Z_2 be two independent standard normal random variables. Then

$$X = \mathbb{P}\left[Z_1 \leq \sqrt{\beta + \beta_x^\epsilon}\left(\rho - \tilde{\sigma}_x(\beta)Z_2\right), Z_2 \leq \xi\right] = \mathbb{P}\left[(Z_2, Z_3) \leq (\xi, 0)\right],$$

where $Z_3 := Z_1 - \sqrt{\beta + \beta_x^\epsilon}\left(\rho - \tilde{\sigma}_x(\beta)Z_2\right) \sim \mathcal{N}\left(-\rho\sqrt{\beta + \beta_x^\epsilon}, 1 + \beta_x^\epsilon/\beta\right)$ and $\mathrm{Cov}(Z_2, Z_3) = \sqrt{\beta_x^\epsilon/\beta}$. It follows that X can be evaluated from the cdf of a bivariate normal distribution. A similar argument can be applied to Y.

EC.11.2. Linear Terminal Payoff

$$\begin{aligned}
&\mathcal{R}_x(\mu,\beta)+c_x=\mathbb{E}\left[h_x\left(\mu+\tilde{\sigma}_x(\beta)Z,\beta+\beta_x^\epsilon\right)\right]-h_x\left(\mu,\beta\right)\\
&=m_{0x}\left\{\mathbb{E}\left[\left(d_x-\mu-\tilde{\sigma}_x(\beta)Z\right)^+\right]-\left(d_x-\mu\right)^+\right\}+m_{1x}\left\{\mathbb{E}\left[\left(\mu+\tilde{\sigma}_x(\beta)Z-d_x\right)^+\right]-\left(\mu-d_x\right)^+\right\}\\
&=m_{0x}\cdot\Delta\left(d_x-\mu,\tilde{\sigma}_x^{-2}(\beta)\right)+m_{1x}\cdot\Delta\left(\mu-d_x,\tilde{\sigma}_x^{-2}(\beta)\right),
\end{aligned}\tag{EC.12}$$

which can be computed using Lemma EC.4.

EC.12. Proof of Table 3

Table EC.3 Example Terminal Payoff Functions with Normal Sampling

	0-1 Terminal Payoff	Linear Terminal Payoff
Payoff Condition 3	Yes	Yes
$\widetilde{H}_x(\mu,\beta)$	$\max\{m_{0x},m_{1x}\}\left[\left(\sqrt{1+\beta_x^\epsilon/\beta}-1\right)A_3+\pi^{-1}\sqrt{\beta_x^\epsilon/\beta}\right]$	$\frac{m_{0x}+m_{1x}}{\sqrt{2\pi\beta}}$
N_x	$\left\lceil\frac{A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}\right\rceil$	$\left\lceil\frac{(m_{0x}+m_{1x})^2}{2\pi c_x^2\beta_x^\epsilon}\right\rceil$
Special Condition 1	Yes	Yes
Special Condition 2	Yes	Yes

where

$$\begin{aligned}
&A_1=A_4^2+2A_4A_5, \qquad A_2=2(A_4+A_5)/\pi, \qquad A_3=\pi^{-2}-A_4^2,\\
&A_4=1+1/\sqrt{2\pi e}, \qquad A_5=c_x/\max\{m_{0x},m_{1x}\}.
\end{aligned}$$

EC.12.1. Preparatory Material

Lemma EC.3. *Let* $u_0:=\Phi^{-1}\left(\frac{m_{1x}}{m_{0x}+m_{1x}}\right)$. *Define functions* $g_x(\cdot)$, $m_x(\cdot)$, $s_x(\cdot)$, $t_x(\cdot)$, $s(\cdot)$ *at* $(-\infty,+\infty)$ *and* $g(\cdot)$ *at* $(-\infty,+\infty)^2$ *by*

$$\begin{aligned}
&g_x(u)=\max\left\{m_{0x}\Phi(u),m_{1x}\left[1-\Phi(u)\right]\right\},\\
&m_x(u)=m_{0x}\mathbf{1}_{\{u\geq u_0\}}+m_{1x}\mathbf{1}_{\{u<u_0\}},\\
&s_x(u)=\begin{cases}\sup_v\left[\frac{m_{1x}\Phi(-v)-m_{0x}\Phi(u)}{u-v}\right] & \text{if } u\geq u_0,\\ \sup_v\left[\frac{m_{0x}\Phi(v)-m_{1x}\Phi(-u)}{v-u}\right] & \text{if } u<u_0,\end{cases}\\
&t_x(u)=\begin{cases}m_{0x}\sup_{v\geq u}\left[\frac{\Phi(v)-\Phi(u)}{v-u}\right] & \text{if } u\geq u_0,\\ m_{1x}\sup_{v\leq u}\left[\frac{\Phi(v)-\Phi(u)}{v-u}\right] & \text{if } u<u_0,\end{cases}\\
&s(u)=\max\left\{s_x(u),t_x(u)\right\},\\
&g(u,v)=g_x(u)+|v-u|s(u).
\end{aligned}$$

Then for all u *and* v,

1. $0 \le s(u) \le \max\{m_{0x}, m_{1x}\}/\sqrt{2\pi}$,
2. $|u|s(u) \le \max\{m_{0x}, m_{1x}\}(1+1/\sqrt{2\pi e})$,
3. $g_x(v) \le g(u,v)$.

Proof of Lemma EC.3. • If $u \ge u_0$, then $g_x(u) = m_{0x}\Phi(u)$ and $m_x(u) = m_{0x}$. Also, if $m_{0x}\Phi(u) \ge m_{1x}$, then $s_x(u) = 0$; otherwise there exists a maximum $v^* < u$ satisfying the first order condition, i.e.,

$$s_x(u) = \frac{m_{1x}\Phi(-v^*) - m_{0x}\Phi(u)}{u - v^*} = m_{1x}\varphi(v^*). \tag{EC.13}$$

If $u \ge 0$, then $t_x(u) = m_{0x}\varphi(u)$; otherwise there exists some $v^{**} > u$ such that

$$t_x(u) = m_{0x}\frac{\Phi(v^{**}) - \Phi(u)}{v^{**} - u} = m_{0x}\varphi(v^{**}). \tag{EC.14}$$

• Similarly if $u < u_0$, then $g_x(u) = m_{1x}\Phi(-u)$ and $m_x(u) = m_{1x}$. If $m_{1x}\Phi(-u) \ge m_{0x}$, then $s_x(u) = 0$; otherwise there exists some $v^* > u$ such that

$$s_x(u) = \frac{m_{0x}\Phi(v^*) - m_{1x}\Phi(-u)}{v^* - u} = m_{0x}\varphi(v^*).$$

If $u \le 0$, then $t_x(u) = m_{1x}\varphi(u)$; otherwise there exists $v^{**} < u$ such that

$$t_x(u) = m_{1x}\frac{\Phi(v^{**}) - \Phi(u)}{v^{**} - u} = m_{1x}\varphi(v^{**}).$$

It follows that $s(u) \le \max\{m_{0x}, m_{1x}\}\varphi(0) = \max\{m_{0x}, m_{1x}\}/\sqrt{2\pi}$. Hence the **first** inequality holds.

For the **second** inequality, we notice that function $u \mapsto |u|\varphi(u)$ is maximized at $u = \pm 1$ with maximum value $\varphi(1) = 1/\sqrt{2\pi e}$. Assume $u \ge u_0$. We first show $|u|s_x(u) \le \max\{m_{0x}, m_{1x}\}(1+1/\sqrt{2\pi e})$. This result holds immediately when $m_{0x}\Phi(u) > m_{1x}$, i.e., $s_x(u) = 0$. Otherwise by (EC.13),

$$|u|s_x(u) \le m_{1x}\Phi(-v^*) - m_{0x}\Phi(u) + m_{1x}|v^*|\varphi(v^*) \le m_{1x}(1+1/\sqrt{2\pi e}) \le \max\{m_{0x}, m_{1x}\}(1+1/\sqrt{2\pi e}).$$

We then show $|u|t_x(u) \le \max\{m_{0x}, m_{1x}\}(1+1/\sqrt{2\pi e})$. This result holds immediately when $u \ge 0$. Otherwise by (EC.14),

$$|u|t_x(u) = m_{0x}\Phi(v^{**}) - m_{0x}\Phi(u) + m_{0x}|v^{**}|\varphi(v^{**}) \le m_{0x}(1+1/\sqrt{2\pi e}) \le \max\{m_{0x}, m_{1x}\}(1+1/\sqrt{2\pi e}).$$

The case when $u < u_0$ is similar.

We now show the **last** inequality. WLOG, suppose $m_{0x} \le m_{1x}$. Then $u_0 \ge 0$.

For $u \ge u_0$, we have $g(u,v) = m_{0x}\Phi(u) + |v-u|\max\{s_x(u), m_{0x}\varphi(u)\}$.

1. For $v \ge u$, by the Mean Value Theorem, $\Phi(v) = \Phi(u) + (v-u)\cdot\varphi(w)$, where $w \in (u,v)$ and $\varphi(w) < \varphi(u)$. Hence $g_x(v) = m_{0x}\Phi(v) < m_{0x}[\Phi(u) + |v-u|\varphi(u)] \le g(u,v)$.

2. For $u_0 \le v < u$, $g_x(v) = m_{0x}\Phi(v) < m_{0x}\Phi(u) < g(u,v)$.

3. For $v < u_0$, $g_x(v) = m_{1x}\Phi(-v) = m_{0x}\Phi(u) + (u-v)\left[\frac{m_{1x}\Phi(-v)-m_{0x}\Phi(u)}{u-v}\right] \le m_{0x}\Phi(u) + |v-u|s_x(u) \le g(u,v)$.

For $0 < u < u_0$ (if $u_0 > 0$), $g(u,v) = m_{1x}\Phi(-u) + |v-u|\max\{s_x(u), t_x(u)\}$.

1. For $v \le u$, $g_x(v) = m_{1x}\Phi(-v) = m_{1x}\left[\Phi(-u) + (u-v)\left[\frac{\Phi(-v)-\Phi(-u)}{u-v}\right]\right] \le m_{1x}\Phi(-u) + |v-u|t_x(u) \le g(u,v)$.

2. For $u < v \le u_0$, $g_x(v) = m_{1x}\Phi(-v) < m_{1x}\Phi(-u) = g_x(u) \le g(u,v)$.

3. For $v > u_0$, $g_x(v) = m_{0x}\Phi(v) = m_{1x}\Phi(-u) + (v-u)\left[\frac{m_{0x}\Phi(v)-m_{1x}\Phi(-u)}{v-u}\right] \le m_{1x}\Phi(-u) + |v-u|s_x(u) \le g(u,v)$.

For $u \le 0$, we can show $g_x(\cdot) \le g(u,\cdot)$ similar to the case when $u \ge u_0$.

Lemma EC.4. *Define* $\Delta\colon (-\infty,+\infty)\times(0,+\infty] \to [0,+\infty)$ *by*

$$\Delta(\mu,\beta) := \left[\mathbb{E}\left(X^+\right) - (\mathbb{E}X)^+ | X \sim \mathcal{N}(\mu, 1/\beta)\right].$$

Then

$$\Delta(\mu,\beta) = \begin{cases} \mu\Phi\left(\mu\sqrt{\beta}\right) + \sqrt{\beta}^{-1}\varphi\left(\mu\sqrt{\beta}\right), & \text{if } \mu \le 0; \\ \mu\left[\Phi\left(\mu\sqrt{\beta}\right) - 1\right] + \sqrt{\beta}^{-1}\varphi\left(\mu\sqrt{\beta}\right). & \text{if } \mu > 0. \end{cases}$$

For any fixed $\beta > 0$, $\Delta(\cdot,\beta)$ *has the following properties:*

- *it is strictly increasing on* $(-\infty, 0]$ *and strictly decreasing on* $[0,+\infty)$*;*
- *it reaches its maximum at* $\mu = 0$*, with maximum value* $1/\sqrt{2\pi\beta}$*;*
- *it converges to* 0 *as* $\mu \to +\infty$ *and* $\mu \to -\infty$*.*

Proof of Lemma EC.4 For any fixed $\beta > 0$ and $\mu \in \mathbb{R}$, suppose $X \sim \mathcal{N}(\mu, 1/\beta)$, then by Clark (1961).

$$\mathbb{E}\left(X^+\right) = \mu\Phi\left(\mu\sqrt{\beta}\right) + \sqrt{\beta}^{-1}\varphi\left(\mu\sqrt{\beta}\right) \tag{EC.15}$$

where Φ and φ are the standard normal cdf and pdf respectively. Since $\Phi'(x) = \varphi(x)$, $\varphi'(x) = -x\varphi(x)$, it follows that

$$\left[\mathbb{E}\left(X^+\right)\right]'_\mu = \Phi\left(\mu\sqrt{\beta}\right) > 0,$$

which indicates that $E(X^+)$ is a strictly increasing function of μ. Similarly,

$$\mathbb{E}\left(X^-\right) = \mathbb{E}\left(X^+\right) - \mu = \mu\left[\Phi\left(\mu\sqrt{\beta}\right) - 1\right] + \sqrt{\beta}^{-1}\varphi\left(\mu\sqrt{\beta}\right), \tag{EC.16}$$

$$\left[\mathbb{E}\left(X^-\right)\right]'_\mu = \Phi\left(\mu\sqrt{\beta}\right) - 1 < 0,$$

hence $E\left(X^-\right)$ is a strictly decreasing function of μ.

Now consider the objective function

$$\Delta(\mu,\beta) = \mathbb{E}\left(X^+\right) - \left[\mathbb{E}\left(X^+\right) - \mathbb{E}\left(X^-\right)\right]^+ = \begin{cases} \mathbb{E}\left(X^+\right) = \mathbb{E}\left(X^-\right), & \text{if } \mu = 0; \\ \mathbb{E}\left(X^+\right), & \text{if } \mathbb{E}\left(X^+\right) < \mathbb{E}\left(X^-\right), \text{ i.e., } \mu < 0; \\ \mathbb{E}\left(X^-\right), & \text{if } \mathbb{E}\left(X^+\right) > \mathbb{E}\left(X^-\right), \text{ i.e., } \mu > 0. \end{cases}$$

It follows that $\Delta(\cdot,\beta)$ is strictly increasing on $(-\infty,0]$, strictly decreasing on $[0,+\infty)$, and maximized at $\mu=0$, where

$$\Delta(0,\beta)=\left[\mathbb{E}\left(X^+\right)|X\sim\mathcal{N}(0,1/\beta)\right]=1/\sqrt{\beta}\cdot\left[\mathbb{E}\left(X^+\right)|X\sim\mathcal{N}(0,1)\right]=1/\sqrt{2\pi\beta}.$$

Finally, since for a standard normal r.v. $Z\in L_1$, we have $0=\lim_{x\to+\infty}x\mathbb{P}(Z>x)=\lim_{x\to-\infty}x\mathbb{P}(Z\le x)=\lim_{x\to+\infty}x(1-\Phi(x))=\lim_{x\to-\infty}x\Phi(x)$. Thus by (EC.15) and (EC.16),

$$\lim_{\mu\to-\infty}\Delta(\mu,\beta)=\lim_{\mu\to-\infty}\left[\mathbb{E}\left(X^+\right)|X\sim\mathcal{N}(\mu,1/\beta)\right]=\sqrt{\beta}^{-1}\left[\lim_{x\to-\infty}x\Phi(x)+\varphi(-\infty)\right]=0;$$
$$\lim_{\mu\to+\infty}\Delta(\mu,\beta)=\lim_{\mu\to+\infty}\left[\mathbb{E}\left(X^-\right)|X\sim\mathcal{N}(\mu,1/\beta)\right]=\sqrt{\beta}^{-1}\left[\lim_{x\to+\infty}x(\Phi(x)-1)+\varphi(+\infty)\right]=0.$$

EC.12.2. 0-1 Terminal Payoff

EC.12.2.1. Payoff Condition 3 Denote $\rho=d_x-\mu$. Applying Lemma EC.3, we have

$$\begin{aligned}
&\mathbb{E}\left[h_x\left(\mu+\tilde{\sigma}_x(\beta)Z,\beta+\beta_x^\epsilon\right)\right]-h_x(\mu,\beta)\\
&=\mathbb{E}\left[g_x\left(\sqrt{\beta+\beta_x^\epsilon}(\rho-\tilde{\sigma}_x(\beta)Z)\right)\right]-g_x\left(\sqrt{\beta}\rho\right)\\
&\le\mathbb{E}\left[g\left(\sqrt{\beta}\rho,\sqrt{\beta+\beta_x^\epsilon}(\rho-\tilde{\sigma}_x(\beta)Z)\right)\right]-g_x\left(\sqrt{\beta}\rho\right)\\
&=\mathbb{E}\left[\left|\left(\sqrt{\beta+\beta_x^\epsilon}-\sqrt{\beta}\right)\rho-\sqrt{\beta+\beta_x^\epsilon}\cdot\tilde{\sigma}_x(\beta)Z\right|\right]\cdot s\left(\sqrt{\beta}\rho\right)\\
&\le\left[\left(\sqrt{1+\beta_x^\epsilon/\beta}-1\right)\sqrt{\beta}|\rho|+\sqrt{\beta_x^\epsilon/\beta}\cdot\mathbb{E}|Z|\right]\cdot s\left(\sqrt{\beta}\rho\right)\\
&=\left(\sqrt{1+\beta_x^\epsilon/\beta}-1\right)\cdot\sqrt{\beta}|\rho|s\left(\sqrt{\beta}\rho\right)+\sqrt{\beta_x^\epsilon/\beta}\cdot\mathbb{E}|Z|s\left(\sqrt{\beta}\rho\right)\\
&\le\max\{m_{0x},m_{1x}\}\left[\left(\sqrt{1+\beta_x^\epsilon/\beta}-1\right)\left(1+1/\sqrt{2\pi e}\right)+\pi^{-1}\sqrt{\beta_x^\epsilon/\beta}\right]\\
&=\widetilde{H}_x(\mu,\beta).
\end{aligned}$$

Now

$$\lim_{n\to\infty}\left[\sup_{(\mu,\beta)\in PS(x;n)}\widetilde{H}_x(\mu,\beta)\right]=\max\{m_{0x},m_{1x}\}\lim_{n\to\infty}\left[\left(\sqrt{1+1/n}-1\right)\left(1+1/\sqrt{2\pi e}\right)+1/\left(\sqrt{n}\pi\right)\right]=0,$$

where $PS(x;n)=\{(\mu,\beta)\in\Lambda:\beta\ge n\cdot\beta_x^\epsilon\}$. Thus Payoff Condition 3 holds. Notice that the expression inside the brackets is a continuous, strictly decreasing function of n. Simple algebra then yields

$$\begin{aligned}
N_x&=\min\left\{n:\max\{m_{0x},m_{1x}\}\left[\left(\sqrt{1+1/n}-1\right)\left(1+1/\sqrt{2\pi e}\right)+1/\left(\sqrt{n}\pi\right)\right]\le c_x\right\}\\
&=\lceil\frac{A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}\rceil.
\end{aligned}$$

EC.12.2.2. Special Condition 1 We write

$$\begin{aligned}
H_x(\mu,\beta)=&\,m_{0x}\cdot\mathbf{1}_{\left\{p_x(\mu,\beta)\le\frac{m_{0x}}{m_{0x}+m_{1x}}\right\}}+m_{1x}\cdot\mathbf{1}_{\left\{p_x(\mu,\beta)\ge\frac{m_{0x}}{m_{0x}+m_{1x}}\right\}}\\
&-\max\left\{m_{0x}\left[1-p_x(\mu,\beta)\right],m_{1x}\cdot p_x(\mu,\beta)\right\}.
\end{aligned}\tag{EC.17}$$

For any fixed β, $p_x(\mu,\beta)=1-\Phi\left(\sqrt{\beta}(d_x-\mu)\right)$ converges to 1 as $\mu\to+\infty$, and converges to 0 as $\mu\to-\infty$. In both cases, we have $H_x(\mu,\beta)\to0$ and thus Special Condition 1 holds.

EC.12.2.3. Special Condition 2 Fix $\beta > 0$. Let $u_0 := \Phi^{-1}\left[\frac{m_{1x}}{m_{0x}+m_{1x}}\right]$. Then

$$p_x(\mu,\beta) \le \frac{m_{0x}}{m_{0x}+m_{1x}} \iff \sqrt{\beta}(d_x-\mu) \ge u_0.$$

Case I $c_x < 2\max\{m_{0x}, m_{1x}\}$.

We know that for any $\alpha \in (0,1)$, $I_\alpha = [-z_\alpha, z_\alpha] := [-\Phi^{-1}(1-\alpha/2), \Phi^{-1}(1-\alpha/2)]$ is the $100(1-\alpha)\%$ confidence interval for the standard normal distribution. By Proposition 3 and (EC.17),

$$\begin{aligned}
&\mathbb{E}\left[V_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right)\right] \le \mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right)\right] \\
&= (1-\alpha)\mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right) \mid Z \in I_\alpha\right] + \alpha\mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right) \mid Z \in I_\alpha^c\right] \qquad \text{(EC.18)} \\
&\le (1-\alpha)\mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right) \mid Z \in I_\alpha\right] + \alpha\cdot\max\{m_{0x}, m_{1x}\}.
\end{aligned}$$

Pick $\alpha = c_x/\left[2\max\{m_{0x}, m_{1x}\}\right]$.

When $\mu \le d_x - \tilde{\sigma}_x(\beta)z_\alpha - u_0/\sqrt{\beta+\beta_x^\epsilon}$, we have $\sqrt{\beta+\beta_x^\epsilon}\left(d_x-\mu-\tilde{\sigma}_x(\beta)Z\right) \ge u_0$ for all $Z \in I_\alpha$, hence

$$\begin{aligned}
&\mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right) \mid Z \in I_\alpha\right] \\
&= m_{0x}\mathbb{E}\left[1-\Phi\left(\sqrt{\beta+\beta_x^\epsilon}(d_x-\mu-\tilde{\sigma}_x(\beta)Z)\right) \mid Z \in I_\alpha\right] \\
&\le m_{0x}\left[1-\Phi\left(\sqrt{\beta+\beta_x^\epsilon}(d_x-\mu-\tilde{\sigma}_x(\beta)z_\alpha)\right)\right],
\end{aligned}$$

which goes to 0 as $\mu \to -\infty$.

Moreover, Special Condition 1 indicates that $\mathcal{R}_x(\mu,\beta) \to -c_x$ as $\mu \to -\infty$.

(15) and (EC.18) then yields

$$\limsup_{\mu\to+\infty} L_x(\mu,\beta,V_x) \le \lim_{\mu\to+\infty}\mathcal{R}_x(\mu,\beta) + \limsup_{\mu\to+\infty}\mathbb{E}\left[V_x(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon)\right] \le -c_x + c_x/2 < 0.$$

We can therefore find some $\underline{\mu}_x(\beta) \le d_x - \tilde{\sigma}_x(\beta)z_\alpha - u_0/\sqrt{\beta+\beta_x^\epsilon}$ such that for all $\mu \le \underline{\mu}_x(\beta)$, $L_x(\mu,\beta,V_x) < 0$ and hence $V_x(\mu,\beta) = 0$.

Similarly, we can find some $\overline{\mu}_x(\beta) \ge d_x + \tilde{\sigma}_x(\beta)z_\alpha$ such that $V_x(\mu,\beta) = 0$ for all $\mu \ge \underline{\mu}_x(\beta)$.

Case II $c_x \ge 2\max\{m_{0x}, m_{1x}\}$. Then

$$\mathbb{E}\left[V\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right)\right] \le \mathbb{E}\left[H_x\left(\mu+\tilde{\sigma}_x(\beta)Z, \beta+\beta_x^\epsilon\right)\right] \le \max\{m_{0x}, m_{1x}\} \le c_x/2.$$

It follows that

$$\limsup_{\mu\to+\infty} L_x(\mu,\beta,V_x) \le -c_x/2 < 0.$$

Hence we can also find $\overline{\mu}_x(\beta)$ and $\underline{\mu}_x(\beta)$ satisfying Special Condition 2.

EC.12.3. Linear Terminal payoff

EC.12.3.1. Special Condition 1 Applying Lemma EC.4,

$$\begin{aligned} H_x(\mu,\beta) &= \mathbb{E}\left[m_{0x}(d_x-\theta_x)^+ + m_{1x}(\theta_x-d_x)^+ \,\middle|\, \theta_x \sim \mathcal{N}(\mu,1/\beta)\right] - \left[m_{0x}(d_x-\mu)^+ + m_{1x}(\mu-d_x)^+\right] \\ &= m_{0x}\cdot\Delta\left(d_x-\mu,\beta\right) + m_{1x}\cdot\Delta\left(\mu-d_x,\beta\right). \end{aligned} \tag{EC.19}$$

Hence for any fixed $\beta>0$, $H_x(\mu,\beta)\to 0$ as $\mu\to+\infty$ and $\mu\to-\infty$.

EC.12.3.2. Payoff Condition 3 Notice that

$$H_x(\cdot,\beta) \le (m_{0x}+m_{1x})/\sqrt{2\pi\beta} = \widetilde{H}_x(\cdot,\beta), \tag{EC.20}$$

hence the first inequality in Payoff Condition 3 holds. Now in the sub-problem with single alternative x, $\beta_{nx}=\beta_{0x}+n\cdot\beta_x^\epsilon$, hence $PS(x;n)=\{(\mu,\beta)\in\Lambda:\beta\ge n\cdot\beta_x^\epsilon\}$. It follows that

$$\lim_{n\to\infty}\left[\sup_{(\mu,\beta)\in PS(x;n)}\widetilde{H}_x(\mu,\beta)\right] = \lim_{n\to\infty}\left[\frac{m_{0x}+m_{1x}}{\sqrt{2\pi n\beta_x^\epsilon}}\right]=0.$$

Thus the second equation in Payoff Condition 3 holds and

$$N_x = \min\left\{n\ge 0: \frac{m_{0x}+m_{1x}}{\sqrt{2\pi n\beta_x^\epsilon}}\le c_x\right\} = \lceil\frac{(m_{0x}+m_{1x})^2}{2\pi c_x^2\beta_x^\epsilon}\rceil. \tag{EC.21}$$

EC.12.3.3. Special Condition 2 Fix $\beta>0$. Let $\rho=\mu-d_x$.

Case I $c_x\sqrt{\beta+\beta_x^\epsilon}/\left(m_{0x}+m_{1x}\right)<1$.

We know that for any $\alpha\in(0,1)$, $I_\alpha=[-z_\alpha,z_\alpha]:=[-\Phi^{-1}(1-\alpha/2),\Phi^{-1}(1-\alpha/2)]$ is the $100(1-\alpha)\%$ confidence interval for the standard normal distribution. By Proposition 3 and (EC.19), (EC.20),

$$\begin{aligned} &\mathbb{E}\left[V_x\left(\mu+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\right] \le \mathbb{E}\left[H_x\left(\mu+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\right] \\ &= (1-\alpha)\mathbb{E}\left[H_x\left(\mu+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\mid Z\in I_\alpha\right] + \alpha\mathbb{E}\left[H_x\left(\mu+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\mid Z\in I_\alpha^c\right] \\ &\le (1-\alpha)\mathbb{E}\left[H_x\left(\mu+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\mid Z\in I_\alpha\right] + \alpha\widetilde{H}_x\left(\cdot,\beta+\beta_x^\epsilon\right) \\ &= (1-\alpha)\mathbb{E}\left[m_{0x}\Delta\left(-\rho-\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)+m_{1x}\Delta\left(\rho+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\mid Z\in I_\alpha\right] + \alpha\widetilde{H}_x\left(\cdot,\beta+\beta_x^\epsilon\right). \end{aligned} \tag{EC.22}$$

where $\widetilde{H}_x(\cdot,\beta)=\left(m_{0x}+m_{1x}\right)/\sqrt{2\pi\beta}$.

Pick $\alpha=c_x\sqrt{\beta+\beta_x^\epsilon}/\left(m_{0x}+m_{1x}\right)$. Then $\alpha\widetilde{H}_x\left(\cdot,\beta+\beta_x^\epsilon\right)=c_x/\sqrt{2\pi}$.

When $\rho\ge\tilde\sigma_x(\beta)z_\alpha$, i.e., $\mu\ge d_x+\tilde\sigma_x(\beta)z_\alpha$, since $\Delta(\cdot,\beta+\beta_x^\epsilon)$ is increasing at $(-\infty,0]$ and decreasing at $[0,+\infty)$, we know

$$\begin{aligned} &\mathbb{E}\left[m_{0x}\Delta\left(-\rho-\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)+m_{1x}\Delta\left(\rho+\tilde\sigma_x(\beta)Z,\beta+\beta_x^\epsilon\right)\mid Z\in I_\alpha\right] \\ &\le m_{0x}\Delta(-\rho+\tilde\sigma_x(\beta)z_\alpha,\beta+\beta_x^\epsilon)+m_{1x}\Delta(\rho-\tilde\sigma_x(\beta)z_\alpha,\beta+\beta_x^\epsilon), \end{aligned}$$

which goes to 0 as $\rho \to +\infty$, i.e., $\mu \to +\infty$.

Moreover, (EC.12) shows that $\mathcal{R}_x(\mu,\beta) \to -c_x$ as $\rho \to +\infty$, i.e., $\mu \to +\infty$.

(15) and (EC.22) then yield

$$\limsup_{\mu\to+\infty} L_x(\mu,\beta,V_x) \le \lim_{\mu\to+\infty} \mathcal{R}_x(\mu,\beta) + \limsup_{\mu\to+\infty} \mathbb{E}\left[V_x(\mu+\tilde{\sigma}_x(\beta)Z,\beta+\beta_x^\epsilon)\right] \le -c_x + c_x/\sqrt{2\pi} < 0.$$

It follows that we can find $\overline{\mu}_x(\beta) \ge d_x + \tilde{\sigma}_x(\beta) z_\alpha$ such that for all $\mu \ge \overline{\mu}_x(\beta)$, $L_x(\mu,\beta,V_x) < 0$ and hence $V_x(\mu,\beta) = 0$.

Similarly, we can find some $\underline{\mu}_x(\beta) \le d_x - \tilde{\sigma}_x(\beta) z_\alpha$ such that $V_x(\mu,\beta) = 0$ for all $\mu \le \underline{\mu}_x(\beta)$.

Case II $c_x\sqrt{\beta+\beta_x^\epsilon}/\left(m_{0x}+m_{1x}\right) \ge 1$. Then

$$\mathbb{E}\left[V\left(\mu+\tilde{\sigma}_x(\beta)Z,\beta+\beta_x^\epsilon\right)\right] \le \widetilde{H}_x\left(\cdot,\beta+\beta_x^\epsilon\right) = \frac{m_{0x}+m_{1x}}{\sqrt{2\pi(\beta+\beta_x^\epsilon)}} \le c_x/\sqrt{2\pi}.$$

It follows that

$$\limsup_{\mu\to+\infty} L_x(\mu,\beta,V_x) \le -c_x + c_x/\sqrt{2\pi} < 0.$$

Hence we can also find $\overline{\mu}_x(\beta)$ and $\underline{\mu}_x(\beta)$ satisfying Special Condition 2.

References

See references list in the main paper.

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