Delay-Differential Equations Applied to Queueing Theory

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Summary. Queueing theory is the mathematical theory of customers waiting in line. The term “customer” could refer to a human customer, but it could also refer to, for example, an automobile waiting in line to pass through a toll plaza. In the case that there are multiple lines, the customer often has the choice of which line to wait in. Delay of queue length information has the potential to influence the decision of a customer as which line to wait in. In this work we introduce the reader to the typical structure of the differential equations of queueing theory, and we study a model involving two lines in which the information regarding the length of a line provided to the customer is not current, but rather is delayed. That is, the provided information gives the state of the system at a previous time rather than at the current time. Thus the governing equations become delay-differential equations. In particular we show that the delay can cause oscillations in the length of the lines due to Hopf bifurcations.

Introduction

Queueing theory represents a new area of application for nonlinear dynamics. It is expected that the typical nonlinear dynamics researcher is unfamiliar with the mathematical models of queueing theory, and so we present a brief summary of the model equations. If the reader is interested in queueing theory, the reader is referred to [1, 2] for an introduction to queueing theory. The term “customer” could refer to a human, but it could also refer to, for example, an automobile waiting in line to pass through a toll plaza. In the case that there are multiple lines, the customer often has the choice of which line to wait in. Delay of queue length information has the potential to influence the decision of a customer as which line to wait in. In this work we introduce the reader to the typical structure of the differential equations of queueing theory, and we study a model involving two lines in which the information regarding the length of a line provided to the customer is not current, but rather is delayed. That is, the provided information gives the state of the system at a previous time rather than at the current time. Thus the governing equations become delay-differential equations. In particular we show that the delay can cause oscillations in the length of the lines due to Hopf bifurcations.

Theorem: For the queueing model (4),(5), the critical delay is given by the following expression

\[ \Delta_{cr} = \frac{2 \arccos(-2\mu/\lambda)}{\sqrt{\lambda^2 - 4\mu^2}}. \] (6)

Proof: The equilibrium in eqs.(4),(5) is given by \( q_1(t) = q_2(t) = \frac{\lambda}{2\mu} \).

Setting \( q_1(t) = \frac{\lambda}{2\mu} + u_1(t) \), \( q_2(t) = \frac{\lambda}{2\mu} + u_2(t) \) (7)

and linearizing in \( u_1 \) and \( u_2 \), we obtain

\[ \dot{u}_1(t) = -\frac{\lambda}{4} \cdot (u_1(t - \Delta) - u_2(t - \Delta)) - \mu \cdot u_1(t) \] (8)

\[ \dot{u}_2(t) = -\frac{\lambda}{4} \cdot (u_2(t - \Delta) - u_1(t - \Delta)) - \mu \cdot u_2(t). \] (9)
Next we uncouple eqs.(8),(9) by setting $v_1(t) = u_1(t) + u_2(t)$ and $v_2(t) = u_1(t) - u_2(t)$ giving:

$$\dot{v}_1(t) = -\mu \cdot v_1(t) \quad \text{(10)}$$
$$\dot{v}_2(t) = -\frac{\lambda}{2} \cdot v_2(t - \Delta) - \mu \cdot v_2 \quad \text{(11)}$$

The general solution to eq.(10) is $v_1(t) = c_1 \exp(-\mu t)$ and is bounded and stable. For eq.(11), we set $v_2(t) = \exp(rt)$ and obtain

$$r = -\frac{\lambda}{2} \cdot \exp(-r\Delta) - \mu \quad \text{(12)}$$

For a Hopf bifurcation, we set $r = i\omega$ giving $i\omega = -\frac{\lambda}{2}(\cos \omega \Delta - i \sin \omega \Delta) - \mu$. Equating the real and imaginary parts to zero, we get:

$$\cos \omega \Delta_{cr} = -\frac{2 \cdot \mu}{\lambda} \quad \text{and} \quad \sin \omega \Delta_{cr} = \frac{2 \cdot \omega}{\lambda} \quad \text{(13)}$$

Squaring both equations and adding them together, we get that $\omega = \frac{1}{2} \sqrt{\lambda^2 - 4\mu^2}$, which when substituted into the first of eqs.(13) yields the required condition (6) stated in the Theorem.

**Conclusions**

In this work we have studied a deterministic queueing model that incorporates customer choice and delayed queue length information. We have derived an expression for an explicit threshold for the critical delay where below the threshold the two queues are balanced and converge to the equilibrium. However, when $\Delta$ is larger than the threshold, the equilibrium point is unstable and an oscillation in queue length results.

This analysis is the first of its kind in the queueing and operations research literature. It is important for businesses and managers to determine and know these thresholds since using delayed information can have such a large impact on the dynamics of the business. Even small delays can cause oscillations and it is of great importance for managers of these service systems to understand when oscillations can arise based on the arrival and service parameters of the queueing model.

The reader is referred to [3] for a more complete treatment of this subject which also analyzes a moving average delay differential equation. The reader is also referred to [4] for an introduction to delay-differential equations.

**References**


