Diffusion Limits for the \((MAP_t/PH_t/\infty)^N\) Queueing Network

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Abstract
In this paper, we prove strong approximations for the \((MAP_t/PH_t/\infty)^N\) queueing network. These strong approximations allow us to derive fluid and diffusion limits for the queue length processes of the network. This extends recent work by Ko and Pender [9], which provides fluid and diffusion limits in the single station setting.

1 Introduction

Queueing networks are very useful for analyzing and approximating real stochastic systems. Many queueing networks assume that the arrivals to the network are of Poisson type. This is a natural assumption when there is no dependence or correlation between arrivals. However, an independence assumption is not warranted in many applications. Stochastic models for describing the dynamics of internet data traffic in telecommunication networks are notoriously difficult since they have dependencies. One example of this is when a user downloads a file from the internet; the arrival of the first packet often indicates that more packets are going to arrive subsequently. Despite the assertions of the Palm-Khintine theorem, which asserts that the superposition of a large number of renewal processes will converge to a Poisson process, it is well known in the teletraffic literature that the arrival traffic is not even a renewal process.
There are many applications and scenarios where data traffic is not renewal, see for example Kang et al. [7], Yang and Tsang [19], Ren and Ramamurthy [18], Horváth et al. [6], Heindl and Telek [5], Andersen and Nielsen [1], Gerhardt and Nelson [4]. To this end, we construct a queueing network where the arrivals are not Poisson and are constructed from Markovian Arrival Processes (MAP’s). MAP’s, unlike phase type distributions, allow one to consider non-renewal processes for the arrival process and offer more flexibility when modelling arrival traffic. One main reason that the MAP is a generalization of a phase type distribution is that the MAP is not restarted independently of its past history. In a MAP, unlike phase type distributions, the next interarrival time is dependent on the exit state of the Markov chain and this feature allows one to capture memory into the arrival process.

In this paper, we study the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network. As a result, we extend recent work by [17, 9], which only considers the single station setting. To this end, we prove strong approximations for the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network and extend the Poisson process representation to the network setting. These strong approximations not only allow us to derive fluid and diffusion limits for the queueing network, but also they provide us with simple differential equations that can be integrated numerically to approximate the sample path behavior of the mean and variance of the queueing network. Lastly, since MAP’s can be fitted using data quite easily, this work is the beginning of a framework for queueing processes that include non-renewal arrival processes.

1.1 Main Contributions of Paper

The contributions of this work can be summarized as follows.

- We derive a Poisson process representation for the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network and prove strong approximations for the network.
- Using strong approximations for Poisson processes, we develop fluid and diffusion limits for the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network to understand the mean and variance dynamics of the network.

1.2 Organization of Paper

The remainder of this paper is organized as follows. Section 2 describes the construction of a MAP and phase type distributions. Section 3 builds a mathematical model for describing the dynamics of the system for the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network via time changed Poisson processes. Using the Poisson representation in Section 3, we also prove the fluid and diffusion limits for the $(\text{MAP}_t/PH_t/\infty)^N$ queueing network. Finally, Section 4 concludes and offers suggestions for future research.

2 Markovian Arrival Processes (MAP’s)

In this section, we give a brief description of MAP’s. The reader should review Section 2 of [9] for a more extensive discussion on MAP’s and their versatility in stochastic modelling.
In order to define a MAP, we will follow the construction given in Chakravarthy [2]. We first consider an irreducible continuous time Markov chain (CTMC) with \( m \) transient states. At the end of a sojourn in state \( i \), that is exponentially distributed with parameter \( \lambda_i \), there are two possible events that can happen. The first possibility corresponds to an event or arrival and the CTMC can visit any of the \( m \) transient states including the state from which this even occurred with probability \( p_{ij} \). The second possible event corresponds to no arrival and the CTMC can visit any of the \( m - 1 \) transient states with probability \( q_{ij} \). Therefore, the CTMC is able to go from state \( i \) to state \( i \) through an arrival. Then, we define matrices 
\[
[D_0]_{ij} = d_{0ij} \quad \text{and} \quad [D_1]_{ij} = d_{1ij}
\]
where 
\[
d_{0ii} = \lambda_i, \quad 1 \leq i \leq m; \quad d_{0ij} = \lambda_i \cdot q_{ij}, \quad j \neq i, \quad 1 \leq i, j \leq m;
\]
\[
d_{1ij} = \lambda_i p_{ij}, \quad 1 \leq i, j \leq m, \quad \text{with} \quad \left( \sum_{j=1}^{m} p_{ij} + \sum_{j \neq i}^{m} q_{ij} \right) = 1, \quad \text{for} \quad 1 \leq i \leq m.
\]
In our description of the MAP, we have suppressed its dependence on time. However, all of our results apply to the time varying setting and we suppress time for notational convenience.

With the above construction, a MAP is described by the two \( m \times m \) matrices \([D_0]_{ij}\) and \([D_1]_{ij}\). The matrix \([D_0]_{ij}\) corresponds to transitions where there is no arrival and \([D_1]_{ij}\) corresponds to the transitions that generate an actual arrival. With this construction, it also obvious why this is more general than phase type distributions. Dependence is created by the fact that when an arrival is generated, then the Markov chain can re-enter the same state, however, when no arrival is generated, it cannot re-enter the same state. Now that we have defined a MAP, it is now important to understand how the MAP is a generalization of some well known stochastic arrival processes.

### 2.1 Phase-Type Distributions

A very special case of MAP’s are phase type distributions. Unlike MAP’s, phase type distributions can only approximate renewal processes with arbitrary precision. A phase-type distribution with \( m \) phases can be viewed as the time taken from an initial state to an absorbing state of a continuous time Markov chain with the following infinitesimal generator matrix:
\[
Q = \begin{pmatrix}
0 & 0 \\
\mathbf{s} & \mathbf{S}
\end{pmatrix},
\]
where \( \mathbf{0} \) is a \( 1 \times m \) zero vector, \( \mathbf{s} = \) is an \( m \times 1 \) vector, and \( \mathbf{S} \) is an \( m \times m \) matrix. Note \( \mathbf{s} = -\mathbf{S} \mathbf{e} \) where \( \mathbf{e} \) is an \( m \times 1 \) vector of ones. The matrix \( \mathbf{S} \) and the initial distribution \( \beta \) which is a \( 1 \times m \) vector identify the phase-type distributions.

We assume that our phase-type distributions for the service times have an initial distribution, \( \beta \) and infinitesimal generator matrix, \( \mathbf{Q_S} \). The number of phases in \( \mathbf{S_S} \) is \( m_S \) and the matrix \( \mathbf{S_S} \) and vector \( \mathbf{s_S} \) can be expressed as:
\[
\mathbf{S_S} = \begin{pmatrix}
\mu_{11} & \cdots & \mu_{1m_S} \\
\vdots & \ddots & \vdots \\
\mu_{m_S1} & \cdots & \mu_{m_Sm_S}
\end{pmatrix}, \quad \mathbf{s_S} = (\mu_{10}, \ldots, \mu_{m_S0})',
\]
where the \( \mu_{it} \)'s agree with the definition of the infinitesimal generator matrix \( \mathbf{Q_S} \). For notational consistency, we use a term phase to indicate the state of CTMC for both the MAPs and phase-type distributions throughout this paper.
3 Poisson Construction of $(MAP_t/PH_t/\infty)^N$ Network

With the MAP’s and phase-type distributions described in Section 2, we now build a mathematical queueing model to describe the dynamics of the $(MAP_t/PH_t/\infty)^N$ queueing network. To this end, we need to provide the primitives of the queueing network. For each station in the network, we assume that the arrivals are generated by a MAP. Given we are at station $m$, the number of states in the MAP is assumed to be $m_{A,m}$. Similarly for service, given that we are in station $m$, we assume that the initial distribution for the phase type service distribution is given by $\beta_m$ where the number phases for service is $m_{S,m}$. Lastly, we assume that the queueing network starts with no customers. Thus, in the infinite server setting, we have the following Poisson process representation for the $(MAP_t/PH_t/\infty)^N$ queueing network,

$$U_{jm}(t) = U_{jm}(0) + \sum_{k\neq j}^{m_{A,m}} \Pi_{kj,m}^0 \left( \int_0^t d_{kj,m}^{A0} U_{km}(s) ds \right)$$

(3.1)

$$+ \sum_{k=1}^{m_{A,m}} \sum_{i=1}^{m_{S,m}} \Pi_{kjm}^{A1} \left( \int_0^t d_{k,j,m}^{A1} U_{jm}(s) ds \right)$$

MAP in station $m$ moves from state $k$ to $j$ (no arrival generated)

$$- \sum_{k\neq j}^{m_{A,m}} \Pi_{jkm}^{A0} \left( \int_0^t d_{j,k,m}^{A0} U_{jm}(s) ds \right)$$

MAP in station $m$ moves from state $j$ to $k$ (no arrival generated)

$$- \sum_{k=1}^{m_{A,m}} \sum_{i=1}^{m_{S,m}} \Pi_{jkm}^{A1} \left( \int_0^t d_{j,k,m}^{A1} U_{jm}(s) ds \right)$$

for $1 \leq j \leq m_{A,m}$.

MAP in station $m$ moves from state $j$ to $k$ (arrival generated)
\[ X_{im}(t) = \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \Pi_{jkim}^{A1} \left( \int_0^t d_{jk,m}^{A1} \beta_{im} U_{jm}(s) ds \right) + \sum_{l \neq i}^{m_{S,m}} \Pi_{lim}^S \left( \int_0^t \mu_{lim} X_{im}(s) ds \right) \]

External arrivals into phase i of service

\[ - \sum_{l \neq i}^{m_{S,m}} \Pi_{ilm}^S \left( \int_0^t \mu_{ilm} X_{im}(s) ds \right) \]

Internal transitions from phase i to phase i

\[ - \sum_{l = 1}^{m_{S,n}} N \sum_{n \neq m}^{N} \Pi_{ilmn}^R \left( \int_0^t \mu_{i0m} p_{mn} \beta_{ln} X_{in}(s) ds \right) \]

Routing transitions from phase i to phase l (station m to n)

\[ + \sum_{l = 1}^{m_{S,m}} N \sum_{n \neq m}^{N} \Pi_{limn}^R \left( \int_0^t \mu_{il0n} p_{nm} \beta_{im} X_{in}(s) ds \right) \]

Routing transitions from phase l to phase i (station n to m)

\[ \text{for } 1 \leq i \leq m_{S,m} \]

where \( U_{jm}(t) \) represents the \( j^{th} \) phase of the MAP in the \( m^{th} \) station of the network and \( X_{im}(t) \) is the \( i^{th} \) phase of the phase type distribution in the \( m^{th} \) station of the network at time \( t \).

We assume that each of the Poisson processes in the representation of the \((MAP_t/PH_t/\infty)^N\) queueing network is of unit rate. First we begin by describing the Poisson processes that help generate the arrival process. \( \Pi_{jkim}^{A0}(\cdot) \) counts the number of transitions that a token will make from phase \( k \) to phase \( j \) of the non-arrival producing part of the MAP in station \( m \). \( \Pi_{jkim}^{A1}(\cdot) \) the number of transitions that a token will make from phase \( k \) to phase \( j \) of the arrival producing part of the MAP in station \( m \). \( \Pi_{jkim}^{A1}(\cdot) \) also counts the number of external arrivals to station \( m \) as well. Note that we have that \( \sum_{j=1}^{m_{A}} U_{jm}(t) = 1 \) and \( \sum_{m=1}^{N} \sum_{j=1}^{m_{A}} U_{jm}(t) = N \). Poisson processes, \( \Pi_{jkim}^{A1}(\cdot) \) count the phase transitions from \( j \) to \( k \) of the MAP process with customer arrivals to phase \( i \) of the service process in station \( m \) according to the initial distribution of station \( m \) given by \( \beta_m \). Poisson processes, \( \Pi_{lim}^S(\cdot) \), count the internal transitions within each station from phase \( l \) to phase \( j \) in station \( m \) of the service process. Poisson processes, \( \Pi_{ilmn}^D(\cdot) \) count the number of departures from phase \( i \) in station \( m \) of the service process completely out of the network. Lastly, the Poisson processes \( \Pi_{limn}^R(\cdot) \), count the routing transitions within each station from phase \( i \) in station \( m \) to phase \( l \) in station \( n \) of the service process. For the remainder of the paper, we will use the following notation for the stochastic queue length process and its fluid version.
\[
Q(t) = Q(0) + \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k \neq j}^{m_{A,m}} l^{A0}_{jkm} \Pi^{A0}_{jkm} \left( \int_0^t f^{A0}_{jkm}(s, Q(s)) ds \right) \\
+ \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \sum_{i=1}^{m_{A,m}} l^{A1}_{jkm} \Pi^{A1}_{jkm} \left( \int_0^t f^{A1}_{jkm}(s, Q(s)) ds \right) \\
+ \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} l^{S}_{ilm} \Pi^{S}_{ilm} \left( \int_0^t f^{S}_{ilm}(s, Q(s)) ds \right) + \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} l^{D}_{im} \Pi^{D}_{im} \left( \int_0^t f^{D}_{im}(s, Q(s)) ds \right) \\
+ \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sum_{l=1}^{m_{S,m}} \sum_{n=1}^{m_{S,n}} l^{R}_{ilmn} \Pi^{R}_{ilmn} \left( \int_0^t f^{R}_{ilmn}(s, Q(s)) ds \right),
\]

where we define

\[
Q_m(t) = (U_{1m}(t), \ldots, U_{mA,mm}(t), X_{1m}(t), \ldots, X_{mSm,m}(t)), \\
q_m(t) = (u_{1m}(t), \ldots, u_{mA,mm}(t), x_{1m}(t), \ldots, x_{mSm,m}(t)), \\
q_m = (u_{1m}, \ldots, u_{mA,mm}, x_{1m}, \ldots, x_{mSm,m}), \\
Q(t) = (Q_1(t)', \ldots, Q_N(t)'), \\
q(t) = (q_1(t)', \ldots, q_N(t)'), \\
q = (q_1', \ldots, q_N').
\]

Moreover, we use the following notation for the rate functions (integrands of the Poisson processes) and the jump vectors (a value determining whether a jump is added or subtracted).

\[
f^{A0}_{jkm}(t, q) : \text{rate function of the (integrand) in } \Pi^{A0}_{jkm}(\cdot), \\
f^{A1}_{jkm}(t, q) : \text{rate function of the (integrand) in } \Pi^{A1}_{jkm}(\cdot), \\
f^{S}_{ilm}(t, q) : \text{rate function of the (integrand) in } \Pi^{S}_{ilm}(\cdot), \\
f^{D}_{im}(t, q) : \text{rate function of the (integrand) in } \Pi^{D}_{im}(\cdot), \\
f^{R}_{ilmn}(t, q) : \text{rate function of the (integrand) in } \Pi^{R}_{ilmn}(\cdot),
\]
\[ I_{jkm}^{A0} : \sum_{m=1}^{N} (m_{A,m} + m_{S,m}) \times 1 \text{ vector}, \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + j \right)^{th} \text{ element is -1,} \]
\[ \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + k \right)^{th} \text{ element is 1, and other elements are 0,} \]
\[ I_{jkm}^{A1} : \sum_{m=1}^{N} (m_{A,m} + m_{S,m}) \times 1 \text{ vector}, \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + j \right)^{th} \text{ element is -1,} \]
\[ \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + k \right)^{th} \text{ element is 1,} \]
\[ \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + m_{A,m} + i \right)^{th} \text{ element is 1, and other elements are 0,} \]
\[ I_{ilm}^{S} : \sum_{m=1}^{N} (m_{A,m} + m_{S,m}) \times 1 \text{ vector}, \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + m_{A,m} + i \right)^{th} \text{ element is -1,} \]
\[ \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + m_{A,m} + l \right)^{th} \text{ element is 1, and other elements are 0,} \]
\[ I_{imn}^{D} : \sum_{m=1}^{N} (m_{A,m} + m_{S,m}) \times 1 \text{ vector}, \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + m_{A,m} + i \right)^{th} \text{ element is -1,} \]
\[ \text{and other elements are 0,} \]
\[ I_{imnn}^{R} : \sum_{m=1}^{N} (m_{A,m} + m_{S,m}) \times 1 \text{ vector}, \left( \sum_{r=1}^{m-1} (m_{A,r} + m_{S,r}) + m_{A,m} + i \right)^{th} \text{ element is -1,} \]
\[ \left( \sum_{r=1}^{n-1} (m_{A,r} + m_{S,r}) + m_{A,n} + l \right)^{th} \text{ element is 1, and other elements are 0.} \]

3.1 Fluid Limit

With the Poisson process representation for the \((\text{MAP}_t/\text{PH}_t/\infty)^N\), we now show how we can use the Poisson representation and strong approximations in order to prove fluid limits for the queue length process. First we, define a sequence of stochastic processes \( \{Q^\eta(t), \eta \in \mathcal{N}, t \in \mathcal{R}^+ \} \):
\[
Q^n(t) = Q^n(0) + \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{l=1}^{m_{A,m}} A^{0}_{jk} F_{\bar{l}}^{s}(s, \bar{Q}^n(s)) ds
\]

\[
+ \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \sum_{k \neq j} A^{0}_{jkm} F_{\bar{k}}^{s}(s, \bar{Q}^n(s)) ds
\]

\[
+ \sum_{m=1}^{N} \sum_{m_{S,m}} \sum_{s=1}^{m_{S,m}} \sum_{l \neq i} I^{s}_{ilm} F_{\bar{l}}^{s}(s, \bar{Q}^n(s)) ds
\]

\[
+ \sum_{m=1}^{N} \sum_{n=1}^{m_{S,m}} \sum_{m_{S,m}} \sum_{l \neq i} I^{R}_{ilmn} F_{\bar{l}}^{R}(s, \bar{Q}^n(s)) ds
\]

where we define
\[
\bar{Q}^n(t) = \frac{1}{\eta} Q^n(t).
\]

Note that we accelerate the arrival rate by setting \(\sum_{j=1}^{m_{A,m}} U^n_{jm}(t) = \eta\) and \(\sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} U^n_{jm}(t) = \eta N\) for \(t \geq 0\). Then, the following proposition describes the fluid limits for the \((MAP_i/PH_i/\infty)^N\) queueing network.

**Theorem 3.1.** Suppose \(Q^n(0)/\eta \to q(0)\) almost surely as \(\eta \to \infty\), then
\[
\lim_{\eta \to \infty} \frac{1}{\eta} Q^n(t) = q(t)\text{ almost surely,}
\]

where \(q(t)\) is the solution to the following system of ordinary differential equations:
\[
\frac{dq(t)}{dt} = F(t, q(t))\text{ where}
\]

\[
F(t, q(t)) = \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{l=1}^{m_{A,m}} A^{0}_{jk} F_{\bar{l}}^{s}(t, q(t)) + \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \sum_{k \neq j} A^{0}_{jkm} F_{\bar{k}}^{s}(t, q(t))
\]

\[
+ \sum_{m=1}^{N} \sum_{m_{S,m}} \sum_{s=1}^{m_{S,m}} \sum_{l \neq i} I^{s}_{ilm} F_{\bar{l}}^{s}(t, q(t)) + \sum_{m=1}^{N} \sum_{m_{S,m}} \sum_{l \neq i} I^{R}_{ilmn} F_{\bar{l}}^{R}(t, q(t))
\]

\[
+ \sum_{m=1}^{N} \sum_{n=1}^{m_{S,m}} \sum_{m_{S,m}} \sum_{l \neq i} I^{R}_{ilmn} F_{\bar{l}}^{R}(t, q(t)).
\]

**Proof.** By adding and subtracting the integrand of each Poisson process, we now have the following bound of the difference of the scaled queue length and the fluid limit,
\[
\left| \frac{1}{\eta} Q^n(t) - q(t) \right| \leq |F(t, \bar{Q}^n(t)) - F(t, q(t))| + |V^n(t)|
\]

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where

\[ \eta V^n(t) = \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k \neq j}^{m_{A,m}} 1_{jkm}^{A0} \Pi_{jkm}^{A0} \left( \eta \int_0^t f_{jkm}^{A0}(s, \bar{Q}^n(s)) \right) \]

\[ + \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k \neq j}^{m_{A,m}} \sum_{i=1}^{m_{A,m}} 1_{jkim}^{Al} \Pi_{jkim}^{Al} \left( \eta \int_0^t f_{jkim}^{Al}(s, \bar{Q}^n(s)) \right) \]

\[ + \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} 1_{ilm}^{S,i} \Pi_{ilm}^{S,i} \left( \eta \int_0^t f_{ilm}^{S,i}(s, \bar{Q}^n(s)) \right) + \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \Pi_{im}^{D,i} \left( \eta \int_0^t f_{im}^{D,i}(s, \bar{Q}^n(s)) \right) \]

\[ + \sum_{m=1}^{N} \sum_{n=1}^{m_{S,n}} \sum_{i=1}^{m_{S,n}} \sum_{l \neq i}^{m_{S,n}} 1_{ilmn}^{R,i} \Pi_{ilmn}^{R,i} \left( \eta \int_0^t f_{ilmn}^{R,i}(s, \bar{Q}^n(s)) \right) \]

and where we define \( \Pi(\cdot) \) as

\[ \Pi \left( \eta \int_0^t f(s, \bar{Q}^n(s))ds \right) = \Pi \left( \eta \int_0^t f(s, \bar{Q}^n(s))ds \right) - \eta \int_0^t f(s, \bar{Q}^n(s))ds. \]

By the Lipschitz continuity of the function \( F(t, \cdot) \), we have that

\[ \left| \frac{1}{\eta} Q^n(t) - q(t) \right| \leq \left| \frac{1}{\eta} Q^n(0) - q(0) \right| + C \int_0^t \left| \frac{1}{\eta} Q^n(s) - q(s) \right| ds + |V^n(t)|. \]

Now by the strong approximation results for Poisson processes in Kurtz [10] and the fact that integrands of the Poisson processes are Lipschitz continuous, we have for \( \eta \) large enough

\[ \varepsilon \geq \left| \frac{1}{\eta} Q^n(0) - q(0) \right| + |V^n(t)|. \]

This implies that

\[ \left| \frac{1}{\eta} Q^n(t) - q(t) \right| \leq \varepsilon + C \int_0^t \left| \frac{1}{\eta} Q^n(s) - q(s) \right| ds, \]

which yields the fluid limit result by using Gronwall’s lemma.

3.2 Diffusion Limit

Now that we have the fluid limit, \( q(t) \), we can derive the diffusion limit as follows:

**Theorem 3.2.** Let \( D^n(t) = \sqrt{\eta} \left( \frac{1}{\eta} Q^n(t) - q(t) \right) \) and suppose that \( \sqrt{\eta} \left( \frac{1}{\eta} Q^n(0) - q(0) \right) \Rightarrow D(0) \) in distribution as \( \eta \to \infty \), then we have that

\[ \lim_{\eta \to \infty} D^n(t) = D(t) \]
Proposition 3.3.
The sequence of stochastic processes equations. The following Proposition 3.3 proves this result.

\[ dD(t) = dH(t, q(t)) + \partial F(t, q(t))D(t)dt, \]

and \( \partial F(t, q(t)) \) is the gradient matrix of \( F(t, q(t)) \) with respect to \( q(t) \). Moreover,

\[
\begin{align*}
  dH(t, q(t)) &= \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k \neq j}^{m_{A,m}} \eta_{jkm} \sqrt{f_{jkm}^{A0}(t, q(t))}dW_{jkm}^{A0}(t) \\
  &+ \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \eta_{jkm} \sqrt{f_{jkm}^{A1}(t, q(t))}dW_{jkm}^{A1}(t) \\
  &+ \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} \sqrt{f_{ilm}^{S}(t, q(t))}dW_{ilm}^{S}(t) + \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sqrt{f_{ilm}^{D}(t, q(t))}dW_{ilm}^{D}(t) \\
  &+ \sum_{m=1}^{N} \sum_{n=1}^{m_{S,m}} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} \sqrt{f_{ilmn}^{R}(t, q(t))}dW_{ilmn}^{R}(t).
\end{align*}
\]

where \( W_{jkm}^{A0}(t), W_{jkm}^{A1}(t), W_{ilm}^{S}(t), W_{ilm}^{D}(t), W_{ilmn}^{R}(t) \) are mutually independent standard Brownian motions.

Proof. In order to construct the diffusion limit, we need to subtract the fluid limit and multiply by \( \sqrt{\eta} \). This yields the following expression for \( D^\eta(t) \)

\[
D^\eta(t) = \sqrt{\eta} \left( \frac{1}{\eta}Q^\eta(t) - q(t) \right) = \sqrt{\eta} \cdot (F(t, Q^\eta(t)) - F(t, q(t))) + \sqrt{\eta} \cdot V^\eta(t).
\]

Now that we have related the centered Poisson processes with time changed Brownian motions, it remains for us to show that the fluid scaled randomly time changed Brownian motion terms converge to Brownian motions time changed with the deterministic fluid equations. The following Proposition 3.3 proves this result.

Proposition 3.3. The sequence of stochastic processes \( \sqrt{\eta} \cdot V^\eta(t) \) converges in distribution to the process \( M(t) \) where

\[
M(t) = \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k \neq j}^{m_{A,m}} \eta_{jkm} \sqrt{f_{jkm}^{A0}(s, q^\eta(s))}f_{jkm}^{A0}(t, q^\eta(s)) \\
+ \sum_{m=1}^{N} \sum_{j=1}^{m_{A,m}} \sum_{k=1}^{m_{A,m}} \sum_{i=1}^{m_{A,m}} \eta_{jkm} \sqrt{f_{jkm}^{A1}(s, q^\eta(s))}f_{jkm}^{A1}(t, q^\eta(s)) \\
+ \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} \sqrt{f_{ilm}^{S}(s, q^\eta(s))}f_{ilm}^{S}(t, q^\eta(s)) + \sum_{m=1}^{N} \sum_{i=1}^{m_{S,m}} \sqrt{f_{ilm}^{D}(s, q^\eta(s))}f_{ilm}^{D}(t, q^\eta(s)) \\
+ \sum_{m=1}^{N} \sum_{n=1}^{m_{S,m}} \sum_{i=1}^{m_{S,m}} \sum_{l \neq i}^{m_{S,m}} \sqrt{f_{ilmn}^{R}(s, q^\eta(s))}f_{ilmn}^{R}(t, q^\eta(s)).
\]
Proof. This follows from the strong approximations for Poisson processes in Kurtz [10] and the fact that Brownian motion is Holder continuous.

**Proposition 3.4.** Suppose that we define \( \tilde{D}^n(t) \) as

\[
\tilde{D}^n(t) \equiv \int_0^t \partial F(s, q(s)) \tilde{D}^n(s) ds + V^n(t), \tag{3.4}
\]

then

\[
\lim_{\eta \to \infty} \sup_{0 \leq t \leq T} |D^n(t) - \tilde{D}^n(t)| = 0 \text{ in probability.} \tag{3.5}
\]

We know by the continuous mapping theorem and Proposition 3.3, which shows that \( \eta \cdot V^n(t) \) converges to \( M(t) \) in Equation 3.3, then we know that that \( \tilde{D}^n(t) \) converges to \( \tilde{D}(t) \). Thus, in order to finally prove our diffusion limit result in Theorem 3.2, it suffices to show that

\[
\lim_{\eta \to \infty} \sup_{0 \leq t \leq T} |D^n(t) - \tilde{D}^n(t)| = 0 \text{ in probability.} \tag{3.6}
\]

To prove this, we define the difference between the two processes as

\[
E^n(t) = D^n(t) - \tilde{D}^n(t)
\]

\[
= \sqrt{\eta} \cdot \left( F(t, \bar{Q}^n(t)) - F(t, q) \right) + V^n(t) - \left( \int_0^t \partial F(s, q(s)) \tilde{D}^n(s) ds + V^n(t) \right)
\]

\[
= \sqrt{\eta} \cdot \left( F(t, \bar{Q}^n(t)) - F(t, q) \right) - \int_0^t \partial F(s, q(s)) \tilde{D}^n(s) ds
\]

\[
= \int_0^t \partial F(u, q(u)) E^n(u) + \sqrt{\eta} \cdot \left( F(t, \bar{Q}^n(t)) - F(t, q(t)) \right) - \int_0^t \partial F(u, q(u)) D^n(u)
\]

\[
= \int_0^t (\partial F(u, \xi^n(u)) - \partial F(u, q(u))) D^n(u) + \int_0^t \partial F(u, q(u)) E^n(u) du.
\]

Thus, by the mean value theorem, the fact that the rate functions in the Poisson representations are continuously differentiable, stochastic boundedness of \( D^n(t) \), and the fluid limit convergence, we obtain our diffusion limit result by applying Gronwall’s lemma.

\[\square\]

4 Conclusion and Future Research

In this paper, we analyze the \( (MAP_t/PH_t/\infty)^N \) queueing network and prove fluid and diffusion limits for the queue length processes. It is critical to fully understanding the behavior of the \( (MAP_t/PH_t/\infty)^N \) queueing network before we can begin to understand networks such as the \( (MAP_t/MAP_t/\infty)^N \) and \( (MAP_t/MAP_t/n_t + MAP_t)^N \) networks since the infinite server case serves as a lower bound and represents the offered load of the system with unlimited resources. As future work in the context of finite server and abandonment settings, it is interesting to apply the methods of Ko and Gautam [8], Massey and Pender [11, 12], Engblom and Pender [3], Pender [13, 14, 15, 16] to these non-Markovian networks since we know the fluid and diffusion limits are not as accurate in this setting.
References


