Stability in a 2-Station Re-Entrant Line under a Static Buffer Priority Policy and the Role of Cross-Docking in the Semiconductor Industry

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Stability in a 2-Station Re-Entrant Line under a Static Buffer Priority Policy and the Role of Cross-Docking in the Semiconductor Industry

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Preface

This dissertation addresses aspects within two different areas of the semiconductor industry: Manufacturing process optimization and supply chain management.

The first part examines re-entrant lines motivated by manufacturing lines in chip production. Re-entrant lines are queueing networks in which jobs are processed several times by the same server. We call the nth processing step in the network the nth class of the network. If jobs from different classes are to be served by the same server, a dispatch policy decides which job is processed next.

Such queueing networks are extremely difficult to study. In particular, the determination of rate stability, or whether the input rate into the system in the long-term equals the output rate of such networks, is often hopeless.

Stochastic queueing networks can be approximated by deterministic fluid models, replacing the discrete job arrivals by continuous fluid flow. The results found in the fluid model can then be interpreted in the original queueing network setting. In particular stability in the fluid setting implies, under some technical conditions, rate stability in the original queueing network. In the fluid model, stability means that starting with any initial fluid levels, the system empties in finite time.

The dissertation gives a first insight how to show stability in a 2-station re-entrant line under a fixed static buffer priority policy.

We examine a 2-station fluid network under a last-buffer-first-served (LBFS) dispatch policy at the first station and a first-buffer-first-served (FBFS) dispatch policy at the second station. We show that the fluid model is stable if it fulfills the so-called usual workload and virtual station conditions.

We hope that the methods we developed can also be applied to other static buffer priority dispatch policies.

The second part of the dissertation examines the motives for cross-docking in semiconductor supply chain management. Cross-docks are distribution centers that allow little or no storage of shipments.

We inspect the peculiar needs of semiconductor distribution networks and show how cross-docking can provide companies with a competitive edge. We illustrate our arguments with examples at Intel Corporation, Agilent Technologies Incorporated and National Semiconductors Corporation.

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Chapter 1

Introduction

While 15 years ago competitive advantage in the $150 billion semiconductor industry was created mostly by being the first with the newest technology and chip design, today the industry is more price competitive, feeling the repercussions of computer chips becoming a commodity product: Profit margins are decreasing while quality demands are on the rise. Hence companies need to do more than just increase the performance of their products in a market in which, according to Moore's Law, the capacity of microchips doubles and prices halve every 18 months. They must also achieve greater efficiencies in production and in the supply chain to remain competitive.

The manufacturing process for chips can be split into 2 parts: Front-end production and back-end production.

In the front-end production the layout of several chips is put on a (usually) silicon disk, the wafer, through a repeated photo-chemical process. This is done in wafer fabs (production facilities) that typically cost around 2 billion dollars (these costs are expected to quintuple by 2010).

In test and assembly facilities the chips are then packed into a ceramic or plastic package and end-tested. This process is called back-end production.

The semiconductor industry is one of the most globalized industries. Blue chip semiconductor companies usually own wafer fabs as well as test and assembly sites in well-developed countries in America, Europe and Asia. From there, chips are sent via airfreight and trucking to customers in telecommunications, automotive, computers and peripherals, consumer electronics, industrial equipment, and others.

Industrial engineering research opportunities in the semiconductor industry are manifold. This dissertation tackles manufacturing process optimization and cross-docking, aspects in two quite different areas within the semiconductor industry.

Manufacturing process optimization

Manufacturing lines in front-end chip production can be roughly modelled as re-entrant queueing networks. In such networks, jobs are processed several times by the same station.

When several concurrent jobs are to be served at the same station, a dispatch policy decides which job is to be served first. Examples of dispatch policies include those that have a fixed ranking at a station depending on the class of a job (we call the nth processing step in the network the nth class of the network). These policies are called static buffer priority policies (SBP).

The re-enterant feature makes such queueing systems very difficult to study. Fluid models of these networks offer the possibility of providing some insight. In particular the stability of the fluid model is linked to the stability of the corresponding queueing network.

Fluid models are continuous, deterministic approximations of discrete, stochastic queueing networks. Discrete jobs are replaced by continuous fluid. In this way, fluid models resemble queueing networks by compressing time and magnifying buffer levels. Although simpler than the original queueing networks, fluid models are complex in their own right.

A fluid model is globally stable if, starting from any given initial fluid levels, the fluid network drains in finite time under all non-idling dispatch policies. For fluid models of 2-station re-entrant lines, it is known that satisfying three sets of conditions known as the usual workload, virtual station and push-start conditions is necessary and sufficient for global stability.

The usual workload conditions state that the total workload at a station should not exceed its capacity. They are independent of the dispatch policy emplyed.

The virtual station conditions reflect that under some policies certain classes cannot be served simultaneously although they are served by different stations (we say that these classes form a "virtual station").

Push-start conditions magnify the effect of virtual stations when a policy gives high priority to classes in the first stages.

In the first part of the dissertation we consider a 2-station re-entrant line with a static buffer priority policy fixed to a last-buffer-first-served policy (LBFS) at station A and a first-buffer-first-served policy (FBFS) at station B. We prove that the usual workload conditions and the LBFS-FBFS-induced virtual station conditions are sufficient to show that the fluid network under this policy is stable.

The tools used in the proof may be useful in analyzing stability under other SBP priority policies.

The role of cross-docks in the semiconductor industry supply chain

In traditional warehouses goods are received, put-away, stored and then — after being ordered — picked, consolidated and shipped. In a cross-dock the goods enter the facility and are directly consolidated and shipped to their destination. Typically, they spend less than 12 hours in the cross-dock.

Cross-docking offers the opportunity for fast freight consolidation and distribution without the expense of storing inventory.

In the second part of the dissertation we discuss the value in cross-docking of finished goods for the semiconductor industry. We start with the drivers that since the second half of the 1990s have forced many semiconductor companies to rethink their distribution network. We then document key features of semiconductor distribution networks.

We discuss the effects of cross-docking on semiconductor supply chain management, in particular on supply chain flexibility, shipping costs, fees/taxes/customs, international hurdles, quality service and security. We illustrate these with examples at Intel Corporation, Agilent Technologies Incorporated and National Semiconductors Corporation.

Finally we take a look at alternatives and hurdles for cross-docking and we consider whether a semiconductor company should own or outsource its cross-dock operations.

Chapter 2

Part I: 2-Station Re-Entrant Lines

2.1 Introduction and overview of part I

The photochemical process in a wafer fab requires a wafer (or, typically, a "lot" of wafers) to be served several times by the same machine (or station) during the course of the production process. Such systems are called re-entrant queueing networks or re-entrant lines and often exhibit counterintuitive behavior. We demonstrate this in a simple 2-station, 5-class setting.

In the example of Figure 1, Jobs (or, in the semiconductor environment, lots) enter the network at an average arrival rate of λ and undergo 5 consecutive processing steps before they leave the system. We refer to jobs waiting for or receiving the ith processing step as class i jobs. Class i jobs reside in buffer i, which has unlimited capacity. Classes 1, 3 and 5 are served by station A while classes 2 and 4 are served by station B. The average processing time of a class i job is $m_i > 0$.

If a station finds jobs in more than one of its classes, a rule is needed to decide which class to serve first. Such rules are called dispatch policies or service disciplines.

We consider only non-idling static buffer priority policies (SBP policies or SBP disciplines), or policies that

Figure 1: A 5-class 2-station re-entrant line

- never let a station idle when there are jobs in its buffers.

- fix a ranking of the classes, and serve jobs of higher ranked classes first.

For example, a last-buffer-first-served (LBFS) gives high priority to higher numbered classes. A first-buffer-first-served (FBFS) policy gives high priority to lower numbered classes. We call a 2 station re-entrant queueing network under a policy that uses a LBFS policy at station A and a FBFS policy at station B a LBFS-FBFS re-entrant line.

We always assume the policy to be preemptive resume, i.e. when higher priority jobs arrive, the station interrupts its work on a lower priority job. The station later returns, continuing its work on the lower priority job from the point when it left off.

An important question in process optimization is whether the system is rate stable, i.e. whether in the long run, the output rate from the system is equal to the input rate — for an exact definition see Dai [4].

A necessary condition for rate stability is that the workload at each station is at most 100%. We call this the usual workload condition for a station. In the example of Figure 1 this means $\lambda(m_1+m_3+m_5) \leq 1$ and $\lambda(m_2+m_4) \leq 1$. Station A must spend on average $m_1 + m_3 + m_5$ time units on each job, limiting the average output rate to $1/(m_1 + m_3 + m_5)$ jobs per time unit. If jobs arrive at average rate $\lambda > \frac{1}{(m_1+m_3+m_5)}$, then the queueing network is not rate stable.

Perhaps surprisingly, the usual workload conditions are not sufficient to ensure that the system is rate stable. For example, suppose the processing times of the network in Figure 1 are $(0.1, 0.6, 0.1, 0.2, 0.6)$ and the arrival rate is $\lambda = 1$. Let station A follow a last-buffer-first-served (LBFS) dispatch policy and station B follow a first-buffer-first-served (FBFS) dispatch policy. Although the network satisfies the usual workload conditions: $\lambda(m_1 + m_3 + m_5) < 1$ and $\lambda(m_2 + m_4) < 1$, the system is not rate stable. Stations A and B can serve classes 2 and 5 simultaneously only during a transient initial period: Once buffer 2 empties, no new class 2 jobs can arrive while there are class 5 jobs and vice versa. Jobs can leave the system at rate of at most $1/(0.6 + 0.6) = 1/1.2$. Since the average arrival rate is 1, work arrives faster than the system can complete it and so the network is not rate stable.

Re-entrant queueing networks are often too complex for successful analysis. Even the determination of rate stability is often hopeless. Fluid models provide a possible way out, at least for some important questions like rate stability of a system.

A fluid model is a continuous, deterministic approximation of the queueing network. Discrete jobs arriving at random time points are replaced by continuous fluid moving deterministically through the system. Buffers containing jobs are replaced by buffers containing fluid, but one has to be careful: Drawing a too strong analogy between queue lengths in the queueing network and buffer levels in the fluid model is misleading and mathematically incorrect.

Many results in the fluid model support analogous conclusions in the queueing network setting.

The fluid model consists of equalities and inequalities derived from queueing network equations, see for example Dai [4]. These equations describe the correlations between the fluid levels of the buffers, the work time of the servers and their idle time.

In our example, at any time the fluid level of a buffer k is the sum of initial fluid in buffer k and the arrival of fluid from the previous buffer less the amount of fluid that was processed in class k and the fluid level is always non-negative. The idle time of any station is the actual time less its work time and it is non-decreasing. The work time is non-decreasing and the work time and idle time of any station are always nonnegative.

Also static buffer priority policies are described through equations stemming from the queueing network. They state in essence that a server assigns its work time not to classes with less priority than the highest priority nonempty buffer, that a server does not idle if there is fluid in one of its buffers and that the fluid input into a buffer is the same as the output if it has higher priority than the actual highest priority non-empty buffer.

A solution to this equation system is called a fluid solution. A fluid solution is generally not unique and may sometimes even not exist.

Taking limits as we expand buffer levels and compress time leads in a queueing network to "fluid limits" which are solutions to the fluid model equations. A detailed description of the connection between queueing systems and their corresponding fluid models is beyond the scope of this dissertation. For reference, see for example Rybko and Stolyar [11], Chen [2], Dai and Meyn [6], Stolyar [12], Meyn [10], Dai [9] and Bramson [1].

We restrict ourselves to the examination of rate stability in a 2-station re-entrant line. In the following we sketch the relevant connections between queueing networks and their corresponding fluid models. Whenever appropriate, we refer to known theorems to interpret fluid model results in the queueing network setting.

The fluid model is said to be stable if, starting from any initial fluid levels, every fluid solution empties the network after some finite time. The fluid model is said to be unstable if it is not stable, i.e. if there are initial fluid levels and a fluid solution such that the system contains fluid at arbitrarily large time points (such a fluid solution is called an unstable fluid solution).

The fluid model is said to be weakly stable if, starting empty, the system remains empty for all future times. The fluid model is said to be weakly unstable if there is some positive time $\delta > 0$ such that starting from an initially empty system, every fluid solution is nonempty at time δ. Note: Being weakly unstable implies not being weakly stable but the converse is generally not true.

Dai shows in [4] under additional technical conditions that if the fluid model of a queueing network is weakly stable then the queueing network is rate stable and if the fluid model is weakly unstable then the queueing network is not rate stable. This result reduces (at least in part) the complicated study of rate stability in re-entrant queueing networks to examining stability in the corresponding fluid models.

In this dissertation, we consider the fluid model of a 2-station re-entrant line, in which a single type of fluid enters the system and passes through classes 1, 2, 3,...,N, N odd, before exiting the system. The odd numbered classes are served by station A, the even numbered classes are served by station B. Assuming the station dedicates 100% of its work time to class k, m_k denotes the processing time for one unit fluid. We normalize the input rate λ into the system to 1.

The usual workload conditions stipulate that the workload at each station be less then 100%, i.e. $\lambda(\sum_{k \text{ class of A}} m_k) = \lambda(m_1 + m_3 + m_5 + \ldots + m_N) < 1$ and $\lambda(\sum_{k \text{ class of B}} m_k) = \lambda(m_2 + m_4 + m_6 + \ldots + m_{N-1}) < 1$.

The virtual station conditions arise from the fact that under some polices certain classes cannot be served simultaneously although they are served by different stations. Such sets of classes form what is called a "virtual station". Different dispatch policies induce different virtual stations.

For example, under the LBFS-FBFS priority policy the 5 class network in Figure 1 exhibits a virtual station consisting of classes 2 and 5. In general, a virtual station in a LBFS-FBFS re-entrant line consists of classes $2,\ 4,\ 6, \ldots,\ k-2,\ k+1,\ k+3, \ldots,\ N$ for any even integer $k > 2$. For a thorough discussion of virtual stations, see Dai and Vande Vate [3] and [7].

Similar to the usual workload conditions, the virtual station conditions for a virtual station $\mathbb{V}$ stipulate that the workload at $\mathbb{V}$ is less then 100%, i.e. $\lambda(\sum_{k \text{ class of } \mathbb{V}} m_k)$ < 1. The virtual station conditions depend on the dispatch policy that is used. In our 5 buffer LBFS-FBFS example $m_2 + m_5$ would be required to be less than 1 and in the N buffer case the virtual station conditions are $\lambda(m_2 + m_4 + m_6 + \ldots + m_{k-2} + m_{k+1} + m_{k+3} + \ldots + m_N) < 1$, for each even numbered class k greater than 2.

Push-start conditions arise when the lowest numbered classes are given highest priority. They magnify the effects of virtual stations. When highest priority is given to the lowest numbered classes, their buffers empty after some initial time and stay empty. Thereafter, fluid passes through the initial classes and enters the remaining classes. The remaining classes behave like a network with processing times that are modified to reflect the effort required by the initial classes (see Dai, Hasenbein and Vande Vate [5] and Dai and Vande Vate [3]). Under an LBFS-FBFS policy there are no push-start conditions since station A gives highest priority to the highest numbered classes.

In [3] Dai and Vande Vate show that the fluid model of a 2-station re-entrant line is stable under any non-idling dispatch policy if and only if the processing times and the arrival rate satisfy the usual workload conditions, and the virtual station and push-start conditions of any SBP policy. This result also suggests the importance of SBP policies to the study of rate stability.

In [7], Dai and Vande Vate show that under a fixed SBP policy, the usual workload conditions, the induced virtual station and push-start conditions are necessary for the stability of the system under this policy. Unfortunately, sufficiency remains a difficult open problem.

This dissertation provides a first insight how to show stability in a 2-station re-entrant line under a fixed static buffer priority policy. We show that under a LBFS-FBFS policy, the usual workload conditions together with the LBFS-FBFS induced virtual station conditions imply the stability of the system (we refer to this statement as the Stability Theorem).

We believe that the techniques we develop for proving the Stability Theorem in Sections 2.4 to 2.8 can be used for proving a similar result for other static buffer priority disciplines.

In Section 2.2 we formalize the fluid model of the LBFS-FBFS re-entrant line and state the Stability Theorem. Section 2.3 outlines the proof of the Stability Theorem. Sections 2.4-2.8 provide a detailed proof. Finally, Section 2.9 concludes the stability proof and discusses possible future research.

2.2 The fluid model of a LBFS-FBFS re-entrant line

In this section we formalize the fluid model of the LBFS-FBFS re-entrant line.

Fluid enters the network through class 1 at station A and is served in classes 2, 3, 4, ..., N consecutively before it leaves the system. The even numbered classes

Figure 2: Layout of a LBFS-FBFS re-entrant line

are served by station A, the odd numbered classes are served by station B, see Figure 2. Fluid that awaits service in class k is referred to as class k fluid. It is stored in buffer k.

We define $A := \{1, 3, 5, \ldots, N\}$ and $B := \{2, 4, 6, \ldots, N-1\}$. Note that we use A and B both for naming the stations and to identify the sets of classes that each station serves. We define by $\sigma(k)$ the station that serves class k, i.e. $\sigma(k) := A$ for $k \in A$ and $\sigma(k) := B$ for $k \in B$.

Each station allocates its capacity among the classes it serves. We denote by $T_k(t) \leq t$ the work time station $\sigma(k)$ devotes to class k in the time interval $[0, t]$. $T_k(t)$ is a non-decreasing function in t.

We say that station Γ works 100% in the time interval $[t_0, t_1]$ if $\sum_{k \in \Gamma}(T_k(t_1) - T_k(t_0)) = t_1 - t_0$.

Given a set of buffers K, we say station Γ spends 100% of its work time on classes in K in the time interval $[t_0, t_1]$ (or, Γ works exclusively and without idling on buffers in K) if $\sum\limits_{j \in K \cap \Gamma} T_j(t_1) - T_j(t_0) = t_1 - t_0$.

Processing one unit of class k fluid ($1 \leq k \leq N$) requires m_k units of work time from station $\sigma(k)$. Therefore, we call m_k the service time for class k. The service rate $\mu_k := 1/m_k$ is the reciprocal of m_k.

Let $Q_k(t)$ denote the amount of class k fluid at time t or, equivalently, the fluid level in buffer k at time t. The buffer level $Q_k(t)$ must always be non-negative. If $Q_k(t) = 0$, we say buffer k is empty (or drained).

We let class 0 represent the exogenous input into the system. We normalize the rate of exogenous input so that $\mu_0 = \frac{1}{m_0} = \lambda = 1$ and set $T_0(t) = t$ to model the constant input of fluid into the network at rate 1.

We denote by $U_\Gamma(t)$ the cumulative idle time at station Γ in the interval $[0,t]$. Thus,

$$\begin{aligned}
&U_A(t) := t - \sum_{k\in A} T_k(t),\\
&U_B(t) := t - \sum_{k\in B} T_k(t) \text{ and}\\
&U_A(t) \text{ and } U_B(t) \text{ are non-decreasing in } t.
\end{aligned}$$

For each class $k \in \{1,\ldots,N\}$, $\mu_{k-1}T_{k-1}(t)$ is the amount of fluid that arrives by time t from class $k-1$ and $\mu_k T_k(t)$ is the amount of class k fluid processed by time t, hence $Q_k(t) = Q_k(0) + \mu_{k-1}T_{k-1}(t) - \mu_k T_k(t)$ defines the relationship between the fluid levels Q and the allocations T.

Definition 2.2.1

Let $Q_k(0) \geq 0$ ($k \in \{1,\ldots,N\}$) and $\mu_k > 0$ ($k \in \{0,\ldots,N\}$) be given.
A pair $(Q(\cdot), T(\cdot))$ where

$$Q : \mathbb{R}^+ \longrightarrow \mathbb{R}^N \textit{ with } Q(\cdot) = (Q_k(\cdot))_{1\leq k\leq N} \tag{1}$$

$$T : \mathbb{R}^+ \longrightarrow \mathbb{R}^{N+1} \textit{ and } T(\cdot) = (T_k(\cdot))_{0\leq k\leq N} \tag{2}$$

satisfying

$$T_0(t) = t$$

$$Q_k(t) = Q_k(0) + \mu_{k-1}T_{k-1}(t) - \mu_k T_k(t) \tag{3}$$

$$Q_k(t) \geq 0 \tag{4}$$

$$T_k(t) \textit{ is non-decreasing} \tag{5}$$

$$U_A(t) = t - \sum_{k \in A} T_k(t) \text{ is non-decreasing} \tag{6}$$

$$U_B(t) = t - \sum_{k \in B} T_k(t) \text{ is non-decreasing} \tag{7}$$

for all $t \geq 0$ and $k = 1, \ldots, N$, is said to be a fluid solution to the 2-station re-entrant line. Restricting (1)–(7) to the time interval $[t_0, t_1]$, we say that $(Q(\cdot), T(\cdot))$ is a fluid solution for the time interval $[t_0, t_1]$.

Remark: Given $Q(0)$, $T(\cdot)$ can be calculated from $Q(\cdot)$ via equation (3). Hence it is sufficient to characterize a fluid solution $(Q(\cdot), T(\cdot))$ by $Q(\cdot)$ or by $T(\cdot)$. We occasionally refer to the fluid solution $Q(\cdot)$ or to the fluid solution $T(\cdot)$.

One special case of this equivalence arises when no fluid enters class k during a time interval (t_0, t_1). In this case $T_k(t_1) - T_k(t_0) = m_k \cdot (Q_k(t_1) - Q_k(t_0))$

Lemma 2.2.2

Let $(Q(\cdot), T(\cdot))$ be a fluid solution. Conditions (3) - (7) imply that T_k and Q_k ($k \in \{1, \ldots, N\}$) are absolutely continuous.

Proof: For a proof that fluid models are absolutely continuous, see for example Dai and Weiss [8].

□

Lemma 2.2.2 implies that for any fluid solution, $T(\cdot)$ and $Q(\cdot)$ are differentiable for almost all t in $(0, \infty)$ — see, for example, Dai and Weiss [8]. We say that t is a regular point of $(Q(\cdot), T(\cdot))$ if $T(\cdot)$ is differentiable at t (otherwise we refer to t as an irregular point of $(Q(\cdot), T(\cdot))$). We denote the derivatives of Q and T at time t with

$\dot{Q}(t)$ and $\dot{T}(t)$. If the fluid solution is clear from context, we simply refer to t as a regular or an irregular point.

Notation: For a regular time point t, let $d_k(t)$ represent the rate of departure from class k,

$$d_k(t) := \mu_k \dot{T}_k(t) \tag{8}$$

Note that by definition, $d_0(t) = \lambda = 1$ for all times $t \geq 0$.

We denote by $d(\cdot) := (d_i(\cdot))_{i \in \{0,\ldots,N\}}$ the departure rates associated with a fluid solution $(Q(\cdot), T(\cdot))$.

Remark: By differentiating equation (3) we see that for each regular time t:

$$\dot{Q}_k(t) = \mu_{k-1}\dot{T}_{k-1}(t) - \mu_k \dot{T}_k(t) = d_{k-1}(t) - d_k(t) \tag{9}$$

for each class $k \in \{1, \ldots, N\}$

Proposition 2.2.3 reformulates (5), (6) and (7) in terms of the departure rates $d(t)$.

Proposition 2.2.3 *Let $(Q(\cdot), T(\cdot))$ be a fluid solution. (5), (6) and (7) imply for all regular points t:*

$$d(t) \geq 0 \tag{10}$$

$$\sum_{k \in A} m_k d_k(t) \leq 1 \tag{11}$$

$$\sum_{k \in B} m_k d_k(t) \leq 1 \tag{12}$$

Proof: Let $(Q(\cdot), T(\cdot))$ be given. For each class $k \in \{1, \ldots, N\}$, $T_k(\cdot)$ is non-decreasing. Therefore for all regular times $t \geq 0$, $0 \leq \dot{T}_k(t) = m_k d_k(t)$. Hence (10) holds for all regular points.

For each class $k \in \{1, \ldots, N\}$ and station $\Gamma \in \{A, B\}$, the fact that $U_\Gamma(\cdot)$ is non-decreasing implies that for regular times $t \geq 0$:

$$0 \leq \dot{U}_\Gamma(t) = 1 - \sum_{k \in \Gamma} \dot{T}_k(t) = 1 - \sum_{k \in \Gamma} m_k d_k(t).$$

Hence (11) and (12) hold for all regular times $t \geq 0$.

$\square$

We define by $M_A := \sum_{k \in A} m_k,\ M_B := \sum_{k \in B} m_k$ the workloads at station A and B.

For each class $k \in \{1, \ldots, N\}$, $M_A(k) := \sum_{l \in A, l < k} m_l$ denotes the workload at station A prior to class k. Similarly, $M_B(k) := \sum_{l \in B, l < k} m_l$ is the workload at station B prior class k.

The LBFS-FBFS priority policy stipulates that station A give higher priority to higher numbered classes (LBFS) while station B give higher priority to lower numbered classes (FBFS). As we shall see, the fluid solution under this policy is largely determined by the highest priority nonempty buffer at each station.

Given a fluid solution $(Q(\cdot), T(\cdot))$, let $a(Q(t))$ be the highest priority (highest numbered) nonempty buffer (hpn buffer) at station A at time t. If all the buffers at station A are empty at time t, we define $a(Q(t)) := 0$. Similarly, we define by $b(Q(t))$ the highest priority (lowest numbered) nonempty buffer at station B at time t. If all buffers at station B are empty at time t, we define $b(Q(t)) := N + 1$.

When we speak about a "state at time t" of a valid fluid solution $(Q(\cdot), T(\cdot))$, we generally mean the buffer levels at time t, $Q(t)$. As we shall see, the behavior of the system is largely dependent on its hpn buffers. Therefore we sometimes refer to a state as "state (a, b) at time t". With this we mean a state $Q(t)$ with hpn buffers $a = a(Q(t)) \in A \cup \{0\}$ and $b = b(Q(t)) \in B \cup \{N+1\}$. Furthermore, when we say "regular state" we mean a state at a regular time point.

Definition 2.2.4 *A fluid solution $(Q(\cdot), T(\cdot))$ is called LBFS-FBFS feasible or a fluid solution under the LBFS-FBFS policy if for all regular times $t \geq 0$, the rates $d(t) = (d_0(t), d_1(t), \ldots, d_N(t))$, satisfy*

$$\begin{aligned}
\sum_{k \in A,\ k \geq a} m_k d_k(t) &= \sum_{k \in A,\ k \geq a} \dot{T}_k(t) = 1 \quad \textit{if } a > 0 && (13) \\
d_k(t) &= 0 \quad \textit{for each class } k \in A,\ k < a && (14) \\
d_k(t) &= d_{k-1}(t) \quad \textit{for each class } k \in A,\ k > a && (15)
\end{aligned}$$

$$\begin{aligned}
\sum_{k \in B,\ k \leq b} m_k d_k(t) &= \sum_{k \in B,\ k \leq b} \dot{T}_k(t) = 1 \quad \textit{if } b < N+1 && (16) \\
d_k(t) &= 0 \quad \textit{for each class } k \in B,\ k > b && (17) \\
d_k(t) &= d_{k-1}(t) \quad \textit{for each class } k \in B,\ k < b && (18)
\end{aligned}$$

where $a = a(Q(t))$ and $b = b(Q(t))$.

Remark: Equations (13) through (18) have the following interpretation:

Constraints (13) and (16) stipulate that if there is fluid in buffer k then station $\sigma(k)$ allocates its entire work time to class k and higher priority classes.

Constraints (14) and (17) state that no work time is allocated to classes of lower priority than the hpn buffer at a station — these classes have no output.

Constraints (15) and (18) ensure that no buffer of priority higher than the hpn buffer of a station can fill — the input rate into each of these classes is equal to the output rate.

Definition 2.2.5 *For each class* $k \in B \setminus \{2\}$ *we define the virtual station* $\mathbb{V}(k) := \{j \in B,\ j < k\} \cup \{j \in A,\ j > k\}$. *Furthermore, we define* $\mathbb{V}(0) = A$ *and* $\mathbb{V}(N+1) = B$.

Definition 2.2.6 *A feasible fluid solution under LBFS-FBFS policy is called valid if for all regular* $t > 0$ *it fulfills the virtual station constraints on the fluid model, i.e. if for all* $l \in B \setminus \{2\}$

$$\sum_{k \in \mathbb{V}(l)} m_k d_k(t) = \sum_{k \in \mathbb{V}(l)} \dot{T}_k(t) = \sum_{k \in A,\ k > l} m_k d_k(t) + \sum_{k \in B,\ k < l} m_k d_k(t) \leq 1 \qquad (19)$$

Similar to Definition 2.2.1, we allow to restrict a valid fluid solution to a time interval $[t_0, t_1]$. *We then say the fluid solution is valid on* $[t_0, t_1]$.

Now we formalize the usual workload conditions and the virtual station conditions for the LBFS-FBFS re-entrant line.

Definition 2.2.7 *The 2-station re-entrant line satisfies the usual workload conditions if*

$$M_A < 1 \text{ and} \qquad (20)$$

$$M_B < 1 \qquad (21)$$

We denote $M_{\mathbb{V}(0)} := M_A$ *and* $M_{\mathbb{V}(N+1)} := M_B$.

The LBFS-FBFS re-entrant line satisfies the virtual station conditions if

$$M_{\mathbb{V}(k)} := M_A - M_A(k) + M_B(k) < 1 \text{ for each class } k \in B \setminus \{2\}. \tag{22}$$

We call $M_{\mathbb{V}(k)}$ *the workload of the virtual station* $\mathbb{V}(k)$.

Now we state the Stability Theorem: If the fluid network satisfies the usual workload and virtual station conditions and some additional technical assumptions, then any valid fluid solution drains the fluid network in finite time.

Theorem 2.2.8 *Under the non-degeneracy assumptions (23)-(26), the LBFS-FBFS fluid network is stable if the processing times satisfy the usual workload conditions (20) and (21), and the virtual station conditions (22).*

The non-degeneracy assumptions (23)-(26) are technical assumptions employed to simplify the proof. In practice, these assumptions can always be fulfilled by slightly perturbing the service times.

Definition 2.2.9 *For each* $a \in A$ *and* $b \in B$, *let* $D(a,b) := m_b(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a))$ *and let* $D(a,b) := 1$ *when* $a = 0$ *or* $b = N+1$.

Non-degeneracy assumptions

We assume that the service times satisfy:

$$D(a,b) \neq 0 \text{ for all } a \in A,\ b \in B \tag{23}$$

$$M_A(b) - M_A(a) \neq M_B(b) - M_B(a) \tag{24}$$

$$\text{for all } a \in A \cup \{0\},\ b \in B \cup \{N+1\}$$

$$m_b \neq m_{b+1} \quad \text{for all } b \in B \tag{25}$$

$$\frac{m_b}{1 - M_B(b)} \neq \frac{m_{b+1}}{1 - M_A(b)} \quad \text{for all } b \in B \tag{26}$$

2.3 Outline of the proof of Theorem 2.2.8

To prove Theorem 2.2.8 it is sufficient to show that starting from any initial fluid levels the system drains in finite time (see Stolyar [12]). Furthermore we show in Theorem 2.7.7 that we can assume without loss of generality that at time 0 there is only fluid in buffer 1 and all other buffers are empty. This reduction significantly simplifies the task of proving stability. Rather than showing that the system drains from all possible initial states, we only need to show that it drains from the single initial state with one unit of fluid in buffer 1 and all other buffers empty. One is tempted to determine how the fluid solution evolves from this initial state. Unfortunately this is not possible, even with this specific initial state we are not able to describe how the fluid solution develops.

Although there are only a limited number of ways in which a fluid solution can evolve as long as there is fluid in station A, if station A is empty, we generally cannot predict how the system proceeds. Furthermore, experimental as well as theoretical results show that such states occur quite often.

If we can show that station A works without idling, the system must drain in finite time since the usual workload conditions guarantee that every unit of fluid that leaves the system received at most $M_A < 1$ units of service at station A and the station never idles. Thus, fluid leaves station A faster than it arrives and so, at some point in time, station A drains. A similar result holds for station B. We denote by τ_A the first time when station A is empty and by τ_B the first time when station B is empty.

The proof that if station A never idles, then the system must drain, can be generalized to any virtual station $\mathbb{V}(l)$ ($l \in B \cup \{N+1\} \setminus \{2\}$): If the virtual station $\mathbb{V}(l)$ never idles (meaning that at all regular times either station A or station B spends 100% of its work time on the classes of the virtual station), then the virtual station condition $M_{\mathbb{V}(l)} = M_A - M_A(l) + M_B(l) < 1$ ensures that the system drains in finite time — see Proposition 2.6.3.

From this perspective, the stations A and B and the virtual stations exhibit the same behavior. Henceforth, we use virtual station to refer to either to station A or station B or a virtual station $M_{\mathbb{V}(l)}$ ($l \in B \setminus \{2\}$).

While we are not able to show how exactly a valid fluid solution evolves, we are able to demonstrate that certain states cannot occur after time τ_A.

In fact, we are able to rule out so many states that we can show that after some finite time there is a virtual station $\mathbb{V}^*$ which is almost never empty as long as there is fluid accumulated in the network. Therefore, after some initial time, station $\mathbb{V}^*$ never idles until the fluid network empties.

With this result, (13) and (16) together with the virtual station condition $M_{\mathbb{V}^*} < 1$ show that the system drains out in finite time. In fact, we characterize $\mathbb{V}^*$ as the virtual station with the maximum workload.

Sections 2.4 to 2.8 provide the details of this proof.

The first goal is to establish how the fluid solution evolves at any given regular time. Lemma 2.4.2 establishes that the fluid solution is only dependent on the highest priority nonempty buffers. The rest of Section 2.4 shows in detail how a valid fluid solution behaves at regular time points, i.e. which buffers are increasing or decreasing their buffer levels and which buffer levels do not change.

Section 2.5 examines how a fluid solution behaves dependent on the hpn buffers at irregular time points.

Section 2.6 defines the notion of total workloads at virtual stations (Definition 2.6.1) and shows that the total workload at each virtual station (and any regular station) eventually reaches 0 (Proposition 2.6.3).

Section 2.7 shows which states are transient and cannot occur after time τ_A in a valid fluid solution. Finally, Section 2.8 proves the Stability Theorem 2.2.8 by showing that the "busiest" virtual station never idles.

The proof of the Stability Theorem is rather long and involves many intermediate results. We mark with a diamond $\diamond$ all results that are not necessary for the proof of the Stability Theorem.

2.4 Behavior at regular times

We henceforth assume that the processing times fulfill the usual workload conditions (20) and (21) and the virtual station conditions (22).

This section describes how a valid fluid solution behaves at regular time points. For this purpose we first determine the departure rates of the classes as a function of the hpn buffers (a, b).

Let us formulate (13)–(18) omitting the time parameter:

$$
\begin{aligned}
d_0 &= 1 && (27)\\
\sum_{k \in A,\ k \geq a} m_k d_k &= 1 \quad \text{if } a > 0 && (28)\\
d_k &= 0 \quad \text{for each class } k \in A,\ k < a && (29)\\
d_k &= d_{k-1} \quad \text{for each class } k \in A,\ k > a && (30)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k \in B,\ k \leq b} m_k d_k &= 1 \quad \text{if } b < N+1 && (31)\\
d_k &= 0 \quad \text{for each class } k \in B,\ k > b && (32)\\
d_k &= d_{k-1} \quad \text{for each class } k \in B,\ k < b && (33)
\end{aligned}
$$

Lemma 2.4.1 shows under the non-degeneracy assumptions, (27)–(33) admits a unique solution.

Lemma 2.4.1 *For each pair $a \in A \cup \{0\}$ and $b \in B \cup \{N+1\}$, the solution d to (27)–(33) is unique if $D(a, b) \neq 0$. The solution is described in Table 1 in Appendix B.*

The proof of Lemma 2.4.1 is straightforward and can be found in Appendix C. Proposition 2.4.1 implies the following lemma.

Lemma 2.4.2 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. Let the a be the hpn buffer of station A and b be the hpn buffer of station B at a regular time t. Then Table 1 describes the departure rates $d_k = d_k(t)$ at time t.*

Proof: Given $a \in A \cup \{0\}$ and $b \in B \cup \{N+1\}$, the solution to (27)–(33) is uniquely determined for $D(a,b) \neq 0$. Therefore the solution to (13)–(18) does not depend on the time parameter but only on the hpn buffers (a, b). Therefore, Table 1 describes the departure rates at t for the hpn buffers a of station A and b of station B.

$\square$

Given a feasible fluid solution $(Q(\cdot), T(\cdot))$, Lemma 2.4.2 implies that under assumption (23) at a regular time point t the hpn buffers $(a(Q(t)), b(Q(t)))$ at stations A and B uniquely determine the departure rates $d_k(t)$ and therefore also $\dot{T}_k(t)$ and $\dot{Q}_k(t)$.

This shows that in a valid fluid solution, the behavior of the system is characterized at regular times by its hpn buffers $a \in A \cup \{0\}$ and $b \in B \cup \{N+1\}$.

We are now introducing the ℓ operator. Given a class k, $\ell(k)$ can be interpreted as the first station B class after k with the property that moving fluid from class k to class $\ell(k)+2$ involves more work for Station B than for Station A.

Definition 2.4.3 *For each class $k \in \{0, \ldots, N\}$, define by $\ell(k)$ to be the lowest numbered class $k' \in B$ such that $k' \geq k$ and*

$$M_B(k'+2) - M_B(k) > M_A(k'+2) - M_A(k)$$

If no such class exists, we define $\ell(k) := N+1$.

Remark: Note that for each class $b \in B$ such that $m_b > m_{b+1}$, $\ell(b) = b$.

A basic property of $\ell(k)$ is that $m_{\ell(k)+1} < m_{\ell(k)}$:

Corollary 2.4.4 *For each class $k \in \{0 \ldots N\}$ with $\ell(k) < N+1$, $m_{\ell(k)+1} < m_{\ell(k)}$.*

Proof: We have

$$\begin{aligned} m_{\ell(k)} + M_B(\ell(k)) - M_B(k) &= M_B(\ell(k)+2) - M_B(k) \\ &> M_A(\ell(k)+2) - M_A(k) \qquad (34) \\ &= m_{\ell(k)+1} + M_A(\ell(k)) - M_A(k). \end{aligned}$$

Now, (34) implies

$$m_{\ell(k)} + M_B(\ell(k)) - M_B(k) - m_{\ell(k)+1} - M_A(\ell(k)) + M_A(k) > 0.$$

But since by the definition of $\ell(k)$,

$$M_B(\ell(k)) - M_B(k) - M_A(\ell(k)) + M_A(k) \leq 0,$$

we get $m_{\ell(k)+1} < m_{\ell(k)}$.

$\square$

The following proposition describes conditions that a state (a, b) at a regular time t has to fulfill in a valid fluid solution. The proof is technical and straightforward and is moved to Appendix C.

Proposition 2.4.5 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. Let t_0 be a regular time and $a = a(Q(t_0))$ and $b = b(Q(t_0))$.*

Case I: *If $0 < a < b < N+1$ then:*

$$\begin{aligned} m_{b+1} &< m_b, && (35) \\ M_A(b) - M_A(a) &> M_B(b) - M_B(a), && (36) \\ D(a,b) &> 0 \text{ and} && (37) \\ b &\leq \ell(a) && (38) \end{aligned}$$

Case II: *If $0 < a > b < N+1$ then $a = b + 1$*

Case III: *If $0 < a < b = N+1$ then $\ell(a) = N+1$*

Case IV: *If $0 = a < b < N+1$ then*

$$\begin{aligned} m_b &> m_{b+1}, && (39) \\ \frac{m_b}{1 - M_B(b)} &> \frac{m_{b+1}}{1 - M_A(b)}, && (40) \end{aligned}$$

and for all $2 < l < b,\ l \in B$:

$$m_{b+1}\frac{1 - M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l) \leq 1 \qquad (41)$$

Remark: If $b \leq \ell(0)$ then (40) implies (39).

Corollary 2.4.6 shows that inequality (41) in Proposition 2.4.5 is always fulfilled for $l \leq \ell(0)$

Corollary 2.4.6 $\diamond$

In Proposition 2.4.5, case IV, inequality (41) is always fulfilled for $l \leq \ell(0) = \ell(1)$.

In particular, if $b \leq \ell(0) + 2$, *then inequality (41) is obsolete.*

Proof: For all $2 < l \leq \ell(0),\ l \in B$, we know by definition of $\ell(0)$ that $M_A(l) \geq M_B(l)$. Then, inequality (40) gives

$$\begin{aligned} 1 &\geq M_A(b) + m_{b+1}\frac{1 - M_B(b)}{m_b} \\ &\geq M_A(b) + m_{b+1}\frac{1 - M_B(b)}{m_b} - M_A(l) + M_B(l). \end{aligned}$$

With this, the first claim is proven. The second claim follows since in this case for all $l < b$, $l \leq \ell(0)$.

$\square$

We now describe what we mean by filling and emptying buffers. Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. Let t be a regular time point (with associated rates $d(t) = (d_i(t))_{i \in \{0,\ldots,N\}}$).

We say a buffer $k \in A \cup B$ is filling or its fluid level is (strictly) increasing if $d_{k-1}(t) - d_k(t) > 0$. If $d_{k-1}(t) - d_k(t) \geq 0$ then we say that the fluid level of buffer k is non-decreasing.

If $d_k(t) - d_{k-1}(t) > 0$ then we say the buffer k is emptying or its fluid level is (strictly) decreasing and $d_k(t) - d_{k-1}(t) \leq 0$ means that the fluid level of buffer k is non-increasing.

The following proposition describes which buffers are filling and which are emptying. Again, the proof is straightforward and moved to the Appendix C.

Proposition 2.4.7 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. Let (a, b) a state at regular time t with departure rates $d(a, b)$. Then:*

Case I: $0 < a < b < N + 1$

If $a \neq 1$, then buffer a is emptying and buffer 1 is filling.

If $a = 1$ and $\frac{m_b}{1-M_B(b)} > \frac{m_{b+1}}{1-M_A(b)}$ then buffer 1 is emptying. Otherwise, buffer 1 is filling.

If $b = \ell(a)$ then buffer b is filling. Otherwise, buffer b is emptying.

Case II: $0 < a > b < N + 1$

Buffer b is emptying.

Buffer a is filling if $m_b < m_{b+1}$ (recall: $a > b$ implies by Proposition 2.4.5 that $a = b + 1$). Otherwise, buffer a is emptying.

Buffer 1 is filling.

Case III: $0 < a < b = N + 1$

Buffer a is emptying.

For $a \neq 1$, buffer 1 is filling

Case IV: $0 = a < b < N + 1$

Buffer b is emptying.

Case V: $0 = a < b = N + 1$

No fluid level in any buffer changes.

Furthermore, buffer $b+2$ is filling if $b+2 < N$.

The fluid levels of all other buffers do not change.

The following lemma shows that at any regular time, one of the hpn buffers is emptying.

Lemma 2.4.8 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. If at a regular time the hpn buffers are (a, b) (and the system is nonempty, i.e. $(a, b) \neq (0, N+1)$), then at least one of buffers a or b is emptying.*

Proof: The statement follows from Proposition 2.4.7 for all cases except case I with $a = 1$. So we have to show that buffer a or buffer b is emptying if $1 = a < b < N+1$.

We assume that the buffer levels in a and b, are both non-decreasing. By Proposition 2.4.5 we know that $D(a, b) > 0$ and that $m_{b+1} < m_b$ and $M_A(b) - M_B(b) > 0$.

The levels of both buffers 1 and b are non-decreasing if and only if $d_b \leq d_1 \leq 1$, which is by Table 1 equivalent to

$$\frac{(M_A(b) - M_A(1)) - (M_B(b) - M_B(1))}{m_b(M_A(b) - M_A(1)) - m_{b+1}(M_B(b) - M_B(1))} \leq \frac{m_b - m_{b+1}}{m_b(M_A(b) - M_A(1)) - m_{b+1}(M_B(b) - M_B(1))} \leq 1$$

Since $D(a, b) > 0$, this inequality holds if and only if

$$M_A(b) - M_B(b) \leq m_b - m_{b+1} \leq m_b M_A(b) - m_{b+1} M_B(b) \qquad (42)$$

Let us restate the first inequality in (42) by introducing a variable $\varepsilon \geq 0$ that measures the difference between the left-hand and the right-hand expression: $M_A(b) - M_B(b) + \varepsilon = m_b - m_{b+1}$.

Since $1 - M_B(b+2) = 1 - m_b - M_B(b) > 0$ (by the usual workload condition (21)) and since $M_A(b) - M_B(b) > 0$, we have

$$\begin{aligned}\varepsilon(1 - M_B(b)) \geq 0 &> (m_b + M_B(b) - 1)(M_A(b) - M_B(b)) \\ &> \varepsilon - \varepsilon M_B(b) > m_b M_A(b) - m_b M_B(b) + M_A(b) M_B(b) \\ &\qquad\qquad -M_B(b)^2 - M_A(b) + M_B(b)\end{aligned}$$

$$\begin{aligned}\text{Hence, } M_A(b) - M_B(b) + \varepsilon &> m_b M_A(b) - m_b M_B(b) + M_B(b)(M_A(b) - M_B(b) + \varepsilon) \\ \text{or, } m_b - m_{b+1} &> m_b M_A(b) - m_{b+1} M_B(b). \qquad (43)\end{aligned}$$

Statement (43) contradicts the second inequality in (42), hence it is impossible that the fluid levels of buffer 1 and b are both non-decreasing at the same time. This concludes the proof.

□

We know that almost all time points are regular by Lemma 2.2.2. The following proposition shows that if the hpn buffers in some neighborhood of a time point t_0 are constant, then t_0 is regular. It is an important preparation for the examination of the behavior of a valid fluid solution at irregular time points.

Proposition 2.4.9 *Let $(Q(\cdot), T(\cdot))$ be a given valid fluid solution and let (a, b) be the hpn buffers at a time point t_0. If there is some $\varepsilon > 0$ such that $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ then t_0 is a regular time point.*

Proof: Lemma 2.2.2 says that $Q_k(\cdot)$ is an absolutely continuous function in time. Hence it is almost everywhere differentiable and the following holds for $t > t_0 - \varepsilon$:

$$Q_k(t) = Q_k(t_0 - \varepsilon) + \int_{t_0-\varepsilon}^{t} \dot{Q}_k(s)\, ds \overset{(9)}{=} Q_k(t_0 - \varepsilon) + \int_{t_0-\varepsilon}^{t} d_{k-1}(s) - d_k(s)\, ds$$

Now, if $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in the regular interval $(t_0 - \varepsilon, t_0 + \varepsilon)$, then by Lemma 2.4.2, $d_{k-1}(\cdot)$ and $d_k(\cdot)$ are constant for all regular points of the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. Since the regular points are dense in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$, $Q(\cdot)$ is affine and therefore differentiable on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

$\square$

Remark: The converse of Proposition 2.4.9 is not necessarily true: There could be a regular time point t with irregular time points in every neighborhood of t. We prove in Proposition 2.7.3 that the converse of Proposition 2.4.9 is true when station A is not empty.

A conclusion of Proposition 2.4.9 and Proposition 2.4.2 is that if the hpn buffers do not change in a time interval, then all times in the interval are regular and the departure rates are constant on the whole interval.

Corollary 2.4.10 *Let (t_0, t_1) be a time interval with constant hpn buffers (a, b). Then all times in (t_0, t_1) are regular and $d(\cdot)$ is constant on (t_0, t_1).*

2.5 Behavior at irregular times

Section 2.4 describes how a valid fluid solution behaves at regular times in dependency of the hpn buffers (a, b). This section examines the behavior of the system at irregular points in dependency of the hpn buffers.

This problem is much more difficult. In fact, for states with station A empty, it will remain unclear how a valid fluid solution behaves.

The core of this section is the study of the *-operator. The *-operator describes how a valid fluid solution behaves an "instant" after an irregular time point. Given a valid fluid solution and a state (a, b) at time t_0, $(a, b)^* = (a^*, b^*)$ is the state succeeding (a, b) for some time.

Definition 2.5.1

Let $(Q(\cdot), T(\cdot))$ *be a valid fluid solution and let* $(a, b) = (a(Q(t_0)), b(Q(t_0)))$ *be the hpn buffers at an arbitrary time point* t_0*. If there are a time* $t_1 > t_0$*, a class* $a^* \in A \cup \{0\}$ *and and a class* $b^* \in B \cup \{N+1\}$ *such that for all* $t \in (t_0, t_1)$*,* $a(Q(t)) = a^*$ *and* $b(Q(t)) = b^*$ *then we say that the *-operator is well-defined (by Corollary 2.4.10 we see that all times in* (t_0, t_1) *are regular). We also write* $(a, b)^*$ *for* (a^*, b^*)*.*

Proposition 2.5.2 shows in which cases (a^*, b^*) is well-defined.

Pretending for a moment that the *-operator is well-defined, let us illustrate with an example that in general $(a^*, b^*) \neq (a, b)$:

Example 1

Assume we start observing a valid fluid solution $(Q(\cdot), T(\cdot))$ at an arbitrary, regular time point t_0 with hpn buffers (a_0, b_0). Let us assume both buffers are emptying via departure rates $d(a_0, b_0)$ until one of the buffers empties at some time $t_1 > t_0$.

Say buffer a_0 empties. Hence the new hpn buffers are $(a_1, b_1) = (a(Q(t_1)), b(Q(t_1)))$ where $a_1 < a_0$ and $b_1 = b_0$. What happens next?

The unique solution $d(a_1, b_1)$ to equations (13)–(18) does not necessarily define feasible departure rates. For example if $b_1 > \ell(a_1)$ then $({a_1}^*, {b_1}^*) = (a_1, b_1)$ would contradict Proposition 2.4.5 which forces ${b_1}^* \leq \ell({a_1}^*)$.

We will see in Proposition 2.5.2 that if ${a_1}^* = a_1$, then ${b_1}^* = \ell(a_1)$. While ${b_1}^*$ is empty at time t_1, ${b_1}^*$ is the hpn buffer in station B in the time interval $(t_1, t_1 + \varepsilon)$ (for some $\varepsilon > 0$).

In Proposition 2.5.2, we describe *-operator in dependency of the hpn buffers of station A and station B.

Proposition 2.5.2 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution.*

Let $(a, b) = (a(Q(t_0)), b(Q(t_0)))$ be the hpn buffers of the system at some time point t_0.

Assume that $a > 0$ (station A is nonempty). Then there exists a time $t_1 > t_0$ such that for all times $t \in (t_0, t_1)$, $(a(Q(t)), b(Q(t)))$ are constant and equal to (a^*, b^*) as described below. Furthermore, at time t_1, buffer a^* or buffer b^* is empty.

Case I: *If $0 < a < b < N + 1$ then*

1. *$m_{b+1} < m_b$ and $\ell(a) > b$ implies $(a^*, b^*) = (a, b)$.*
2. *$m_{b+1} < m_b$ and $\ell(a) \leq b$ implies $(a^*, b^*) = (a, \ell(a))$.*
3. *$m_{b+1} > m_b$ and $\ell(a) > b$ implies $(a^*, b^*) = (b + 1, b)$.*
4. *$m_{b+1} > m_b$ and $\ell(a) \leq b$ implies $(a^*, b^*) = (a, \ell(a))$ or $(a^*, b^*) = (b + 1, b)$.*

Case II: *If $0 < a > b < N + 1$ then $a = b + 1$ and $(a^*, b^*) = (b + 1, b)$.*

Case III: *If* $0 < a < b = N+1$ *then* $(a^*, b^*) = (a, \ell(a))$.

Assume that $a = 0$ *and that the* *-*operator is well-defined, i.e. that there exists a* $t_1 > t_0$ *such that for all* $t_0 < t < t_1$, (a^*, b^*) *are the (not changing) hpn buffers of the system. Then* (a^*, b^*) *are of the following form:*

Case IV: *If* $0 = a < b < N+1$ *then*

1. *if* $m_b < m_{b+1}$ *then* $(a^*, b^*) = (b+1, b)$.
2. *if* $m_b > m_{b+1}$ *then we have the following subcases*
   (a) *If* $\frac{m_b}{1-M_B(b)} \geq \frac{m_{b+1}}{1-M_A(b)}$ *and for all* $2 < l < b,\ l \in B$:
   $$m_{b+1}\frac{1-M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l) \leq 1$$
   then $(a^*, b^*) = (0, b)$.
   (b) *If* $\ell(1) > b$ *and* $\frac{m_b}{1-M_B(b)} < \frac{m_{b+1}}{1-M_A(b)}$ *then* $(a^*, b^*) = (1, b)$.
   (c) *Otherwise there is no possible valid fluid solution.*

Case V: *If* $0 = a < b = N+1$ *then no fluid accumulates in any buffer and hence* $(a^*, b^*) = (a, b) = (0, N+1)$.

Remark: On the other hand, if at time t_0, (a, b) are the hpn buffers of the system, then (a^*, b^*) define a valid fluid solution on the interval $[t_0, t_1]$.

Proof: In this proof we frequently use that $Q_k(\cdot)$ is an absolutely continuous function. Hence $Q_k(\cdot)$ is almost everywhere differentiable and the following holds for any time $\tilde{t}$ and $t > \tilde{t}$:

$$Q_k(t) = Q_k(\tilde{t}) + \int_{\tilde{t}}^{t} \dot{Q}_k(s)\,ds \overset{(9)}{=} Q_k(\tilde{t}) + \int_{\tilde{t}}^{t} d_{k-1}(s) - d_k(s)\,ds \tag{44}$$

where we define $\dot{Q}_k(\cdot)$, $d_k(\cdot)$ and $d_{k-1}(\cdot)$ to be 0 wherever Q_k is not differentiable.

The general approach of the proof is as follows:

First we determine t_1 to be the first time when buffer a or buffer b empties.

Then we show that most of the station A buffers with higher priority than a and most of the station B buffers with higher priority than b cannot fill during the interval (t_0, t_1).

Now we are only left with a few possibilities for potential hpn buffer constellations in (t_0, t_1). We then conclude the proof by ruling out impossible constellations.

We are using (44) together with the results of Section 2.4 to prove the proposition.

Case I and III: $0 < a < b \leq N + 1$:

In the following, if $b = N + 1$, set $Q_b(\cdot)$ to be greater than 0.

Let t_1 be the first time, when buffer a or b empties, i.e. $t_1 = \sup\{t > t_0 \;:\; Q_a(s) > 0,\; Q_b(s) > 0 \;\text{ for all } s \text{ with } t_0 \leq s \leq t\}$ ($t_1 > t_0$ since $Q_k(t)$ is continuous).

The length of the proof for this case requires an outline of the steps. We prove the following statements for the time interval $\{t_0, t_1\}$:

1. First, we show that all station A buffers above $b + 1$ and all station B buffers below a stay empty.
2. Then we prove that during the all buffers between a and $\min\{\ell(a), b\}$ stay empty.

3. Now we consider the case $b < \ell(a)$. Steps 1 and 2 imply that the only possible hpn buffer constellations are (a, b) and $(b+1, b)$. We consider the following subcases:

   (a) $m_b > m_{b+1}$: We show that $(a^*, b^*) = (a, b)$.

   (b) $m_b < m_{b+1}$: We show that $(a^*, b^*) = (b + 1, b)$.

4. We consider now the case $b \geq \ell(a)$ (this includes case III, $b = N + 1$). We show that all station A buffers between $\ell(a)$ and b stay empty. We consider the subcases:

   (a) $m_b > m_{b+1}$:

   We show that no station A buffer above a can fill. This implies with step 4 that $(a^*, b^*) = (a, \ell(a))$.

   (b) We show that $(a^*, b^*) = (a, \ell(a))$.

   (c) $m_b < m_{b+1}$:

   We show that $(a^*, b^*) = (b + 1, b)$ or that $(a^*, b^*) = (a, \ell(a))$.

This concludes the proof for cases I and III. We discuss now the outlined steps in detail.

1. We show that all station A buffers above $b + 1$ and all station B buffers below a stay empty.

   Let t be an arbitrary time point in (t_0, t_1) and let k be a station B class below a ($k \in B$, $k < a$). We have $d_{k-1}(s) = 0$ for all regular $s \in (t_0, t)$ since $k - 1 \in A$ has a lower priority than a and hence does not have any output. Therefore, (44) shows that $Q_k(t) = 0$ for each class $k \in B$, $k < a$.

Similarly, let k be a station A class above $b+1$ ($k \in A$ and $k > b+1$). We have $d_{k-1}(s) = 0$ for all regular $s \in (t_0, t)$ ($k-1 \in B$ has a lower priority than b and does not have any output). Therefore, (44) shows that $Q_k(t) = 0$ for each class $k \in A$, $k > b+1$.

2. We show now that all buffers between a and $\min\{\ell(a), b\}$ stay empty, i. e. for all $t \in (t_0, t_1)$, $Q_k(t) = 0$ for all $a < k < \min\{\ell(a), b\}$.

   Assume this is not true. Then is a $k \in A \cup B$, $\min\{b, \ell(a)\} > k > a$, k chosen to be minimal, and a time $\delta_1 \in (t_0, t_1)$ such that $Q_k(\delta_1) > 0$.

   Let δ_0 be the time when buffer k began to fill, i.e. $\delta_0 := \inf\{t \ : \ Q_k(s) > 0 \text{ for all } s \text{ with } t \leq s \leq \delta_1 < t_1\}$. Clearly $\delta_0 \geq t_0$, otherwise either a or b could not be the hpn buffer at time t_0. The continuity of Q_k assures that $\delta_0 < \delta_1$ and that $Q_k(\delta_0) = 0$.

   Setting $\tilde{t} = \delta_0$ in (44) gives:

$$Q_k(t) = Q_k(\delta_0) + \int_{\delta_0}^{t} d_{k-1}(s) - d_k(s)\, ds = \int_{\delta_0}^{t} d_{k-1}(s) - d_k(s)\, ds \qquad (45)$$

   We are going to show that the integrand in (44) is less than or equal to 0 at all regular times and hence the integral in (45) is equal to 0.

   Assume now that $k \in A$: For all regular s, $d_{k-1}(s) = d_{k-2}(s) = 0$ because $k-1 \in B$ has high priority in B and there is no fluid leaving from station A below k.

   Assume now that $k \in B$. For any given regular $s \in (\delta_0, \delta_1)$ there are two possibilities for the hpn buffer $a'(s)$ at station A:

$a'(s) > k$.

This implies $d_{k-1}(s) = 0$ (same argument as above) and therefore the integrand in (45) is less than or equal to 0.

$a'(s) < k$.

This implies $a'(s) = a$ (since $s \in (t_0, t_1)$ and k was chosen to be minimal). With Table 1 we conclude for regular $s \in (\delta_0, \delta_1)$:

$d_{k-1}(s) = d_{k-2}(s) = \ldots = d_a(s) = \frac{m_k - m_{k+1}}{D(a,k)}$

and $d_k(s) = \frac{(M_A(k) - M_A(a)) - (M_B(k) - M_B(a))}{D(a,k)}$.

Proposition 2.4.7 and the non-degeneracy assumptions (24)-(26) imply that the fluid level of buffer k is non-increasing at regular time points, i.e. for regular $s \in (\delta_0, \delta_1)$, $d_{k-1}(s) - d_k(s) \leq 0$.

In both cases we arrive at the conclusion that (45) is less than or equal to 0 and buffer k does not fill (contradiction).

3. Consider now the case where $b < \ell(a)$.

   Step 2 shows that all buffers between a and b stay empty. So the only hpn buffers that can occur in the time interval (t_0, t_1) are (a, b) and $(b+1, b)$ since b and $b+1$ are the only possible highest priority filling buffers.

   (a) If $m_b > m_{b+1}$ then we show that buffer $b+1$ stays empty in (t_0, t_1).

      Assume that $Q_{b+1}(\delta_1) > 0$ for some $\delta_1 \in (t_0, t_1)$ then define δ_0 to be the time when buffer $b+1$ started to fill:

      $\delta_0 := \inf\{t \ : \ Q_{b+1}(s) > 0 \ \text{ for all } s \text{ with } t \leq s \leq \delta_1 < t_1\}$. We see that from equation (44) for $k = b + 1$

$$Q_{b+1}(\delta_1) = Q_{b+1}(\delta_0) + \int_{\delta_0}^{\delta_1} d_b(s) - d_{b+1}(s)\, ds, \tag{46}$$

where $Q_{b+1}(\delta_0) = 0$. The integrand is less than or equal to 0 at regular times s since the service rate at class $b+1$ is $\frac{1}{m_{b+1}}$ ($b+1$ is the hpn buffer at station A, so station A works exclusively and without idling on class $b+1$), while $d_b(s) \leq \frac{1}{m_b} < \frac{1}{m_{b+1}}$ since all station B buffers below b are empty and do not receive any fluid.

Hence there is no higher priority station A buffer than a and no higher priority station B buffer than b containing fluid at some time in (t_0, t_1). Therefore, in case I.1 (a^*, b^*) can only be equal to (a, b). Proposition 2.4.7 shows that $(a^*, b^*) = (a, b)$ is a valid fluid solution for the time interval (t_0, t_1) since the fluid levels in both buffers decrease until one of them drains, i.e. until we reach time t_1.

(b) If $m_b < m_{b+1}$, then for all regular times $t \in (t_0, t_1)$ we have $Q_{b+1}(t) > 0$: If the hpn buffers were (a, b) for some regular time then there would be a contradiction to case I of Proposition 2.4.5. Hence for all regular times $\tilde{t} \in (t_0, t_1)$, $Q_{b+1}(\tilde{t}) > 0$. Then for arbitrary $t \in (t_0, t_1)$:
Choose $\tilde{t}$ such that $t_0 < \tilde{t} < t$. Then

$$\begin{aligned} Q_{b+1}(t) &= Q_{b+1}(\tilde{t}) + \int_{\tilde{t}}^{t} d_b(s) - d_{b+1}(s)\, ds \\ &= Q_{b+1}(\tilde{t}) + \int_{\tilde{t}}^{t} \frac{1}{m_b} - \frac{1}{m_{b+1}}\, ds > 0. \end{aligned}$$

Hence for all $t \in (t_0, t_1)$ the hpn buffers are $(b+1, b)$. Proposition 2.4.7 shows that $(a^*, b^*) = (b+1, b)$ is valid through the time period (t_0, t_1): Buffer $b+1$ is filling and buffer b is emptying until it has 0 fluid level.

4. Consider now the case $b \geq \ell(a)$.

   Step 2 proves that all buffers between a and $\ell(a)$ stay empty for all times in the interval (t_0, t_1).

   We show now that all station A buffers between $\ell(a)$ and b stay empty, i.e. $Q_k(t) = 0$ for all $\ell(a) < k < b, k \in A$ and for all $t \in (t_0, t_1)$.

   We prove this by contradiction — assume there exists a buffer k, $\ell(a) < k < b$, $k \in A$ chosen to be minimal, which is nonempty at time $\delta_1 \in (t_0, t_1)$. Let $\delta_0 := \inf\{t \ : \ Q_k(s) > 0 \ \text{ for all } s \text{ with } t \leq s \leq \delta_1 < t_1\}$.
   Clearly $\delta_0 \geq t_0$, otherwise a could not be the hpn buffer at time t_0. Also, $\delta_0 < \delta_1$ by the continuity of Q_k. We have

   $$Q_k(t) = Q_k(\delta_0) + \int_{\delta_0}^{t} d_{k-1}(s) - d_k(s)\, ds \tag{47}$$

   with $Q_k(\delta_0) = 0$. Now, if buffer $k-1$ is empty for all regular times in (δ_0, δ_1), then $d_{k-1}(s) = 0$ for all regular times in (δ_0, δ_1) ($d_{k-1}(s) = 0$ if $k-1 > b(Q(s))$ and $d_{k-1}(s) = d_{k-2}(s) = 0$ if $k-1 < b(Q(s))$ by assumption (18)). But this implies that (47) is equal 0 and buffer k could not fill.

   Therefore, there is a regular time $\delta_3 \in (\delta_0, \delta_1)$, for which $k-1$ is the hpn buffer at station B.

   Furthermore, for all regular $t \in (\delta_0, \delta_1)$, $Q_l(t) = 0$ for each class $l \in B$, $l < k-1$. Otherwise there is a regular t such that $a(Q(t)) > b(Q(t))+1$, which contradicts case II in Proposition 2.4.5. The continuity of Q_l extents this statement to the whole interval (δ_0, δ_1), i.e. for all $t \in (\delta_0, \delta_1)$, $Q_l(t) = 0$ for each class $l \in B,\ l < k-1$.

Let δ_2 be the time when buffer $k-1$ started to fill,
$\delta_2 := \inf\{t \ : \ Q_{k-1}(s) > 0 \ \text{ for all } s \text{ with } t \leq s \leq \delta_3\}$. Then, $\delta_2 \geq t_0$; otherwise b could not be the hpn buffer at station B at time t_0.
For all $t \in (\delta_0, \delta_1)$ there is no output from class $k-2$ (buffer k has positive fluid level and has higher priority than $k-2$). Therefore there cannot be any fluid accumulating in buffer $k-1$ in the time interval (δ_0, δ_1) and we necessarily have $\delta_2 < \delta_0$ (indicated in Figure 3).

Figure 3: Position of the intervals $[\delta_2, \delta_3]$ and $[\delta_0, \delta_1]$ on the time line

Assume now that for all regular times $t \in (\delta_2, \delta_0)$ there is a station A buffer $l \geq k$ such that $Q_l(t) > 0$ (Proposition 2.4.5, case II, would then imply that $l = k$ for regular t).

$$\begin{aligned} \text{Then } Q_{k-1}(t) &= Q_{k-1}(\delta_2) + \int_{\delta_2}^{t} d_{k-2}(s) - d_{k-1}(s)\, ds \\ &= \int_{\delta_2}^{t} -d_{k-1}(s)\, ds = 0. \end{aligned}$$

This is a contradiction since $Q_{k-1}(t) > 0$ for all $t \in (\delta_2, \delta_0)$.

Therefore, there exists a *regular* time $\delta \in (\delta_2, \delta_0)$ with $Q_l(\delta) = 0$ for each class $l \in A$, $l \geq k$. Since $k > \ell(a)$ was chosen to be minimal and since $Q_l(\delta) = 0$ for all $a < l < \ell(a)$ by step 2, the hpn buffer at station A at time δ is a. Proposition 2.4.5 implies that the hpn buffer at station B at time δ is less than or equal to $\ell(a)$. By step 2, $k - 1 < \ell(a)$ is not possible, therefore $k - 1 = \ell(a)$.

By assumption, for all regular $t \in (\delta_0, \delta_3)$, $k = \ell(a) + 1$ is nonempty. Furthermore, it is the hpn buffer at station A, otherwise we get a contradiction to Proposition 2.4.5, case II, since $a(Q(t))$ has to be less than or equal to $b(Q(t)) + 1$.

Similarly, Proposition 2.4.5, case II, shows that the hpn buffer at station B for regular times $t \in (\delta_0, \delta_3)$ is greater than or equal to $k - 1 = \ell(a)$, so it is equal to $k - 1$ since $Q_{k-1}(t) > 0$.

Therefore, for $t \in (\delta_0, \delta_3)$ (47) gives

$$\begin{aligned} Q_k(t) &= Q_k(\delta_0) + \int_{\delta_0}^{t} d_{k-1}(s) - d_k(s)\, ds \\ &= \int_{\delta_0}^{t} \frac{1}{m_{k-1}} - \frac{1}{m_k}\, ds. \end{aligned} \tag{48}$$

$k - 1 = \ell(a)$ implies by Corollary 2.4.4 that $m_k < m_{k-1}$, hence (48) is less or equal to 0. Contradiction.We showed so far that in the time interval

(t_0, t_1), the only possible hpn buffers in station A are a and $b + 1$. We also showed that no station B buffer below $\ell(a)$ can fill.

If at a regular time, buffer $b+1 \in A$ is filled then no station B buffer below b can be filled by Proposition 2.4.5. Vice versa, if at a regular time in (t_0, t_1), buffer $b+1$ is empty, then buffer $\ell(a)$ has to be filled. Therefore, at regular times in (t_0, t_1), only the states $(a, \ell(a))$ and $(b+1, b)$ are possible. To finalize the proof, we consider the following subcases:

(a) $b < N+1$ and $m_{b+1} < m_b$.

We want to prove by contradiction that the hpn buffers $(b+1, b)$ are not possible in the time interval (t_0, t_1).

Assume that buffer $b+1$ is nonempty at some time in (t_0, t_1), i.e. there is some $\delta_1 \in (t_0, t_1)$ such that $Q_{b+1}(\delta_1) > 0$. Choose $\delta_0 := \inf\{t \ : \ Q_{b+1}(s) > 0 \ \text{ for all } t \leq s \leq \delta_1\} > t_0$ (by continuity $\delta_0 < \delta_1$).

Proposition 2.4.5, case II, implies that for regular time points in (δ_0, δ_1), the hpn buffers of the system are $(b+1, b)$, since both buffers contain fluid and $a(Q(t)) \leq b(Q(t)) + 1$. Hence we see that

$$
\begin{aligned}
Q_{b+1}(t) &= Q_{b+1}(\delta_0) + \int_{\delta_0}^{t} d_b(s) - d_{b+1}(s)\, ds \\
&= \int_{\delta_0}^{t} \frac{1}{m_b} - = \frac{1}{m_{b+1}}\, ds \leq 0,
\end{aligned}
$$

so buffer $b+1$ in fact cannot fill (i.e. will not have positive fluid level in the time interval (t_0, t_1)). Contradiction.

Therefore, Proposition 2.4.5 shows that the only applicable case for regular time points is case I and so $Q_{\ell(a)}(\tilde{t}) > 0$ for regular $\tilde{t} \in (t_0, t_1)$.

Then for an arbitrary $t \in (t_0, t_1)$:

Choose a regular time $\tilde{t} \in (t_0, t)$. Then we see, using table 1:

$$\begin{aligned} Q_{\ell(a)}(t) &= Q_{\ell(a)}(\tilde{t}) + \int_{\tilde{t}}^{t} d_{\ell(a)-1}(s) - d_{\ell(a)}(s)\, ds \\ &= Q_{\ell(a)}(\tilde{t}) + \int_{\tilde{t}}^{t} \frac{m_{\ell(a)} - m_{\ell(a)+1}}{D(a,\ell(a))} \\ &\quad - \frac{(M_A(\ell(a)) - M_A(a)) - (M_B(\ell(a)) - M_B(a))}{D(a,\ell(a))}\, ds \end{aligned} \tag{49}$$

$Q_{\ell(a)}(\tilde{t})$ and the integrand are greater than 0, hence (49) is greater than 0. Therefore, for all $t \in (t_0, t_1)$ the hpn buffers are $(a, \ell(a))$. Proposition 2.4.7 shows that in the time interval (t_0, t_1) the fluid solution $(a, \ell(a))$ is valid with buffer a emptying and buffer $\ell(a)$ filling until buffer a drains at time t_1, so $(a^*, b^*) = (a, \ell(a))$ is well-defined.

(b) $b = N + 1$

We use the same argumentation as in subcase 4a to show that $(a^*, b^*) = (a, \ell(a))$ (the only difference is that we do not need to show that buffer $b + 1$ does not fill).

(c) $b < N + 1$ and $m_{b+1} > m_b$

In this case, $m_{b+1} > m_b$ and $b \geq \ell(a)$ imply $b > \ell(a)$ (follows from Corollary 2.4.4). The hpn buffers at regular times in (t_0, t_1) can only be $(a, \ell(a))$ or $(b + 1, b)$.

We now prove now by contradiction that not both hpn buffer constellations can occur in the time interval (t_0, t_1):

Assume that there is some time $\delta_1 \in (t_0, t_1)$ such that the hpn buffers at time δ_1 are $(a, \ell(a))$. Assume further that there is some time $\delta_2 \in (t_0, t_1)$ such that the hpn buffers at time δ_2 are $(b+1, b)$.

Recall that t_1 is the first time when buffer a or b drains. Then by the previous steps and since buffer $\ell(a)$ is filling (see Proposition 2.4.7), $(a, \ell(a))$ are the hpn buffers on the whole time interval (δ_1, t_1).

Similarly, by case II (which we prove below) and since buffer $b+1$ is filling (see Proposition 2.4.7), $(b+1, b)$ are the hpn buffers on the whole time interval (δ_2, t_1).

Now, for $t = \max\{\delta_1, \delta_2\}$, buffers a and $b+1$ are at the same time hpn buffers at station A, a contradiction.

Roughly speaking, both possibilities, $(a,b)^* = (a, \ell(a))$ and $(a,b)^* = (b+1, b)$, can occur, but: Once the fluid solution "decides" which possibility to choose, it will "stick" to its choice until either buffer a or buffer b drains.

Case II: $0 < a > b < N+1$.

Let t_1 be the first time, when buffer a or b empties, i.e.

$t_1 = \sup\{t > t_0 \ : \ Q_a(s) > 0, \ Q_b(s) > 0 \ \text{ for all } t_0 \le t \le t_1\} > t_0.$

Assume $a = b+1$.

Proposition 2.4.5, case II, shows that for no regular time point $t \in (t_0, t_1)$ can any station A buffer numbered higher than a nor any station B buffer numbered lower than b contain fluid — the fluid solution for this situation would then not be valid (since then $a(Q(t)) > b(Q(t)) + 1$ for some regular time point). The density of regular time points and the continuity of Q extend this statement to the whole interval (t_0, t_1).

Hence in (t_0, t_1), (a, b) are the hpn buffers of the system and $(a^*, b^*) = (a, b) = (b + 1, b)$ is well-defined: Proposition 2.4.7 shows that buffer b is emptying and buffer a is emptying or filling dependent on whether $m_b > m_{b+1}$ or not, until buffer a or b drains at time t_1.

Assume now that $a > b + 1$.

For any time point $t \in (t_0, t_1)$ we have $b(Q(t)) + 1 \leq b + 1 < a \leq a(Q(t))$. Therefore a valid fluid solution would contradict Proposition 2.4.5 at regular points in (t_0, t_1) (recall that the regular points are dense in (t_0, t_1)).

Case III: If $0 < a < b = N + 1$.

We have $(a^*, b^*) = (a, \ell(a))$ (possibly $\ell(a) = N + 1$). See case I.

Case IV: $0 = a < b < N + 1$.

We assume that (a^*, b^*) is well-defined, i.e. there is some $t_1 > t_0$ such that $(a(Q(t)), b(Q(t))) = (a^*, b^*)$ for all $t \in (t_0, t_1)$.

First, we are going to show that no buffer between buffer 1 and buffer b is filling: Clearly either $a^* = 0$ or $b^* = b$ since otherwise buffers a^* and b^* would both be filling at regular time points, in contradiction to Lemma 2.4.8. Proposition 2.4.7 shows that $b^* < b$ is impossible since otherwise in a state $(0, b^*)$ with $b^* < b$, buffer b^* is emptying at regular points and therefore

$$Q_{b^*}(t) = Q_{b^*}(t_0) + \int_{t_0}^{t} d_{b^*-1}(s) - d_{b^*}(s)\, ds = 0. \tag{50}$$

Similarly, Proposition 2.4.7 shows that a^* can only be 0,1 or $b+1$.

We showed that no buffers between 1 and b fill during (t_0, t_1). The only possible hpn buffer constellations for the time interval (t_0, t_1) are therefore $(0, b)$, $(1, b)$ and $(b+1, b)$. Consider the following cases:

1. If $m_{b+1} > m_b$, then for all regular times $t \in (t_0, t_1)$, Proposition 2.4.5 does not admit hpn buffers $(0, b)$ and $(1, b)$ and therefore for all regular times $t \in (t_0, t_1)$, the hpn buffers of the system can only be $(b+1, b)$. This implies that $(0, b)^* = (b+1, b)$ for all times in (t_0, t_1).

   Proposition 2.4.7 shows that $(0^*, b^*) = (b+1, b)$ is well-defined in time interval (t_0, t_1): Buffer b is emptying while buffer $b+1$ is filling until buffer b drains at time t_1.

2. If $m_{b+1} < m_b$, then Proposition 2.4.7 shows that $(a^*, b^*) = (b+1, b)$ cannot occur at regular times since buffer $b+1$ would be emptying in that case. Now, we examine whether buffer 1 fills in (t_0, t_1) or not.

   (a) Let us consider the subcase $\frac{m_b}{1-M_B(b)} \geq \frac{m_{b+1}}{1-M_A(b)}$.

   Then

$$
\begin{aligned}
Q_1(t) &= Q_1(t_0) + \int_{t_0}^{t} 1 - d_1(s)\, ds \\
&= \int_{t_0}^{t} 1 - d_{b-1}(s)\, ds.
\end{aligned}
$$

   Proposition 2.4.7, case I, shows that if $a^* = 1$, then the integrand is less than or equal to 0 for all regular $s \in (t_0, t_1)$. Therefore the fluid level in buffer 1 cannot be greater than 0 for $t \in (t_0, t_1)$, so $(0^*, b^*) = (0, b)$ is the only possible solution.

Proposition 2.4.5 shows that the solution $(0^*, b^*) = (0, b)$ is valid for $t \in (t_0, t_1)$ if we satisfy condition (41). Proposition 2.4.7 shows that in this case buffer b is emptying until its fluid level is equal to 0.

If condition (41) is not satisfied, then $(0, b)^* = (0, b)$ contradicts Proposition 2.4.5. Hence this situation cannot occur in a valid fluid solution.

(b) Let us consider the subcase $\frac{m_b}{1-M_B(b)} < \frac{m_{b+1}}{1-M_A(b)}$.

At any regular time point in the time interval (t_0, t_1), state $(0, b)^* = (0, b)$ is impossible by Proposition 2.4.5. So $(1, b)$ is the only possible hpn buffer option left in the interval (t_0, t_1).

If $b > \ell(1)$,

then state $(0, b)^* = (1, b)$ is impossible for the time interval (t_0, t_1) by Proposition 2.4.5, case I. Hence this situation cannot occur in a valid fluid solution.

For $b = \ell(1)$,

then

$$Q_1(t) = Q_1(t_0) + \int_{t_0}^{t} 1 - d_1(s)\, ds.$$

We have $Q_1(t_0) = 0$ and the integrand is less than or equal to 0 by Proposition 2.4.7 and by Lemma 2.4.8 (buffer b is filling hence buffer 1 is emptying).

Hence $b > \ell(1)$ cannot occur in a valid fluid solution.

If $b < \ell(1)$,

then $(0^*, b^*) = (1, b)$ defines a valid fluid solution in (t_0, t_1): Buffer 1 is filling by Proposition 2.4.7, buffer b is emptying (by Lemma 2.4.8) until it drains at time t_0. Proposition 2.4.5 shows that $(0^*, b^*) = (1, b)$ is valid.

Case V: $0 = a < b = N + 1$.

Assuming that (a^*, b^*) is well-defined (i.e. there is some $t_1 > t_0$ such that $(a(Q(t)), b(Q(t))) = (a^*, b^*)$ for all $t \in (t_0, t_1)$) we argue similar to case IV that the only possibility for (a^*, b^*) is $(1, N + 1)$ or $(0, N + 1)$:

Clearly either $a^* = 0$ or $b^* = N + 1$ since otherwise buffers a^* and b^* are filling simultaneously at regular time points, in contradiction to Lemma 2.4.8.

Proposition 2.4.7 shows that $b^* < N + 1$ is impossible since in state $(0, b^*)$ buffer b^* is emptying. Proposition 2.4.7 also shows that a^* can only be 0 or 1.

Assume that $Q_1(\cdot) > 0$ in (t_0, t_1). Table 1, case III, shows that in this case, for regular $t > t_0$, $d_1(t)$ is equal to $\frac{1}{M_A - M_A(1)} = \frac{1}{M_A} > 1$, hence buffer 1 cannot fill.

Therefore, once the fluid solution arrives in state $(0, N+1)$, it will stay there if (a^*, b^*) is well-defined.

$\square$

Table 2 on page 110 gives an overview of the results of Propositions 2.4.7 and 2.5.2. In most of the cases, (a^*, b^*) is uniquely defined. We discuss the following exceptions:

Case I.4: $m_{b+1} > m_b$ and $\ell(a) < b$ (for $\ell(a) = b$, Corollary 2.4.4 implies $m_{b+1} < m_b$):

In this case the fluid solution can evolve in two different ways. We will see in Proposition 2.7.6 that $\ell(a) < b$ cannot occur for $a > 1$.

Case II: $a > b + 1$:

In this case there is no valid fluid solution.

Station A is empty (cases IV and V). In this case it is not clear how a valid fluid solution evolves. If (a^*, b^*) is defined, then it is uniquely defined except for case IV.2.(c) which cannot occur in a valid fluid solution.

The situations in which station A is empty (cases IV and V) are crucial for our further analysis of valid fluid solutions. While we are not able to show that $(0, b)^*$ is well-defined, the next sections provide enough understanding of this "empty station A" – situation to complete the stability proof.

The following example demonstrates what makes case IV, $(0, b)^*$, so difficult.

Example 2 In this example we show that in a fluid solution (that does not satisfy the virtual station conditions), (a^*, b^*) is not necessarily well-defined:

Let us consider a 7 class fluid example with the following service times: $m_1 = 0.1$, $m_2 = 0.6$, $m_3 = 0.2$, $m_4 = 0.1$, $m_5 = 0.5$, $m_6 = 0.2$, $m_7 = 0.1$. We construct a fluid solution that cycles over and over again through the same states (i.e. states that have the same hpn buffers) with increasing cycle times. We conclude there exists a time t_0 with hpn buffers $(0, 6)$ for which $(0, 6)^*$ is not defined: In any time interval (t_0, t) (for $t > t_0$) there are infinitely many cycles.

Notation in the figures:

$\blacksquare$	Buffer contains fluid
$\square$	Empty buffer
$\boxplus$	Empty buffer (filling)
$\uparrow$	Buffer is filling
$\downarrow$	Buffer is emptying
$\rightarrow$	Buffer level is not changing
$\odot$	Buffer has no output

We start with fluid in buffers 1 and 6 at some time t_1. Suppose that buffer 6 has much more fluid than buffer 1 (at this point we do not specify the amount of fluid in buffer 6). According to Proposition 2.5.2 a valid fluid solution fills buffer $\ell(1) = 2$ while it empties buffer 1. This is illustrated in Figure 4.

Step 1, time t_1

At the end of this step, when buffer 1 empties at time t_2, there is only fluid in buffers 2, 4 and 6, as depicted in Figure 5.

Step 2, time t_2

Although the *-operator in this step is not well-defined by Proposition 2.5.2, in our specific fluid solution we assume it is well-defined. We conclude that $(0, 2)^* = (0, 2)$: Buffer 2 is emptying while buffer 4 is filling.

Let us calculate t_3, the time when buffer 2 empties: In the time interval (t_1, t_3) the station B processes all the fluid that was at time t_1 in buffer 1, $Q_1(t_1)$, plus the

A	B
$\downarrow$ 1■	
	⊞2 $\uparrow$
$\rightarrow$ 3□	
	⊞4 $\uparrow\odot$
$\odot$ 5□	
	■6 $\odot$
$\odot$ 7□	

Figure 4: The fluid levels at the beginning of step 1

A	B
$\rightarrow$ 1□	
	■2 $\downarrow$
$\rightarrow$ 3□	
	■4 $\uparrow\odot$
$\odot$ 5□	
	■6 $\odot$
$\odot$ 7□	

Figure 5: The fluid levels at the beginning of step 2

fluid that entered the system during the time interval (t_1, t_3), $t_3 - t_1$. During (t_1, t_3), station B worked exclusively and without idling on class 2. Hence we calculate: $t_3 - t_1 = m_2 \cdot (Q_1(t_1) + (t_3 - t_1))$, or $t_3 - t_1 = \frac{Q_1(t_1)}{1-m_2} = \frac{Q_1(t_1)}{0.4} = 2.5 \cdot Q_1(t_1)$.

At time t_3, all the fluid station B worked on in (t_1, t_3) is in buffer 4. Hence, $Q_4(t_3)$ is equal to $Q_1(t_1) + (t_3 - t_1) = Q_1(t_1) + \frac{Q_1(t_1)}{0.4} = 3.5 \cdot Q_1(t_1)$.

We arrive at step 3 with fluid only in buffers 4 and 6, see Figure 6.

Step 3, time t_3

A	B
$\odot\uparrow 1\boxplus$	
	$\square 2\ \odot$
$\odot\ 3\square$	
	$\blacksquare 4\ \downarrow$
$\uparrow 5\boxplus$	
	$\blacksquare 6\ \uparrow\odot$
$\odot\ 7\square$	

Figure 6: The fluid levels at the beginning of step 3

Again, Proposition 2.5.2 does not show what happens next. In our example we assume that the *-operator is well-defined at t_3. Then the next regular state is $(5, 4)$, with buffers 1, 5 and 6 filling (since class 5 is slow). At the end of this step, at time t_4, buffer 4 drained: All the fluid is now in buffers 1, 5 and 6.

How much time does this step take? Station B works exclusively and without idling on fluid in buffer 4. Hence, $t_4 - t_3 = m_4 \cdot Q_4(t_3) = 0.35 \cdot Q_1(t_1)$. The fluid level in buffer 5 at time t_4 is

$$\begin{aligned} Q_5(t_4) &= \int_{t_3}^{t_4} d_4(s) - d_5(s)\ ds \\ &= \int_{t_3}^{t_4} \frac{1}{m_4} - \frac{1}{m_5}\ ds = (t_4 - t_3) \cdot (\frac{1}{0.1} - \frac{1}{0.5}) \\ &= 8 \cdot 0.35 \cdot Q_1(t_1) = 2.8 \cdot Q_1(t_1) \end{aligned} \tag{51}$$

A	B
$\odot\uparrow 1\blacksquare$	
	$\square 2\ \odot$
$\odot\ 3\square$	
	$\square 4\ \odot$
$\downarrow 5\blacksquare$	drains first
drains last	$\blacksquare 6 \downarrow$
$\rightarrow 7\square$	

Figure 7: The fluid levels at the beginning of step 4

Step 4, time t_4

In this stage buffer 7 is not filling since it is fast. Therefore buffers 5 and 6 are both emptying. Since we assumed that buffer 6 had initially a lot of fluid, buffer 5 drains first. Hence we arrive at the end of this step (at time t_5) back at step 1 with fluid in buffers 1 and 6 and the cycle is completed. The difference between the current situation and the beginning of the cycle is that the fluid levels in buffers 1 and 6 changed.

During the time interval (t_4, t_5), station A worked on class 5 on $Q_5(t_4)$ amount of fluid. But it does not work exclusively on class 5, since class 7 takes away some work time: $\dot{T}_5(s) = 1 - \dot{T}_7(s)$ (for $s \in (t_4, t_5)$). During this step we have $\frac{1}{m_7} \cdot \dot{T}_7(s) = d_7(s) = d_6(s) = \frac{1}{m_6} \cdot \dot{T}_6(s) = \frac{1}{m_6}$ implying $\dot{T}_5(s) = 1 - \frac{m_7}{m_6} = 0.5$.

Now we calculate $t_5 - t_4$:

$$\begin{aligned} 0 = Q_5(t_5) = Q_5(t_4) + \int_{t_4}^{t_5} d_4(s) - d_5(s)\ ds \\ = \quad 2.8 \cdot Q_1(t_1) - \int_{t_4}^{t_5} \frac{1}{m_5}\dot{T}_5(s)\ ds \qquad \text{(by (51))} \\ = \quad 2.8 \cdot Q_1(t_1) - (t_5 - t_4) \end{aligned}$$

Hence $(t_5 - t_4) = 2.8 \cdot Q_1(t_1)$.

At the end of step 4 in station A there is fluid only in buffer 1. This fluid accumulated during steps 3 and 4. The system did not work at all on this fluid. Hence, $Q_1(t_5) = t_5 - t_3 = t_5 - t_4 + t_4 - t_3 = 2.8 \cdot Q_1(t_1) + 0.35 \cdot Q_1(t_1) = 3.15 \cdot Q_1(t_1)$. The time $t_5 - t_1$ can then be calculated as $(2.5 + 0.35 + 2.8) \cdot Q_1(t_1) = 5.65 \cdot Q_1(t_1)$.

From this we conclude the following: Going backwards in time we go over and over through steps 1-5, reducing in each cycle the amount of fluid in buffer 1 (at the beginning of each cycle) by a factor of 1/3.15. Each cycle lasts 5.65 times the amount of fluid in buffer 1. Note that this process is independent of the (large) amount of fluid we had in buffer 6. In this way, at some time point t_0 the system must have had hpn buffers $(0, 6)$: No fluid in station A and some fluid in buffer 6 at station B. At time t_0, $(0, 6)^*$ is not well-defined. In fact, since $(0, 6)^* = (0, 6)$ would be a valid fluid solution, the fluid solution at time t_0 is not unique. Figure 8 shows how the irregular time points are located relative to time t_0.

We conclude this section by making some useful observations about Proposition 2.5.2. We formulate them as corollaries.

Figure 8: Location of the irregular time points starting at time t_0

Corollary 2.5.3 *Let* $(Q(\cdot), T(\cdot))$ *be a valid fluid solution. Given* (a, b) *with* $a > 0$*, we know that* $a^* \geq a$ *and* $b^* \leq b$*. Furthermore, if* $a^* \neq a$ *then* a^* *must be filling. Similarly, if* $b^* \neq b$ *then* b^* *is filling.*

Proof: The first statement holds since a and b are non-empty buffers and their fluid cannot "vanish" immediately after t_0 by the continuity of the fluid solution.

The other statements are conclusions from Proposition 2.5.2 and Table 2.

□

Corollary 2.5.4 *Let* $(Q(\cdot), T(\cdot))$ *be a valid fluid solution. Let* (a, b) *representthe hpn buffers at some time point* t_0*. If the *-operator is well-defined, there are only a few possibilities for filling new buffers:*

The only way to fill a new station B buffer l *is either via* $l = b + 2$ *(in this case* $l > b$ *is not the hpn buffer at station B), or via* $l = \ell(a)$*.*

The only way to fill a new station A buffer k *is via* $k = 1$ *or* $k = b + 1$*.*

Proof: This follows from Proposition 2.5.2 and table 2.

□

2.6 Total workload and transience

In this section we introduce the notion of the total workload at a (virtual) station. The total workload is the minimal amount of time a (virtual) station needs to spend on the fluid in the system if the input of fluid is cut off.

We use the total workload of a virtual station to show that the buffers of the virtual station will drain in finite time. This shows that virtual stations and regular stations are in this sense on equal footing.

Definition 2.6.1 *The total workload at station $\mathbb{V}(k)$ ($k \in \{0,\ 4,\ 6,\ 8,\ \ldots,\ N+1\}$) is defined as*

$$f_k(t) := \sum_{l \in \mathbb{V}(k)} m_l \cdot \sum_{j \leq l} Q_j(t).$$

Remark: Note that for $k \neq N+1$, $f_k(t) = 0$ implies that the system is empty. $f_{N+1}(t) = 0$ implies that there is no fluid in buffers $\{1,\ 2,\ \ldots,\ N-1\}$.

In any case, $f_k(t) = 0$ implies $\sum_{j \in \mathbb{V}(k)} Q_j(t) = 0$ ($k \in \{0,\ 4,\ 6,\ 8,\ \ldots,\ N+1\}$).

In Proposition 2.6.3 we are going to show that every virtual station has to drain at some point. For the proof of Proposition 2.6.3, we need the following technical lemma:

Lemma 2.6.2 *For any time t_0 and $t > t_0$,*

$$f_k(t) = f_k(t_0) + M_{\mathbb{V}(k)} \cdot t - \sum_{l \in \mathbb{V}(k)} T_l(t)$$

Proof: The proof is straightforward using that $\sum_{j \leq l} Q_j(t) = \sum_{j \leq l} Q_j(t_0) + t - \frac{1}{m_l} T_l(t)$ (this equation follows from (3)).

□

Proposition 2.6.3 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution. Let $t_0 \geq 0$ be some arbitrary time point. For any virtual station $\mathbb{V}(k)$ there is a time $\tau_k(t_0) \geq t_0$ such that the total amount of fluid in that virtual station, $\sum\limits_{l \in \mathbb{V}(k)} Q_l(\tau_k(t_0))$, is equal to 0.*

In particular, station A drains after some time $\tau_0(t_0) \geq t_0$. Similarly, at some time after $\tau_{N+1}(t_0)$, station B drains.

Proof: Without loss of generality we can assume that $t_0 = 0$. We prove the claim by contradiction.

Let us assume there is a virtual station $\mathbb{V}(k)$ such that $\mathbb{V}(k)$ never drains. We use that $\sum_{l \in \mathbb{V}(k)} \dot{T}_l(t) = 1$ if $Q_j(t) > 0$ for some $j \in \mathbb{V}(k)$ (this follows from equations (13) and (16)). Hence as long as there is fluid in the virtual station $\mathbb{V}(k)$, we have $\sum_{l \in \mathbb{V}(k)} T_l(t) = t$.

Since the virtual station $\mathbb{V}(k)$ is nonempty for all times $t > t_0$, then the virtual condition (22) and Lemma 2.6.2 show that f_k is decreasing with derivative $M_{\mathbb{V}(k)} - 1 < 0$. Then there must be a finite time τ_k with $f_k(\tau_k) = 0$. Hence the virtual station $\mathbb{V}(k)$ is empty at time τ_k. Contradiction.

□

Definition 2.6.4 *We define $\tau_k(t) \geq t$ to be the minimal time such that the virtual station $\mathbb{V}(k)$ is empty, ($\tau_k(t) := \min\{s \geq t : \sum\limits_{l \in \mathbb{V}(k)} Q_l(s) = 0\}$ — the minimum is achieved by the continuity of Q).*

We define $\tau_A(t) := \tau_0(t)$, the minimal time when station A drains after time t. ($\tau_A(t) = \min\{s \geq t : a(Q(s)) = 0\}$).

$\tau_B(t) := \tau_{N+1}(t)$*, the minimal time when station B drains after time* t *(*$\tau_B(t) = \min\{s \geq t : b(Q(s)) = N+1\}$*).*
We denote $\tau_A := \tau_A(0)$ *and* $\tau_B := \tau_B(0)$*.*

After time τ_A*, certain (regular or irregular) states cannot occur anymore. We call those states transient.*

2.7 Transient states

Since transient states do not occur after time τ_A, they can be ignored for the proof of the Stability Theorem.

This section determines certain transient states (in Propositions 2.7.4, 2.7.6 and 2.7.8). The transience of these states enables us to show that (after time τ_A) a state with station B empty leads to a state with fluid only in buffer 1 (Theorem 2.7.7). Together with Proposition 2.6.3 this shows that any valid fluid solution reaches a state with fluid only in buffer 1 in finite time.

First, we examine sequences of states created through the *-operator. Proposition 2.7.2 shows that any infinite sequence contains states where station A is empty.

Recall that "state (a, b) at time t_0" means that a valid fluid solution has hpn buffers (a, b) at the (regular or irregular) time point t_0.

Proposition 2.5.2 shows that starting from some time t_0 with hpn buffers (a, b) and $a > 0$, the fluid solution is affine in the time interval $(t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

So in the set of irregular time points where station A is nonempty there are no right-handed accumulation points as in example 2 (on page 52), and the following sequence of states starting at some regular time t_0 is defined:

$$(a(Q(t_0)), b(Q(t_0))) =: (a_0, b_0)(\text{where } a_0 > 0) \longrightarrow$$
$$\longrightarrow ({a_0}^*, {b_0}^*) =: (a_1, b_1) \longrightarrow (a_2, b_2)\ [\neq (a_1, b_1)] \longrightarrow ({a_2}^*, {b_2}^*) =: (a_3, b_3) \longrightarrow$$
$$\longrightarrow (a_4, b_4)\ [\neq (a_3, b_3)] \longrightarrow \ldots \longrightarrow (0, \tilde{b})$$

(Proposition 2.7.2 shows that the state $(0, \tilde{b})$ is reached after finitely many nonempty station A states.)

Let us introduce the notion of limit states and limit points. Roughly speaking, limit points are accumulation points of irregular points (like time t_0 in example 2) and limit states are states at limit points. Proposition 2.7.2 shows that at a limit point, station A is empty.

Definition 2.7.1 *Given a valid fluid solution $(Q(\cdot), T(\cdot))$ and a time t_0.*

A state $(a(t_0), b(t_0))$ is called a left-handed limit state, if for all times $t < t_0$, there is some time $t_1 \in (t, t_0)$ such that $(a(t), b(t)) \neq (a(t_1), b(t_1))$. t_0 is called a left-handed limit point.

Correspondingly, a state $(a(t_0), b(t_0))$ is called a right-handed limit state, if for all times $t > t_0$, there is some time $t_1 \in (t_0, t)$ such that $(a(t), b(t)) \neq (a(t_1), b(t_1))$. t_0 is called a right-handed limit point.

Remark: Corollary 2.4.10 implies that a left-handed (right-handed) accumulation point of irregular time points is necessarily a left-handed (right-handed) limit point.

Proposition 2.5.2 shows that in a valid fluid solution, a right-handed limit state can only be a state with empty station A buffers (otherwise the *-operator is well-defined). The following proposition shows that left-handed limit states can only be states with empty station A buffers.

Proposition 2.7.2 *Every infinite sequence of states in a valid fluid solution contains states where station A is empty.*
As a consequence, in a left-handed limit state, station A is empty.

Proof: The following arguments are based on the fact that at the end of every regular state with nonempty station A (that is not a right handed limit point) the system has to drain at least one buffer (this follows from Lemma 2.4.8).

We also know from Proposition 2.5.2 that in odrder to fill some buffer $k > 1$, a lower numbered buffer $l < k$ is emptying (again under the assumption that station A is nonempty).

Another observation is that if a state (a^*, b^*) with $a^* > 0$ is draining station A, then a^* has to be 1, because otherwise there will be fluid in buffer 1 at the end of state (a^*, b^*). On the other hand, if buffer 1 drains, then station A is empty.

We prove the claim by contradiction. Assume there is an infinite sequence of states starting at time t_0 that does not contain a state where station A is empty.

How often can each buffer be drained during the sequence without draining buffer 1 and hence draining station A?

Consider buffer 2: At time t_0 buffer 2 may or may not contain fluid. Therefore buffer 2 can be drained at most once: To fill it again buffer 1 would need to empty, leading to an empty station A.

Buffer 3: At time t_0 it can contain fluid, so it can drain once. To drain it again it has to refill by emptying one of the previous buffers, i.e. buffer 1 or 2 (and emptying buffer 1 leads to an empty station A). Therefore, refilling buffer 3 can happen only once. So without draining buffer 1 we can drain buffer 3 at most 2 times.

Inductively, without draining buffer 1, buffer k can be drained at most $1 + \sum_{i=2}^{k-1} \rho(l)$ times (where $\rho(l)$ is the maximal possible number of times buffer l can be emptied without draining buffer 1).

In the end, before draining buffer 1 there can only be a finite number of states. Contradiction and the first part of the proposition is proven.

We prove also the second part by contradiction. Assume there is a left-handed limit state (a_0, b_0) at time t_0 and $a_0 \geq 1$ (hence $Q_{a_0}(t_0) > 0$). By the continuity of the fluid solution, there is some time $t_1 < t_0$, such that $Q_{a_0}(t) > 0$ for all $t \in (t_1, t_0)$.

Since station A is nonempty in (t_0, t_1), the *-operator is well-defined for all times in this interval. Starting at some time within (t_0, t_1), by the definition of a left-handed limit state there must be an infinite sequence of states with nonempty station A, which contradicts the first part of the proposition.

$\square$

We show now that for a nonempty station A, the converse of Proposition 2.4.9 is true: At a regular time t_0, the hpn buffers at station A and B are constant in some open time interval containing t_0.

Proposition 2.7.3 *Let $(Q(\cdot), T(\cdot))$ be a given valid fluid solution and let (a, b) be the hpn buffers at time t_0 with $a > 0$. If t_0 is a regular time, then there is some $\varepsilon > 0$ such that $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.*

This shows that if $(a, b) \neq (a, b)^*$ with $a > 0$ at time t, then t is irregular.

Proof: Since station A contains fluid at time t_0, there is some $\delta > 0$ such that there is fluid in station A for all times in $[t_0 - \delta, t_0 + \delta]$. This implies that there is only a finite sequence of different states in $[t_0 - \delta, t_0 + \delta]$. Furthermore, the *-operator is well-defined in this interval.

Assume now that there is no ε with $0 < \varepsilon < \delta$ such that $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in $[t_0 - \varepsilon, t_0 + \varepsilon]$. Since in $[t_0 - \delta, t_0 + \delta]$ there is only a finite sequence of states, there must be a $\delta' > 0$ such that $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in $[t_0 - \delta', t_0)$ and in $(t_0, t_0 + \delta']$. But since t_0 was a regular point, we know (by the Mean Value Theorem and since $Q(\cdot)$ is an affine function in $[t_0 - \delta', t_0)$ and in $(t_0, t_0 + \delta'])$ that $\dot{Q}(\cdot)$ is constant in $[t_0 - \delta', t_0]$ and in $[t_0, t_0 + \delta']$, hence constant in $[t_0 - \delta', t_0 + \delta']$.

$\dot{Q}$ being constant in $[t_0 - \delta', t_0 + \delta']$ implies that $(a(Q(\cdot)), b(Q(\cdot)))$ is constant in the interval $(t_0 - \delta', t_0 + \delta')$. Contradiction.

$\square$

The following proposition shows that fast station A buffers (buffers that are faster than their preceding station B buffer) are transient.

Proposition 2.7.4 *For any $k \in A \setminus \{1\}$, if $Q_k(t_0) > 0$ for some time $t_0 \geq \tau_A$, then $m_k > m_{k-1}$. Hence any state (a, b) with $m_a < m_{a-1}$ is transient.*

Proof: Since $Q(\cdot)$ is absolutely continuous, we know that

$$\begin{aligned} Q_k(t) &= 0 + \int_{\tau_A}^{t} d_{k-1}(s) - d_k(s) ds \\ &= \int_{\tau_A}^{t} \frac{1}{m_{k-1}} \dot{T}_{k-1}(s) - \frac{1}{m_k} \dot{T}_k(s)\, ds. \end{aligned} \tag{52}$$

We show that the integrand is smaller than or equal to 0 for all regular times s.

If $k > a(Q(s))$ then the integrand is equal to 0 by equation (15).

If $k < a(Q(s))$, then by Proposition 2.4.5, $k < b(Q(s))$ (since $b(Q(s)) \geq a(Q(s)) - 1$ and $k \leq a(Q(s)) - 2$). Therefore (14) and (18) imply $d_{k-1}(s) = 0$, which also shows that the integrand is equal to 0.

It remains to show that the integrand is equal to 0 for the case $k = a(Q(s))$. If $b(Q(s)) > k-1$, then $d_{k-1}(s) = d_{k-2}(s) = 0$. If $b(Q(s)) \leq k-1$, then Proposition 2.4.5 implies $b(Q(s)) = k - 1$. Furthermore, Proposition 2.4.7 shows that the integrand is less than or equal to 0.

Therefore, the integrand is almost everywhere less than or equal to 0, hence the integral is equal to 0 and the claim is proven.

$\square$

For the proof of the next proposition we need the following technical lemma.

Lemma 2.7.5 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution and let (t_0, t_1) be a time interval in which station A is not empty. Let K be a set of consecutive classes with $K = \{k, k+1, k+2, \dots, l\}$ ($k \in B$ and $l \in A$).*

Then station A and station B work exclusively and without idling on classes $k, k+1, k+2, \dots, l$ if and only if the hpn buffers $a(Q(t))$ and $b(Q(t))$ are both in K for all regular times $t \in (t_0, t_1)$.

The proof of Lemma 2.7.5 is purely technical. It is moved to Appendix C.

Proposition 2.7.6 shows that states with fluid in buffer $k \in A \setminus \{1\}$ (k is not necessarily the hpn buffer at station A) and hpn buffer b at station B with $b > \ell(k)$, are transient. Hence the filled station A buffers cannot be "too far away" from the hpn buffer b at station B.

Proposition 2.7.6 *For any valid fluid solution $(Q(\cdot), T(\cdot))$ the following holds for all times $t_0 \geq \tau_A$: For all $k \in A \setminus \{1\}$ with $Q_k(t_0) > 0$, we have $b(Q(t)) \leq \ell(k)$.*

Proof: Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution and let $t_0 \geq \tau_A$ be given.

We prove Proposition 2.7.6 by, starting at time t_0, following the states of the system backwards in time. After finitely many steps, we reach at time t_{i_0} an empty station A where the *-operator is well-defined. Starting at time t_{i_0}, we prove the claim by induction, following the states of the system along its irregular time points t_i until we reach time t_0.

If at time t_0 station A is empty, there is nothing to show.

Hence we consider the case that $a_0 := a(Q(t_0)) > 0$. By Proposition 2.7.2, t_0 is not a left-handed limit point. Therefore there is some (minimal) time $t_1 < t_0$ such that the hpn buffers $(a(Q(\cdot)), b(Q(\cdot)))$ at stations A and B do not change during the time interval (t_1, t_0) $(t_1 = \inf\{s < t_0 : (a(Q(\cdot)), b(Q(\cdot)))$ is constant on $(t_1, t_0)\})$. Clearly, $t_1 \geq \tau_A$. We define $a_1 := a(Q(t_1))$ and $b_1 := b(Q(t_1))$. By definition of a_1 and b_1, we have $(a_1, b_1)^* = ({a_1}^*, {b_1}^*) = (a(Q(t)), b(Q(t)))$ for all $t \in (t_1, t_0)$.

As long as station A is nonempty, we can inductively find times $t_2 > t_3 > t_4 > \ldots$ with the hpn buffers at station A and B, $(a(Q(\cdot)), b(Q(\cdot)))$, being constant on the intervals (t_{i+1}, t_i) $(i = 0, 1, 2, 3, \ldots)$ and $(a_{i+1}, b_{i+1})^* := (a(Q(t_{i+1})), b(Q(t_{i+1})))^* = (a(Q(t)), b(Q(t)))$ for all $t \in (t_{i+1}, t_i)$.

Proposition 2.7.2 shows that after finitely many steps we reach time $t_{i_0} \geq \tau_A$ with $a_{i_0} = 0$.

Note that for time t_{i_0}, $(a_{i_0}, b_{i_0})^*$ is well-defined although we encountered an empty station A state. We prove the claim by induction, starting with state (a_{i_0}, b_{i_0}) at time t_{i_0} and ending at time t_0 with state (a_0, b_0). In every step, assuming that the claim holds for all times in $[t_{i_0}, t_i]$, we prove the claim for state $(a_i, b_i)^*$ on time interval (t_i, t_{i-1}) and then for state (a_{i-1}, b_{i-1}) at time t_{i-1}.

Sometimes we refer to "state (a_h, b_h) at time t_h" short as "state (a_h, b_h)". Similarly, we sometimes refer to "state $(a_h, b_h)^*$ in time interval (t_h, t_{h+1})" short as "state $(a_h, b_h)^*$".

Induction beginning:

Trivially the claim is true for the state $(a_{i_0}, b_{i_0}) = (0, b_{i_0})$. It is also true for the following regular state $(0, b_{i_0})^*$ which is by Table 2 either $(0, b_{i_0})$, $(1, b_{i_0})$ or $(b_{i_0}+1, b_{i_0})$.

Finally, the claim is true for the irregular state following $(0, b_{i_0})^*$ at time t_{i_0+1}: By Proposition 2.4.7, such a state is either $(0, b_{i_0}+2)$, $(1, b_{i_0}+2)$ or $(b_{i_0}+1, b_{i_0}+2)$. For the first two trivially the claim is true. For the latter state, we have by Proposition 2.7.4 $m_{b_{i_0}} < m_{b_{i_0}+1}$, implying $\ell(b_{i_0}+1) \geq b_{i_0}+2$, so the claim holds here as well.

Induction step:

Assume now that the claim is true until time t_i with the (irregular) state (a_i, b_i).

We are going to prove the claim for the time interval (t_i, t_{i-1}) and for time t_{i-1}.

There are the following cases to consider:

1. $b_i = N+1$:

   Then by the induction hypothesis $\ell(a_i) = N+1$. Table 2 shows that $(a_i, b_i)^* = (a_i, N+1)$ with buffer a_i emptying. Therefore, the claim holds by the induction hypothesis. Hence $(a_{i-1}, b_{i-1}) = (a', N+1)$ with some $1 \leq a' < a_i$ and the claim holds for (a_{i-1}, b_{i-1}) by the induction hypothesis, since all station A buffers that are filled at time t_{i-1} were also filled at time t_i. For this case, the induction step is proven.

2. $b_i < N+1$ and $m_{b_i+1} > m_{b_i}$:

   First we show that $(a_i, b_i)^* = (b_i+1, b_i)$. We distinguish between the following subcases:

   $a_i = 1$ and $\ell(a_i) \leq b_i$:

   This implies that $\ell(a_i) < b_i$ by Corollary 2.4.4.

   According to Table 2 there are 2 possibilities for $(a_i, b_i)^*$: Either it is equal to $(1, \ell(1))$ or it is equal to (b_i+1, b_i). The first option leads to an empty station

A by Proposition 2.4.7 (buffer $\ell(1)$ is filling and hence buffer 1 is emptying). This contradicts the choice of the index i_0 (for all $i < i_0$, we have $a_i > 0$ and ${a_i}^* > 0$), so $(a_i, b_i)^* = (b_i + 1, b_i)$.

$a_i = 1$ and $\ell(a_i) > b_i$:

In this case, Table 2 shows that $(a_i, b_i)^* = (b_i + 1, b_i)$.

$a_i > 1$:

In this case we have $\ell(a_i) \geq b_i$ by induction hypothesis. Together with $m_{b_i+1} > m_{b_i}$, this implies that $\ell(a_i) > b_i$ and hence $(a_i, b_i)^* = (b_i + 1, b_i)$.

So $(a_i, b_i)^* = (b_i+1, b_i)$ is proven. Now, by Proposition 2.4.7, buffer $a_i^* = b_i+1$ is filling while buffer b_i is emptying. That means that in the transition from state (a_i, b_i) to state $(b_i + 1, b_i)$ we do not get a different hpn buffer at station B and the claim holds for each class $k \in A$, $1 < k < b_i$ by the induction hypothesis. For buffer b_i+1 the claim is trivially true since (by definition) $\ell(b_i+1) \geq b_i+2$.

Proof of the claim for the irregular state $(a_{i-1}, b_{i-1}) = (b_i+1, b_i+2)$ that follows $(a_i, b_i)^* = (b_i + 1, b_i)$ at time t_{i-1}:

The claim is trivial for buffer b_i+1. For all other station A classes $1 < k < b_i+1$ with $Q_k(t_0) > 0$: We know by the induction hypothesis $\ell(k) \geq b_i$. That means (see definition of $\ell(k)$): For each class $l \in B$, $k < l \leq b_i$: $M_B(l) - M_B(k) \leq M_A(l) - M_A(k)$.

Using that $m_{b_i} < m_{b_i+1}$, we conclude that for each class $l \in B$, $k < l \leq b_i + 2$: $M_B(l) - M_B(k) \leq M_A(l) - M_A(k)$ and hence $\ell(k) \geq b_i + 2$. Hence for this case, the induction step is proven.

3. $b_i < N+1$ and $m_{b_i+1} < m_{b_i}$:

   In this case by Table 2, no previously empty station A buffer can fill except buffer 1.

   Therefore the claim is trivial if $a_i = 1$, since then for $(a_i, b_i)^*$ and for the subsequent irregular state (a_{i-1}, b_{i-1}) there is no filled station A buffer numbered higher than 1.

   If $a_i > 1$ then by the induction hypothesis ($b_i \leq \ell(a_i)$), we have $(a_i, b_i)^* = (a_i, b_i)$ with buffer b_i either filling or emptying (depending on whether b_i is equal to $\ell(a_i)$ or not). No new station A buffer fills, so during state $(a_i, b_i)^*$ the claim holds by the induction hypothesis.

   If in this state buffer a_i drains before buffer b_i (including the case that buffer b_i is actually filling), the subsequent irregular state after $(a_i, b_i)^*$ is $(a_{i-1}, b_{i-1}) = (a', b_i)$ with $a' < a_i$ and hence the claim is proven by the induction hypothesis (no newly filled station A buffer, no different hpn buffer at station B).

   It remains to prove that the claim holds for the irregular state (a_{i-1}, b_{i-1}) after $(a_i, b_i)^*$ if buffer b_i drains first or a_i and b_i drain simultaneously (the subsequent irregular state in this subcase is $(a_{i-1}, b_{i-1}) = (a', b_i + 2)$ with some $a' \leq a_i$). We necessarily have in this case that $\ell(a_i) \geq b_i + 2$, otherwise buffer b_i could not have been emptying.

   We refer to the time when buffer a_i started to fill as t', $t' = \inf\{t < t_i \ : \ Q_{a_i}(s) > 0 \text{ for all } t < s < t_i\}$. Since $t' \geq t_{i_0}$ and since in the time interval (t_{i_0}, t'), station A is empty, there is some index $j_0 \geq i_0$, such that $t_{j_0} = t'$.

The following statement is intuitively clear but the proof is somewhat tedious. We refer the proof to the end.

Claim A: In the time interval (t_{j_0}, t_{i-1}), stations A and B work exclusively and without idling on the classes $a_i - 1,\ a_i,\ a_i + 1,\ \dots,\ b_i + 1$.

By Lemma 2.7.5 this is equivalent to saying that the hpn buffers of the fluid solution have numbers between $a_i - 1$ and $b_i + 1$ at all regular times in (t_{j_0}, t_{i-1}).

By Table 2, the irregular state (a_{j_0}, b_{j_0}) (before state $(a_{j_0}, b_{j_0})^*$ when we started to fill buffer a_i), necessarily is equal to $(a', a_i - 1)$ with $a' < a_i$ (a' is possibly equal to 0). So at time t_{j_0}, when the fluid solution started to fill buffer a_i, there was no fluid in the buffers $a_i,\ a_i + 2,\ a_i + 4,\ \dots$ at station A.
Hence from time t_{j_0} of state (a_{j_0}, b_{j_0}) until time t_i of state (a_i, b_i) station A and B were working exclusively and without idling on the classes $a_i - 1,\ a_i,\ a_i + 1,\ a_i + 2\ \dots,\ b_i + 1$ on fluid that was at time t_{j_0} in the buffers $a_i - 1,\ a_i + 1,\ a_i + 3,\ a_i + 5,\ \dots\ b_i$.

For draining this fluid from buffer b_i out of the buffers $a_i - 1,\ a_i,\ a_i + 1,\ a_i + 2,\ \dots,\ b_i + 1$, station B needs m_{b_i} time per unit of fluid, and station A needs m_{b_i+1} time per unit of fluid.

For draining the fluid from buffer $b_i - 2$ out of the area $a_i - 1,\ a_i,\ a_i + 1,\ a_i + 2,\ \dots,\ b_i + 1$, station B needs $m_{b_i} + m_{b_i-2}$ time per unit of fluid, and station A needs $m_{b_i+1} + m_{b_i-1}$ time per unit of fluid.

We continue in this manner for buffers $b_i - 4$, $b_i - 6$, ...

At the end we get:

For draining the fluid from buffer $a_i - 1$ out of the area $a_i - 1,\ a_i,\ a_i + 1,\ a_i + 2,\ \ldots,\ b_i + 1$, station B needs $m_{b_i} + m_{b_i - 2} + \ldots + m_{a_i - 1}$ time per unit of fluid, and station A needs $m_{b_i+1} + m_{b_i-1} + \ldots + m_{a_i}$ time per unit of fluid.

Since station B finishes this task at least as fast as station A (otherwise buffer a_i would empty first) and since $m_{b_i} > m_{b_i+1}$, there must be some $\kappa \in \{a_i, \ldots b_i - 1\},\ \kappa \in A$, such that the following inequality holds:

$$m_{b_i} + m_{b_i-2} + \ldots + m_{\kappa-1} < m_{b_i+1} + m_{b_i-1} + \ldots + m_{\kappa} \tag{53}$$

Now, let $k \in A\backslash\{1\}$ with positive buffer level at the irregular state $(a_{i-1}, b_{i-1}) = (a', b_i + 2)$ at time t_{i-1}. We are going to show that $\ell(k) \geq b_{i-1} = b_i + 2$.

By the remark before claim A, we already know that $\ell(a_i) \geq b_i + 2$.

For any other k, we know by the induction hypothesis that $\ell(k) \geq b_i$. In other words:

$$\text{For each class } l \in B,\ k < l \leq b_i:\ M_B(l) - M_B(k) \leq M_A(l) - M_A(k), \tag{54}$$

in particular (setting $l = \kappa - 1$):

$$M_B(\kappa - 1) - M_B(k) \leq M_A(\kappa - 1) - M_A(k). \tag{55}$$

Adding (53) and (55) gives

$$m_{b_i} + m_{b_i-2} + \ldots + m_{\kappa-1} + M_B(\kappa - 1) - M_B(k)$$
$$\leq m_{b_i+1} + m_{b_i-1} + \ldots + m_{\kappa} + M_A(\kappa - 1) - M_A(k),$$

which shows

$$M_B(b_i+2) - M_B(k) \le M_A(b_i+2) - M_A(k). \tag{56}$$

But (56) together with (54) gives

$$\text{For each class } l \in B,\ k < l \le b_i + 2:\ M_B(l) - M_B(k) \le M_A(l) - M_A(k),$$

in other words, $\ell(k) \ge b_i + 2$.

Therefore, at time t_{i-1}, we have for all $k \in A \setminus \{1\}$ that $\ell(k) \le b_{i-1}$. Hence the induction hypothesis is proven for this case.

We prove now claim A:
In the time interval (t_{j_0}, t_{i-1}), stations A and B work exclusively and without idling on the classes $a_i - 1,\ a_i,\ a_i + 1,\ \ldots,\ b_i + 1$.

Observe that it is sufficient to prove the claim for all open intervals (t_j, t_{j-1}) starting at $j = j_0 \ge i_0$ and ending at $j = i + 1$ (this follows from the absolute continuity of $Q(\cdot)$ and $T(\cdot)$).

We prove the claim on these open intervals by induction, starting with state $(a_{j_0}, b_{j_0})^*$ in the time interval (t_{j_0}, t_{j_0-1}) (with $j_0 \ge i_0$), and ending at state $(a_i, b_i)^*$ in the time interval (t_i, t_{i-1}).

Induction beginning:

The only way to fill the previously empty buffer $a_i > 1$ is, by Corollary 2.5.4, via state $(a_{j_0}, b_{j_0})^* = (a_i, a_i - 1)$ (and having $m_{a_i} > m_{a_i - 1}$ by Proposition 2.7.4). By Table 1, during (t_{j_0}, t_{j_0-1}), stations A and B are working exclusively and without idling on classes $a_i - 1$ and a_i. Buffer $a_i + 1 \in B$ is filling. At time $t_{j_0} - 1$, we arrive at state $(a_{j_0-1}, b_{j_0-1}) = (a_i, a_i + 1)$.

Induction step:

We assume by induction hypothesis stations A and B work exclusively and without idling on buffers $a_i - 1,\ a_i,\ a_i + 1,\ \ldots,\ b_i + 1$ during time intervals (t_j, t_{j-1}).

We are going to show that if $j - 1 \geq i$, then stations A and B work also exclusively and without idling on buffers $a_i - 1,\ a_i,\ a_i + 1,\ \ldots,\ b_i + 1$ during time intervals (t_{j-1}, t_{j-2}).

Since stations A and B work during the time interval (t_j, t_{j-1}) exclusively and without idling on buffers $a_i - 1,\ a_i,\ a_i + 1,\ \ldots,\ b_i + 1$, we have by Lemma 2.7.5 $b_i + 1 \geq {a_j}^* \geq a_i$ and $a_i - 1 \leq {b_j}^* \leq b_i$.

What happens during state $({a_j}^*, {b_j}^*)$ in time interval (t_j, t_{j-1})? Clearly, buffer ${b_j}^* + 2$ is filling. At the end of this state, buffer ${a_j}^*$ or buffer ${b_j}^*$ drain (maybe both simultaneously).

In any case, if this was not already state $(a_i, b_i)^*$ (i.e. $j = i$), then the next consecutive irregular state is (a_{j-1}, b_{j-1}) with $a_{j-1} \geq a_i$ (since we do not empty buffer a_i during interval (t_{j_0}, t_{i-1})). We also have $b_{j-1} \leq b_i$: If not then for some time in (t_{j-1}, t_i) some buffer b' with $b' \leq b_i$ would need to refill (since at time t_i, buffer b_i is nonempty). We show that this is impossible:

Assume there is such a $b' \leq b_i$ which refills in (t_{j-1}, t_i). Without loss of generality we can choose b' to be minimal (i.e. buffer $b' \leq b_i$ is the minimal buffer in station

B that refills in (t_{j-1}, t_i)). By Corollary 2.5.4, b' can only start to fill via state $(a_h, b_h)^* = (a_h, b_h - 2)$ or via state $(a_h, b_h)^* = (a_h, \ell(a_h))$ and $j - 1 \geq h \geq i$. The first case contradicts the minimality of b'. The second case implies that $\ell(a_h) < b_h$. But since $h \geq i$, the induction hypothesis of the outer induction (the induction over i) implies that $\ell(a_h) \geq b_h$.

We showed that $a_{j-1} \geq a_i$ and that $b_{j-1} \leq b_i$.

The regular state $(a_{j-1}, b_{j-1})^* = ({a_{j-1}}^*, {b_{j-1}}^*)$ during time interval (t_{j-1}, t_{j-2}) has the property that $a_i - 1 \leq {b_{j-1}}^* \leq b_{j-1} \leq b_i$ and $a_i \leq a_{j-1} \leq {a_{j-1}}^* \leq {b_{j-1}}^* + 1 \leq b_i + 1$ (by Proposition 2.4.5 and Corollary 2.5.3).

Therefore, Lemma 2.7.5 shows that station A and B work during time interval (t_{j-1}, t_{j-2}) exclusively and without idling on buffers $a_i - 1,\ a_i,\ a_i + 1,\ \ldots,\ b_i + 1$ and the induction step is proven.

Therefore, claim A and hence the proposition is proven.

$\square$

Remark: The statement of Proposition 2.7.6 is very strong. It implies that (after some finite initial time) we can never encounter a state with hpn buffers (a, b) where $b > \ell(a)$ (for $a > 1$).

The most important conclusion we can draw from Proposition 2.7.6 is that to examine whether a valid fluid solution is stable, we can assume without loss of generality that there is only fluid in buffer 1 and all other buffer levels are 0:

Theorem 2.7.7 *Starting from any initial fluid levels at time 0, there is a time $t \geq 0$ such that $Q_k(t) = 0$ for each class $k > 1$.*

Proof: At at time $\tau_B(\tau_A)$, station B is empty. By Proposition 2.7.6, for each class $k \in A \setminus \{1\}$ with $Q_k(\tau_B(\tau_A)) > 0$, we have $\ell(k) = N+1$. Table 2 shows that starting from time $\tau_B(\tau_A)$ we drain the hpn buffer a at station A via state $(a, N+1) = (a, N+1)^*$ without filling any other buffer except buffer 1. As soon as a drains, we drain the next station A buffer that contained fluid, a', via state $(a', N+1) = (a', N+1)^*$, without filling any previously empty buffer except buffer 1. Continuing in this fashion we drain one station A buffer after another without ever filling any new buffer until there is fluid only in buffer 1.

$\square$

The next proposition shows that states with fluid in buffer $k \in A \setminus \{1\}$ (k is not necessarily the hpn buffer at station A) and hpn buffer b at station B with $b > \ell(k-1)$, are transient. Although similar to Proposition 2.7.6, Proposition 2.7.8 is not implied by Proposition 2.7.6 or vice versa.

Proposition 2.7.8 *For any valid fluid solution $(Q(\cdot), T(\cdot))$ the following holds for all times $t_0 \geq \tau_A$: For each class $k \in A \setminus \{1\}$ with $Q_k(t_0) > 0$, we have $b(Q(t)) \leq \ell(k-1)$.*

Proof: We follow the proof of Proposition 2.7.6. Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution and $t_0 \geq \tau_A$ be given. As in Proposition 2.7.6, if at time t_0 station A is empty, there is nothing to show.

In the same way as in the proof of Proposition 2.7.6 we construct a sequence $t_1 > t_2 > t_3 > \ldots$ of times with $(a(Q(\cdot)), b(Q(\cdot)))$ being constant on the intervals (t_{i+1}, t_i) $(i = 0, 1, 2, 3, \ldots)$. We define $a_i := a(Q(t_i))$ and $b_i := b(Q(t_i))$. Then $(a_{i+1}, b_{i+1})^* = (a(Q(t)), b(Q(t)))$ for all $t \in (t_{i+1}, t_i)$.

After finitely many steps we arrive at time t_{i_0} in some state (a_{i_0}, b_{i_0}) with $a_{i_0} = 0$.

For time t_{i_0}, $(a_{i_0}, b_{i_0})^*$ is well-defined although it is an empty station A state. We prove the proposition by an induction beginning with state $(a_{i_0}, b_{i_0})^*$ at time t_{i_0} and ending with state (a_0, b_0) at time t_0.

Induction beginning:

Trivially the claim is true for the state $(a_{i_0}, b_{i_0}) = (0, b_{i_0})$. It is also true for the following state $(0, b_{i_0})^*$ which is either $(0, b_{i_0})$, $(1, b_{i_0})$ or $(b_{i_0} + 1, b_{i_0})$ by Table 2.

Finally, the claim is true for the irregular state following $(0, b_{i_0})^*$:
By Proposition 2.4.7 this state is either $(0, b_{i_0} + 2)$, $(1, b_{i_0} + 2)$ or $(b_{i_0} + 1, b_{i_0} + 2)$. For the first two trivially the claim is true, for the latter state we have $m_{b_{i_0}} < m_{b_{i_0}+1}$, implying $\ell(b_{i_0}) \geq b_{i_0} + 2$ by Proposition 2.7.4, so the claim holds also here.

Induction step:

Assume the claim is true until the (irregular) state (a_i, b_i) is reached.
There are the following cases to consider:

1. $b_i = N + 1$.

   Then $(a_i, b_i)^* = (a_i, N + 1)$ and hence the claim is true during the time interval (t_i, t_{i-1}) by the induction hypothesis. For the following state $(a_{i-1}, b_{i-1}) = (a', N + 1)$ with $a' < a_i$, the claim also holds by the induction hypothesis.

2. $b_i < N + 1$ and $m_{b_i+1} > m_{b_i}$.

   As in the proof of Proposition 2.7.6, it can be shown that $(a_i, b_i)^* = (b_i + 1, b_i)$. Then by Proposition 2.4.7, buffer $a_i^* = b_i + 1$ is filling while buffer b_i is emptying. That means that the hpn buffer at station B does not change and the claim holds for all filled classes k with $1 < k < b_i$ by the induction hypothesis. For buffer $b_i + 1$ the claim is trivially true since by definition $\ell(b_i) \geq b_i$.

Proof of the claim for the irregular state $(a_{i-1}, b_{i-1}) = (b_i+1, b_i+2)$ that follows $(a_i, b_i)^* = (b_i + 1, b_i)$ at time t_{i-1}:

The claim is trivial for buffer b_i+1 since $m_{b_i+1} > m_{b_i}$ and hence $\ell(b_{i_0}) \geq b_{i_0}+2$.

For all other station A classes $1 < k < b_i+1$ with $Q_k(t_0) > 0$: By the induction hypothesis, $b_i \leq \ell(k-1)$ holds. Therefore (see definition of $\ell(k-1)$) for each class $l \in B$, $k-1 < l \leq b_i$, we have $M_B(l) - M_B(k-1) \leq M_A(l) - M_A(k-1)$. Using that $m_{b_i} < m_{b_i+1}$, we conclude that for each class $l \in B$, $k < l \leq b_i + 2$: $M_B(l) - M_B(k) \leq M_A(l) - M_A(k)$ and hence $\ell(k) \geq b_i + 2$, qed.

3. $b_i < N + 1$ and $m_{b_i+1} < m_{b_i}$:

   In this case, by Table 2, no previously empty station A buffer can fill except buffer 1.

   Therefore the claim is trivial if $a_i = 1$, since then for $(a_i, b_i)^*$ and for the subsequent irregular state (a_{i-1}, b_{i-1}) there is no filled station A buffer numbered higher than 1.

   If $a_i > 1$ then by Proposition 2.7.6, $(a_i, b_i)^* = (a_i, b_i)$ and buffer b_i is either filling or emptying (depending on whether $b_i = \ell(a_i)$ or not). No new station A buffer fills, so for state $(a_i, b_i)^*$ the claim holds by induction hypothesis.

   If in this state buffer a_i drains before buffer b_i (including the case that buffer b_i is actually filling), then the subsequent irregular state after $(a_i, b_i)^*$ is $(a_{i-1}, b_{i-1}) =$

(a', b_i) with some $a' < a_i$. Hence the claim is proven by the induction hypothesis (no newly filled station A buffer, no different hpn buffer at station B).

It remains to prove the claim for the subsequent irregular state (a_{i-1}, b_{i-1}) after $(a_i, b_i)^*$ if buffer b_i drains first or a_i and b_i drain simultaneously (the subsequent irregular state is in this case $(a_{i-1}, b_{i-1}) = (a', b_i + 2)$ with some $a' \leq a_i$).

Exactly as in the proof of claim A in Proposition 2.7.6, we show that from the time when the fluid solution started to fill buffer a_i until the end of state (a_i, b_i) at time t_{i-1}, station A and B are working exclusively and without idling on the classes $a_i - 1,\ a_i,\ a_i + 1,\ a_i + 2\ \ldots,\ b_i + 1$ on fluid that was initially in the buffers $a_i - 1,\ a_i + 1,\ a_i + 3,\ a_i + 5,\ \ldots\ b_i$.

As in the proof of Proposition 2.7.6 we conclude that, since b_i drains first, there must be some $\kappa \in \{a_i, \ldots b_i - 1\},\ \kappa \in A$, such that the following inequality holds:

$$m_{b_i} + m_{b_i - 2} + \ldots + m_{\kappa - 1} < m_{b_i + 1} + m_{b_i - 1} + \ldots + m_\kappa \tag{57}$$

Now: Let $k \in A$ with positive buffer level at the irregular state $(a_i, b_i + 2)$. By induction hypothesis we have $\ell(k-1) \geq b_i$. In other words: For each class $l \in B,\ k < l \leq b_i$:

$$M_B(l) - M_B(k-1) \leq M_A(l) - M_A(k-1), \tag{58}$$

in particular (setting $l = \kappa - 1$):

$$M_B(\kappa - 1) - M_B(k-1) \leq M_A(\kappa - 1) - M_A(k-1). \tag{59}$$

Adding (57) and (59) gives

$$m_{b_i} + m_{b_i-2} + \ldots + m_{\kappa-1} + M_B(\kappa-1) - M_B(k-1)$$
$$\leq m_{b_i+1} + m_{b_i-1} + \ldots + m_{\kappa} + M_A(\kappa-1) - M_A(k-1).$$

Hence,

$$M_B(b_i+2) - M_B(k-1) \leq M_A(b_i+2) - M_A(k-1). \tag{60}$$

But (60) together with (58) gives for each class $l \in B,\ k < l \leq b_i + 2:$

$$M_B(l) - M_B(k-1) \leq M_A(l) - M_A(k-1),$$

In other words, $\ell(k-1) \geq b_i + 2$, and the induction hypothesis is proven for (a_{i-1}, b_{i-1}).

This concludes the proof of the proposition.

□

2.8 Fluid sections and stability

In this section the classes of the system will be split into fluid sections. Proposition 2.8.3 shows that fluid in station A (except buffer 1) can only be accumulated in buffers of the same fluid section as the hpn buffer at station B — this follows from Proposition 2.7.4 and Proposition 2.7.8.

With Proposition 2.8.3 the proof of the Stability Theorem 2.2.8 can be completed by showing that the slowest virtual station $\mathbb{V}^*$ of the network (i.e. $M_{\mathbb{V}^*}$ is maximum among all virtual stations) works 100% as long there is fluid accumulated on some buffer.

The fluid sections are defined through the buffers β_i and β^i. β^i is the first station B class after β_i that is faster than the succeeding station A class, and β_i is the first station B buffer with the property that moving fluid from class β^{i-1} to class β_i involves more work for Station B than for Station A:

Definition 2.8.1

Define $\beta_1 := \ell(1)$ (implying that $m_{\beta_1+1} < m_{\beta_1}$).

Continue to define β^i and β_i inductively:

Given β_i, β^i is the first station B class numbered higher than β_i such that $m_{\beta^i+1} > m_{\beta^i}$. Given β^i, $\beta_{i+1} := \ell(\beta^i)$.

Observe that $\beta^1 \geq 4$.

This process can end in 3 different ways:

1. *The process stops directly with $\ell(1) = N+1$. In this case we define $\theta := 0$.*

2. *The process ends with $\beta^\nu = N+1$ (there is no station B class numbered higher than $\beta_{\nu-1}$ that is faster then the consecutive station A class). In this case we define $\theta := N+1$*

3. *The process ends with a β^ν ($\beta^\nu \in B \setminus \{2\}$) such that $\ell(\beta^\nu) = N+1$. In this case, we define $\theta := \beta^\nu$.*

For easier notation we define the classes (in station A and B) between β_i and β^i to be the i-section of the network. Figure 9 shows a graphical representation of the β_i, β^i.

Figure 9: Illustration of the definitions of β_i and β^i

The following corollary shows that the virtual station induced by class θ is the slowest virtual station.

Corollary 2.8.2 ($\diamond$) *Let β_i, β^i be defined as above. Then for all $k \in B \cup \{N+1\} \setminus \{2\}$, we have*

$$M_{\mathbb{V}(\theta)} = M_A - M_A(\theta) + M_B(\theta) \leq M_{\mathbb{V}(k)} = M_A - M_A(k) + M_B(k)$$

The proof of Corollary 2.8.2 is purely technical. It is moved to the Appendix C.

The following proposition shows the connection between the sections of the fluid network and a valid fluid solution:

Proposition 2.8.3 *Let $(Q(\cdot), T(\cdot))$ be a valid fluid solution and let t be some time greater than τ_A. Let (a, b) be the hpn buffers of the system at time t and let $b > \beta_j$ for some β_j. Then $Q_k(t) = 0$ for each class $k \in A \setminus \{1\}$, $k < \beta_j$.*

This implies that for $b \in \{\beta_i + 2, \beta_i + 4, \ldots \beta^i - 2\}$, a is equal to 0 or to 1.

Proof: Because of Proposition 2.7.8 and Proposition 2.7.4 it is sufficient to show that for each class $k \in A \setminus \{1\}$ with $m_k > m_{k-1}$ and $k < \beta_j$ we have $\ell(k-1) \leq \beta_j$.

Let such a $k \in A \setminus \{1\}$ with $m_k > m_{k-1}$ and $k < \beta_j$ be given. Then there is some $i < j$ such that $\beta^i \leq k \leq \beta_{i+1} - 2$. The definitions of β^i and β_{i+1} imply

$$M_A(k-1) - M_A(\beta^i) \geq M_B(k-1) - M_B(\beta^i) \text{ and} \quad (61)$$

$$M_A(\beta_{i+1}+2) - M_A(\beta^i) < M_B(\beta_{i+1}+2) - M_B(\beta^i). \quad (62)$$

Inequality (62) implies

$$M_A(\beta_{i+1}+2) - M_A(k-1) + M_A(k-1) - M_A(\beta^i) <$$

$$M_B(\beta_{i+1}+2) - M_B(k-1) + M_B(k-1) - M_B(\beta^i),$$

which shows together with (61) that

$$M_A(\beta_{i+1} + 2) - M_A(k-1) < M_B(\beta_{i+1} + 2) - M_B(k-1).$$

Therefore $\ell(k-1) \leq \beta_{i+1} \leq \beta_j$, and the proposition is proven.

$\square$

With this result, we can prove the Stability Theorem 2.2.8:

Proof: By Theorem 2.7.7 we can assume without loss of generality that we start at time 0 with fluid only in buffer 1. We are going to show that the system drains in finite time.

We distinguish three cases: $\theta = 0$, $\theta = N + 1$ and $2 < \theta < N + 1$.

1. $\theta = 0$:

   Then we have $\ell(1) = N + 1$. In this case by Table 2, $(1, N + 1)^* = (1, N + 1)$ and buffer 1 is emptying without filling any other buffer. At the end of this state the system is empty.

2. $\theta = N + 1$:

   We show first that if at some regular time t there is fluid in some station A buffer, then there is fluid in some station B buffer.

   Assume this is not true. Let t be a regular time with hpn buffers $(a, N+1)$ and $a > 0$. Since $\theta = N+1$, by the definition of θ and by Proposition 2.8.3, we have $a \leq 1$, hence $a = 1$. The definition of θ also implies $\ell(1) < N+1$. Therefore we have $(1, N + 1)^* = (1, \ell(1))$ and hence t was not regular by Proposition 2.7.3. Contradiction.

We showed that for all regular times t, if there is fluid in the system, then there is fluid at station B. Therefore, for all regular times when the system is nonempty, $\sum_{k \in B} \dot{T}(t) = 1$, which means that station B is working 100%. This implies that the buffers $\{1,\ 2,\ 3,\ \ldots,\ N-1\}$ drain after a finite amount of time by Lemma 2.6.2 and the remark below Definition 2.6.1 (using $M_B < 1$).

Since buffer N never fills ($m_N < m_{N-1}$!), the whole systems drains in a finite amount of time.

3. $2 < \theta < N+1$:

   We have $\ell(\theta) = N+1$. By Lemma 2.6.2 and the remark below Definition 2.6.1 it is sufficient to show that the virtual station $\mathbb{V}(\theta)$ works 100% for all regular times in which there is fluid in some buffer of the network.

   We show that for all regular times for which station B does not work 100% on fluid buffers $2,\ 4,\ \ldots,\ \theta-2$, there is fluid in the station A buffers $\theta+1,\ \theta+3,\ \ldots, N$.

   We prove this in 3 steps

   (a) First we define t_1 to be the first time when the station B starts to work on buffer θ. We show that before time t_1, station B works 100% on fluid in buffers $2, 4, 6, \ldots, \theta-2$ and after time t_1, buffer $\theta+1 \in A$ starts to fill. Furthermore, at time t_1, there is no fluid in any buffer except buffers θ and possibly 1.

   (b) We choose t_2 to be the minimum over all times after t_1 when station A or station B empties. We show that at time t_2, station B empties.

(c) Now we know by Proposition 2.7.6, that all station A buffers drain one after another until at time t_3 there is fluid only in buffer 1 and all other buffers are empty.

Setting $t_2 = 0$, we see that these steps are repeated as long there is fluid in the system.

During $[0, t_1]$ Station B works exclusively and without idling on classes 2, 4, $\ldots$, $\theta - 2$. During $[t_1, t_2]$, stations A and B both work exclusively and without idling on fluid in buffers $\theta,\ \theta+1,\ \theta+2,\ \ldots,\ N$ and during during $[t_2, t_3]$, station A works exclusively and without idling classes $\theta+1,\ \theta+3,\ \theta+5,\ \ldots,\ N$ while station B possibly idles. At time t_3 there is fluid only in buffer 1.

Therefore the system works at all regular times 100% on fluid in buffers of the virtual station $\mathbb{V}(\theta)$. This proves the theorem.

We are now showing the steps in detail:

(a) Clearly, at time 0 in state $(1, N+1)$, we have $(1, N+1)^* = (1, \ell(1))$. That means time 0 is irregular by Proposition 2.7.3 and during state $(1, \ell(1))$ there is fluid in buffer $\ell(1) < \theta$.

Define t_1 to be the time when station B starts to work on buffer θ, i.e. $t_1 = \inf\{t > 0 \ :\ T_\theta(t) > 0\}$. By (3) and (4) we see that
$t_1 = \inf\{t > 0 : \text{there is a } k \in \{\theta,\ \theta+2, \ldots, N-1\} \text{ such that } T_k(t) > 0\}$.
As long as there is fluid in the actual hpn buffer b at station B, the system does not dedicate any work time to classes $b+2$ and below, so the system

works exclusively and without idling on buffers $2,\ 4,\ \ldots,\ \theta - 2$ until they drain.

This shows that at time t_1, buffers $2, 4, 6, \ldots, \theta - 2$ are empty.

Furthermore, at time t_1 there is no fluid in buffers $\theta+1, \theta+2, \theta+3, \ldots, N$ since before time t_1, buffer θ hat no output.

Propositions 2.8.3 and 2.7.4 show that at time t_1 there is also no fluid in any station A buffer below θ except possibly buffer 1.

Hence we showed that at time t_1, there is fluid only in buffer θ and possibly in buffer 1 ($Q_\theta(t_1) > 0$, otherwise there is a contradiction to the definition of t_1).

Now we show that $t_1 = \inf\{t > 0\ :\ Q_{\theta+1}(t) > 0\}$:

Clearly $t_1 \leq \inf\{t > 0\ :\ Q_{\theta+1}(t) > 0\}$ since buffer $\theta + 1$ cannot fill as long as there is no output from class θ.

It is left to show that $t_1 \geq \inf\{t > 0\ :\ Q_{\theta+1}(t) > 0\}$.
Observe that for every regular point with $Q_\theta(t) > 0$, we have $\dot{T}_{\theta+1}(t) \leq \dot{T}_\theta(t)$ — this follows from equation (16) and the virtual station constraint (19) (setting $l = \theta$ and noting that $k > \theta + 1$ and $Q_\theta(t) > 0$ imply $m_k d_k(t) = \dot{T}_k(t) = 0$ by (15) and (17)).

Since $Q_\theta(t_1) > 0$ and since $Q(\cdot)$ is continuous, we can choose $t' > t_1$ such that $Q_\theta(t) > 0$ for all $t \in (t_1, t')$. By the definition of t_1 and by the monotonicity of T_θ, we have $T_\theta(t) > 0$ for $t \in (t_1, t')$.

We conclude that for all $t \in (t_1, t')$:

$$\begin{aligned} Q_{\theta+1}(t) &= \int_{t_1}^{t} d_\theta(s) - d_{\theta+1}(s)\, ds \\ &= \int_{t_1}^{t} \frac{1}{m_\theta}\dot{T}_\theta(s) - \frac{1}{m_{\theta+1}}\dot{T}_{\theta+1}(s)\, ds \\ &\geq \int_{t_1}^{t} \frac{1}{m_\theta}\dot{T}_\theta(s) - \frac{1}{m_{\theta+1}}\dot{T}_\theta(s)\, ds \\ &= (\frac{1}{m_\theta} - \frac{1}{m_{\theta+1}}) \cdot \int_{t_1}^{t} \dot{T}_\theta(s)\, ds > 0 \end{aligned}$$

Therefore $t_1 \geq \inf\{t > 0 \ : \ Q_{\theta+1}(t) > 0\}$ is proven.

Hence at time t_1, $(a(Q(t_1)), b(Q(t_1)))^* = (0, \theta)^* = (\theta + 1, \theta)$ is is well-defined. For times greater than t_1, the *-operator is well-defined until station A drains again.

(b) Let t_2 the minimal time when either the station A buffers above buffer 1 or station B drains. We prove that station B drains at time t_2, i.e. $b(Q(t_2)) = N + 1$.

Starting at time t_1, both stations work exclusively and without idling (until t_2) on fluid that was accumulated at time t_1 in buffer θ. For any state (a, b) in time interval (t_1, t_2) (with $a, b \geq \theta$), the system worked since time t_1 exclusively and without idling on classes $\theta,\ \theta + 1,\ \theta + 2,\ \ldots,\ b,\ b + 1$ (recall: a is smaller than or equal to $b + 1$ as long as buffer b is filled!) on fluid that was accumulated in buffer θ at time t_1 — compare to a similar argument in the proof of claim A in the proof of Proposition 2.7.6.

For this task Station A needs $(M_A(b+2) - M_A(\theta)) \cdot Q_\theta(t_1)$ amount of time and station B needs $(M_B(b + 2) - M_B(\theta)) \cdot Q_\theta(t_1)$ amount of time. But the definition of θ implies $\ell(\theta) = N + 1$ and hence $M_A(b+2) - M_A(\theta) \geq M_B(b+2) - M_B(\theta)$, so station B finishes this task first. Therefore in the time interval (t_1, t_2), the following holds: If there is fluid in station B, then there is fluid in station A in some of the buffers $\theta + 1,\ \theta + 3,\ \ldots,\ N$. So we showed that indeed station B drains before the station A buffers above 1.

(c) Hence at time t_2 there is fluid only in station A in the buffers $1,\ \theta+1,\ \theta+3,\ \ldots,\ N$ (not necessarily in all of them).

Let $K := \{k \in A \setminus \{1\}\ :\ Q_k(t_1) > 0\}$.

Proposition 2.7.6 implies that for each class $k \in K$, $\ell(k) = N + 1$. This implies by Table 2 that we drain those buffers one after another via states $(k, N+1) = (k, N+1)^*$. At the end we reach time t_3 with fluid in buffer 1 and all other buffers empty.

□

Remark: Observe that the virtual station that works 100% is the slowest virtual station of Corollary 2.8.2.

2.9 Conclusion and outlook

The stability proof of the fluid model of a LBFS-FBFS re-entrant line that satisfies the usual workload conditions and the virtual station conditions provides some insight into how a fluid solution in the network evolves, and hence gives a starting point for examination of 2-station re-entrant lines under other SBP policies, at least for other SBP policies that do not induce push-start conditions. The crucial points for such an extension are to define the $\ell(\cdot)$ operator (Definition 2.4.3) for such a policy and to transfer the proofs of Propositions 2.4.2, 2.7.6 and 2.7.8 into the new setting.

It is apparent that in our special case the virtual stations induced by the LBFS-FBFS policy play a key role in determining the evolution of a fluid solution. The slowest virtual station turns out to be the bottleneck of the system. It would be interesting to see if for other SBP policies the slowest virtual station also determines the pace of the system (after some transient initial time).

Chapter 3

Part II: Cross-Docking in the Semiconductor Industry

3.1 Introduction

The semiconductor industry has faced significant market changes since the second half of the 1990s: Chips have become increasingly a commodity product with lower profit margins. Distribution strategies are exceedingly influenced by these challenges as e-commerce and e-fulfillment offer new supply chain potential.

Intel has countered these developments by changing its distribution strategy in Europe from warehousing/direct shipping to cross-docking.

In this second part of the dissertation we show the benefits of cross-docking in the semiconductor industry and discuss how these benefits can be enhanced by the shipper's ownership of the cross-dock operations. We also take a look at the obstacles semiconductors face in implementing such a model.

Semiconductor market changes in the late 90s

In the second half of the 1990s, the differences between competing semiconductor products blurred as products became increasingly exchangeable. This even occurred in the high-end PC microprocessor market. As Intel compatibility issues diminished to an insignificant level, products differed principally in terms of processor speed and price. Breaking the 100Mhz barrier made most software applications run with reasonable speeds, putting forth price as the prevalent sales argument.

Figure 10: Semiconductor customers conversion between computers and telecommunications — Source: Electronic Business, December 1999 [13]

At the same time, the demand curve for products that incorporate semiconductors flattened and price competition between semiconductor customers increased. While high-growth computer demand used to drive the chip business, the target markets have shifted since 1995 to the faster growing consumer electronics and telecommunications markets (see Figure 10). These are experiencing more recent slowdowns. The severe competition between semiconductor customers also resulted in higher demands on the supply chain quality service.

The introduction of e-commerce and e-fulfillment enabled the semiconductor industry to implement new distribution strategies but also demanded flexibility in the supply chain management structures.

These developments had severe effects on the semiconductor industry:

- A rigorous price competition in accordance with Moore's law, intensified through thinner profit margins. Moore's law states that within 6 months the chips' capacities double and the prices halve.

- An adverse supply/demand ratio induced by high customer demand volatility and by the lag of the industry's reaction to market changes. A wafer fab (wafer production facility) today is a 2 billion dollar investment (expected to quintuple by 2010) and takes roughly two years to build. In 1998 the demand/supply ratio went down to 62% — a ratio of 70% could be considered normal.

- Increased customer service demands in the form of faster delivery, tight delivery windows, shorter cancellations lead times etc.

The industry's 10% revenue plunge of 1998 made it abundantly clear that the gold rush of the early 90s was over.

Semiconductor companies countered in several ways, including collaborations, outsourcing, product proliferation and supply chain management improvements.

Collaborations and mergers & acquisitions helped spread the risk and investment costs to increase production capacities and the costs of product development. A special form of semiconductors collaboration is the foundry business, the outsourcing of production and the provision of excess production capacities to other companies. The foundry business enables semiconductor companies to trade fabrication capacities. There are even pure foundry companies like Taiwan Semiconductor Manufacturing Corporation or Midwest Microelectronics.

According to Electronic Trend publications, Inc., the capacity of the foundry business capacities increased nearly 70% between 1996 and 1998, despite the obvious challenges of design ownership, design portability, yield and cycle time issues.

Many companies answered the adverse market conditions by entering new markets. Intel set foot in businesses like the electronic toy end-consumer market and Internet services. It achieved these capabilities by extended merger & acquisition activities.

Distribution networks management was for a long time neglected in semiconductors business strategy. While the industry always required high quality service in order fulfillment, the high profit margins made it affordable to pay premium prices for such service. In particular, since the second half of the 1990s semiconductor companies are facing even higher demands on order fulfillment, at the same time encountering a difficult business environment. DHL quantifies National Semiconductors logistics network reliability to be 85%, aiming for 98%.

At the same time, collaborations, product proliferation and mergers and acquisitions put more stress on the supply chain, demanding flexibility towards capacity, shipping routes and freight variety.

Distribution factors

Blue chip players in the semiconductor industry run globally distributed wafer fabs ("front-end") in America, Europe and Asia. From there the dices (processed wafers) are transported to test & assembly facilities ('back-end"), where they are packaged and tested, and sent to distribution centers or directly to customers. The locations of the production and assembly sites are generally decoupled from the market locations. Pull supply chain strategies are predominant, minimizing transportation times by airfreight/truck delivery.

The high demand fluctuations, the short life cycles and high value of semiconductor products force minimal inventory levels. The demand for newly introduced semiconductor products is usually high so that any delay in delivery for filling storage space is absurd. In the microprocessor market companies engage in proactive pricing policies, artificially inflating prices to tone down the demand peaks for new products.

Due to their sensitivity to environmental conditions and their attractiveness to thieves, storage of semiconductor products is avoided wherever possible. Fewer warehouses are easier to protect and require lower investments for providing appropriate climate and security conditions.

3.2 Advantages of cross-docking in comparison to direct shipping

Semiconductor products can be distributed in two ways: Either via direct shipments or via cross-dock distribution centers, where freight arrives, is broken down and re-consolidated to destinations, and shipped out (usually within 12 hours). Cross-docks provide little or no storage.

Intel moved from direct shipments to its European, Middle Eastern and African (EMEA) customers from a European warehouse to a system where freight from world-wide Intel origins is distributed via a cross-docking distribution center in Amsterdam, Netherlands (see Appendix D). Intel even took over the cross-dock operations (previously run by a major freight forwarder). Intel hoped in this way to amplify the advantages of this cross-dock / pooling strategy.

A big problem in the distribution processes of semiconductors is the different evaluation of the freight through the shipper and the freight forwarder/carrier. While for the freight forwarder it is "just another box", for the semiconductor company it is a high value-low weight asset. Its loss or wrong delivery can have serious repercussions for the shipper, seriously damaging customer relationships.

Supply chain flexibility

In a highly competitive global environment with extensive merger & acquisition activities, flexible distribution structures are crucial. On first sight, direct shipping seems to be advantageous. The possibility to redirect shipments "en route" in a distribution facility controverts this assertion. Shipment redirection permits handling of last minute order cancellations, shipment damages or other transportation mishaps.

Changing supply chain processes is usually tedious and expensive since it requires negotiation with all involved parties. During this negotiation, semiconductor companies often experience the strong bargaining power of the freight forwarder they depend on.

For example, FedEx forced National Semiconductors to send slow moving shipments via their most expensive overnight express shipping option.

Intel felt it was difficult to optimize cross-dock internal procedures when the principal freight forwarder ran the cross-dock operations. There was no cost-saving incentive and any change had to be tiresomely negotiated.

Cross-docking enables leg-based distribution routes that significantly reduce the shipper's dependency on the freight forwarder. At the same time it increases the bargaining power towards the freight forwarder and hence supply chain process flexibility.

Shipping costs

The standard argument for cross-docking is reduced transportation costs through freight consolidation. This enables shipping on pallet or truckload level instead on order size level and hence reduces transportation costs.

But this is only a part of the shipping cost savings. Often, globally operating freight forwarders are focused on geographic regions. Having a distribution center between the origins and destinations enables the choice of freight forwarders on leg-level instead of itinerary-level. This allows opting for the regionally best price-service performer.

For example, Intel is moving in 2001 in its Amsterdam facility from a single outbound carrier to a competitive bidding on its outbound routes. Embedding transportation costs as well as service levels into the biding, Intel strives for quality enhancement at lower costs.

Fees, taxes and customs

While shipping costs are the most obvious reasons to run operations through a cross-dock, the taxes and customs reductions are financially compelling, "Just the tax breaks make it worthwhile to distribute through the AMS distribution center" says Neil Fergusson at Intel.

For a number of fees the economics of scale apply. For example, certain airport charges are raised per shipment — so consolidated freight reduces the per-item expenditure.

Intel's European distribution center serves as a legal entity that holds title to the goods flowing through the facility. Customers order from their local Intel subsidiary, but the product is delivered from the Intel Amsterdam legal entity. Regularly, the Intel subsidiaries then receive income according to a distribution key that resembles the orders received through the subsidiary. To avoid high fluctuations due to anomalously big orders the key varies only within certain ranges. This enables relatively balanced income statements of Intel's subsidiaries, reducing the overall long-term taxes.

Although goods can move freely across Europe once they have cleared customs, differences in local tax rates and customs terms can be significant at the point of entry. Hence the selection of the location of the importing distribution center is crucial. The tax and income advantages have to be carefully weighed against regional disadvantages we discuss later.

In many countries tax terms are to some extent negotiable. The company's bargaining power emerges out of its free choice of location as well as out of the facility's yearly turn-over.

International hurdles

In a global industry, overcoming international hurdles such as international law or language barriers is a crucial competitive advantage. In a direct shipping environment legal support, communications etc. between all origin-destination pairs is costly and resource consuming. With a distribution center the number of supported routes decreases significantly. Let us illustrate this with a simple example of 4 origins serving 10 countries. In a direct shipping environment there are up to 40 different legal environments, while with a distribution center there are at most 14.

Peculiarly, embargoes significantly restrain a multinational company's operations. Once more the right choice of location is crucial to expunge such issues.

Security

The National Cargo Security Council estimates the business impact through cargo theft at \$30-\$60 billion dollars per year. The semiconductor industry is a primary target and cargo theft is an exceedingly important issue in semiconductor logistics. Main losses occur through truck stealing/hijacking, warehouse theft/robbery and theft en route (when freight is touched).

Traditionally freight is insured based on weight not on value. According to the Technology Asset Protection Association TAPA, the compensation at industry rate for a stolen 30 pounds box being worth tens of thousands of dollars is less than \$300. Bear in mind that the loss suffered through a stolen box is not only its nominal value but also the potential damage in customer relationships through delayed delivery.

Common freight forwarding facilities, in particular facilities used for various kinds of goods, cannot provide accurate protection. A pure semiconductor cross-dock serves as a "safe haven" for freight, making assets easier to guard and enabling better shipment tracking, which plays a central role in freight security.

Quality service

The exceeding exchangeability of semiconductor products and the severe price competition between semiconductor customers drive rising claims on delivery quality. Semiconductor clients desire one-day delivery windows, order cancellation close delivery date and full supply chain visibility. Issues need to be investigated and resolved as quickly as possible.

We described how cross-docking permits the rerouting of shipments and hence improved quality services. This argument can be strengthened when the shipper itself runs the cross-dock operations.

Cross-docking also makes it easier to track shipments. In this way, it supports shipping issue investigation and it supports the installation of quality metrics.

Finally, due to the shorter route to the destination than in a direct shipping environment, cross-docking facilitates meeting 1 day delivery windows – of course presuming mini-buffer capabilities in the cross-dock operations.

3.3 Alternatives

Despite the impressive advantages of embedding cross-docking into the semiconductor supply chain network, cross-docking is not for every company. A critical question is whether the shipping volume is high enough to support cross-dock operations. The decision depends also on shipment sizes, and the individual evaluation of freight security. Also the need for flexibility and quality service in the distribution network needs to be taken into account. National Semiconductors, for example, sticks to direct shipments, praising the somewhat faster delivery times and the complete supply chain visibility provided by its today freight forwarder, UPS.

The choice of location of the distribution center is vital. Not only tax and customs laws must be taken into account, but also infrastructure connection to the customers. Safety, ergonomics and labor laws can compromise operations significantly.

An even more delicate question is the ownership of the cross-dock operations.

Before 2000, in Intel's Amsterdam distribution center there were two inbound and one outbound freight forwarder who also ran the cross-dock operations. Being dependent on close cooperation with the cross-dock operator made it difficult to introduce process improvements and to introduce new freight forwarders. The take-over of the operations supported Intel's efforts towards a competitive bidding environment and a higher delivery quality.

Ownership of the cross-dock operations increases the shipper's supply chain flexibility. The short delivery time frame of usually less than 48 hours makes real time decisions in accordance with the customer needs imperative. Effective communication channels between the shipper's customer service department and cross-dock operations are easier to achieve if the shipper owns the cross-dock operations.

Intel's ownership of its European cross-dock distribution center also helps to identify and analyze shipping issues, providing valuable metrics for Intel's Quality Operating System.

Another example of the increased flexibility and quality service in the Intel owned cross-dock facility is the warehouses that Intel runs next to the cross-dock operations in Amsterdam. These warehouses mainly serve the delivery of extremely urgent orders. Freight leaving the warehouses enters the distribution channels of the cross-dock facility. In this way Intel is able to provide different delivery service levels using the same distribution channels.

Despite these advantages, it is a strategic question whether a semiconductor company should leave its core competency and invest in logistics know-how. Working closely with the freight forwarders and having sophisticated shipping IT solutions in place can somewhat mitigate any lack of knowledge and experience.

Appendix A

Semiconductor Basics

Semiconductor sales were 149 billion US-dollars in 1999. The top 10 players in this industry are Intel Corp., NEC Corp., Toshiba Corp., Texas Instruments Inc., Motorola Inc., Samsung Electronics Co., Ltd. Hitachi, Ltd. Infineon (former Siemens AG), Hyundai Electronics Industries Co. (now Hynix Semiconductor), ST Microelectronics (order of sales in semiconductors 1998).

To compensate high volatilities in chip demand and to utilize synergy effects, the industry has turned more and more to foundry production (outsourced production) especially in low-end and highly standardized semiconductor products. Fabless semiconductor companies focus only on chip design and outsource their entire production. Still most of the semiconductor companies are integrated device manufacturers (IDMs) that design and manufacture chips.

The production process for chips can be split into 2 parts: Front-end production and back-end production.
In front-end production the layout of several chips is put on a (usually) silicon disk, the wafer, through a repeated photo-chemical process. This is done in wafer fabs (production facilities) that typically cost around 2 billion dollars (these costs are expected to quintuple by 2010). The size of the wafers nowadays are 200-300 mm

(8-12 inch), the unit measure for building chips is 0.13-0.18 micron (1/500th of the seize of a human hair).

In back-end production the chips are packed into a ceramics or plastics package and end-tested.

We distinguish between the following different semiconductor product types:

1. Micro Processor Unit (MPU)
2. Micro Peripherical Processor (MPR), which are subdivided into the subclasses systems support, data storage, graphics, communications and voice processing.
3. Micro Controller Unit (MCU) and Digital Signal Processors (DSP)
4. Dynamic Random Access Memory (DRAM)
5. Static Random Access Memory (SRAM)
6. Nonvolatile Memory, subdivided into ROM, EPROM, flash memory, EEPROM and memory cards.
7. Metal Oxide Semiconductor (MOS)
8. Analog
9. Digital Bipolar (replaced more and more by complementary metal oxide semiconductor (CMOS))

Appendix B

Tables Describing the Behavior of the LBFS-FBFS Re-Entrant Line

Table 1 describes the departure rates of all classes in a valid fluid solution at a regular point t. The hpn buffers at time t are (a, b). The time parameter in $d(t) = (d_k)_{k\in\{1\ldots N\}}$ is omitted.

Table 1: Departure rates at regular points in a valid fluid solution as a function of the hpn buffers (a, b)

Case I: $0 < a < b < N+1$:

$$d_0 = 1$$

$$d_k = 0 \qquad \text{for all } 0 < k < a$$

$$d_k = d_a \qquad \text{for all } a \le k < b$$

$$\text{with } d_a = \frac{m_b - m_{b+1}}{D(a,b)}$$

$$d_{b+1} = d_b \qquad \text{with } d_b = \frac{(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))}{D(a,b)}$$

$$d_k = 0 \qquad \text{for all } k > b+1$$

Case II: $0 < a > b < N+1$:

$$d_0 = 1$$

$$d_k = 0 \qquad \text{for all } \; k \notin \{0, a, b\}$$

$$d_a = 1/m_a$$

$$d_b = 1/m_b$$

Case III: $0 < a < b = N+1$:

$$d_0 = 1$$

$$d_k = 0 \qquad \text{for all } 0 < k < a$$

$$d_k = d_a \qquad \text{for all } k \ge a$$

$$\text{with } d_a = \frac{1}{M_A - M_A(a)}$$

Case IV: $0 = a < b < N+1$:

$$d_k = 1 \qquad \text{for all } \; k < b$$

$$d_{b+1} = d_b \qquad \text{with } d_b = \frac{1 - M_B(b)}{m_b}$$

$$d_k = 0 \qquad \text{for all } \; k > b+1$$

Case V: $0 = a < b = N+1$:

$$d_k = 1 \qquad \text{for all } 0 \le k < N+1$$

Table 2 is a summary of the results of Proposition 2.4.7 and Proposition 2.5.2 under the usual workload and virtual station conditions (20), (21) and (22) and under assumptions (23)-(26).

In Table 2, we assume that at some given time t_0, the hpn buffers at stations A and B are $a \in A$ and $b \in B$ ($a = 0$ means that station A is empty, $b = N + 1$ means that station B is empty). The Table shows how the fluid solution behaves in a time interval $(t_0, t_0 + \varepsilon)$ for some (possibly small) $\varepsilon > 0$.

The first column distinguishes between 5 main cases (they depend on whether a is greater than b). The function $\ell(\cdot)$ maps any station A or station B class k $(1 \leq k \leq N)$ onto some class $\ell(k)$ at station B – see Definition 2.4.3.

The second column of the Table shows the hpn buffers of the system a^* and b^* during the time interval $(t_0, t_0 + \varepsilon)$ (they are not necessarily equal to a and b).

The last column shows which buffers are filling and which are emptying in the time interval $(t_0, t_0 + \varepsilon)$ — $\uparrow$ means increasing fluid level, $\downarrow$ means decreasing fluid level and $\rightarrow$ means constant fluid level. The fluid level of any buffer other than 1, a^*, b^* and $b^* + 2$ does not change.

Note that in the Table, for some states there is no fluid solution, for some states there are multiple solutions, and for some states (namely where station B is empty, case IV) we only can predict a^*, b^* if they are well-defined, i.e. if there exists some small $\varepsilon > 0$ such that the fluid solution is differentiable in the whole time interval $(t_0, t_0 + \varepsilon)$.

Table 2: Behavior of a valid fluid solution — summary of the results of Proposition 2.4.7 and Proposition 2.5.2

case / condition	(a^*, b^*)	filling / emptying buffers			
		1	a^*	b^*	$b^*+2^{(2)}$
$I.1:\ 1 = a < b < N+1$					
$m_{b+1} < m_b,\ \ell(1) > b,\ \frac{m_b}{1-M_B(b)} > \frac{m_{b+1}}{1-M_A(b)}$	$(1, b)$	$\downarrow$		$\downarrow$	$\uparrow$
$m_{b+1} < m_b,\ \ell(1) > b,\ \frac{m_b}{1-M_B(b)} < \frac{m_{b+1}}{1-M_A(b)}$	$(1, b)$	$\uparrow$		$\downarrow^{(1)}$	$\uparrow$
$m_{b+1} < m_b,\ \ell(1) \leq b$	$(1, \ell(1))$	$\downarrow^{(1)}$		$\uparrow$	$\uparrow$
$m_{b+1} > m_b,\ \ell(1) > b$	$(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$m_{b+1} > m_b,\ \ell(1) \leq b$	$(1, \ell(1))$	$\downarrow^{(1)}$		$\uparrow$	$\uparrow$
	or $(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$I.2:\ 1 < a < b < N+1$					
$m_{b+1} < m_b,\ \ell(a) > b$	(a, b)	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
$m_{b+1} < m_b,\ \ell(a) \leq b$	$(a, \ell(a))$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
$m_{b+1} > m_b,\ \ell(a) > b$	$(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$m_{b+1} > m_b,\ \ell(a) \leq b$	$(a, \ell(a))$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
	or $(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$II.:\ 0 < a > b < N+1$					
$a = b+1, m_{b+1} < m_b$	$(b+1, b)$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
$a = b+1, m_{b+1} > m_b$	$(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$a > b+1$	NO FLUID SOLUTION				
$III.:\ 0 < a < b = N+1$					
$\ell(a) < N+1$	$(a, \ell(a))$	$\uparrow^{(4)}$	$\downarrow^{(1)}$	$\uparrow$	$\uparrow$
$\ell(a) = N+1$	$(a, N+1)$	$\uparrow^{(4)}$	$\downarrow$	$\times$	$\times$
$IV.:\ 0 = a < b < N+1^{(3)}$					
$m_{b+1} > m_b$	$(b+1, b)$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
$m_{b+1} < m_b,\ \frac{m_b}{1-M_B(b)} \geq \frac{m_{b+1}}{1-M_A(b)}$, for all $2 < l < b,\ l \in B:$ $m_{b+1}\frac{1-M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l) \leq 1$	$(0, b)$	$\rightarrow$	$\times$	$\downarrow$	$\uparrow$
$m_{b+1} < m_b, b < \ell(1),\ \frac{m_b}{1-M_B(b)} < \frac{m_{b+1}}{1-M_A(b)}$	$(1, b)$	$\uparrow$		$\downarrow^{(1)}$	$\uparrow$
otherwise	NO FLUID SOLUTION				
$V.:\ 0 = a < b = N+1^{(3)}$					
	$(0, N+1)$	$\rightarrow$	$\times$	$\times$	$\times$

(1) 1 is emptying because of Lemma 2.4.8 (2) if $b^* + 2 < N$
(3) if $(a^*, b^*) = (0^*, b^*)$ is well-defined (4) for $a^* \neq 1$

Appendix C

Technical proofs of Part I

Lemma 2.4.2

Proof: We prove Lemma 2.4.2 by solving (13)–(18) omitting the time parameter:

We distinguish the following cases

Case I: $0 < a < b < N+1$.

From (29), (30), (32) and (33) we see that

$$\begin{aligned} d_0 &= 1 \\ d_k &= 0 \qquad \text{for all } 0 < k < a, && (63) \\ d_k &= d_a \qquad \text{for all } a \leq k < b, && (64) \\ d_{b+1} &= d_b \qquad \text{and} && (65) \\ d_k &= 0 \qquad \text{for all } k > b+1 && (66) \end{aligned}$$

Using (64), (65) and (66), (28) implies

$$\begin{aligned} 1 = \sum_{k\in A,\ k\geq a} m_k d_k &= \sum_{k\in A,\ b>k\geq a} m_k d_k + m_{b+1} d_{b+1} \\ &= d_a \cdot \sum_{k\in A,\ b>k\geq a} m_k + m_{b+1} d_b. \qquad (67) \end{aligned}$$

$$\text{Therefore, } m_b = m_b d_a \cdot \sum_{k\in A,\ b>k\geq a} m_k + m_b m_{b+1} d_b \qquad (68)$$

Similarly, using (63)and (64), (31) shows

$$1 = \sum_{k\in B,\ k\leq b} m_k d_k = \sum_{k\in B,\ b>k>a} m_k d_k + m_b d_b = d_a \cdot \sum_{k\in B,\ b>k>a} m_k + m_b d_b,$$

$$\text{so, } m_{b+1} = m_{b+1} d_a \cdot \sum_{k\in B,\ b>k>a} m_k + m_{b+1} m_b d_b. \tag{69}$$

Subtracting (69) from (68) we see that

$$\begin{aligned} m_b - m_{b+1} &= d_a \cdot \left[m_b \cdot \sum_{k\in A,\ b>k\geq a} m_k \ - \ m_{b+1} \cdot \sum_{k\in B,\ b>k>a} m_k \right] \\ &= d_a \cdot [(M_A(b) - M_A(a)) \ - \ m_{b+1} \cdot (M_B(b) - M_B(a))]. \end{aligned}$$

Thus,

$$d_a = \frac{m_b - m_{b+1}}{m_b(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a))} = \frac{m_b - m_{b+1}}{D(a,b)} \tag{70}$$

Now, (70) and (67) give

$$\begin{aligned} m_{b+1} d_b &= 1 - \frac{m_b - m_{b+1}}{D(a,b)}(M_A(b) - M_A(a)) \\ &= \frac{m_b(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a))}{D(a,b)} \\ &\qquad - \frac{(m_b - m_{b+1}) \cdot (M_A(b) - M_A(a))}{D(a,b)} \\ &= m_{b+1} \cdot \frac{(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))}{D(a,b)} \end{aligned}$$

$$\text{which shows } d_b = \frac{(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))}{D(a,b)}$$

Note that since $D(a,b) \neq 0$, this this solution is unique.

Case II: $0 < a > b < N + 1$

Using (27)-(33), we see that

$$\begin{aligned} d_0 &= 1 \\ d_k &= 0 \qquad \text{for all} \quad k \notin \{0, a, b\} \\ d_a &= 1/m_a \qquad \text{and} \\ d_b &= 1/m_b \end{aligned}$$

Note that this solution is unique.

Case III: $0 < a < b = N + 1$

From (29), (30), (32) and (33) we see that

$$d_0 = 1$$

$$d_k = 0 \qquad \text{for all } 0 < k < a, \tag{71}$$

$$d_k = d_a \qquad \text{for all } k \geq a \tag{72}$$

Further, (28) implies that

$$d_a = \frac{1}{\sum_{k \in A,\ k \geq a} m_k} = \frac{1}{M_A - M_A(a)} > 0.$$

Note that also this solution is unique.

Case IV: $0 = a < b < N + 1$

From (27)-(33) and since $d_k = 1$ for $k < b$, we see that

$$1 = \sum_{k \in B,\ k \leq b} m_k d_k = m_b d_b + \sum_{k < b,\ k \in B} m_k d_k = m_b d_b + M_B(b),$$

$$\text{hence, } d_b = \frac{1 - M_B(b)}{m_b}.$$

Therefore we see that

$$
\begin{aligned}
d_k &= 1 && \text{for all} \quad k < b \\
d_b = d_{b+1} &= \frac{1 - M_B(b)}{m_b} && \\
d_k &= 0 && \text{for all} \quad k > b + 1
\end{aligned}
\tag{73}
$$

Note that also this solution is unique.

Case V: $0 = a < b = N + 1$

In this case (27)-(33) just give $d_k = 1$ for all $1 \leq k \leq N$. Note that also this solution is unique.

□

Proposition 2.4.5

Proof: Recall the conditions for $(Q(\cdot), T(\cdot))$ being a valid fluid solution:It is a fluid solution, i.e. it satisfies (3) -(7) — we use (3), (4) and (10)-(12).

It is a LBFS-FBFS feasible fluid solution, i.e. it satisfies (13)-(18).

The departure rates $d(\cdot)$ satisfy the virtual station constraints, i.e. (19) holds for each class $l \in B$.

Let us fix the regular time point $t_0 > 0$ with hpn buffers (a, b). Let $d = (d_k)_{1 \leq k \leq N}$ be the solution to equations (27)-(33) at time t_0.

Case I: $0 < a < b < N+1$.

We are first going to show that $D(a,b)$, $m_b - m_{b+1}$ and $[(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))]$ are all strictly greater than 0.

From Table 1, we see that $d_a \geq 0$ if and only if either $m_b - m_{b+1}$ and $D(a,b)$ have the same sign or if $m_b = m_{b+1}$ — the latter case is excluded by the non-degeneracy assumption (25).

Similarly, Table 1 shows that $d_b \geq 0$ if and only if either $(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))$ and $D(a,b)$ have the same sign or if $(M_A(b) - M_A(a)) = (M_B(b) - M_B(a))$ — the latter case is excluded by the non-degeneracy assumption 24. In other words,

$$\text{(10) is equivalent to } \left\{ \begin{array}{c} d_a \geq 0 \text{ and} \\ d_b \geq 0 \end{array} \right\} \text{ which holds if and only if}$$

$$\left\{ \begin{array}{c} m_b - m_{b+1},\ (M_A(b) - M_A(a)) - (M_B(b) - M_B(a)) \\ \text{and } D(a,b) \text{ are all either non negative or non positive} \end{array} \right\}. \quad (74)$$

Now let us transform the virtual station constraints (19). Then for each class $l \in B$, we have

$$\sum_{k \in A,\ l<k} m_k d_k + \sum_{k \in B,\ k<l} m_k d_k \leq 1$$

(for $l = 2$ this is implied by (11)).

Using Table 1, we see that for $l \in B$, $a < l \leq b$,

$$\begin{aligned}
1 &\geq \sum_{k \in A,\ l<k} m_k d_k \ + \sum_{k \in B,\ k<l} m_k d_k \\
&= m_{b+1} d_b \ + \ d_a \sum_{k \in A,\ l<k<b} m_k \ + \ d_a \sum_{k \in B,\ a<k<l} m_k \\
&= m_{b+1} \frac{(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))}{D(a,b)} \\
&\quad + \frac{m_b - m_{b+1}}{D(a,b)} \cdot (M_A(b) - M_A(l)) + \frac{m_b - m_{b+1}}{D(a,b)} (M_B(l) - M_B(a)).
\end{aligned}$$

Multiplying this equation with

$D(a,b) = m_b(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a))$, we see that,

if $D(a,b) > 0$,

$$\begin{aligned}
& m_b(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a)) \\
& \quad \geq \ m_{b+1}(M_A(b) - M_A(a)) - m_{b+1}(M_B(b) - M_B(a)) \\
& \qquad + (m_b - m_{b+1})(M_A(b) - M_A(l)) \\
& \qquad + (m_b - m_{b+1})(M_B(l) - M_B(a)).
\end{aligned}$$

Or, equivalently,

$$\begin{aligned}
0 &\geq (m_{b+1} - m_b)(M_A(b) - M_A(a)) - (m_{b+1} - m_b)(M_A(b) - M_A(l)) \\
&\quad - (m_{b+1} - m_b)(M_B(l) - M_B(a)) \\
&= (m_{b+1} - m_b)(M_A(l) - M_A(a)) - (m_{b+1} - m_b)(M_B(l) - M_B(a)) \\
&= (m_{b+1} - m_b) \left[(M_A(l) - M_A(a)) - (M_B(l) - M_B(a)) \right]. \qquad (75)
\end{aligned}$$

If $D(a,b) < 0$, we get (75) with a reversed inequality:

$$0 \leq (m_{b+1} - m_b)\left[(M_A(l) - M_A(a)) - (M_B(l) - M_B(a))\right]. \tag{76}$$

Letting $l = b$ in (76) we see that

$$0 \leq (m_{b+1} - m_b)\left[(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))\right].$$

But then, by (74) we have $m_{b+1} - m_b > 0$

and $(M_A(b) - M_A(a)) - (M_B(b) - M_B(a)) < 0$, which gives a contradiction.

Hence $D(a,b)$, $m_b - m_{b+1}$ and $[(M_A(b) - M_A(a)) - (M_B(b) - M_B(a))]$ are all strictly greater than 0.

We are now going to show that $\ell(a) \geq b$.

Assume that $\ell(a) < b$. Then (75) gives for $l = \ell(a) + 2 \leq b$:

$$\begin{aligned} 0 \geq\ & (m_{b+1} - m_b) \\ & \cdot \left[(M_A(\ell(a)+2) - M_A(a)) - (M_B(\ell(a)+2) - M_B(a))\right], \end{aligned}$$

where $m_{b+1} - m_b < 0$ and

$(M_A(\ell(a)+2) - M_A(a)) - (M_B(\ell(a)+2) - M_B(a)) < 0$ by the definition of $\ell(a)$.

This gives a contradiction. Therefore, $\ell(a) \geq b$ and the claim is proven.

Case II: $0 < a > b < N + 1$

To see that $a = b + 1$, we observe that the departure rates of table 1 can only satisfy the virtual station constraint (19) for $l = b + 2$, if $a = b + 1$.

Suppose $a > b+1$, then since $a \in A$, and therefore odd, $a > b+2 = l$ and

$$\sum_{k\in A,\ k>b+2} m_k d_k + \sum_{k\in B,\ k<b+2} m_k d_k = m_a \frac{1}{m_a} + m_b \frac{1}{m_b} > 1,$$

showing that the departure rates violate the virtual station constraint (19) for $l = b+2$. Thus, if $a > b$, we must have $a = b+1$.

Case III: $0 < a < b = N+1$

We are going to show that $\ell(a) = N+1$.

Assume this is not true, i.e. $\ell(a) < N+1$. Then, by the definition of $\ell(a)$,

$$M_B(\ell(a)+2) - M_B(a) + M_A(a) - M_A(\ell(a)+2) > 0. \tag{77}$$

Now, the LBFS-FBFS feasibility equation (12) with $l = \ell(a)+2 \le N+1$, and equations (19) and (11) can be transformed to:

$$\begin{aligned} 1 &\ge \sum_{k\in A,\ k>l} m_k d_k + \sum_{k\in B,\ k<l} m_k d_k \\ &= d_a \left(\sum_{k\in A,\ k>l} m_k \ + \sum_{k\in B,\ a<k<l} m_k \right) \\ &= d_a \cdot [M_A - M_A(l) \ + \ M_B(l) - M_B(a)] \le 1. \end{aligned} \tag{78}$$

Then, Table 1 shows that

$$\begin{aligned} M_A - M_A(l) + M_B(l) - M_B(a) &\le M_A - M_A(a), \\ \text{or } M_A(a) - M_A(l) + M_B(l) - M_B(a) &\le 0. \end{aligned} \tag{79}$$

But (79) contradicts (77). Therefore, $\ell(a) = N+1$.

Case IV: $0 = a < b < N + 1$

First, we show that $\frac{m_b}{1-M_B(b)} \geq \frac{m_{b+1}}{1-M_A(b)}$.

The LBFS-FBFS feasibility equation (11) can be transformed as follows:

$$\begin{aligned} 1 &\geq \sum_{k \in A} m_k d_k \\ &= m_{b+1} d_{b+1} + M_A(b) \\ &= m_{b+1} \frac{1 - M_B(b)}{m_b} + M_A(b). \end{aligned}$$

Multiplying with m_b leads to

$$m_{b+1} - m_b + m_b M_A(b) - m_{b+1} M_B(b) \leq 0,$$

so we see that

$$\frac{m_b}{1 - M_B(b)} \geq \frac{m_{b+1}}{1 - M_A(b)}. \tag{80}$$

Now we are proving for all $2 < l < b,\ l \in B$:

$$m_{b+1} \frac{1 - M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l) \leq 1.$$

The virtual station constraints (19) can for $2 < l \leq b,\ l \in B$ be transformed to

$$\begin{aligned} 1 &\geq \sum_{k \in A,\ k > l} m_k d_k \ + \sum_{k \in B,\ k < l} m_k d_k \\ &= m_{b+1} d_b + M_A(b) - M_A(l) \ + \ M_B(l) \\ &= m_{b+1} \frac{1 - M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l). \end{aligned}$$

This results in

$$0 \geq m_{b+1} - m_b + m_b M_A(b) - m_{b+1} M_B(b) + m_b M_B(l) - m_b M_A(l). \tag{81}$$

Hence, $m_{b+1}\frac{1-M_B(b)}{m_b} + M_A(b) - M_A(l) + M_B(l) \leq 1$ is proven for $\ell(0) < l \leq b,\ l \in B$.

It is left to show that $m_{b+1} - m_b \leq 0$.

For $l = b$, (81) gives

$$\begin{aligned} 0 &\geq m_{b+1} - m_b - M_B(b)(m_{b+1} - m_b) \\ &= (m_{b+1} - m_b)(1 - M_B(b)) \end{aligned}$$

The usual workload condition at station B, (21), shows that this holds if and only if $m_{b+1} - m_b \leq 0$.

Case V: $0 = a < b = N + 1$

In this case there is nothing to prove.

$\square$

Proposition 2.4.7

Proof: For the proof, we mainly use Table 1 and Proposition 2.4.5.

Observe that in each case of Proposition 2.4.5, $d_k = d_{k-1}$ except possibly for class $k \in \{1,\ a,\ b,\ b+2\}$.

Hence the fluid levels in those buffers are constant.

Furthermore, we see that $d_{b+2} = 0$ for $b < N - 1$. Since $d_{b+1} > 0$ if $b < N$, we see that buffer $b + 2$ is always filling (if $b + 2 < N$).

Now we discuss the different cases.

Case I: $0 < a < b < N+1$:

We show that the fluid level of buffer 1 is strictly increasing for $1 \neq a$:

If $a > 1$ then $d_1 = 0$ and $d_0 = \lambda = 1$, hence buffer 1 is filling.

Now we show that buffer a is emptying if $a > 1$.

Table 1 shows $d_{a-1} = 0$. We also see that $d_a > 0$ since the non-degeneracy assumption (25) implies $m_b \neq m_{b+1}$. Hence the level in buffer a is decreasing.

Consider now the case $a = 1$:

By Proposition 2.4.5 we know that $D(a,b) > 0$. The fluid level of buffer 1 is decreasing if and only if $d_a > d_0 = 1$, which can be transformed to

$$\begin{aligned}
d_a \frac{m_b - m_{b+1}}{D(a,b)} &> 1, \\
\text{or} \qquad m_b - m_{b+1} &> m_b(M_A(b) - M_A(1)) - m_{b+1}(M_B(b) - M_B(1)) \\
&= m_b M_A(b) - m_{b+1} M_B(b), \\
\text{or} \qquad \frac{m_b}{1 - M_B(b)} &> \frac{m_{b+1}}{1 - M_A(b)}.
\end{aligned}$$

Similarly, buffer $a = 1$ is filling if and only if $\frac{m_b}{1-M_B(b)} > \frac{m_{b+1}}{1-M_A(b)}$.

Fluid level of buffer b:

To determine whether buffer b is filling we examine whether $d_a > d_b$. Since

$D(a,b) > 0$ by Proposition 2.4.5, we can use Table 1 to transform $d_a > d_b$ to

$$\begin{aligned} m_b - m_{b+1} &> (M_A(b) - M_A(a)) - (M_B(b) - M_B(a)), \\ \text{or} \qquad 0 &> (m_{b+1} + M_A(b) - M_A(a)) - (m_b + M_B(b) - M_B(a)) \\ &= (M_A(b+2) - M_A(a)) - (M_B(b+2) - M_B(a)). \end{aligned}$$

By Proposition 2.4.5 this holds if and only if $b = \ell(a)$.

If $b < \ell(a)$ then $(M_A(b+2) - M_A(a)) - (M_B(b+2) - M_B(a)) \geq 0$. By the non-degeneracy assumption (24) the inequality is strict, and hence buffer b is emptying.

Case II: $0 < a > b < N+1$:

By Proposition 2.4.5 we know that $a = b+1$. Buffer b is emptying since $d_b = \frac{1}{m_b} > 0$ and $d_{b-1} = 0$.

Buffer a is filling if $d_a < d_b$. Table 1 shows that this is equivalent to $m_b < m_a$. Similarly, buffer a is emptying if $m_b > m_a$. Buffer 1 is filling since $a > b > 1$, and therefore $d_1 = 0$ and $d_0 = 1$.

Case III: $0 < a < b = N+1$:

If $a > 1$ then buffer a is emptying since by Table 1, $d_{a-1} = 0$ and $d_a > 0$. Buffer 1 is filling since $d_1 = 0$ and $d_0 = 1$.

If $a = 1$ then buffer a is emptying if and only if $d_a > 1$. By Table 1, this is equivalent to $1 < \frac{1}{M_A - M_A(a)} = \frac{1}{M_A}$, which holds by the usual workload condition (20).

Case IV: $0 = a < b < N + 1$:

Buffer b is emptying if and only if $d_b > d_{b-1} = 1$. By Table 1 this is equivalent to $m_b < 1 - M_B(b)$, or $0 < 1 - M_B(b+2)$. This holds by the usual workload condition at station B (21).

Case V: $0 = a < b = N + 1$:

The statement follows from $1 = d_0 = d_k$ for each class $k \in A \cup B$.

□

Proposition 2.7.5

Proof: Assume first that during the time interval (t_0, t_1) the hpnb buffers at any regular time point are in K. This means, for a given regular t, $k < a \leq l$ and $k \leq b < l$. Then, (13), (17) and (15) imply that $\sum_{j \in K \cap A} \dot{T}_j(t) = 1$. Similarly, (16), (14) and (18) imply that $\sum_{j \in K \cap B} \dot{T}_j(t) = 1$.

Since $T(\cdot)$ is absolutely continuous, we have

$$\sum_{j \in K \cap A} T_j(t_1) - T_j(t_0) = \int_{t_0}^{t_1} \sum_{j \in K \cap A} \dot{T}_j(t)\, dt = 1.$$

Hence station A works during the time interval (t_0, t_1) exclusively and without idling on classes $k, k+1, k+2, \ldots, l$. Similarly,

$$\sum_{j \in K \cap B} T_j(t_1) - T_j(t_0) = \int_{t_0}^{t_1} \sum_{j \in K \cap B} \dot{T}_j(t)\, dt = 1.$$

Hence station B works during the time interval (t_0, t_1) exclusively and without idling on classes $k, k+1, k+2, \ldots, l$.

Now, we assume that there is a regular time t where the hpn buffer in station A, a, or the hpn buffer of station B is not in K.

Say, we have $a \notin K$. Then, by Proposition 2.7.3, there is a whole interval (t_2, t_3) on which a is the hpn buffer. Then, for all $s \in (t_2, t_3)$, we see by table 1, $0 < \frac{1}{m_a} d_a(s) = \dot{T}_a(s)$. This implies that $T_a(t_1) - T_a(t_0) \geq \int_{t_2}^{t_3} \dot{T}_a(s)\, ds > 0$ and hence $\sum_{j \in K \cap A} T_j(t_1) - T_j(t_0) < 1$. Therefore station A does not work exclusively and without idling on classes $k, k+1, k+2, \ldots, l$ during the time interval (t_0, t_1). Similarly, if $b \notin K$ then station B does not work exclusively and without idling on classes $k, k+1, k+2, \ldots, l$ during the time interval (t_0, t_1).

$\square$

Corollary 2.8.2

Proof: We distinguish the following cases:

1. $\theta = 0$:

   Then, by definition we have $\beta_1 = \ell(1) = N + 1$. Hence, by the definition of $\ell(1)$, we see that $M_A(k) \geq M_B(k)$ for each class $k \in B \cup \{N+1\} \setminus \{2\}$. Hence for each class $k \in B$, $M_A - M_A(k) + M_B(k) \leq M_A$, proving the claim.

2. $\theta < N + 1$:

   First we prove the claim, $M_{\mathbb{V}(k)} \leq M_\theta$ for all classes $k \in K = \{\beta^1,\, \beta^1 + 2,\, \ldots,\, \beta_2,\, \beta_2 + 2,\, \beta_2 + 4,\, \ldots,\, \beta^2,\, \ldots,\, \beta_\nu,\, \beta_\nu + 2,\, \ldots,\, \beta^\nu\}$.

In each induction step we show that if the claim holds for some buffer β^i, then it also holds for buffers β^{i-1}, $\beta^{i-1}+2$, $\beta^{i-1}+4, \ldots$, $\beta^i - 2$. The induction index is i and we start with $i = \nu$.

Induction beginning:

For $k = \beta^\nu = \theta$ the claim is trivially true.

Induction step:

We assume the claim holds for class β^i. The induction step consists of three parts:

(a) First we show the claim for classes $k \in \{\beta_i,\ \beta_i + 2,\ \ldots,\ \beta^i - 2\}$.

(b) Then we show the claim for class $k = \beta^{i-1}$

(c) Finally we show the claim for classes $k \in \{\beta^{i-1}+2,\ \beta^{i-1}+4,\ \ldots,\ \beta_i - 2\}$.

This concludes the induction.

(a) If the claim holds for some β^i then it is true for $k \in \{\beta_i,\ \beta_i+2,\ \ldots,\ \beta^i-2\}$:

Since $m_{\beta^i-2} > m_{\beta^i-1}$ (by definition of β^i), we have

$$\begin{aligned} M_{\mathbb{V}(\beta^i-2)} &= M_A - M_A(\beta^i-2) + M_B(\beta^i-2) \\ &< M_A - M_A(\beta^i-2) + M_B(\beta^i-2) + m_{\beta^i-2} - m_{\beta^i-1} \\ &= M_A - M_A(\beta^i) + M_B(\beta^i) = M_{\mathbb{V}(\beta^i)}. \end{aligned}$$

This argument can be continued inductively to show that $M_{\mathbb{V}(k-2)} < M_{\mathbb{V}(k)}$ for each $k \in \{\beta_i,\ \beta_i + 2,\ \ldots,\ \beta^i - 2\}$. This concludes step 2a.

(b) Now we show the claim for class $k = \beta^{i-1}$:

Since $\ell(\beta^{i-1}) = \beta_i$, we have

$$M_B(\beta_i + 2) - M_B(\beta^{i-1}) > M_A(\beta_i + 2) - M_A(\beta^{i-1}). \tag{82}$$

Furthermore, by step 2a and since $\beta_i + 2 \le \beta^i$, we see that

$$\begin{aligned} M_{\mathbb{V}(\beta_i+2)} &= M_A - M_A(\beta_i + 2) + M_B(\beta_i + 2) \\ &\le M_{\mathbb{V}(\beta^i)} = M_A - M_A(\beta^i) + M_B(\beta^i). \end{aligned} \tag{83}$$

Equations (82) and (83) resolve in

$$\begin{aligned} M_A - M_A(\beta^{i-1}) + M_B(\beta^{i-1}) &< M_A - M_A(\beta^i) + M_B(\beta^i), \\ \text{or } M_{\mathbb{V}(\beta^{i-1})} &< M_{\mathbb{V}(\beta^i)}. \end{aligned} \tag{84}$$

This concludes step 2b.

(c) Finally, we prove the claim each class $k \in \{\beta^{i-1}+2,\ \beta^{i-1}+4,\ \ldots,\ \beta_i - 2\}$: The definition of $\ell(\beta^{i-1})$ implies $M_B(k) - M_B(\beta^{i-1}) \le M_A(k) - M_A(\beta^{i-1})$.

This implies that $M_A - M_A(k) + M_B(k) \le M_A - M_A(\beta^{i-1}) + M_B(\beta^{i-1})$. Then, (84) proves the claim.

So the induction is proven. Using the arguments of step 2a for $i = 1$, we see that $M_{\mathbb{V}(k)} \le M_{\mathbb{V}(\theta)}$ also holds for each class $k \in \{\beta_1,\ \beta_1 + 2,\ \ldots,\ \beta^1 - 2\}$.

We showed the claim for each class $k \in \{\beta_1,\ \beta_1+2,\ \beta_1+4,\ \ldots,\ \beta^\nu\}$. It remains to prove the claim for each class $k \in \{0,\ 4,\ ,\ 6,\ 8,\ \ldots,\ \ell(1) - 2\}$ and for each class $k \in \{\theta + 2,\ \theta + 4,\ \ldots,\ N + 1\}$.

Note that by the definition of $\ell(1)$ and by the induction,

$$M_{\mathbb{V}(0)} = M_A < M_A - M_A(\ell(1)+2) + M_B(\ell(1)+2) = M_{\mathbb{V}(\ell(1)+2)} \leq M_{\mathbb{V}(\theta)}.$$

For each class k with $2 < k < \ell(1) = \beta_1$ we use that $M_A \geq M_A - M_A(k) + M_B(k)$ to show that $M_{\mathbb{V}(k)} \leq M_A$.

For each class $k \in \{\theta+2,\ \theta+4,\ \dots,\ N+1\}$ we use that $\ell(\beta^\nu) = N+1$ (hence $M_B(k) - M_B(\theta) \leq M_A(k) - M_A(\theta)$) to show that $M_A - M_A(k) + M_B(k) \leq M_A - M_A(\beta^\nu) + M_B(\beta^\nu)$.

Hence the claim is proven for this case.

3. If $\beta_\nu = N+1$, then we use the same induction as in the previous case, starting with $k = N+1$.

$\square$

Appendix D

Intel Corporation

Intel is the largest semiconductor manufacturer, with revenues of $33.7 billion dollars in 2000. 45% of Intel's revenues come from North America, 28% come from Europe, 20% from the Asia/Pacific Region and 7% from Japan.

Intel's principal products can be divided into

- Intel architecture platform products such as microprocessors and motherboards
- Computing enhancement products such as chipsets, Flash Memory (reprogrammable memory), embedded control chips
- Network communications products

The major customers of Intel are distributors (like Arrows and AVNet), OEM manufacturers of computers and peripheral devices and other manufacturers of industrial and telecommunications equipment.

Intel operates about 20 production and test/assembly sites in America, Europe and Asia, serving around 600 customers around the world. Intel delivers its products through a pull supply chain management.

The greater European distribution center in Amsterdam, Netherlands, Intel's largest distribution center, accounts for one third of Intel's shipment (in revenues). The Amsterdam facility serves customers in Europe, Mid-East and Africa.

Before 1996, Intel's European distribution system was a hybrid of warehousing and direct shipments. Intel shipped its goods to customers directly from its Swindon (England) warehouse facility.

In spring 1996 Intel changed to a pooling / cross-dock distribution network: Intel ships chips via airfreight from its production and test/assembly sites to a cross-dock facility in Schiphol (Amsterdam, Netherlands). From there, products are distributed via truck or airfreight to customers in Europe, Africa and Mid-East.

In July 2000, Intel Corporation took over the Amsterdam cross-dock operations from the main freight forwarder Danzas AEI. The transition period was only approximately two months.

In 2001, a competitive bidding on the outbound carriers was introduced to reduce transportation costs and issues, to enhance shipment quality service and to improve supply chain flexibility.

Next to the cross-dock operations in Amsterdam, Intel is also running a small assembly facility and two small warehouses serving extremely urgent orders.

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Vita

Jozo Acksteiner was born on August 22, 1969 in Remscheid, Germany. After his military service 1989-1990 in Heide and Kiel, Germany, he enrolled as an undergraduate student in Mathematics at the University of Wuppertal, Germany. He received his “Vordiplom” (equivalent to a Bachelors degree) in 1992. In 1996 he received his “Diplom” (equivalent to a Masters degree) in Mathematics at the University of Wuppertal in the speciality Discrete Dynamical Systems (Chaos Theory).

Jozo joined the Ph.D. program in the School of Mathematics of the Georgia Institute of Technology in January 1997. He was awarded a stipend of the German Academic Exchange Service (DAAD) in 1997. In 1998 he received his Masters degree in Applied Mathematics. In 1998 Jozo switched to the Ph.D. program at the School of Industrial and Systems Engineering. In August 2001 Jozo received his Ph.D. degree from the School of Industrial and Systems Engineering.

After receiving his degree, Jozo will be working for Booz · Allen & Hamilton Consulting in Munich, Germany.