Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution

Jim Dai

The H. Milton Stewart School of Industrial and Systems Engineering

Joint work with Masakiyo Miyazawa

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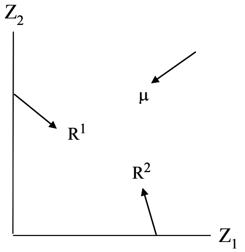
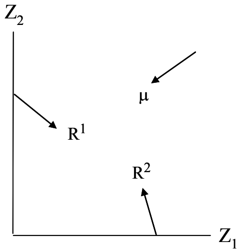
Semimartingale reflecting Brownian motion (SRBM)

$$Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0,$$
X is a (μ, Σ) Brownian motion,
$Z(t) \in \mathbb{R}_+^n$ for all $t \geq 0$,
$Y(\cdot)$ is continuous and nondecreasing with $Y(0) = 0$,
$Y_i(\cdot)$ only increases when $Z_i(\cdot) = 0, \quad i = 1, \ldots, n.$



Semimartingale reflecting Brownian motion (SRBM)

$$Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0,$$
$$X \text{ is a } (\mu, \Sigma) \text{ Brownian motion},$$
$$Z(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0,$$
$$Y(\cdot) \text{ is continuous and nondecreasing with } Y(0) = 0,$$
$$Y_i(\cdot) \text{ only increases when } Z_i(\cdot) = 0, \quad i = 1, \ldots, n.$$

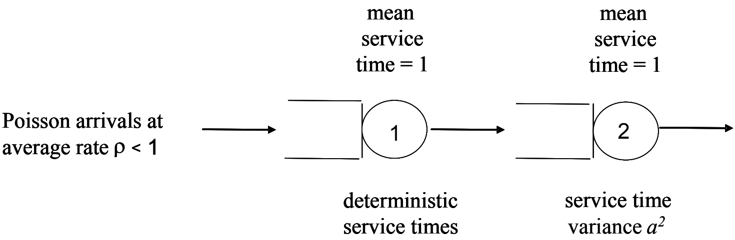


- We focus on $n = 2$.
- R is assumed to be completely-$\mathcal{S}$

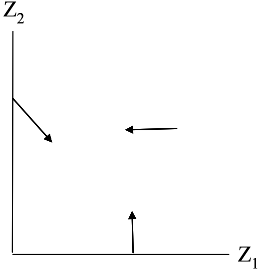
$$R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}.$$

- Z exists and is unique in distribution; (Taylor-Williams 92)

A queueing network



Its Approximating SRBM



$$Z_1(t) = X_1(t) + Y_1(t),$$
$$Z_2(t) = X_2(t) - Y_1(t) + Y_2(t)$$
drift of X is $\mu = (\rho - 1, 0)$,
covariance of X is $\Sigma = \begin{pmatrix} \rho & 0 \\ 0 & a^2 \end{pmatrix}$

- R. J. Williams (95), Semimartingale reflecting Brownian motions in the orthant, in Stochastic Networks, eds. F. P. Kelly and R. J. Williams, the IMA Volumes in Mathematics and its Applications, Vol. 71 (Springer, New York)
- R. J. Williams (96), On the approximation of queueing networks in heavy traffic, in Stochastic Networks: Theory and Applications, eds. F. P. Kelly, S. Zachary and I. Ziedens, Royal Statistical Society (Oxford Univ. Press, Oxford)

Proposition (Hobson-Rogers 94, Harrison-Hasenbein 09)

Assume that

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \tag{1}$$

$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad \text{and} \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \tag{2}$$

Then SRBM Z has a unique stationary distribution π.

- Condition (1): R is a $\mathcal{P}$-matrix.
- Condition (2): $R^{-1}\mu < 0$.
- Basic adjoint relationship: for each $f \in C_b^2(\mathbb{R}_+^2)$

$$\int_{\mathbb{R}_+^2} \Big(\frac{1}{2}\langle \nabla, \Sigma \nabla f\rangle + \langle \mu, \nabla f\rangle\Big)\pi(dx) + \sum_{i=1}^{2}\int_{\mathbb{R}_+^2} \langle R^i, \nabla f\rangle \nu_i(dx) = 0,$$

$\nu_i(A) = \mathbb{E}_\pi\left[\int_0^1 1_{\{Z(u)\in A\}}\, dY_1(u)\right]$ defines the ith boundary measure.

Tail asymptotics of the stationary distribution

- Let $Z(\infty) = (Z_1(\infty), Z_2(\infty))$ be the random vector that has the stationary distribution π.
- For any $c \in \mathbb{R}_+^2$, set

$$\langle c, Z(\infty) \rangle = c_1 Z_1(\infty) + c_2 Z_2(\infty).$$

- We are interested in finding a function $f_c(x)$ that satisfies

$$\lim_{x \to \infty} \frac{\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}}{f_c(x)} = b$$

  for some constant $b > 0$.
- The function $bf_c(x)$ is said to be the exact asymptotic of $\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}$.

An overview

Main Result

- *For an SRBM satisfying (1) and (2), the exact asymptotic is given by*

$$f_c(x) = x^{\kappa_c} e^{-\alpha_c x}.$$

- *The decay rate α_c and the constant κ_c can be computed explicitly from the primitive date (μ, Σ, R).*
- *The constant κ_c must take one of the values $-3/2$, $-1/2$, 0 or 1.*

Let $p_c(x)$ be the density of $\langle c, Z(\infty)\rangle$. In most cases, we have

$$\lim_{x\to\infty} \frac{p_c(x)}{f_c(x)} = b \text{ for some } b > 0.$$

A diversion: Laplace transform

- Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and integrable function. Define

$$\tilde{f}(z) = \int_0^\infty e^{zx} f(x)dx, \quad \Re z < \alpha_f,$$

  where $\alpha_f = \sup\{\theta \geq 0 : \tilde{f}(\theta) < \infty\}$. $\tilde{f}$ is analytic in $\Re z < \alpha_f$.
- When f is a Gamma(k, α) density, namely,

$$f(x) = \frac{c}{\Gamma(k)} x^{k-1} e^{-\alpha x},$$

  then $\alpha_f = \alpha$ and

$$\tilde{f}(z) = \frac{c}{(\alpha - z)^k} \equiv g(z) \quad \text{for } \Re z < \alpha.$$

- Complex function g is an "analytic" extension of $\tilde{f}$.

A diversion: complex inversion

Proposition (When k is a positive integer)

Suppose that the analytic extension g of $\tilde{f}$ satisfies that

$$g(z) - \frac{c_0}{(\alpha_0 - z)^k}$$

is analytic for $\Re z < \alpha_1$ for some $\alpha_1 > \alpha_0$. Then, under some mild conditions on g,

$$f(x) \sim \frac{c_0}{\Gamma(k)} x^{k-1} e^{-\alpha_0 x} \quad \text{as } x \to \infty. \tag{3}$$

When α_0 is the kth order pole of $g(z)$, then (3) holds.

Convergence domain

- Define

$$\varphi(\theta_1, \theta_2) = \mathbb{E}\Big[e^{\langle \theta, Z(\infty)\rangle}\Big].$$

Then the transform of $\langle c, Z(\infty)\rangle$ has the following expression

$$\psi_c(z) = \mathbb{E}\big(e^{z\langle c, Z(\infty)\rangle}\big) = \varphi(zc).$$

- Define the convergence domain of φ

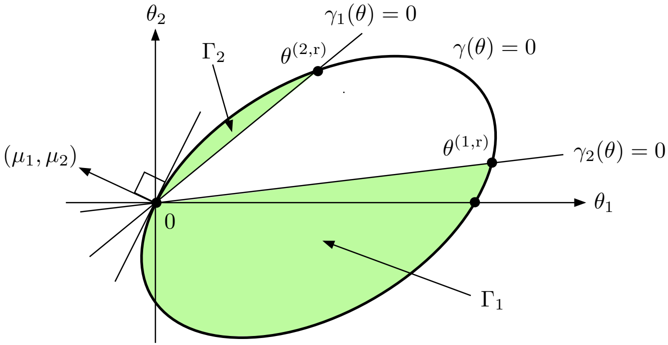
$$\mathcal{D} = \text{ interior of } \{\theta \in \mathbb{R}^2 : \varphi(\theta) < \infty\}.$$

- When $\theta \in \mathcal{D}$, BAR gives the key relationship

$$\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1). \quad (4)$$

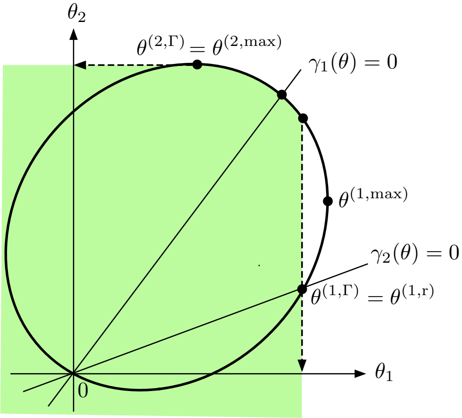
Geometric objects: Γ, Γ_1 and Γ_2

$$\gamma(\theta) = -\langle \theta, \mu \rangle - \frac{1}{2}\langle \theta, \Sigma\theta \rangle,$$
$$\gamma_1(\theta) = r_{11}\theta_1 + r_{21}\theta_2 = \langle R^1, \theta \rangle,$$
$$\gamma_2(\theta) = r_{12}\theta_1 + r_{22}\theta_2 = \langle R^2, \theta \rangle,$$



Theorem 1: Convergence domain characterization

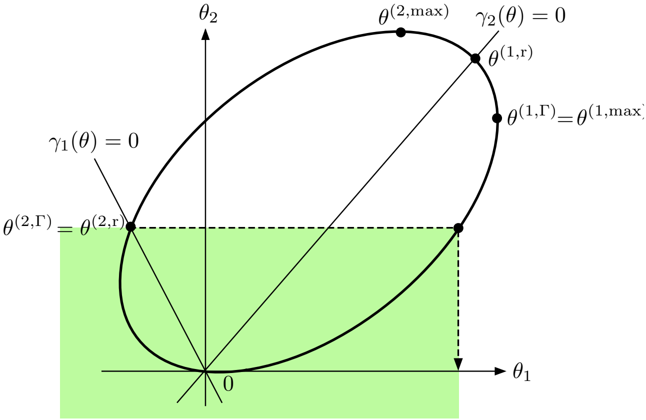
Convergence domain: Category I



- $\theta^{(2,\Gamma)}$ the highest point in Γ_2; $\theta^{(1,\Gamma)}$ the right-most point in Γ_1
- Category I:

$$\theta_1^{(2,\Gamma)} < \theta_1^{(1,\Gamma)} \text{ and } \theta_2^{(1,\Gamma)} < \theta_2^{(2,\Gamma)},$$

Convergence domain: Category II

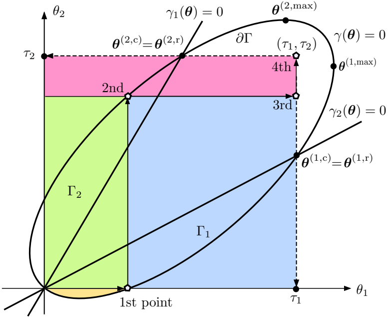


Category II: $\theta^{(2,\Gamma)} \leq \theta^{(1,\Gamma)}$,

Category III: $\theta^{(1,\Gamma)} \leq \theta^{(2,\Gamma)}$.

Key steps in proving Theorem 1

- $\varphi(\theta) < \infty$ implies $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$.
- $\theta \in \Gamma_1$ and $\varphi_1(\theta_2) < \infty$ imply $\varphi(\theta) < \infty$.
- $\theta \in \Gamma$, $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$ imply $\varphi(\theta) < \infty$.



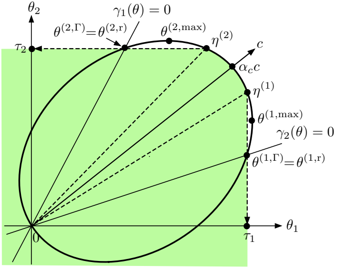
An illustration for proving Theorems 2 and 3

- Key relationship (4) gives

$$\gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2 z) + \gamma_2(zc)\varphi_2(c_1 z) \quad \text{for } \Re z < \alpha_c.$$

- Letting $\gamma(zc) = z\zeta_c(z)$, we have

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$



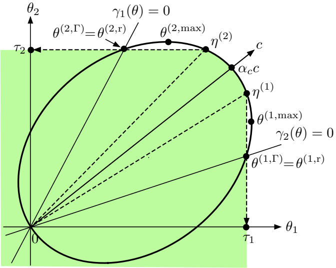
- $\varphi_2(z)$ is analytic on $\Re z < \tau_1$.
- $\varphi_1(z)$ is analytic on $\Re z < \tau_2$.
- $\alpha_c c \in \partial\Gamma$.
- $\alpha_c < \min(\tau_1/c_1, \tau_2/c_2)$.
- $\psi_c(z)$ has a single pole at $z = \alpha_c$.
- $f_c(x) = e^{-\alpha_c x}$.

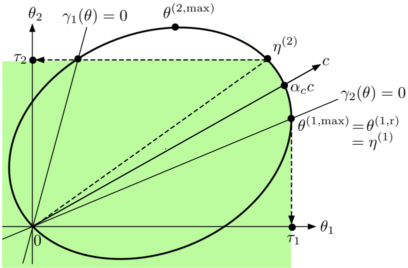
Singularity properties of $\varphi_2(z)$ at $z = \tau_1$

- In Category I when $\tau_1 < \theta_1^{(1,max)}$, single pole
- In Category I when $\tau_1 = \theta_1^{(1,max)}$
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, 1/2 "analytic" or −1/2 "pole"
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, 1/2 "pole"
- In Category II when $\tau_1 < \theta_1^{(1,max)}$,
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, single pole
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, double pole
- In Category II when $\tau_1 = \theta_1^{(1,max)}$
  - When $\theta^{(1,max)} \neq \theta^{(1,r)}$, 1/2"pole"
  - When $\theta^{(1,max)} = \theta^{(1,r)}$, 1 "pole"

Theorem 2: exact asymptotics for Category I

Category I: $0 \le \beta_1 < \beta < \beta_2$





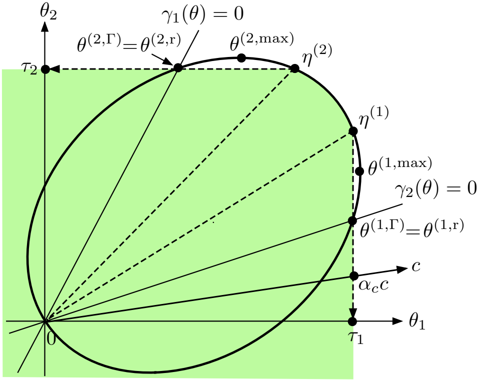
□

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$

$\varphi_2(z)$ is analytic for $\Re z < \tau_1$ $\qquad$ $\varphi_1(z)$ is analytic for $\Re z < \tau_2$

$$f_c(x) = e^{-\alpha_c x}$$

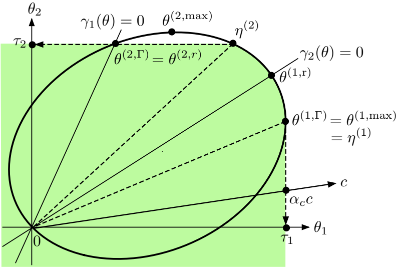
Category I: $0 \leq \beta < \beta_1$, $\tau_1 < \theta_1^{(1,\max)}$

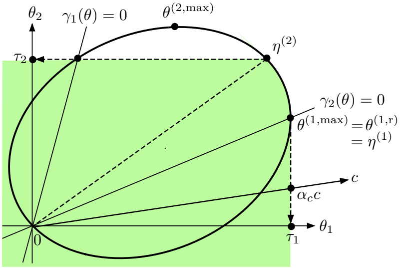


- single pole at τ_1 for $\varphi_2(z)$
- $\alpha_c c \notin \partial\Gamma$
- $$f_c(x) = e^{-\alpha_c x}.$$

$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$

Category I: $0 \le \beta < \beta_1$, $\tau_1 = \theta_1^{(1,\max)}$

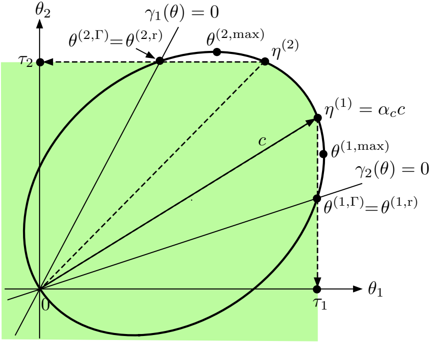




"$-1/2$" pole at τ_1 for $\varphi_2(z)$ "$1/2$" pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = \begin{cases} x^{-3/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} \neq \theta^{(1,r)}, \\ x^{-1/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} = \theta^{(1,r)}. \end{cases}$$

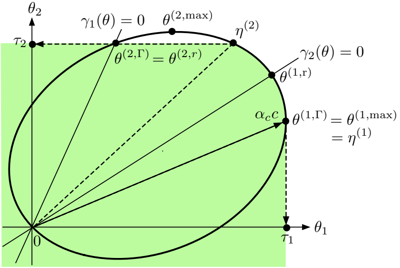
Category I: $0 \leq \beta = \beta_1 < \beta_2$, $\tau_1 < \theta_1^{(1,\max)}$

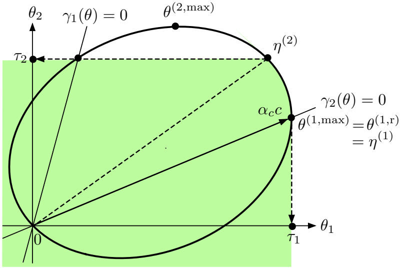


$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$

$$f_c(x) = x e^{-\alpha_c x}$$

Category I: $0 \le \beta = \beta_1 < \beta_2$, $\tau_1 = \theta_1^{(1,\max)}$

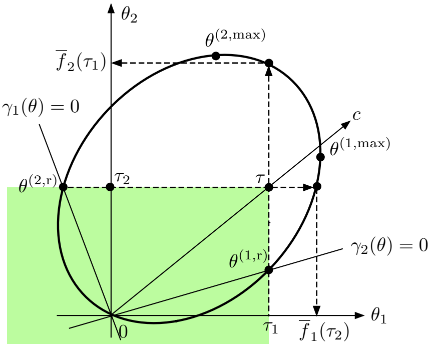




"$-1/2$" pole at τ_1 for $\varphi_2(z)$ "$1/2$" pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = \begin{cases} x^{-1/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} \neq \theta^{(1,r)}, \\ e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} = \theta^{(1,r)}. \end{cases}$$

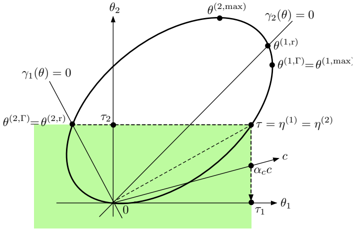
Category I: $0 \leq \beta = \beta_1 = \beta_2$



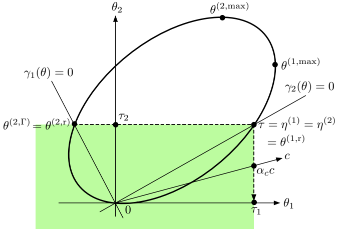
$$\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2 z) + \gamma_2(c)\varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$

$$f_c(x) = \begin{cases} x e^{-\alpha_c x} & \text{if } \eta^{(1)} = \eta^{(2)} = \tau \in \partial\Gamma, \\ e^{-\alpha_c x} & \text{if } \eta^{(1)} = \eta^{(2)} = \tau \in \text{ interior of } \Gamma. \end{cases}$$

Theorem 3: exact asymptotics for Category II



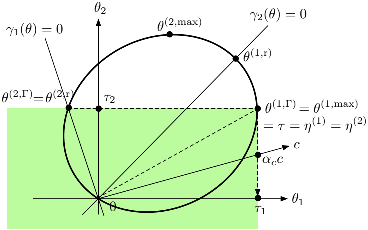
single pole at τ_1 for $\varphi_2(z)$

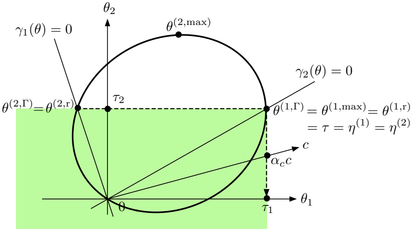


double pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = \begin{cases} e^{-\alpha_c x} & \text{if } \tau \neq \theta^{(1,\mathrm{r})} \\ x e^{-\alpha_c x} & \text{if } \tau = \theta^{(1,\mathrm{r})} \end{cases}$$

Category II: $0 \le \beta < \beta_1$: $\tau_1 = \theta_1^{(1,max)}$

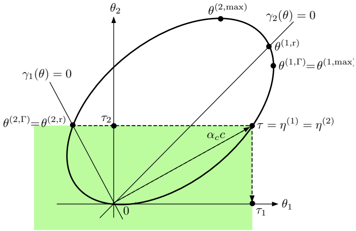


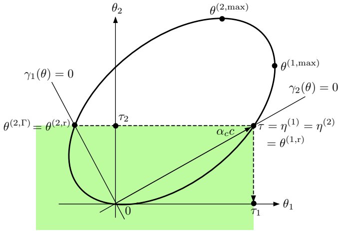


"1/2" pole at τ_1 for $\varphi_2(z)$ "1" pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = \begin{cases} x^{-1/2} e^{-\alpha_c x}, & \text{if } \tau \neq \theta^{(1,r)}, \\ e^{-\alpha_c x}, & \text{if } \tau = \theta^{(1,r)}. \end{cases}$$

Category II: $0 \leq \beta = \beta_1$, $\tau_1 < \theta_1^{(1,max)}$

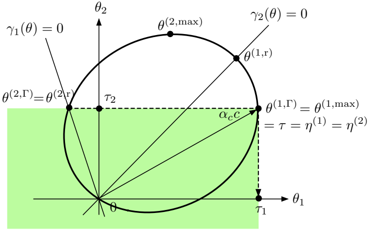


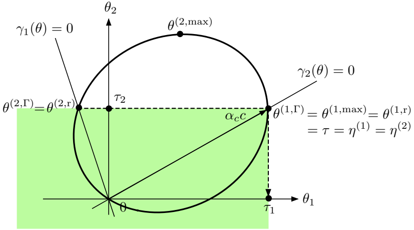


single pole at τ_1 for $\varphi_2(z)$ | double pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = xe^{-\alpha_c x}.$$

Category II: $0 \le \beta = \beta_1$, $\tau_1 = \theta_1^{(1,max)}$



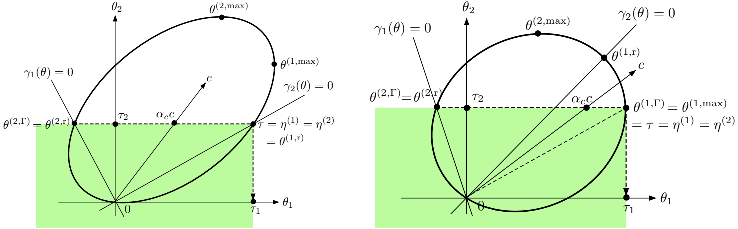


"1/2" pole at τ_1 for $\varphi_2(z)$

"1" pole at τ_1 for $\varphi_2(z)$

$$f_c(x) = xe^{-\alpha_c x}.$$

Category II: $\beta_1 < \beta \le \pi/2$



$$f_c(x) = e^{-\alpha_c x}.$$

References

- J. G. Dai and M. Miyazawa (2011), Semimartinagle Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution, *Stochastic Systems*, to appear.
- M. Miyazawa and M. Kobayashi, M. (2011). Conjectures on tail asymptotics of the stationary distribution for a multidimensional SRBM. *Queueing Systems*, to appear.