Reflecting Brownian motion in two dimensions:
Exact asymptotics for the stationary distribution

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Semimartingale reflecting Brownian motion (SRBM)

\[ Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0, \]
\[ X \text{ is a } (\mu, \Sigma) \text{ Brownian motion}, \]
\[ Z(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0, \]
\[ Y(\cdot) \text{ is continuous and nondecreasing with } Y(0) = 0, \]
\[ Y_i(\cdot) \text{ only increases when } Z_i(\cdot) = 0, \quad i = 1, \ldots, n. \]
Semimartingale reflecting Brownian motion (SRBM)

\[ Z(t) = X(t) + RY(t) \quad \text{for all } t \geq 0, \]

\( X \) is a \((\mu, \Sigma)\) Brownian motion,

\( Z(t) \in \mathbb{R}_+^n \) for all \( t \geq 0, \)

\( Y(\cdot) \) is continuous and nondecreasing with \( Y(0) = 0, \)

\( Y_i(\cdot) \) only increases when \( Z_i(\cdot) = 0, \quad i = 1, \ldots, n. \)

\begin{itemize}
  \item We focus on \( n = 2. \)
  \item \( R \) is assumed to be completely-\( S \)
  \[ R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}. \]
  \item \( Z \) exists and is unique in distribution; (Taylor-Williams 92)
\end{itemize}
A queueing network

Poisson arrivals at average rate $\rho < 1$

mean service time $= 1$

1

deterministic service times

mean service time $= 1$

2

service time variance $a^2$
Its Approximating SRBM

\[ Z_1(t) = X_1(t) + Y_1(t), \]
\[ Z_2(t) = X_2(t) - Y_1(t) + Y_2(t) \]

Drift of \( X \) is \( \mu = (\rho - 1, 0) \),

Covariance of \( X \) is \( \Sigma = \begin{pmatrix} \rho & 0 \\ 0 & a^2 \end{pmatrix} \)

- **R. J. Williams (95)**, Semimartingale reflecting Brownian motions in the orthant, in Stochastic Networks, eds. F. P. Kelly and R. J. Williams, the IMA Volumes in Mathematics and its Applications, Vol. 71 (Springer, New York)

Proposition (Hobson-Rogers 94, Harrison-Hasenbein 09)

Assume that

\[ r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad (1) \]
\[ r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad \text{and} \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \quad (2) \]

Then SRBM \( Z \) has a unique stationary distribution \( \pi \).

- Condition (1): \( R \) is a \( \mathcal{P} \)-matrix.
- Condition (2): \( R^{-1}\mu < 0 \).
- Basic adjoint relationship: for each \( f \in C_b^2(\mathbb{R}^2_+) \)

\[
\int_{\mathbb{R}^2_+} \left( \frac{1}{2} \langle \nabla, \Sigma \nabla f \rangle + \langle \mu, \nabla f \rangle \right) \pi(dx) + \sum_{i=1}^{2} \int_{\mathbb{R}^2_+} \langle R^i, \nabla f \rangle \nu_i(dx) = 0,
\]

\[ \nu_i(A) = \mathbb{E}_\pi \left[ \int_0^1 1\{Z(u) \in A\} \, dY_1(u) \right] \] defines the \( i \)th boundary measure.
Let $Z(\infty) = (Z_1(\infty), Z_2(\infty))$ be the random vector that has the stationary distribution $\pi$.

For any $c \in \mathbb{R}^2_+$, set

$$\langle c, Z(\infty) \rangle = c_1 Z_1(\infty) + c_2 Z_2(\infty).$$

We are interested in finding a function $f_c(x)$ that satisfies

$$\lim_{x \to \infty} \frac{\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}}{f_c(x)} = b$$

for some constant $b > 0$.

The function $b f_c(x)$ is said to be the exact asymptotic of $\mathbb{P}\{\langle c, Z(\infty) \rangle > x\}$. 
An overview

Main Result

- For an SRBM satisfying (1) and (2), the exact asymptotic is given by
  \[ f_c(x) = x^{κ_c} e^{-α_c x}. \]

- The decay rate $α_c$ and the constant $κ_c$ can be computed explicitly from the primitive date $(μ, Σ, R)$.

- The constant $κ_c$ must take one of the values $-3/2, -1/2, 0$ or $1$.

Let $p_c(x)$ be the density of $\langle c, Z(∞) \rangle$. In most cases, we have

\[ \lim_{x \to ∞} \frac{p_c(x)}{f_c(x)} = b \text{ for some } b > 0. \]
A diversion: Laplace transform

- Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous and integrable function. Define

\[
\tilde{f}(z) = \int_0^{\infty} e^{zx} f(x) \, dx, \quad \Re z < \alpha_f,
\]

where \( \alpha_f = \sup\{ \theta \geq 0 : \tilde{f}(\theta) < \infty \} \). \( \tilde{f} \) is analytic in \( \Re z < \alpha_f \).

- When \( f \) is a Gamma\((k, \alpha)\) density, namely,

\[
f(x) = \frac{c}{\Gamma(k)} x^{k-1} e^{-\alpha x},
\]

then \( \alpha_f = \alpha \) and

\[
\tilde{f}(z) = \frac{c}{(\alpha - z)^k} \equiv g(z) \quad \text{for} \quad \Re z < \alpha.
\]

- Complex function \( g \) is an “analytic” extension of \( \tilde{f} \).
Proposition (When $k$ is a positive integer)

Suppose that the analytic extension $g$ of $\tilde{f}$ satisfies that

$$g(z) - \frac{c_0}{(\alpha_0 - z)^k}$$

is analytic for $\Re z < \alpha_1$ for some $\alpha_1 > \alpha_0$. Then, under some mild conditions on $g$,

$$f(x) \sim \frac{c_0}{\Gamma(k)} x^{k-1} e^{-\alpha_0 x} \quad \text{as } x \to \infty. \quad (3)$$

When $\alpha_0$ is the $k$th order pole of $g(z)$, then (3) holds.
Define

\[ \varphi(\theta_1, \theta_2) = \mathbb{E} \left[ e^{\langle \theta, Z(\infty) \rangle} \right]. \]

Then the transform of \( \langle c, Z(\infty) \rangle \) has the following expression

\[ \psi_c(z) = \mathbb{E} \left( e^{z \langle c, Z(\infty) \rangle} \right) = \varphi(zc). \]

Define the convergence domain of \( \varphi \)

\[ D = \text{interior of } \{ \theta \in \mathbb{R}^2 : \varphi(\theta) < \infty \}. \]

When \( \theta \in D \), BAR gives the key relationship

\[ \gamma(\theta) \varphi(\theta) = \gamma_1(\theta) \varphi_1(\theta_2) + \gamma_2(\theta) \varphi_2(\theta_1). \quad (4) \]
Geometric objects: $\Gamma$, $\Gamma_1$ and $\Gamma_2$

\[
\gamma(\theta) = -\langle \theta, \mu \rangle - \frac{1}{2} \langle \theta, \Sigma \theta \rangle,
\]

\[
\gamma_1(\theta) = r_{11}\theta_1 + r_{21}\theta_2 = \langle R^1, \theta \rangle,
\]

\[
\gamma_2(\theta) = r_{12}\theta_1 + r_{22}\theta_2 = \langle R^2, \theta \rangle,
\]

\[
\theta_1 \quad \theta_2
\]

\[
(\mu_1, \mu_2)
\]

\[
0
\]

\[
\Gamma_1
\]

\[
\Gamma_2
\]

\[
\theta^{(1,r)}
\]

\[
\theta^{(2,r)}
\]

\[
\gamma_1(\theta) = 0
\]

\[
\gamma_2(\theta) = 0
\]

\[
\gamma(\theta) = 0
\]
Theorem 1: Convergence domain characterization
**Convergence domain: Category I**

- $\theta^{(2,\Gamma)}$ the highest point in $\Gamma_2$; $\theta^{(1,\Gamma)}$ the right-most point in $\Gamma_1$
- Category I:
  
  $\theta^{(2,\Gamma)} < \theta^{(1,\Gamma)}_{1}$ and $\theta^{(1,\Gamma)}_{2} < \theta^{(2,\Gamma)}_{2}$,
Convergence domain: Category II

\[ \gamma_1(\theta) = 0 \]
\[ \gamma_2(\theta) = 0 \]
\[ \theta^{(1,\max)} = \theta^{(1,r)} \]
\[ \theta^{(2,\Gamma)} = \theta^{(2,r)} \]

Category II: \[ \theta^{(2,\Gamma)} \leq \theta^{(1,\Gamma)} \]
Category III: \[ \theta^{(1,\Gamma)} \leq \theta^{(2,\Gamma)} \]
Key steps in proving Theorem 1

- $\varphi(\theta) < \infty$ implies $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$.
- $\theta \in \Gamma_1$ and $\varphi_1(\theta_2) < \infty$ imply $\varphi(\theta) < \infty$.
- $\theta \in \Gamma$, $\varphi_1(\theta_2) < \infty$ and $\varphi_2(\theta_1) < \infty$ imply $\varphi(\theta) < \infty$. 
Key relationship (4) gives

\[ \gamma(zc)\psi_c(z) = \gamma_1(zc)\varphi_1(c_2z) + \gamma_2(zc)\varphi_2(c_1z) \quad \text{for } \Re z < \alpha_c. \]

Letting \( \gamma(zc) = z\zeta_c(z) \), we have

\[ \psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2z) + \gamma_2(c)\varphi_2(c_1z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c. \]

- \( \varphi_2(z) \) is analytic on \( \Re z < \tau_1 \).
- \( \varphi_1(z) \) is analytic on \( \Re z < \tau_2 \).
- \( \alpha_c c \in \partial \Gamma \).
- \( \alpha_c < \min(\tau_1/c_1, \tau_2/c_2) \).
- \( \psi_c(z) \) has a single pole at \( z = \alpha_c \).
- \( f_c(x) = e^{-\alpha_c x} \).
Singularity properties of $\varphi_2(z)$ at $z = \tau_1$

- In Category I when $\tau_1 < \theta_1^{(1,\text{max})}$, single pole
- In Category I when $\tau_1 = \theta_1^{(1,\text{max})}$
  - When $\theta_1^{(1,\text{max})} \neq \theta_1^{(1,r)}$, $1/2$ “analytic” or $-1/2$ “pole”
  - When $\theta_1^{(1,\text{max})} = \theta_1^{(1,r)}$, $1/2$ “pole”
- In Category II when $\tau_1 < \theta_1^{(1,\text{max})}$,
  - When $\theta_1^{(1,\text{max})} \neq \theta_1^{(1,r)}$, single pole
  - When $\theta_1^{(1,\text{max})} = \theta_1^{(1,r)}$, double pole
- In Category II when $\tau_1 = \theta_1^{(1,\text{max})}$
  - When $\theta_1^{(1,\text{max})} \neq \theta_1^{(1,r)}$, $1/2$ “pole”
  - When $\theta_1^{(1,\text{max})} = \theta_1^{(1,r)}$, $1$ “pole”
Theorem 2: exact asymptotics for Category I
Category I: $0 \leq \beta_1 < \beta < \beta_2$

\[
\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2z) + \gamma_2(c)\varphi_2(c_1z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.
\]

\[\varphi_2(z) \text{ is analytic for } \Re z < \tau_1 \quad \varphi_1(z) \text{ is analytic for } \Re z < \tau_2\]

\[
f_c(x) = e^{-\alpha_c x}
\]
Category I: $0 \leq \beta < \beta_1$, $\tau_1 < \theta_1^{(1,\text{max})}$

- single pole at $\tau_1$ for $\varphi_2(z)$
- $\alpha_c c \not\in \partial \Gamma$

$$f_c(x) = e^{-\alpha_c x}.$$ 

$$\psi_c(z) = \frac{\gamma_1(c) \varphi_1(c_2 z) + \gamma_2(c) \varphi_2(c_1 z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.$$
Category I: $0 \leq \beta < \beta_1$, $\tau_1 = \theta_1^{(1,\text{max})}$

\[ f_c(x) = \begin{cases} 
  x^{-3/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\text{max})} \neq \theta^{(1,r)}, \\
  x^{-1/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\text{max})} = \theta^{(1,r)}. 
\end{cases} \]
Category I: $0 \leq \beta = \beta_1 < \beta_2$, $\tau_1 < \theta_1^{(1,\text{max})}$

\[
\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2z) + \gamma_2(c)\varphi_2(c_1z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.
\]

\[
f_c(x) = xe^{-\alpha_c x}
\]
**Category I:** \(0 \leq \beta = \beta_1 < \beta_2, \tau_1 = \theta_1^{(1,\max)}\)

\[
f_c(x) = \begin{cases} 
  x^{-1/2} e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} \neq \theta^{(1,r)}, \\
  e^{-\alpha_c x} & \text{if } \eta^{(1)} = \theta^{(1,\max)} = \theta^{(1,r)}. 
\end{cases}
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Category I: $0 \leq \beta = \beta_1 = \beta_2$

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\psi_c(z) = \frac{\gamma_1(c)\varphi_1(c_2z) + \gamma_2(c)\varphi_2(c_1z)}{\zeta_c(z)} \quad \text{for } \Re z < \alpha_c.
\]

\[
f_c(x) = \begin{cases} 
    xe^{-\alpha_c x} & \text{if } \eta^{(1)} = \eta^{(2)} = \tau \in \partial \Gamma, \\
    e^{-\alpha_c x} & \text{if } \eta^{(1)} = \eta^{(2)} = \tau \in \text{interior of } \Gamma.
\end{cases}
\]
Theorem 3: exact asymptotics for Category II
**Category II:** $0 \leq \beta < \beta_1: \tau_1 < \theta_1^{(1,\text{max})}$

single pole at $\tau_1$ for $\varphi_2(z)$

double pole at $\tau_1$ for $\varphi_2(z)$

$$f_c(x) = \begin{cases} 
  e^{-\alpha_c x} & \text{if } \tau \neq \theta^{(1,r)} \\
  xe^{-\alpha_c x} & \text{if } \tau = \theta^{(1,r)}
\end{cases}$$
Category II: $0 \leq \beta < \beta_1$: $\tau_1 = \theta_1^{(1,\text{max})}$

“1/2” pole at $\tau_1$ for $\varphi_2(z)$

“1” pole at $\tau_1$ for $\varphi_2(z)$

\[ f_c(x) = \begin{cases} 
 x^{-1/2} e^{-\alpha_c x}, & \text{if } \tau \neq \theta^{(1,r)}, \\
 e^{-\alpha_c x}, & \text{if } \tau = \theta^{(1,r)}.
\end{cases} \]
Category II: $0 \leq \beta = \beta_1$, $\tau_1 < \theta_1^{(1,max)}$

**single** pole at $\tau_1$ for $\varphi_2(z)$

**double** pole at $\tau_1$ for $\varphi_2(z)$

\[
f_c(x) = xe^{-\alpha_c x}.
\]
Category II: $0 \leq \beta = \beta_1$, $\tau_1 = \theta_1^{(1,\text{max})}$

\[\gamma_1(\theta) = 0\]
\[\theta^{(2,\text{max})} = \theta^{(2,r)}\]
\[\alpha_c\]

"1/2" pole at $\tau_1$ for $\varphi_2(z)$

\[f_c(x) = xe^{-\alpha_c x}\]
Category II: $\beta_1 < \beta \leq \pi/2$

\[
f_c(x) = e^{-\alpha_c x}.
\]
References
