On the Gap between the Stability of Fluid and Queueing Networks

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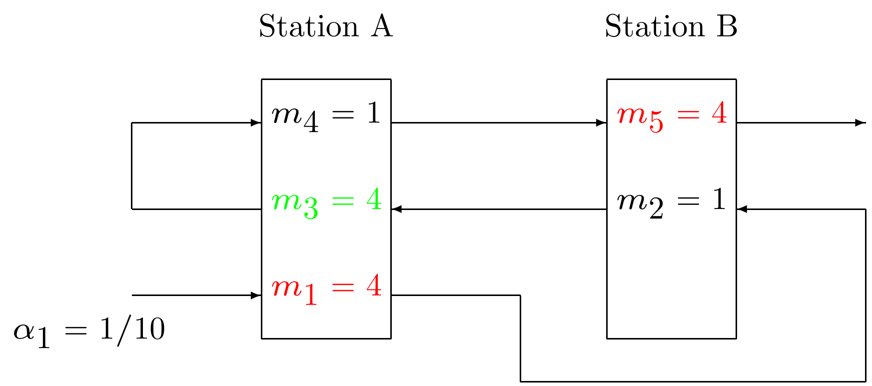
A 2-Station, 5-Class Stochastic Network



Static buffer priority (SBP) discipline: $\{(1, 3, 4), (5, 2)\}$. Red buffers have the highest priority. Black buffers have the lowest priority.

$$\rho_1 = \alpha_1(m_1 + m_3 + m_4) = 0.9,$$

$$\rho_2 = \alpha_1(m_2 + m_5) = 0.5,$$

$$\rho_{\text{ps}} = \alpha_1\left(\frac{m_3}{1 - \alpha_1 m_1} + m_5\right) = \frac{16}{15} > 1.$$

Main Results

Theorem.
(a) When all the distributions are deterministic, from any initial state, the system converges to a limit cycle. In particular, $\sup_{t\geq 0}|Z(t)|<\infty$.
(b) When all the distributions are exponential, the Markov chain $Z=\{Z(t),t\geq 0\}$ *is not positive recurrent.*
(c) When all distributions are (non-degenerate) uniform with certain choice of parameters, for any initial state and sample path, $\sup_{t\geq 0}|Z(t)|<\infty$.

Stability Regions

Positive Harris recurrence:

Stochastic Boundedness: $\lim_{M\to\infty}\lim_{t\to\infty}\mathbb{P}(|Z(t)| > M) = 0.$

Corollary 1. *The region of stability depends on the interarrival and service time distributions, not just their means.*

Rate Stability

Definition. *The network is said to be* rate stable *if for any initial state, with probability* 1, $\lim_{t\to\infty} D_k(t)/t = \alpha_1$, $k = 1, \ldots, 5$, *where* $D_k(t)$ *is the number of departures from class* k *by time* t.

Corollary 2. *When all distributions are either deterministic or uniform, the network is rate stable.*

When all distributions are exponential, a simulation study shows that, with probability 1, $|Z(t)|$ grows linearly as $t \to \infty$. Thus, the network is not rate stable.

Fluid Model

A *fluid model* is defined through a set of equations.

$$Z_k(t) = Z_k(0) + D_{k-1}(t) - D_k(t),$$

$$Z_k(t) \geq 0,$$

$$D_k(t) = \mu_k T_k(t),$$

$$T_k(0) = 0,\ T_k(\cdot) \text{ is non-decreasing,}$$

additional equations that are specific to a discipline.

Under the SBP discipline defined earlier, e.g., $Z_2(t) + Z_5(t) > 0$ implies that $\dot{T}_2(t) + \dot{T}_5(t) = 1$.

A fluid model is said to be *stable* if there exists a $\delta > 0$ such that for each fluid model solution, $Z(t) = 0$ for $t \geq \delta$.

Fluid Limit Model

For each initial state x and $r > 0$, let

$$\mathbb{X}^{x,r}(t,\omega) = \frac{1}{r}(Z^x(rt,\omega), T^x(rt,\omega))$$

where x is the initial state of a stochastic network.

Definition. *A pair of functions (Z,T) is said to be a* fluid limit *if there exists a "good" ω, a sequence $r_n \to \infty$ and a sequence of initial states x_n such that $\lim_{n\to\infty} |x_n|/r_n \le 1$ and $\mathbb{X}^{x_n,r_n} \to (Z,T)$ as $n \to \infty$.*

The set of all fluid limits is said to be the *fluid limit model* of a stochastic network.

Definition. *The fluid limit model is said to be stable if there exists a $\delta > 0$ such that for each fluid limit, $Z(t) = 0$ for $t \ge \delta$.*

Stability Relationship

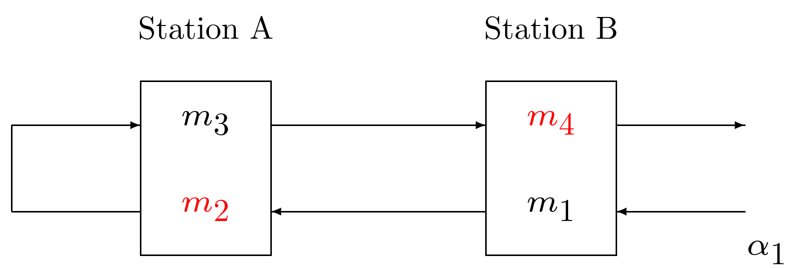
Corollary 3. *A fluid model may never be able to detect the exact stability region of the corresponding stochastic queueing network, no matter how many fluid model equations one adds.*

It is known that the stability of a fluid limit model implies the stability of the corresponding queueing network. Rybko-Stolyar (92), Dai (95), Stolyar (95), Chen (95).

Partial converse in Meyn (95), Dai (96).

See also Bramson (98), Foss (98).

Virtual Station Conditions



Red buffers have high priority.

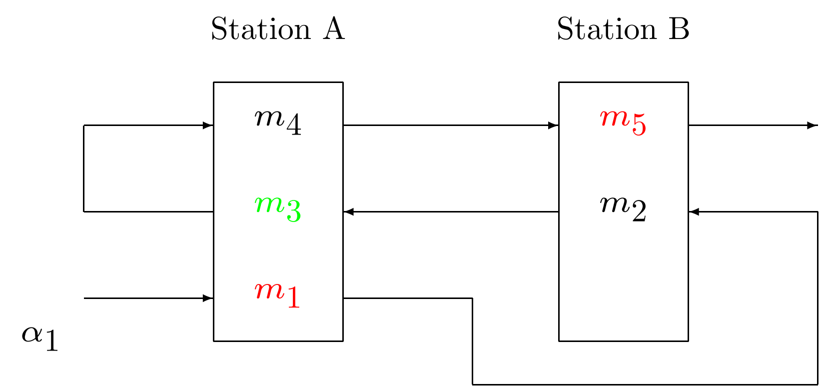
Lemma 1. *If* $Z_2(0)Z_4(0) = 0$*, then* $Z_2(t)Z_4(t) = 0$ *for* $t \geq 0$*.*

Harrison-Nguyen (95), Dumas (97), Dai-Vande Vate (97). Virtual station condition:

$$\rho_v = \alpha_1(m_2 + m_4) < 1.$$

If $\rho_v > 1$, with probability 1, $|Z(t)|$ grows linearly as $t \to \infty$.

Push Start Conditions



Push start condition:

$$\rho_{\mathrm{ps}} = \alpha_1 \Big(\frac{m_3}{1 - \alpha_1 m_1} + m_5 \Big) < 1.$$

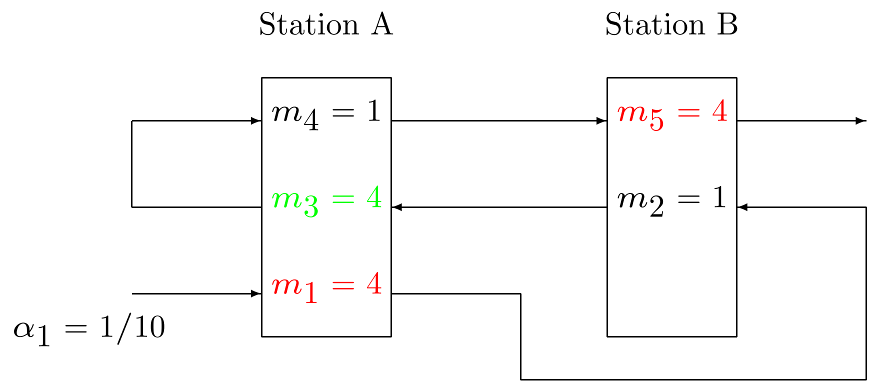
2-Station Fluid Networks

Theorem. [Dai-Vande Vate 97] *Consider a 2-station, multi-type queueing network. The corresponding fluid model is stable under any non-idling service discipline iff the usual traffic, virtual station and push start conditions are all satisfied. Furthermore, the extreme disciplines are SBP disciplines.*

Corollary. *If the usual traffic, virtual station and push start conditions are all satisfied, the stochastic queueing network is stable under any SBP discipline.*

Theorem. *If the usual traffic or the virtual station condition is strictly violated, then there exists an SBP discipline under which, for any initial state, with the probability* 1, $|Z(t)| \to \infty$ *as* $t \to \infty$.

Push Start Conditions Are Not Always Necessary!



$\rho_{\text{ps}} = 16/15 > 1$. Under the SBP discipline: $\{(1, 3, 4), (5, 2)\}$, the push start condition is necessary if all distributions are exponential.

However, we have

Corollary 4. *The push start condition is* not *necessary if all distributions are deterministic or uniform.*

Summary

- Stability region of a stochastic network may depend on the distributions of interarrival and service times. Mean values are not enough.

- Fluid model (not fluid limit model) may never be able to detect the stability region of a stochastic network.

- Push start condition may not be necessary for the stability of 2-station stochastic networks.