Queues in service systems: customer abandonment and diffusion approximations

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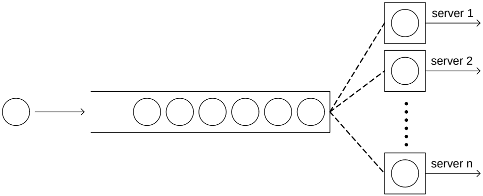
Outline

- Single-server queues vs many-server queues
- The QED regime and the square-root staffing rule
- Need of modeling customer abandonment
- Distributional sensitivity
- Diffusion models for many-server queues

Multi-server queues

A $G/GI/n + GI$ queue



- n identical servers working in parallel (single-server $n = 1$; many-server $n \gg 1$)
- first-in-first out buffer of infinite size
- a general arrival process (G)
- iid service times (GI)
- iid patience times ($+GI$)

Large-scale service systems

A many-server queue serves as a building block for modeling large-scale service systems

- Call centers
  - Bank of America, over one thousand agents
  - UPS, several hundred agents
- Hospital beds
  - hundreds of beds
- Web farms/computer clusters
  - up to several thousand servers/CPUs

Single-server queues vs many-server queues

Insight

The performance of many-server queues is qualitatively different from that of single-server queues or queues with a small number of servers

Key performance measures

- delay probability P_w
- mean customer waiting time w
- fraction of abandonment P_A

Performance of a single-server queue

Consider an $M/GI/1$ queue without abandonment

- Poisson arrival process with rate λ
- mean service time m
- traffic intensity $\rho := \lambda m$

Assume $\rho < 1$. By PASTA and Pollaczek-Khinchine,

$$P_w = \rho \quad \text{and} \quad w = m\left(\frac{\rho}{1-\rho}\right)\left(\frac{1+c_s^2}{2}\right),$$

- SCV of service times c_s^2, and waiting time factor

$$f := w/m = \left(\frac{\rho}{1-\rho}\right)\left(\frac{1+c_s^2}{2}\right).$$

Since $P_w \to 1$ as $\rho \to 1$, almost all have to wait before being served

Single-server queues, painful choice: quality OR efficiency?

- quality: no waiting or very short waiting
- efficiency: $\rho \to 1$

However,

$$f \text{ is proportional to } \frac{\rho}{1-\rho}$$

Insight

In a single-server queue, one cannot maintain high server utilization to achieve good quality of service

Quality OR efficiency?

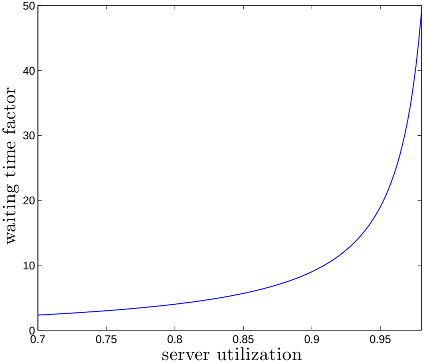


FIGURE: Waiting time factor f vs server utilization ρ in an $M/M/1$ queue

Performance of a multi-server queue

Consider an $M/M/n$ queue. Traffic intensity $\rho := \lambda m/n$.
By Erlang-C,

$$P_w = \frac{(n\rho)^n}{n!}\left((1-\rho)\sum_{k=0}^{n-1}\frac{(n\rho)^k}{k!} + \frac{(n\rho)^n}{n!}\right)^{-1}$$

The waiting time factor

$$f = \frac{w}{m} = \frac{P_w}{(1-\rho)n} \le \frac{1}{(1-\rho)n}$$

With ρ fixed, $f \to 0$ as $n \to \infty$

Many-server queues: quality AND efficiency!

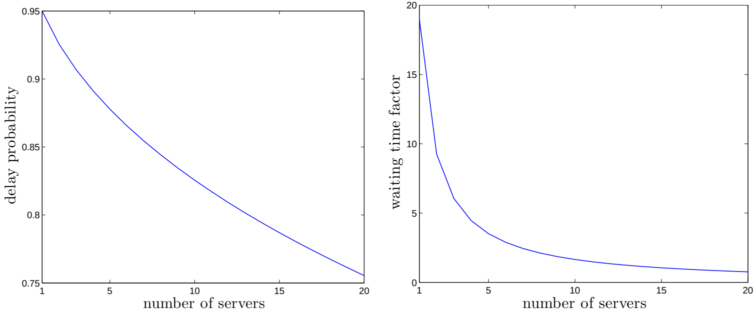


FIGURE: Delay probability P_w and waiting time factor f vs number of servers n, for $M/M/n$ queues with $\rho = 0.95$

If one increases n to 100, then $P_w = 50.7\%$ and $f = 0.101$

Waiting time factor

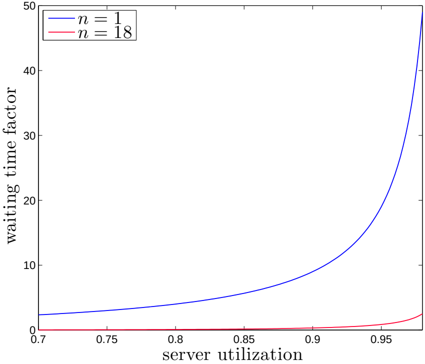


FIGURE: Waiting time factor f vs server utilization ρ in an $M/M/1$ queue and an $M/M/18$ queue

The QED regime

The above $M/M/100$ queue with $\rho = 0.95$ achieves both high quality of service and operational efficiency:

- the server utilization close to 1 (efficiency)
- only a fraction of customers need to wait (quality)
- waiting times are relatively short (quality)

The system is operated in the quality- and efficiency-driven (QED) regime, also called the rationalized regime.

The square-root staffing rule in the $M/M/n$ setting

Let $R := \lambda m$ be the offered load. The square-root safety staffing rule recommends

$$n \approx R + \beta\sqrt{R} \quad \text{for some } \beta > 0$$

Erlang-C

$$P_w = \frac{(n\rho)^n}{n!}\left((1-\rho)\sum_{k=0}^{n-1}\frac{(n\rho)^k}{k!} + \frac{(n\rho)^n}{n!}\right)^{-1}$$

Halfin and Whitt (1981) proved that

$$P_w \to \gamma = \frac{1}{\beta\Phi(\beta)/\phi(\beta) + 1} \quad \text{as } R \to \infty \tag{1}$$

- ϕ is the standard normal probability density
- Φ is the standard normal cumulative distribution function

The square-root staffing rule in the $M/M/n$ setting

Fix $\beta > 0$ and set $n \approx R + \beta\sqrt{R}$. As R increases,

- P_w stabilizes at $\gamma \in (0, 1)$
- $\rho = R/n \to 1$
- $f = P_w/(\sqrt{n}\beta)$ is on the order of $1/\sqrt{n}$

The system is operated in the QED regime!

Performance analysis using formula (1)

Given staffing level n and utilization level $\rho < 1$, set

$$\beta = \sqrt{n}(1 - \rho)$$

Then,

$$P_w \approx \frac{1}{\beta\Phi(\beta)/\phi(\beta) + 1} \quad \text{and} \quad f = \frac{P_w}{\sqrt{n}\beta}$$

	Exact	Approx. by (1)
P_w	50.7%	50.5%
f	0.101	0.101

TABLE: Performance measures in an $M/M/100$ queue with $\rho = 0.95$

Staffing using formula (1)

Suppose P_w is required to be less than some $\gamma \in (0,1)$. First solve for β by

$$\gamma = \frac{1}{\beta\Phi(\beta)/\phi(\beta)+1}$$

Then,

$$n \approx R + \beta\sqrt{R}$$

The square-root staffing rule for $GI/GI/n$ queue

Consider a $GI/GI/n$ queue. Let

$$n = R + \beta\sqrt{R} \quad \text{for some } \beta > 0$$

Reed (2009) proved that as R increases, this staffing level drives the queue to the QED regime

Historical remarks

- The origin of that can be traced back to Erlang (1923)
- Erlang reported that it had been in use at the Copenhagen Telephone Company since 1913
- Advocated by Newell (1973,1982), Kolesar (1986), Grassmann (1986,1988), among others
- Whitt (1992) formally proposed and analyzed this rule

Customer abandonment

- Human's patience is always limited!
- Customer abandonment is present in most service systems

INSIGHT

For a service system with significant customer abandonment, any queueing model that ignores the abandonment phenomenon is likely irrelevant to operational decisions

One must model abandonment explicitly!

	$M/M/50+M$	$M/M/50$
Mean service time	1	1
Mean patience time	2	N/A
Arrival rate	55	$55 \times (1 - 10.2\%) = 49.39$
Abandonment fraction	10.2%	N/A
Server utilization	98.8%	98.8%
Mean waiting time (in sec.)	12.5	87.7
Mean queue length	11.2	72.2

TABLE: Queues with and without customer abandonment

Why customer abandonment matters?

- Customers who experience long waiting tend to abandon the system
- With abandonment, the system can reach a steady state even if the arrival rate is larger than the service capacity
- Some performance measures in a queue with abandonment is better than in a queue without abandonment
- To meet certain service levels without considering abandonment, one tends to overestimate the staffing level

The square-root staffing rule is still applicable

Insight

In the presence of customer abandonment, the square-root safety staffing rule can still lead the system to the QED regime and yield high server utilization, short waiting times, and a very small abandonment fraction

The square-root staffing rule in the $M/M/n+M$ setting

As argued by Garnett et al. (2002), with

$$n \approx R + \beta\sqrt{R} \quad \text{for } \beta \in \mathbb{R} \text{ and } R \text{ large,}$$

one has

$$P_w \approx \left(1 + \frac{h(\beta\sqrt{\mu/\alpha})}{\sqrt{\mu/\alpha}\,h(-\beta)}\right)^{-1}$$

- $1/\alpha$ is the mean patience time
- $h(x) = \phi(x)/(1-\Phi(x))$ is the standard normal hazard rate

The fraction of abandonment

$$P_A \approx \frac{1}{\sqrt{R}}\left(\sqrt{\alpha/\mu}\,h(\beta\sqrt{\mu/\alpha}) - \beta\right)\left(1 + \frac{h(\beta\sqrt{\mu/\alpha})}{\sqrt{\mu/\alpha}\,h(-\beta)}\right)^{-1}$$

Waiting times in the QED regime

Fix $\beta = \sqrt{n}(1-\rho)$. As n increases,

- the mean waiting time decreases at rate $1/\sqrt{n}$ in $M/M/n$ queues
- Garnett et al (2002) confirmed the same decreasing rate in $M/M/n+M$ queues

When n is large,

- waiting times are relatively short
- the patience time distribution F, outside a small neighborhood of the origin, barely has any influence on the system dynamics

Sensitivity on F with fixed $\alpha = F'(0+)$

Consider an $M/M/100 + GI$ queue

- with different F
- but with the same $\alpha = F'(0+)$
- $\lambda = 105$ and $m = 1$

	Abandonment fraction			Mean queue length		
	Exp	Uniform	H_2	Exp	Uniform	H_2
$\alpha = 0.1$	0.0497	0.0498	0.0496	52.18	50.59	54.19
$\alpha = 0.5$	0.0603	0.0607	0.0599	12.67	12.06	13.43
$\alpha = 1$	0.0670	0.0676	0.0662	7.031	6.585	7.592
$\alpha = 2$	0.0739	0.0748	0.0730	3.882	3.547	4.313
$\alpha = 10$	0.0886	0.0902	0.0869	0.9301	0.7540	1.172

TABLE: Performance insensitivity to patience time distributions F

Sensitivity on F with mean patience time m_p fixed

	Abandonment fraction			Mean queue length		
	Exp	Uniform	H_2	Exp	Uniform	H_2
$m_p = 0.1$	0.0886	0.0840	0.0926	0.9301	1.505	0.5840
$m_p = 0.5$	0.0739	0.0676	0.0794	3.882	6.585	2.455
$m_p = 1$	0.0670	0.0608	0.0730	7.031	12.06	4.313
$m_p = 2$	0.0603	0.0550	0.0682	12.67	22.10	6.438
$m_p = 10$	0.0497	0.0481	0.0543	52.18	98.07	24.52

Table: Mean patience time is a wrong statistic!

Patience insensitivity to patience time distributions

INSIGHT

In the QED regime, the system performance is generally invariant with the patience time distribution as long as its density at the origin is fixed and positive

- For a $G/GI/n+GI$ queue in the QED regime, it is generally accurate to replace F with an exponential distribution with rate $\alpha = F'(0+)$
- The matrix-analytic method can be used to evaluate $GI/Ph/n+M$ queues

Customer abandonment in the QED regime

Dai and He (2010) proved that

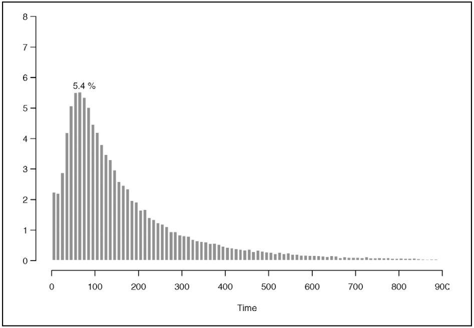
$$A(t) \approx \alpha \int_0^t Q(s)\,\mathrm{d}s$$

- $A(t)$ is the number of abandonments by time t
- $Q(t)$ is the number of waiting customers at time t

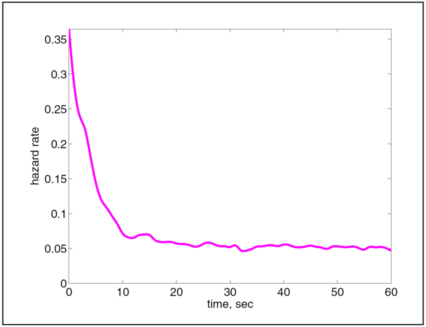
This justifies the replacement of $+GI$ with $+M$.

Non-exponential distributions

The exact analysis of a many-server queue has largely been limited to the $M/M/n+M$ model. However...



Service time distribution of a call center, by Brown et al (2005)



Patience time hazard rate of a call center, by Mandelbaum and Zeltyn (2004)

Challenges

A $GI/GI/n + GI$ queue is difficult to be analyzed because of

- general interarrival/service/patience time distributions
- a large number of servers

As a consequence,

- no analytical solution and no numerical algorithms for performance measures
- usually evaluated by simulation

We use diffusion processes to approximate many-server queues

A Poisson sample path with $\lambda = 1$

Let $\{E(t) : t \geq 0\}$ be a Poisson process with rate λ

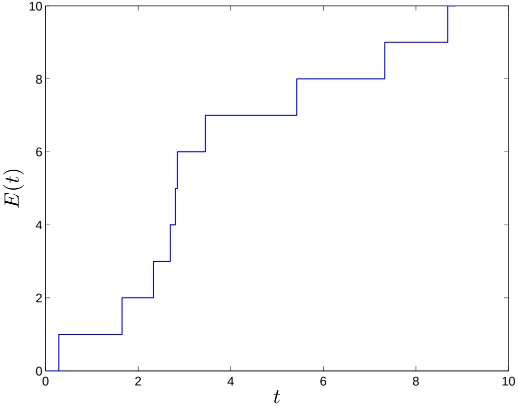


FIGURE: A Poisson sample path with rate $\lambda = 1$

The centered sample path with $\lambda = 1$

Then, $\{E(t) - \lambda t : t \geq 0\}$ is the centered process

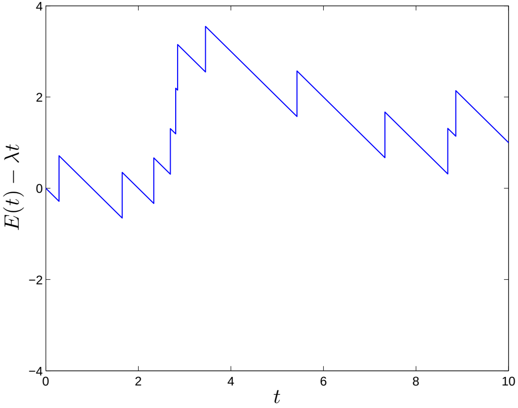


FIGURE: The sample path of the centered process with $\lambda = 1$

A Poisson sample path with $\lambda = 100$

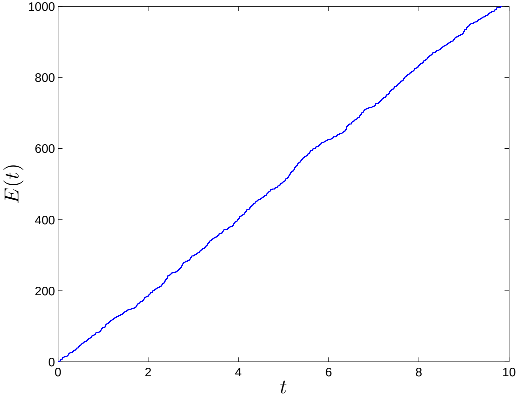


FIGURE: A Poisson sample path with rate $\lambda = 100$

The centered sample path with $\lambda = 100$

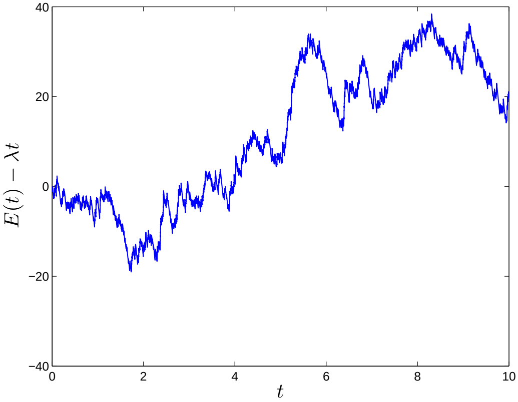


FIGURE: The sample path of the centered process with $\lambda = 100$

A Poisson sample path with $\lambda = 10,000$

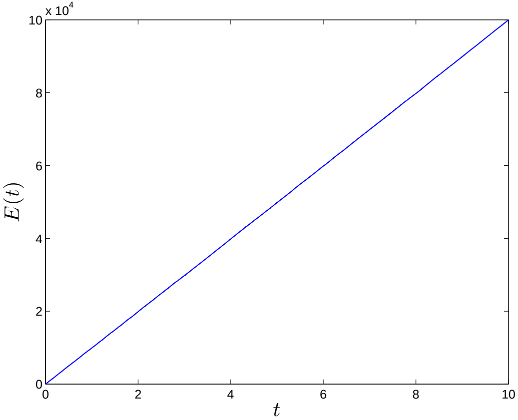


FIGURE: A Poisson sample path with rate $\lambda = 10,000$

The centered sample path with $\lambda = 10,000$

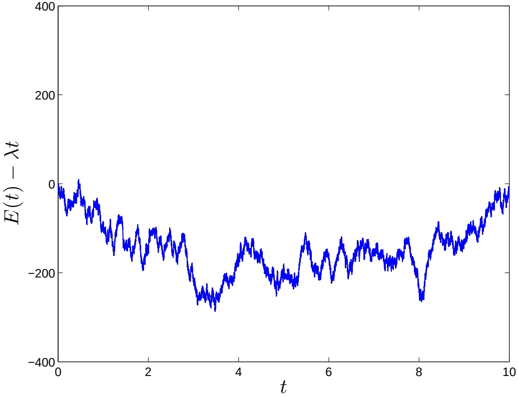


FIGURE: The sample path of the centered process with $\lambda = 10,000$

Brownian motion and Donsker's theorem

Let

$$\tilde{E}_\lambda(t) = \frac{E(t) - \lambda t}{\sqrt{\lambda}}$$

Donsker's theorem implies that the process $\tilde{E}_\lambda$ is close to a standard Brownian motion when λ is large

Definition

A process $B = \{B(t) : t \geq 0\}$ is said to be a (μ_B, σ_B^2)-Brownian motion if

- $B(0) = 0$ and almost every sample path is continuous
- $\{B(t) : t \geq 0\}$ has stationary, independent increments
- $B(t)$ is normally distributed with mean $\mu_B t$ and variance $\sigma_B^2 t$ for every $t > 0$

B is a standard Brownian motion if $\mu_B = 0$ and $\sigma_B^2 = 1$

System equation for an $M/M/n+GI$ queue

$$X(t) = X(0) + E(t) - S\Big(\mu \int_0^t Z(s)\, ds\Big) - A(t)$$

- $X(t)$ is the number of customers in system at time t
- $E(t)$ is the number of arrivals by time t
- $\{S(t) : t \geq 0\}$ is a Poisson process with rate one
- $\mu = 1/m$ is the service rate
- $Z(t)$ is the number of busy servers at time t
- $A(t)$ is the number of abandonments by time t

Brownian approximation

Let

$$\tilde{E}(t) = \frac{E(t) - \lambda t}{\sqrt{n}} \quad \text{and} \quad \tilde{S}(t) = \frac{S(nt) - nt}{\sqrt{n}}$$

By Donsker's theorem

$$\tilde{E} \approx B_E \quad \text{and} \quad \tilde{S} \approx B_S$$

- B_E is a $(0, \rho\mu)$-Brownian motion
- B_S is a $(0, 1)$-Brownian motion
- B_E and B_S are independent

Approximation of the abandonment process

Recall that

$$\alpha = F'(0+)$$

The abandonment process is approximated by

$$A(t) \approx \alpha \int_0^t Q(s)\,ds = \alpha \int_0^t (X(s) - n)^+\,ds$$

Scaled system equations

$$\tilde{X}(t) = \frac{X(t) - n}{\sqrt{n}}, \quad \beta = \sqrt{n}(1 - \rho), \quad \tilde{A}(t) = \frac{A(t)}{\sqrt{n}}$$

The scaled system equation

$$\tilde{X}(t) = \tilde{X}(0) + \tilde{E}(t) - \tilde{S}\Big(\mu \int_0^t \frac{Z(s)}{n}\, ds\Big) - \beta\mu t + \mu \int_0^t \tilde{X}(s)^- \, ds - \tilde{A}(t)$$

where

$$\tilde{E} \approx B_E$$

$$\tilde{S} \approx B_S$$

$$\tilde{A} \approx \alpha \int_0^t \tilde{X}(s)^+ \, ds$$

$$\frac{Z(t)}{n} \approx \rho \wedge 1$$

A diffusion model for an $M/M/n + GI$ queue

The scaled system equation

$$\tilde{X}(t) = \tilde{X}(0) + \tilde{E}(t) - \tilde{S}\Big(\mu \int_0^t \frac{Z(s)}{n}\, ds\Big) - \beta\mu t + \mu \int_0^t \tilde{X}(s)^-\, ds - \tilde{A}(t)$$

The diffusion model

$$Y(t) = \tilde{X}(0) + B_E(t) - B_S\big((\rho \wedge 1)\mu t\big) - \beta\mu t + \mu \int_0^t Y(s)^-\, ds - \alpha \int_0^t Y(s)^+\, ds$$

A piecewise OU process

The diffusion model

$$
\begin{aligned}
Y(t) = \tilde{X}(0) + B_E(t) - B_S\big((\rho \wedge 1)\mu t\big) \\
- \beta\mu t + \mu \int_0^t Y(s)^- \, ds - \alpha \int_0^t Y(s)^+ \, ds
\end{aligned}
$$

- a piecewise linear drift

$$
b(x) = \begin{cases} -\beta\mu - \alpha|x| & \text{when } x \geq 0 \\ -\beta\mu + \mu|x| & \text{when } x \leq 0 \end{cases}
$$

- the mean-reverting property

Y is a piecewise Ornstein-Uhlenbeck (OU) process. It becomes an OU process when $\alpha = \mu$

Stationary distribution of an OU process

A process $Y = \{Y(t) : t \geq 0\}$ is called an OU process if it satisfies

$$Y(t) = Y(0) + \sigma B(t) - \beta\mu t - \mu \int_0^t Y(s)\, ds$$

- a linear drift
$$b(x) = -\beta\mu - \mu x$$
- a normal stationary distribution
$$g(z) = \sqrt{\frac{\mu}{\pi\sigma^2}} \exp\left(- \frac{\mu(z+\beta)^2}{\sigma^2} \right) \quad \text{for } z \in \mathbb{R}$$

Stationary distribution of a piecewise OU process

The diffusion model has a piecewise linear drift

$$b(x) = \begin{cases} -\beta\mu - \alpha|x| & \text{when } x \geq 0 \\ -\beta\mu + \mu|x| & \text{when } x \leq 0 \end{cases}$$

It admits a piecewise normal stationary distribution

$$g(z) = \begin{cases} a_1 \exp\left(- \dfrac{\alpha(z + \alpha^{-1}\mu\beta)^2}{\sigma_B^2} \right) & \text{when } z \geq 0, \\ a_2 \exp\left(- \dfrac{\mu(z + \beta)^2}{\sigma_B^2} \right) & \text{when } z < 0, \end{cases}$$

- $\sigma_B^2 = \mu\big(\rho + \rho \wedge 1\big)$
- a_1 and a_2 are constants such that

$$\int_{-\infty}^{\infty} g(z)\, dz = 1 \quad \text{and} \quad g(0-) = g(0+)$$

Performance approximations for $M/M/n + GI$ queues

- the long-run average queue length

$$\bar{Q} \approx \sqrt{n} \cdot \mathbb{E}[Y(\infty)^+] = \sqrt{n} \int_0^\infty x g(x)\, \mathrm{d}x$$

- the long-run average number of idle servers

$$\bar{I} \approx \sqrt{n} \cdot \mathbb{E}\big[Y(\infty)^-\big] = -\sqrt{n} \int_{-\infty}^0 x g(x)\, \mathrm{d}x.$$

- the abandonment fraction

$$P_A \approx 1 - \frac{\mu(n - \bar{I})}{\lambda}$$

Diffusion approximation for the $M/M/100+M$ queue

	Abandonment fraction		Mean queue length	
	Exp	Diffusion	Exp	Diffusion
$\alpha = 0.1$	0.0497	0.0497	52.18	52.19
$\alpha = 0.5$	0.0603	0.0603	12.67	12.66
$\alpha = 1$	0.0670	0.0669	7.031	7.022
$\alpha = 2$	0.0739	0.0738	3.882	3.877
$\alpha = 10$	0.0886	0.0886	0.9301	0.9302

TABLE: Performance estimates for the $M/M/100+M$ queue

Beyond exponential service distributions

A two-phase hyperexponential distribution (H_2)

- $$V = \begin{cases} \text{Exp}(\nu_1) & \text{with probability } p_1 \\ \text{Exp}(\nu_2) & \text{with probability } p_2 = 1 - p_1 \end{cases}$$
- fraction of phase j workload
$$\theta_j = \frac{p_j/\nu_j}{p_1/\nu_1 + p_2/\nu_2}, \quad \theta_1 + \theta_2 = 1$$
- a special case of phase-type distributions

A diffusion model for an $M/H_2/n + GI$ queue

Let $X_j(t)$ be the number of type j customers in system at time t

$$\tilde{X}_j(t) = \frac{X_j(t) - n\theta_j}{\sqrt{n}}$$

A two-dimensional process (Y_1, Y_2) is used to approximate $(\tilde{X}_1, \tilde{X}_2)$

A diffusion model for an $M/H_2/n + GI$ queue

$$
\begin{aligned}
Y_j(t) = \tilde{X}_j(0) - \beta\mu p_j t + p_j B_E(t) + (-1)^{j-1} B_M(\rho\mu t) - B_j((\rho \wedge 1)\theta_j \nu_j t) \\
+ \nu_j \int_0^t (p_j(Y_1(t) + Y_2(t))^+ - Y_j(t))\, \mathrm{d}s \\
- p_j \alpha \int_0^t (Y_1(s) + Y_2(s))^+\, \mathrm{d}s
\end{aligned}
$$

- B_E is a $(0, \rho\mu)$-Brownian motion
- B_1 and B_2 are $(0, 1)$-Brownian motions
- B_M is a $(0, p_1 p_2)$-Brownian motion
- they are all independent

See He and Dai (2011) for diffusion models for a $GI/Ph/n + GI$ queue

Computing the stationary distribution of the diffusion model

Let Y be a d-dimensional diffusion process. Assume that Y has a unique stationary density g on $\mathbb{R}^d$. The basic adjoint relationship (BAR) says

$$\int_{\mathbb{R}^d} \mathcal{G}f(x)g(x)\,\mathrm{d}x = 0 \quad \text{for all } f \in C_b^2(\mathbb{R}^d)$$

- $\mathcal{G}$ is the generator of Y
- He and Dai (2011) designed an algorithm to solve the BAR

Example: an $M/H_2/500 + M$ queue

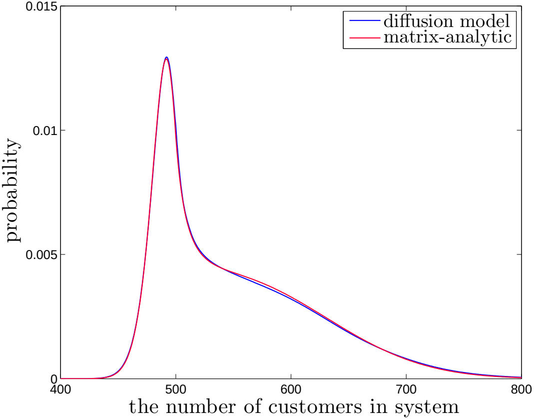


FIGURE: $\rho = 1.045$, $p = (0.9351, 0.0649)$, $1/\nu = (0.1069, 13.89)$, mean patience time $= 2$

Example: an $M/H_2/20 + M$ queue

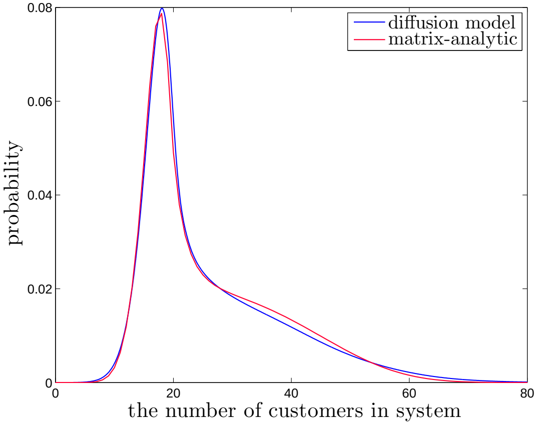


FIGURE: $\rho = 1.112$, $p = (0.9351, 0.0649)$, $1/\nu = (0.1069, 13.89)$, mean patience time $= 2$.

Limitations of the abandonment approximation

The approximation $A(t) \approx \alpha \int_0^t Q(s)\,\mathrm{d}s$ is not always good

- The abandonment process still depends on F in a neighborhood of the origin, not just the origin
- When the patience time changes rapidly near the origin, this abandonment approximation can be inaccurate
- When $\alpha = 0$ and $\rho > 1$, the queue can still reach a steady state thanks to abandonment, but the diffusion model does not have a stationary distribution

How to improve the abandonment approximation?

Consider a neighborhood of the origin rather than the origin itself!

- Exploiting the idea of scaling the patience time hazard rate, proposed by Reed and Ward (2008)
- Assume F has a bounded hazard function

$$h(t) = \frac{f(t)}{1 - F(t)} \quad \text{for } t \geq 0.$$

The scaled abandonment process is approximated by

$$\tilde{A}(t) \approx \int_0^t \int_0^{\frac{Q(s)}{\sqrt{n}}} h\Big(\frac{\sqrt{n}u}{\lambda}\Big) \, du \, ds.$$

Intuition on the abandonment rate

- By time s, the ith customer from the back of the queue has been waiting around i/λ minutes
- This customer will abandon the queue during the next δ minutes with probability $h(i/\lambda)\delta$
- The abandonment rate at time s is around $\sum_{i=1}^{Q(s)} h(i/\lambda)$
- The scaled abandonment rate

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{Q(s)} h\Big(\frac{i}{\lambda}\Big) \approx \int_0^{\frac{Q(s)}{\sqrt{n}}} h\Big(\frac{\sqrt{n}u}{\lambda}\Big)\, \mathrm{d}u$$

Refined diffusion model for the $M/H_2/n + GI$ queue

The refined diffusion model

$$Y_j(t) = \tilde{X}_j(0) - \beta\mu p_j t + p_j B_E(t) + (-1)^{j-1} B_M(\rho\mu t) - B_j((\rho \wedge 1)\theta_j \nu_j t)$$
$$+ \nu_j \int_0^t (p_j(Y_1(t) + Y_2(t))^+ - Y_j(t))\, \mathrm{d}s$$
$$- p_j \int_0^t \int_0^{(Y_1(s)+Y_2(s))^+} h\Big(\frac{\sqrt{n}u}{\lambda}\Big)\, \mathrm{d}u\, \mathrm{d}s$$

Example: an $M/H_2/500 + H_2$ queue

	Simulation	Diffusion	Refined diffusion
Mean queue length	6.413	1.475	6.359
Abandonment fraction	0.05512	0.05863	0.05517
$\mathbb{P}[X(\infty) > 480]$	0.8881	0.8663	0.8929
$\mathbb{P}[X(\infty) > 500]$	0.4720	0.3192	0.4822
$\mathbb{P}[X(\infty) > 520]$	0.1050	9.274×10^{-5}	0.1074

Table: Performance measures of the $M/H_2/500 + H_2$ queue.

- traffic intensity: $\rho = 1.045$
- service time distribution: $p = (0.5915, 0.4085)$ and $\nu = (5.917, 0.454)$
- patience time distribution: $p = (0.9, 0.1)$ and $\nu = (1, 200)$

Example: an $M/H_2/500+E_3$ queue

	Simulation	Refined diffusion
Mean queue length	119.1	119.5
Abandonment fraction	0.04337	0.04340
$\mathbb{P}[X(\infty) > 480]$	0.9940	0.9946
$\mathbb{P}[X(\infty) > 500]$	0.9756	0.9770
$\mathbb{P}[X(\infty) > 600]$	0.6645	0.6733

TABLE: Performance measures of the $M/H_2/n+E_3$ queue.

- $\rho = 1.045$ and $\alpha = 0$, the first diffusion model fails!
- service time distribution: $p = (0.5915, 0.4085)$ and $\nu = (5.917, 0.454)$
- mean patience time 1 minute

Summary

- Single-server queues and many-server queues are qualitatively different
- Follow the square-root staffing rule to operate your system in the QED regime
- Model customer abandonment explicitly
- In the QED regime, the patience density at the origin has the most impact on system performance
- Diffusion models is a useful tool to evaluate a many-server queue's performance

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