Fluid models and Stability of Multiclass Queueing Networks

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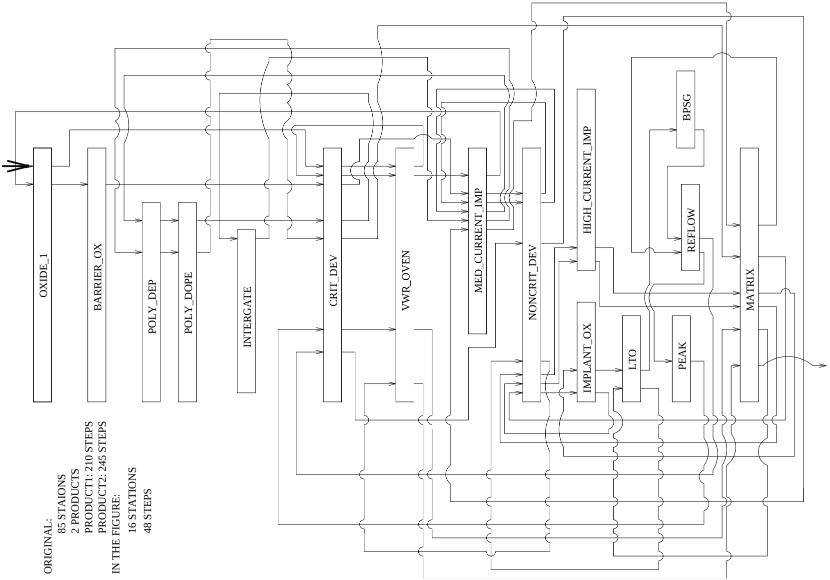
Outline

- Part I: Importance of an operational policy in a wafer fab
- Part II: Fluid models and their stability
- Part III: For a queueing network operating under a service policy, its stability region can depend on
  - its distributions, not just means;
  - the preemption mechanism.

Flow in a Wafer Fab



Performance Measures

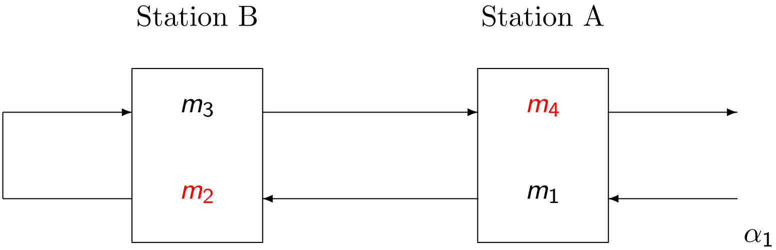
First order ones:

- Throughput: rate at which entities leave a system
- Utilization

Second order ones:

- Cycle time: processing times plus waiting time of an entity; average and variance of cycle time

An Re-Entrant Line (Lu-Kumar network)

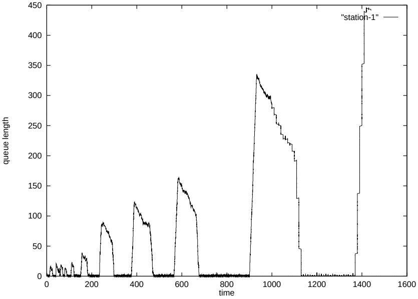


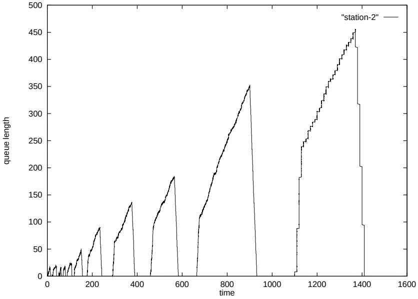
$$\alpha_1 = 1, \quad m_1 = .2, \quad m_2 = .6, \quad m_3 = .1, \quad m_4 = .6.$$

Operational policy: LBFS at Station A, FBFS at Station B.

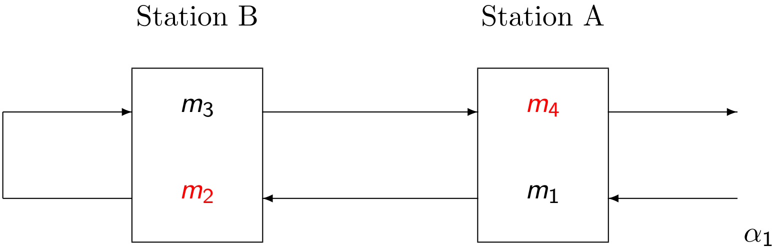
$$\rho_1 = 80\%, \quad \rho_2 = 70\%.$$

WIP Levels at Two Stations



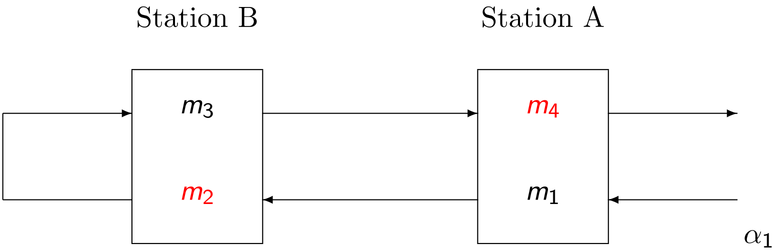


Utilization and Cycle Time



# departed	100	1,000	10,000	100,000
cycle time	15.8	183.7	1740.2	17043.7
utilization A	0.65	0.60	0.61	0.65
utilization B	0.59	0.68	0.67	0.61

Theorem

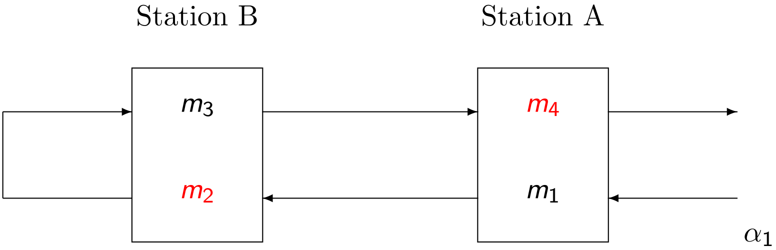


Under the operational policy, the system is "stable" if and only if

$$\rho_1 = \alpha_1(m_1 + m_4) \leq 1,$$
$$\rho_2 = \alpha_1(m_2 + m_3) \leq 1,$$
$$\rho_v = \alpha_1(m_2 + m_4) \leq 1.$$

Dai and Vande Vate, *Operations Research*, 721–744, 2000.

Maximum Throughput

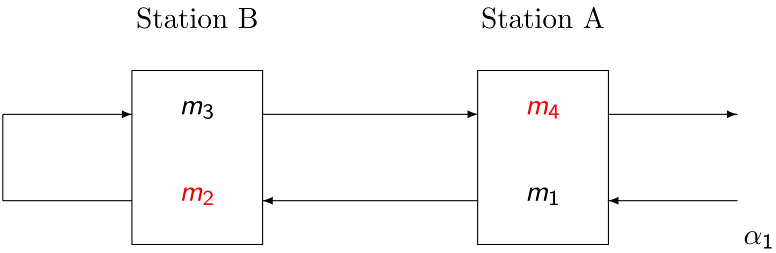


If the static-buffer-priority policy is used, the maximum throughput is

$$\min\left\{\frac{1}{m_1+m_4}, \quad \frac{1}{m_2+m_3}, \quad \frac{1}{m_2+m_4}\right\}.$$

In our example, the maximum throughput is 0.83 instead of 1.25, a 50% relative difference.

Virtual Station



Lemma (Harrison-Nguyen 95, Dumas)

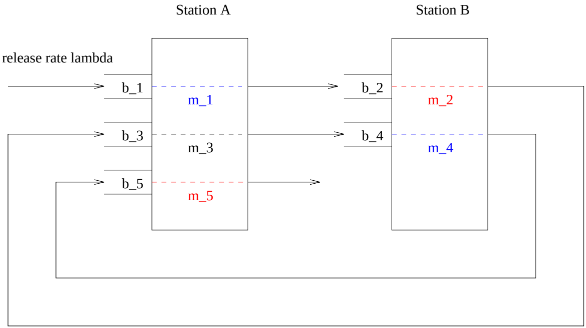
Under the operational policy,

$$Z_2(t)Z_4(t) = 0 \quad \textit{for all } t \geq 0$$

if $Z_2(0)Z_4(0) = 0$. *Thus, classes* $\{2, 4\}$ *form a virtual station.*

If $\rho_v = \alpha_1(m_2 + m_4) > 1$, with probability one, the total number of jobs goes to infinity.

Virtual Station



If the red buffers have higher priority than the blue buffers, jobs in buffer 2 and buffer 5 can *never* be processed simultaneously. Mathematically,

$$Z_2(t)Z_5(t) = 0, \quad t \geq 0 \quad \text{if} \quad Z_2(0)Z_5(0) = 0.$$

Steps 2 and 5 form a virtual station under the priority dispatch policy.

Virtual Bottleneck?

Phenomenon:

- WIP is high, and
- bottleneck machines are underutilized

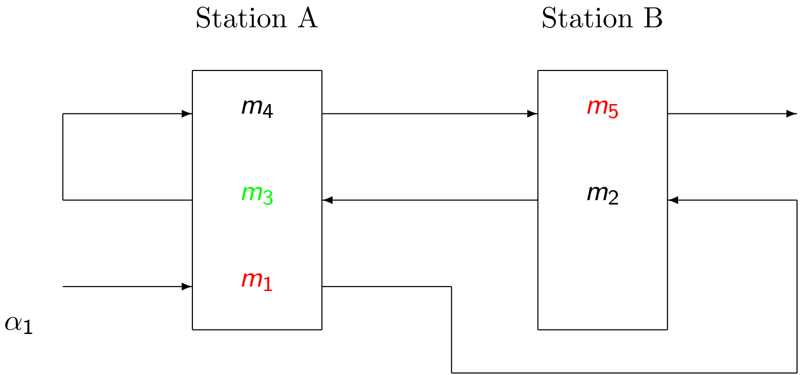
Efficient and Inefficient Policies

- Inefficient policies:
  - First-in-first-out (FIFO) (Bramson 1994, Seidman 1994)
  - Static buffer priority (Lu-Kumar 1992)
  - Shortest processing time first
  - Shortest remaining processing time first
  - Exhaustive service (Kumar-Seidman 1990)
  - ...
- Under an efficient policy, the throughput is constrained by actual machine speed.
- Many operational policies have been discovered and proved to be efficient.
- Fluid model is the main tool for proofs.

Motivation I: Power of Fluid Models

- Sufficiency: a queueing network is stable if the corresponding fluid model is stable. [Rybko-Stolyar (92), Dai (95), Stolyar (95), Dai-Meyn]
  Powerful in showing "good policies" for a stochastic network are indeed "good" . [generalized HL processor sharing, HL proportional processor sharing (Bramson), global LIFO (Rybko-Stolyar-Suhov) , global FIFO (Bramson), ...]
- Partial converses: Meyn (95), Dai (96), Rybko-Pulhaski (99)

Push Start



$$\frac{m_3}{1 - \alpha_1 m_1}$$

$$\alpha_1 \left(\frac{m_3}{1 - \alpha_1 m_1} + m_5 \right) \leq 1.$$

Motivation II: Fluid Model Stability Region Characterization

- the usual traffic conditions: $\rho_i < 1$
- virtual station conditions: $\rho_v < 1$
- push start conditions: $\rho_{\mathrm{ps}} < 1$

Dai-Vande Vate (00) for general 2-station fluid networks

Fluid Model

- deterministic, continuous analog
- defined through a set of equations
- non-unique fluid model solutions

Definition

A fluid model is said to be stable if every fluid solution model empties eventually.

Fluid Model Equations

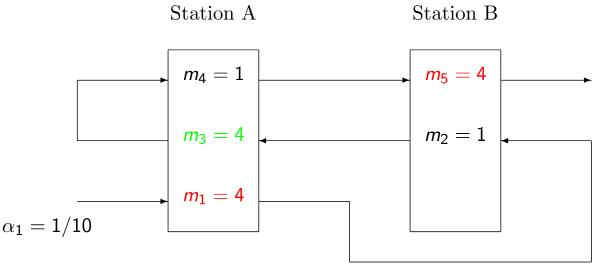
$$Z_k(t) = Z_k(0) + \mu_{k-1} T_{k-1}(t) - \mu_k T_k(t), \tag{1}$$

$$T_k(t) \text{ is nondecreasing}, \tag{2}$$

$$Z_5(t) > 0 \Rightarrow \dot{T}_5(t) = 1, \tag{3}$$

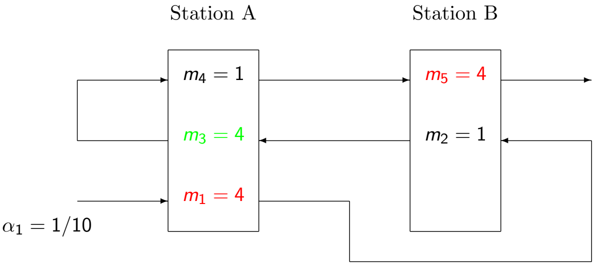
$$Z_2(t) + Z_5(t) > 0 \Rightarrow \dot{T}_2(t) + \dot{T}_5(t) = 1, \tag{4}$$

$$\ldots$$



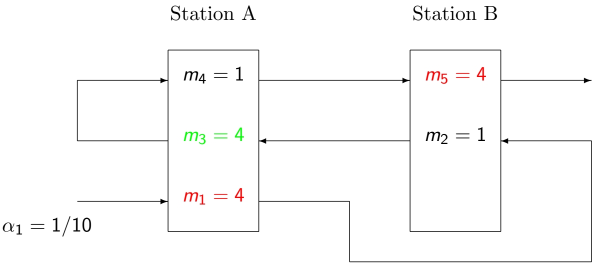
An Unstable Fluid Model Solution: Part I

Let $d_k(t) = \mu_k \dot{T}_k(t)$ be the departure rate from buffer k. Assume that $Z(0) = (0, 0, 0, 1, 0)$.
$d_5(t) = \mu_5 = 1/4$, $d_4(t) = \mu_4(1 - \alpha_1 m_1) = \mu_4(0.6) > d_5(t)$.
Buffer 2 accumulates as long as buffer 5 is non-empty. Buffer 5 empties at time $t_1 = m_5$.



Unstable Fluid Model Solution: Part II

$Z(t_1) = (0, \alpha_1 m_5, 0, 0, 0)$.
$d_2(t) = \mu_2 = 1,\ d_3(t) = \mu_3(1 - \alpha_1 m_1) = 0.28 < d_2(t)$.
Buffer 4 accumulates until buffer 3 empties.
Buffers 1-3 empty at time $t_1 + t_2$ with $t_2 = \frac{\alpha_1 m_5}{d_3(t) - \alpha_1}$.



Unstable Fluid Model Solution: Back to Part I

$Z(t_1 + t_2) = (0, 0, 0, \square, 0)$ with

$$
\begin{aligned}
\square &= \alpha_1 t_1 + \alpha_1 t_2 \\
&= \frac{\alpha_1 m_5}{1 - \frac{\alpha_1 m_3}{(1-\alpha_1 m_1)}}.
\end{aligned}
$$

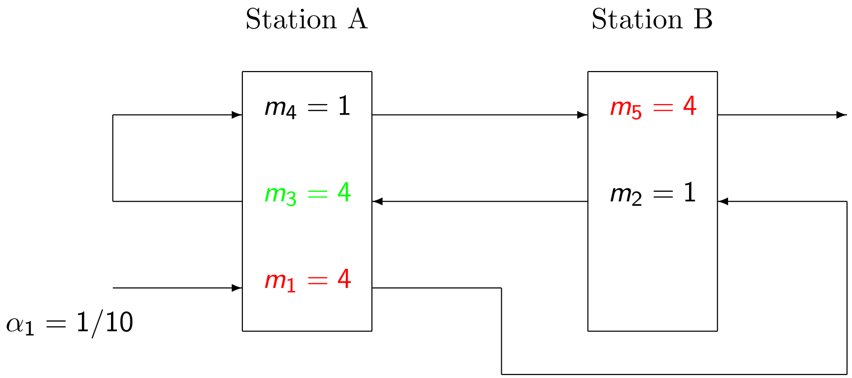
The last expression > 1 if and only if the push start condition is violated, i.e.,

$$
\rho_{\mathrm{ps}} \equiv \frac{\alpha_1 m_3}{1 - \alpha_1 m_1} + \alpha_1 m_5 = \frac{16}{15} > 1.
$$

Summary of Part II

- For a queueing network operating under a HL service policy, its stability region can depend on
  - its distributions, not just means;
  - the preemption mechanism.
- stability
  - total number of jobs being stochastically bounded
  - rate stability
  - positive recurrence

The 2-Station, 5-Class Queueing Network



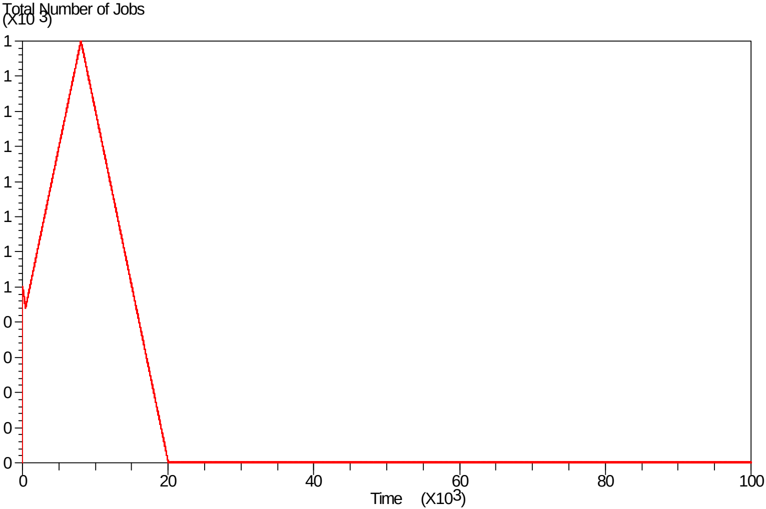
Static buffer priority (SBP) policy: $\{(1, 3, 4), (5, 2)\}$. Red buffers have the highest priority. Black buffers have the lowest priority.

$$\rho_1 = \alpha_1(m_1 + m_3 + m_4) = 0.9,$$
$$\rho_2 = \alpha_1(m_2 + m_5) = 0.5.$$

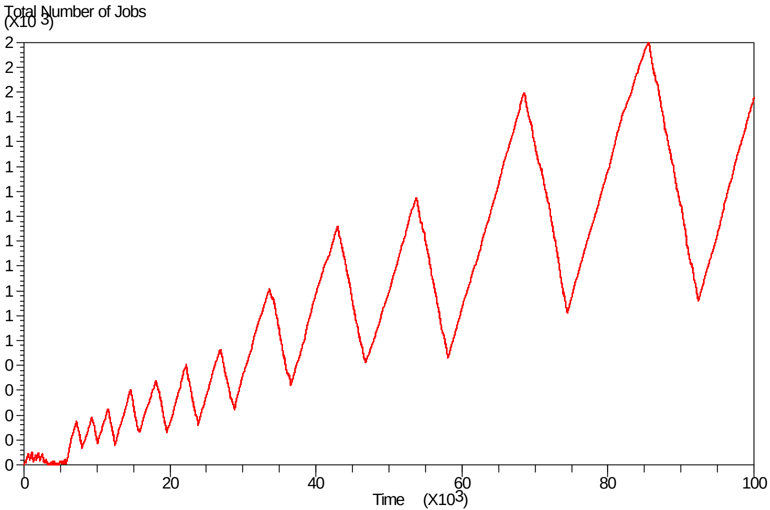
Distributions Matter

- deterministic
- exponential
- uniform with small width
- uniform with large width

$Z(0) = (100, 100, 100, 100, 100)$.

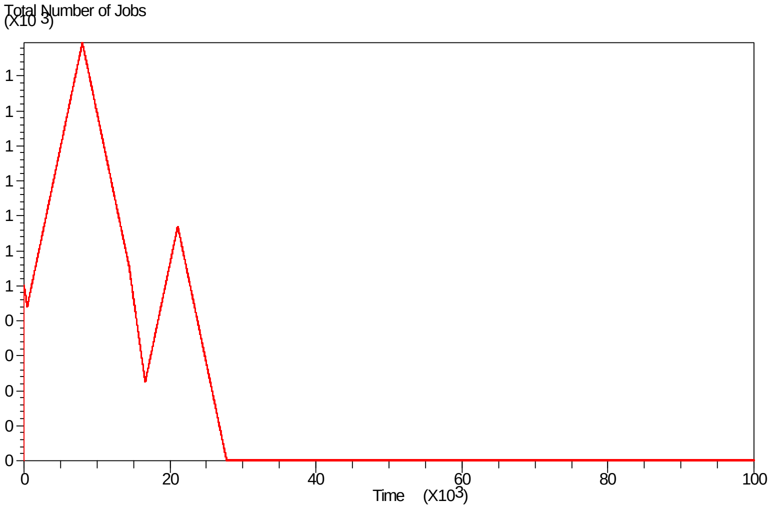


$Z(0) = 0.$


Total Number of Jobs
(X10 3)
2
2
2
1
1
1
1
1
1
1
1
1
1
0
0
0
0
0
0
20
40
60
80
100
Time (X10 3)

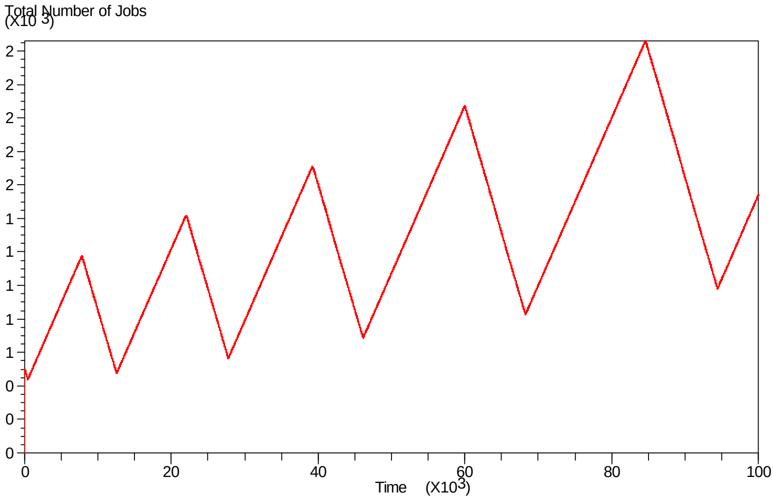
Uniform Distribution: $\epsilon = 0.01$

$Z(0) = (100, 100, 100, 100, 100)$.



Uniform Distribution: $\epsilon = 1.0$

$Z(0) = (100, 100, 100, 100, 100)$.

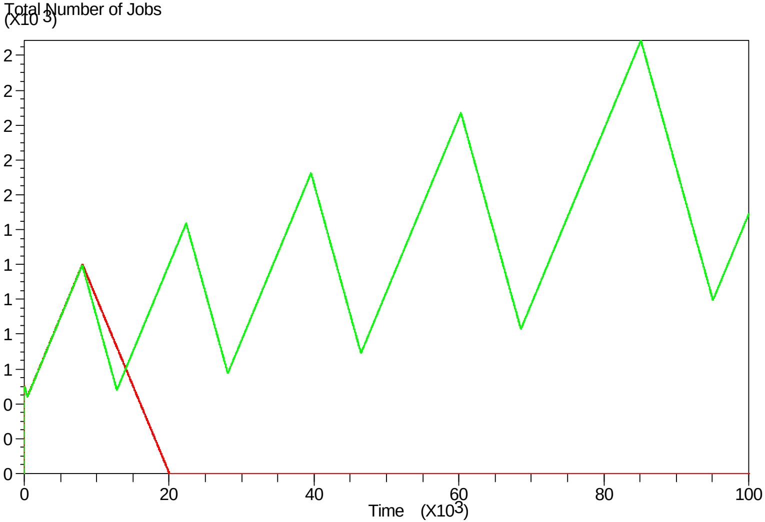


Preemption Matters

- non-preemptive
- preemptive

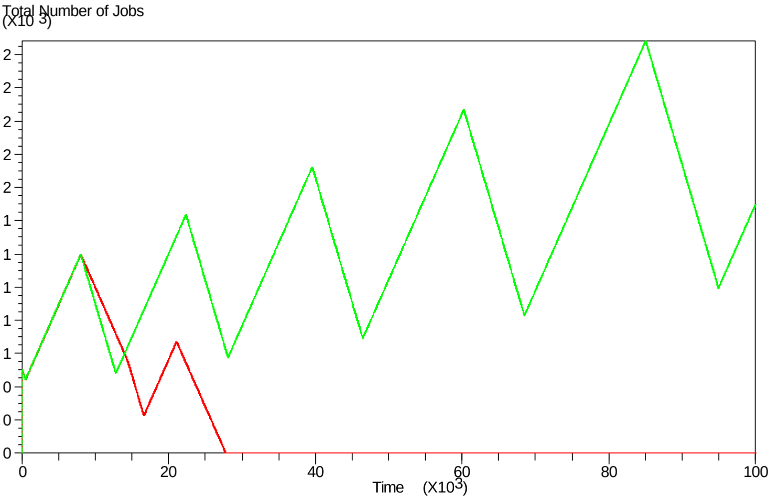
Deterministic Case

$Z(0) = (100, 100, 100, 100, 100)$.



Uniform Distribution with $\epsilon = 0.01$

$Z(0) = (100, 100, 100, 100, 100).$



Theorem 1: Instability for Exponential Case

Theorem

Assume that all distributions are exponential and the non-preemptive SBP policy is used.
Starting from any state, with probability one, the total number of jobs goes to infinity.

Theorem 2: Stability for Deterministic Case

Theorem

Assume that all distributions are deterministic, and the non-preemptive SBP policy is used.
Starting from any state, $Z(t)$ reaches a limit cycle in finite time.
Furthermore, the limit cycle is unique with at most two jobs in the system.

Theorem 3: Instability for Preemptive Case

Theorem

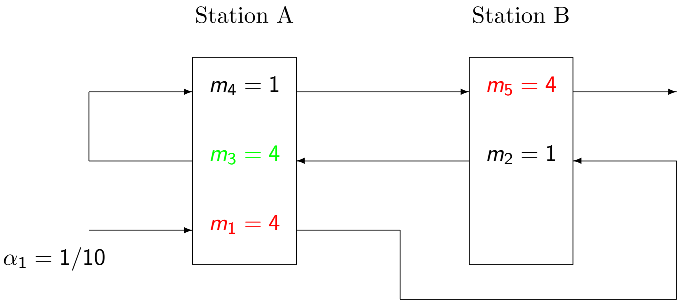
Assume that all distributions are deterministic (or random with small enough supports), and the preemptive SBP policy is used.
Starting from state $Z(0) = (0, n, 0, 0, 0)$ with large enough n, $Z(t)$ cycles to infinity as $t \to \infty$.

Proofs

Follow fluid model solutions!
But which solution?

A Stable Fluid Model Solution

- In Part II, when $Z(t_1) = (0, \alpha_1 m_5, 0, 0, 0)$, do we have to have $d_2(t) = \mu_2 = 1$, $d_3(t) = \mu_3(1 - \alpha_1 m_1) = 0.28$?
- No. One can verify that $d_2(t) = 1/(m_2 + m_5)$, and $d_5(t) = d_4(t) = d_3(t) = d_2(t)$ is another solution.



Which Solution?

- Exponential network follows the unstable fluid model solution.
- Deterministic network follows the stable fluid model solution.
- Deterministic network with preemption follows the unstable fluid model solution.

Let $X^x(t)$ be the state of a queueing network at time t with initial state x.

$$\bar{X}^{x,r}(t,\omega) = \frac{1}{r} X^x(rt,\omega)$$

If there exist sequences $r_n \to \infty$ and x_n with $\limsup_n |x_n|/r_n \leq 1$ such that as $n \to \infty$

$$\bar{X}^{x_n,r_n} \to \bar{X},$$

$\bar{X}$ is then said to be a fluid limit.

Fluid limits can be defined pathwise or distributionally.

Fluid Model v.s. Fluid Limits

- Each fluid limit is a fluid model solution.
- Which fluid model equation should one add?
- Practical fluid models should depend on means only, not on distributions.

The Fluid Model Fails

- Bramson (99): there is a stable exponential queueing network whose fluid model is unstable.
- No matter how many fluid model equations one adds, the fluid model cannot determine the stability of our queueing network.

Proof of Theorem 1: Exponential Case

Proposition. Suppose that $Z(0) = (0, z_2, 0, n, 0)$. There exist $\theta > 1$ and $\delta > 0$ such that for all large n and any z_2,

$$\mathbb{P}\{Z_4(T) \geq \theta n\} \geq 1 - \exp(-\delta\sqrt{n}),$$

where T is some random time with $Z(T) = (0, Z_2(T), 0, Z_4(T), 0)$. Furthermore,

$$\mathbb{P}\{|Z(t)| \geq \kappa n \text{ for all } t \in [0, T]\} \geq 1 - \exp(-\delta\sqrt{n}).$$

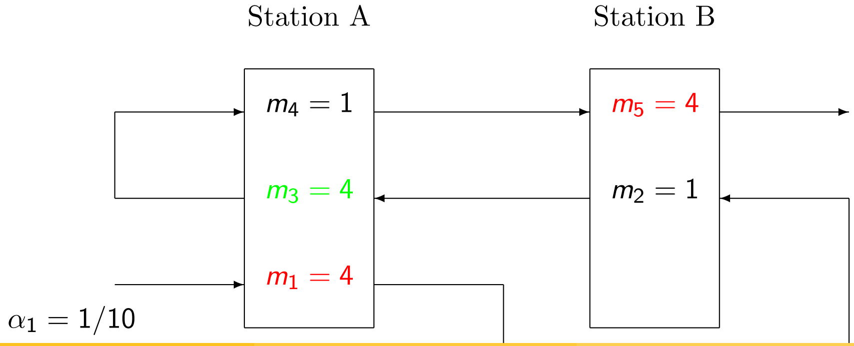
Follows the unstable fluid model solution with high probability!

Proof of Theorem 2: Deterministic without Preemption

State $(0, 0, 0, 1, 0; 1)$ starts a period.

$$Z(0) = (0, 0, 0, n, 0; 1), \qquad Z(1) = (1, 0, 0, n-1, 1; 10),$$
$$Z(5) = (0, 1, 0, n-1, 0; 6), \qquad Z(6) = (0, 0, 1, n-2, 1; 5),$$
$$Z(10) = (0, 0, 0, n-1, 0; 1).$$

Follows the stable fluid model solution.



Proof of Theorem 3: Almost Deterministic with Preemption

With probability one, follow the unstable fluid model solution.

Summary

- For a queueing network operating under some service policy, its stability region can depend on
  - its distributions,
  - the preemption mechanism,
  - the way that simultaneous events are handled.
- Practical fluid models cannot capture these fine factors, and hence cannot be used to sharply determine stability of the corresponding queueing network.
- Fluid model is useful in designing and verifying good policies.
- Fluid model may still be possible to determine sharply the global stability of a queueing network.

References

- Dai, Hasenbein and VandeVate, Stability and instability of a two-station queueing network, *Annals of Applied Probability*, 2004.
- Dai, Hasenbein and VandeVate, Stability of a Three-Station Fluid Network, *Queueing Systems*, 1999
- Bramson, A Stable Queueing network with unstable fluid network, *Annals of Applied Probability*, 1999.