Reflecting Brownian motion in three dimensions: A new proof of sufficient conditions for positive recurrence

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Abstract

Let $Z = \{Z(t), t \geq 0\}$ be a semimartingale reflecting Brownian motion that lives in the three-dimensional non-negative orthant. A 2002 paper by El Kharroubi, Ben Tahar and Yaacoubi gave sufficient conditions for positive recurrence of Z. Recently Bramson, Dai and Harrison have shown that those conditions are also necessary for positive recurrence. In this paper we provide an alternative proof of sufficiency, the salient feature of which is its use of a linear Lyapunov function.

Keywords: reflecting Brownian motion, Skorohod problem, fluid model, positive recurrence, queueing networks, heavy traffic, diffusion approximation

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1 Introduction

This paper considers a problem related to the recurrence classification of semimartingale reflecting Brownian motions (SRBMs). We will eventually focus on the three-dimensional case, but it is useful at the outset to consider a general SRBM $Z = \{Z(t), t \geq 0\}$ with state space $\mathbb{R}^d_+$. The data of the process Z are a drift vector $\theta \in \mathbb{R}^d$, a non-singular $d \times d$ covariance matrix Σ , and a $d \times d$ matrix R that specifies boundary behavior. In the interior of the orthant Z behaves as an ordinary Brownian motion with parameters θ and Σ, and roughly speaking, Z is pushed in direction R^j whenever the boundary $\{z \in \mathbb{R}^d_+ : z_j = 0\}$ is struck, where R^j is the jth column of R.

For reasons that will become clear in Section 2, it is not necessary for our purposes to state the precise definition of SRBM, which is simple in its essence but technical in its details. However, to articulate a key restriction on the process data, the following definition is indispensable. (Here and later, for a vector v, we write $v > 0$ to mean that each component of v is positive, and we write $v \geq 0$ to mean that each component of v is non-negative.)

Definition 1. A $d \times d$ matrix R is said to be an $\mathcal{S}$*-matrix* if there exists a d-vector $w \geq 0$ such that $Rw > 0$ (or equivalently, if there exists $w > 0$ such that $Rw > 0$), and R is said to be *completely*-$\mathcal{S}$ if each of its principal sub-matrices is an $\mathcal{S}$-matrix.

The definition of an $\mathcal{S}$-matrix appears on page 140 of Cottle et al. (1992), and throughout that book the prefix "completely" is used to indicate that a matrix property is shared by all principal submatrices. Theorem 3.10.7 of Cottle et al. (1992) gives a number of equivalent ways to characterize completely-$\mathcal{S}$ matrices.

The definition of SRBM is reviewed in Appendix A of Bramson et al. (2009), and in the survey paper by Williams (1995), which also recapitulates the following foundational result: there exists an SRBM Z with data (θ, Σ, R) if and only if R is completely-$\mathcal{S}$, in which case Z is unique in distribution. We say that Z is positive recurrent if the expected time to hit an open neighborhood of the origin is finite for every starting state; the words "stable" and "stability" will occasionally be used as synonyms for "positive recurrent" and "positive recurrence," respectively.

The next section lays out a set of conditions involving θ and R that are necessary and sufficient for positive recurrence of Z when $d = 3$. The sufficiency of those conditions was established by El Kharroubi et al. (2002), and the necessity by Bramson et al. (2009) However, the proof of sufficiency by El Kharroubi et al. (2002) is not entirely rigorous, involving verbal passages that mask significant technical difficulties; see comments immediately following Definition 2 below. In this paper we provide an alternative proof of sufficiency, the salient feature of which is its use of a linear Lyapunov function, a proof technique that was originally promulgated by Chen (1996) in the context of SRBMs. Readers are referred to Bramson et al. (2009) for a more complete discussion of the problem context, and a more extensive compilation of references.

The rest of the paper is structured as follows. First, Section 2 introduces the important notion of a fluid path associated with the process data (θ, R), allowing a general value of d. There we also lay out the stability conditions identified by El Kharroubi et al. (2002) when $d = 3$, and state the main result to be proved. Section 3 is devoted to construction of linear Lyapunov functions in a number of distinct cases, and then the main result is proved in Section 4, again through consideration of different cases.

2 Fluid paths and the main result

Let $\theta \in \mathbb{R}^d$ and let R be a $d \times d$ completely-$\mathcal{S}$ matrix. In the following definition, (2.4) says that $y_j(\cdot)$ only increases when $z_j(\cdot) = 0$.

Definition 2. A fluid path associated with the data (θ, R) is a pair of continuous functions $y, z : [0, \infty) \to \mathbb{R}^d$ that satisfy the following conditions:

$$z(t) = z(0) + \theta t + Ry(t) \text{ for all } t \geq 0, \tag{2.1}$$

$$z(t) \in S \text{ for all } t \geq 0, \tag{2.2}$$

$$y(\cdot) \text{ is continuous and nondecreasing with } y(0) = 0, \tag{2.3}$$

$$\int_0^\infty z_j(t)\, dy_j(t) = 0, \quad (j = 1, \dots, d). \tag{2.4}$$

A significant source of difficulty in fluid-based arguments is that the fluid path emanating from a given initial state $z(0)$ need not be unique. Specifically, if $d = 3$ and $z(0)$ has exactly one positive element, there may be uncountably many fluid paths emanating from $z(0)$, even over a small initial time interval $0 \leq t \leq \epsilon$, or there may be just one, depending on the structure of θ and R. In our view, El Kharroubi et al. (2002) are not sufficiently attentive to questions of fluid path uniqueness, as in the first six lines on page 254 of their paper; also, phrases like "two consecutive times of face change" (at the top of page 256) implicitly assume that fluid paths are piecewise linear with just finitely many pieces in any finite time span, but the authors do not actually establish that property.

Definition 3. We say that a fluid path (y, z) is *attracted to the origin* if $z(t) \to 0$ as $t \to \infty$.

Dupuis and Williams (1994) showed that if all fluid paths associated with (θ, R) are attracted to the origin, and if Σ is a non-singular $d \times d$ covariance matrix, then the SRBM with data (θ, Σ, R) is positive recurrent. Armed with that result, we can and will concern ourselves only with the behavior of fluid paths. That is, it will suffice to show that under the conditions identified by El Kharroubi et al. (2002), all fluid paths are attracted to the origin. To state those conditions, a number of preliminary definitions are needed.

Definition 4. A fluid path (y, z) is said to be *linear* if it has the form $y(t) = ut$ and $z(t) = vt$, $t \geq 0$, where $u, v \geq 0$.

Linear fluid paths are in one-to-one correspondence with solutions of the following linear complementarity problem (LCP): Find vectors $u = (u_i)$ and $v = (v_i)$ in $\mathbb{R}^d$ such that

$$u, v \geq 0, \tag{2.5}$$

$$v = \theta + Ru, \tag{2.6}$$

$$u \cdot v = 0, \tag{2.7}$$

where $u \cdot v = \sum_i u_i v_i$ is the inner product of u and v. See Cottle et al. (1992) for a systematic account of the theory associated with the general problem (2.5)-(2.7).

Definition 5. A solution (u, v) of the LCP (2.5)-(2.7) is said to be *stable* if $v = 0$ and to be *divergent* otherwise.

As mentioned above, we study fluid paths primarily to understand the recurrence of a corresponding SRBM. It is proved in Appendix C of Bramson et al. (2009) that a necessary condition for the positive recurrence of the SRBM is the following:

$$R \text{ is non-singular and } R^{-1}\theta < 0. \tag{2.8}$$

If (2.8) holds, then clearly $(u^*, 0)$ is the unique stable solution of the LCP, where

$$u^* = -R^{-1}\theta > 0. \tag{2.9}$$

Hereafter we specialize to dimension $d = 3$. Let C_1 be the set of (θ, R) pairs that satisfy

$$\theta < 0, \tag{2.10}$$

$$\theta_1 > \theta_2 R_{12} \quad \text{and} \quad \theta_3 < \theta_2 R_{32}, \tag{2.11}$$

$$\theta_2 > \theta_3 R_{23} \quad \text{and} \quad \theta_1 < \theta_3 R_{13}, \tag{2.12}$$

$$\theta_3 > \theta_1 R_{31} \quad \text{and} \quad \theta_2 < \theta_1 R_{21}, \tag{2.13}$$

and for $(\theta, R) \in C_1$ define

$$\beta_1(\theta, R) = \left(\frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3}\right)\left(\frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1}\right)\left(\frac{\theta_3 - \theta_1 R_{31}}{\theta_1 R_{21} - \theta_2}\right) > 0. \tag{2.14}$$

If (2.10)-(2.13) hold, then, starting from any point on the boundary of $\mathbb{R}^3_+$, there is a piecewise linear fluid path that spirals on the boundary, and $\beta_1(\theta, R)$ is the *single-cycle gain* for that path, cf. Section 3 of Bramson et al. (2009). Figure 1 pictures such a fluid path generated by

$$\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}.$$

and initial state $z(0) = (0, 0, \kappa)'$. The corresponding single-cycle gain is $\beta_1(\theta, R) = 8$.

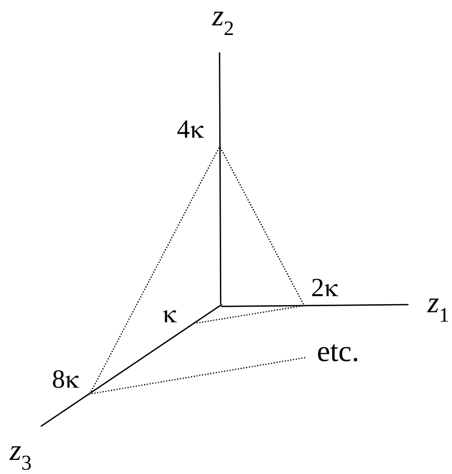

Figure 1: A fluid path that spirals on the boundary

Now let C_2 consist of all (θ, R) pairs that satisfy (2.10) and further satisfy (2.11) through (2.13) *with all six of the inequalities reversed*, and for $(\theta, R) \in C_2$ define

$$\beta_2(\theta, R) = \frac{1}{\beta_1(\theta, R)} = \left(\frac{\theta_3 - \theta_2 R_{32}}{\theta_2 R_{12} - \theta_1}\right)\left(\frac{\theta_1 - \theta_3 R_{13}}{\theta_3 R_{23} - \theta_2}\right)\left(\frac{\theta_2 - \theta_1 R_{21}}{\theta_1 R_{31} - \theta_3}\right) > 0. \tag{2.15}$$

Then C_2 consists of (θ, R) pairs giving rise to *clockwise* spiral fluid paths, and $\beta_2(\theta, R)$ is the single-cycle gain for such paths. Finally, as in Section 3 of Bramson et al. (2009), let $C = C_1 \cup C_2$ and define $\beta(\theta, R) = \beta_1(\theta, R)$ for $(\theta, R) \in C_1$ and $\beta(\theta, R) = \beta_2(\theta, R)$ for $(\theta, R) \in C_2$. Our goal in this note is to prove the following.

Theorem 1. *Assume that R is completely-$\mathcal{S}$, that* (2.8) *holds, and that one of the following additional hypothesises is satisfied: (a) $(\theta, R) \in C$ and $\beta(\theta, R) < 1$; or (b) $(\theta, R) \notin C$ and the unique solution of the LCP (2.5)-(2.7) is the stable solution $(u^*, 0)$ given by* (2.9)*. Then each fluid path associated with (θ, R) is attracted to the origin.*

Theorem 1 will be proved in the next two sections, throughout which (2.8) is assumed to hold. Hereafter we also assume, without loss of generality, that the problem data (θ, R) have been normalized to satisfy the following conditions, cf. Appendix B of Bramson et al. (2009):

$$R_{ii} = 1 \text{ for } i = 1, 2, 3, \tag{2.16}$$
$$\theta_i \in \{-1, 0, 1\} \quad \text{for } i = 1, 2, 3. \tag{2.17}$$

3 Linear Lyapunov function

This section proves the following lemma.

Lemma 1. *Under the hypotheses of Theorem 1, there exists* $h = (h_1, h_2, h_3) > 0$ *that satisfies all the following inequalities:*

$$h_1(\theta_1 - R_{13}\theta_3) + h_2(\theta_2 - R_{23}\theta_3) < 0 \textit{ if } \theta_3 \le 0, \tag{3.1}$$

$$h_2(\theta_2 - R_{21}\theta_1) + h_3(\theta_3 - R_{31}\theta_1) < 0 \textit{ if } \theta_1 \le 0, \tag{3.2}$$

$$h_1(\theta_1 - R_{12}\theta_2) + h_3(\theta_3 - R_{32}\theta_2) < 0 \textit{ if } \theta_2 \le 0, \tag{3.3}$$

$$h_1\theta_1 + h_2\theta_2 + h_3\theta_3 < 0. \tag{3.4}$$

The vector h will be used in Section 4 to define a linear Lyapunov function that is the key to our proof of Theorem 1. As a preliminary to the proof of Lemma 1, let

$$V_{ij} = \theta_i - R_{ij}\theta_j \text{ for } i, j = 1, 2, 3. \tag{3.5}$$

Thus, V is a 3×3 matrix with zeros on the diagonal.

Lemma 2. *Assume that (2.8) holds and that the unique solution of the LCP (2.5)-(2.7) is the stable solution* $(u^*, 0)$ *given by* (2.9)*. Then,* $\theta_i \le 0$ *implies that the* i*th column of* V *has at least one negative element,* $i = 1, 2, 3$*.*

Proof. Arguing by contradiction, suppose that $\theta_1 \le 0$ and that both $V_{21} \ge 0$ and $V_{31} \ge 0$. Defining $u = (-\theta_1, 0, 0)'$ and $v = (0, V_{21}, V_{31})'$, it follows that (u, v) is a solution of (2.5)-(2.7), and because u has two zero components, it is distinct from the vector u^* in (2.9), which contradicts the hypotheses of the lemma to be proved. This establishes the desired conclusion for $i = 1$, and it is established for $i = 2$ and $i = 3$ in exactly the same way. □

Lemma 3. *Assume that the hypotheses of Lemma 2 are satisfied, that* $\theta = (-1, -1, -1)'$*, that* $(\theta, R) \notin C$*, and that none of the off-diagonal elements of* V *is zero. Then*

$$\textit{at least one row of } V \textit{ has both off-diagonal elements negative.} \tag{3.6}$$

Proof. Arguing by contradiction, suppose (3.6) does *not* hold. It will be shown that $(\theta, R) \in C$, which establishes the desired conclusion. We know from Lemma 2 that the first column of V has at least one negative element; suppose for concreteness that $V_{21} < 0$. Then $V_{23} > 0$, because (3.6) is assumed not to hold. Then $V_{13} < 0$ by Lemma 2, implying that $V_{12} > 0$ because (3.6) is assumed not to hold. In exactly the same way, $V_{32} < 0$ by Lemma 2, implying that $V_{31} > 0$. Thus we have

$$V_{12} > 0, \quad V_{23} > 0, \quad V_{31} > 0,$$
$$V_{21} < 0, \quad V_{13} < 0, \quad V_{32} < 0,$$

which is precisely the condition that $(\theta, R) \in C_1$. One can prove similarly that if $V_{21} > 0$, then $(\theta, R) \in C_2$. □

Proof of Lemma 1. Recall that we assume the conventions (2.16) and (2.17). Also, we assume (2.8) holds. Suppose first that condition (a) of Theorem 1 holds. In this case, $\theta = (-1, -1, -1)'$ by convention (2.17), and conditions (3.1)-(3.3) are equivalent to finding an $h = (h_1, h_2, h_3) > 0$ such that

$$hV < 0. \tag{3.7}$$

Assume $(\theta, R) \in C_1$ and $\beta_1(\theta, R) < 1$. The following proof of (3.7) is a simple modification of the proof of Lemma 2 in Bramson et al. (2009). From (2.17) we have that

$$V = \begin{pmatrix} 0 & a_2 & -b_3 \\ -b_1 & 0 & a_3 \\ a_1 & -b_2 & 0 \end{pmatrix}, \tag{3.8}$$

where $a_i, b_i > 0$ for $i = 1, 2, 3$. In this notation, the definition (2.14) is as follows:

$$\beta_1(\theta, R) = \frac{a_1 a_2 a_3}{b_1 b_2 b_3} < 1. \tag{3.9}$$

Setting

$$h_1 = \alpha_1, \quad h_2 = \frac{a_1 a_2}{b_1 b_2} \quad \text{and} \quad h_3 = \alpha_3 \frac{a_2}{b_2},$$

for some constants $\alpha_1 > 0$ and $\alpha_3 > 0$ that satisfy

$$\beta_1(\theta, R) < \alpha_1 < \alpha_3 < 1,$$

it is easy to verify that $hV^1 = -(1-\alpha_3)a_1 a_2/b_2 < 0$, $hV^2 = -(\alpha_3 - \alpha_1)a_2 < 0$, and $hV^3 = -b_3\big(\alpha_1 - a_1 a_2 a_3/(b_1 b_2 b_3)\big) < 0$. Similarly, when $(\theta, R) \in C_2$ and $\beta_2(\theta, R) < 1$, we can find $h > 0$ such that (3.7) is satisfied.

In the rest of the proof, we assume that condition (b) of Theorem 1 is satisfied. Then at least one component of θ must be -1, because otherwise the LCP (2.5)-(2.7) would have a solution (u, v) that is distinct from $(u^*, 0)$, where u^* is given in (2.9). We separate the proof of the lemma into three categories: in Category I, exactly one component of θ is 1; in Category II, exactly two components of θ are 1; in Category III, none of the components of θ is 1.

Category I. Exactly one component of θ_i is equal to 1. We separate this category into two cases: (a) among the remaining two components of θ, one component is -1 and the other is 0, and (b) the other two components of θ are both -1.

Case Ia. Assume without loss of generality that $\theta = (-1, 0, 1)'$. Constraint (3.1) does not apply. Setting $h_3 = 1$, constraints (3.3) and (3.4) are satisfied for any $h_1 > 1$. By Lemma 2, either $\theta_2 - R_{21}\theta_1 < 0$ or $\theta_3 - R_{31}\theta_1 < 0$. If $\theta_2 - R_{21}\theta_1 < 0$, one can choose h_2 large enough so that (3.2) is satisfied. If $\theta_3 - R_{31}\theta_1 < 0$, one can choose h_2 small enough so that (3.2) is satisfied.

Case Ib. This turns out to be the most difficult case to prove. Without loss of generality, we assume that $\theta = (-1, 1, -1)'$. In this case, the constraint (3.3) does not apply. Setting $h_2 = 1$, we now argue that there exists $(h_1, h_3) > 0$ that satisfies (3.1), (3.2) and (3.4). Using the specific values of θ and h_2, these constraints become

$$h_1(R_{13} - 1) + (R_{23} + 1) < 0, \tag{3.10}$$

$$(R_{21} + 1) + h_3(R_{31} - 1) < 0, \tag{3.11}$$

$$h_1 + h_3 > 1. \tag{3.12}$$

We now show that there is a pair $(h_1, h_3) > 0$ that satisfies (3.10)-(3.12), thus proving the lemma for Case Ib.

We first assume $R_{13} - 1 < 0$. In this case one can choose h_1 large enough to satisfy (3.10) and (3.12) for any $h_3 > 0$. By Lemma 2, either $R_{21} + 1 < 0$ or $R_{31} - 1 < 0$. If $R_{21} + 1 < 0$, one can choose h_3 small enough to satisfy (3.11). If $R_{31} - 1 < 0$, one can choose h_3 large enough to satisfy (3.11). Thus there exists $(h_1, h_3) > 0$ that satisfies (3.10)-(3.12).

Next we assume $R_{13} - 1 = 0$. Then $R_{23} + 1 < 0$ by Lemma 2. Thus (3.10) is satisfied for any $h_1 > 0$. Setting $h_1 = 2$, one sees that (3.12) is satisfied for any $h_3 > 0$. Now we can choose $h_3 > 0$ small enough or large enough to satisfy (3.11), just as in the case where $R_{13} - 1 < 0$. Thus there exists $(h_1, h_3) > 0$ that satisfies (3.10)-(3.12).

Similarly, one can argue that if $R_{31} - 1 \le 0$, there exists $(h_1, h_3) > 0$ that satisfies (3.10)-(3.12).

Finally, we assume that $R_{13} - 1 > 0$ and $R_{31} - 1 > 0$. By Lemma 2, $R_{23} + 1 < 0$ and $R_{21} + 1 < 0$. Note that there exists an $(h_1, h_3) > 0$ satisfying (3.10)-(3.12) if and only if

$$-\frac{R_{23} + 1}{R_{13} - 1} - \frac{R_{21} + 1}{R_{31} - 1} > 1. \tag{3.13}$$

or equivalently

$$-(R_{23}R_{31} + R_{21}R_{13}) + R_{23} + R_{21} + 1 > R_{13}R_{31}. \tag{3.14}$$

On the other hand, let

$$\hat{R} = \begin{pmatrix} 1 & R_{13} \\ R_{31} & 1 \end{pmatrix}, \quad \hat{\theta} = (\theta_1, \theta_3)', \quad \hat{u} = -(\hat{R})^{-1}\hat{\theta},$$

and

$$\beta = \theta_2 + R_{21}\hat{u}_1 + R_{23}\hat{u}_3. \tag{3.15}$$

It must be true that $\beta < 0$, because otherwise the LCP (2.5)-(2.7) would have a solution (u, v) with $u = (\hat{u}_1, 0, \hat{u}_3)'$ and $v = (0, \beta, 0)'$, so (u, v) would be distinct from the unique stable solution $(u^*, 0)$. After deriving an explicit expression for $\hat{u}$, the condition $\beta < 0$ becomes

$$1 + \frac{1}{R_{31}R_{13} - 1}\Big(R_{21}(R_{13} - 1) + R_{23}(R_{31} - 1)\Big) < 0,$$

which is equivalent to

$$R_{31}R_{13} + R_{21}(R_{13} - 1) + R_{23}(R_{31} - 1) < 1. \tag{3.16}$$

We have already established that $\beta < 0$, so (3.16) is satisfied, which implies that (3.14) holds. Therefore, there exists $(h_1, h_3) > 0$ that satisfies (3.10)- (3.12).

Category II. Exactly two components of θ are 1 and the other component is -1. Without loss of generality, assume $\theta = (-1, 1, 1)'$. Setting $h_1 = 1$, constraints (3.1)-(3.4) become

$$h_2(R_{21} + 1) + h_3(R_{31} + 1) < 0, \tag{3.17}$$

$$h_2 + h_3 < 1. \tag{3.18}$$

Because either $(R_{21} + 1) < 0$ or $(R_{31} + 1) < 0$, there exists $(h_2, h_3) > 0$ satisfying (3.17)-(3.18).

Category III. No component of θ is 1. We separate this category into three cases: (a) exactly one component is -1, (b) exactly two components are -1, and (c) all three component are -1.

Case IIIa. Assume without loss of generality that $\theta = (0, 0, -1)'$. Clearly, (3.4) is satisfied for any $(h_i) > 0$. Constraints (3.2) and (3.3) become $h_3(-1) < 0$, which is satisfied for any $h_3 > 0$. By Lemma 2, either $\theta_1 - R_{13}\theta_3 < 0$ or $\theta_2 - R_{23}\theta_3 < 0$. Thus there exists $(h_1, h_2) > 0$ that satisfies (3.1). Therefore, we have $(h_1, h_2, h_3) > 0$ that satisfies (3.1)-(3.4).

Case IIIb. Assume without loss of generality that $\theta = (0, -1, -1)'$. Again it is clear that (3.4) is satisfied for any $(h_i) > 0$. Constraint (3.2) becomes $-h_2 - h_3 < 0$, which is satisfied for any $(h_2, h_3) > 0$. Set $h_1 = 1$. If $\theta_1 - R_{13}\theta_3 < 0$, one can choose h_2 small enough to satisfy (3.1). Otherwise, $\theta_2 - R_{23}\theta_3 < 0$ by Lemma 2, and thus one can choose h_2 large enough to satisfy (3.1). Similarly, one can prove that there exists $h_3 > 0$ that satisfies (3.3). Thus, we have $(h_1, h_2, h_3) > 0$ that satisfies (3.1)-(3.4).

Case IIIc. In this case $\theta = (-1, -1, -1)'$. Clearly, (3.4) is satisfied for any $(h_i) > 0$, and (3.1)-(3.3) become

$$h_1V_{13} + h_2V_{23} < 0, \tag{3.19}$$

$$h_2V_{21} + h_3V_{31} < 0, \tag{3.20}$$

$$h_1V_{12} + h_3V_{32} < 0. \tag{3.21}$$

We first prove that if one off-diagonal element of V is zero, there exists an $h = (h_1, h_2, h_3) > 0$ that satisfies (3.19)-(3.21). Assume, for example, that $V_{12} = 0$. By Lemma 2, $V_{32} < 0$. Thus (3.21) is satisfied for any $h_1 > 0$ and $h_3 > 0$. We take $h_2 = 1$. If $V_{21} < 0$, one can choose h_3 small enough so that (3.20) is satisfied. Otherwise, it must be true that $V_{31} < 0$ by Lemma 2. One can choose h_3 large enough so that (3.20) is satisfied. It follows similarly that one can choose h_1 so that (3.19) is satisfied.

Now we assume that none of the off-diagonal elements of V is zero. By Lemma 3, there exists at least one row whose off-diagonal elements are both negative. We assume for concreteness that the first row of V has both off-diagonal elements negative. That is,

$$V_{12} < 0 \quad \text{and} \quad V_{13} < 0. \tag{3.22}$$

Fix $h_2 = 1$. If $V_{21} < 0$ we can choose h_3 small enough so that (3.20) is satisfied. Otherwise, it must be true that $V_{31} < 0$ by Lemma 2. Thus we can choose h_3 large enough so that (3.20) is satisfied. Because (3.22) is assumed to hold, one can choose h_1 large enough so that (3.19) and (3.21) are satisfied. Therefore, we have found $(h_i) > 0$ that satisfies (3.19)-(3.21).

□

4 Proof of Theorem 1

Let (y, z) be a fluid path. We would like to show that there exists a $t_0 \geq 0$ such that $z(t) = 0$ for $t \geq t_0$. From Lemma 1, there exists $(h_1, h_2, h_3) > 0$ that satisfies (3.1)-(3.4). Let

$$f(t) = h_1 z_1(t) + h_2 z_2(t) + h_3 z_3(t)$$

for $t \geq 0$. Clearly, $f(t) \geq 0$; furthermore, $f(t) > 0$ when $z(t) \neq 0$. The function f is known as a Lyapunov function that is linear in the state variable $z(t)$. We would like to show that there exists $t_0 \geq 0$ such that

$$f(t) = 0 \text{ for } t \geq t_0, \tag{4.1}$$

which proves the theorem.

To prove (4.1), we work with derivatives of f. Clearly, $\dot{f}(t) = h_1 \dot{z}_1(t) + h_2 \dot{z}_2(t) + h_3 \dot{z}_3(t)$ when the derivative $\dot{z}(t)$ exists. However, $\dot{z}(\cdot)$ does not always exist, so one needs to be careful in dealing with $\dot{f}(t)$.

Because θt, viewed as a function of t, is Lipschitz continuous, it follows from the oscillation inequality (cf. Taylor and Williams (1993)) that (y, z) is also Lipschitz in t. In particular, (y, z) is absolutely continuous and hence it has derivatives for almost every time $t \in (0, \infty)$ with respect Lebesgue measure.

A time $t > 0$ is said to be a *regular point* for (y, z) if (y, z) is differentiable at time t. At a regular point t, we use $(\dot{y}(t), \dot{z}(t))$ to denote the derivative of (y, z). Note that $z_i(t) \geq 0$.

When $z_i(t) = 0$ and t is a regular point, it is necessarily true that $\dot{z}_i(t) = 0$. Also, because y_i is nondecreasing, $\dot{y}_i(t) \geq 0$ for a regular point t. It follows from condition (2.4) that $z_i(t) > 0$ implies $\dot{y}_i(t) = 0$ when t is a regular point. At a regular point t, we have from (2.1) that

$$\dot{z}(t) = \theta + R\dot{y}(t). \tag{4.2}$$

To prove (4.1), it suffices to prove that there exists $\epsilon > 0$ such that

$$\dot{f}(t) \leq -\epsilon \text{ for each regular } t \text{ with } z(t) \neq 0. \tag{4.3}$$

The rest of this section is devoted to proving (4.3) by examining three separate cases, depending on the signs of $(z_1(t), z_2(t), z_3(t))$.

Case 1: All three components of $z(t)$ are positive. In this case, $\dot{y}_1(t) = \dot{y}_2(t) = \dot{y}_3(t) = 0$. Therefore, $\dot{z}_i(t) = \theta_i$ for $i = 1, 2, 3$. Thus,

$$\dot{f}(t) = h_1\theta_1 + h_2\theta_2 + h_3\theta_3,$$

which is negative by (3.4).

Case 2: Exactly two components of $z(t)$ are positive. Without loss of generality, assume that $z_1(t) = 0$, $z_2(t) > 0$, $z_3(t) > 0$. In this case, $\dot{y}_2(t) = \dot{y}_3(t) = 0$ and $\dot{y}_1(t) = -\theta_1 \geq 0$. Thus

$$\dot{z}_2(t) = \theta_2 - R_{21}\theta_1,$$
$$\dot{z}_3(t) = \theta_3 - R_{31}\theta_1.$$

Therefore,

$$\dot{f}(t) = h_2(\theta_2 - R_{21}\theta_1) + h_3(\theta_3 - R_{31}\theta_1),$$

which is negative by (3.2).

Case 3: Exactly one component of $z(t)$ is positive. Without loss of generality, assume that $z_1(t) > 0$, $z_2(t) = z_3(t) = 0$. Then $\dot{y}_1(t) = 0$, and $\dot{z}_2(t) = \dot{z}_3(t) = 0$. Let

$$\tilde{R} = \begin{pmatrix} 1 & R_{23} \\ R_{32} & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\theta} = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}.$$

Also, define $\tilde{u} = (\dot{y}_1(t), \dot{y}_2(t))'$. Combining (4.2) with our current hypotheses, one has

$$\tilde{\theta} + \tilde{R}\tilde{u} = 0 \text{ where } \tilde{u} \geq 0. \tag{4.4}$$

In the case where condition (a) of Theorem 1 is satisfied, one has $\tilde{\theta} = (-1, -1)'$, and one of the following holds, depending on whether $(\theta, R) \in C_1$ or $(\theta, R) \in C_2$: either $R_{23} > 1$ and $R_{32} < 1$, or else $R_{23} < 1$ and $R_{32} > 1$. In either of these scenarios readers can easily verify that (4.4) is impossible, so hereafter we restrict attention to the case where condition (b) of Theorem 1 is satisfied.

(i) Assume that $\det(\tilde{R}) \neq 0$. It follows from (4.2) that $\tilde{u} = -\tilde{R}^{-1}\tilde{\theta}$ and hence

$$\dot{z}_1(t) = \theta_1 - (R_{12}, R_{13})\tilde{R}^{-1}\tilde{\theta} \equiv \beta_1.$$

Now it must be true that $\beta_1 < 0$, because otherwise $(\dot{y}(t), \dot{z}(t))$ would be a solution of the LCP (2.5)-(2.7) that is different from the stable solution $(u^*, 0)$, contradicting condition (b) of Theorem 1. Thus in this case

$$\dot{f}(t) = h_1\beta_1 < 0.$$

(ii) Assume $\det(\tilde{R}) = 0$. Then $R_{23} = R_{32} = 1$. It follows from (4.2) that $\dot{y}_2(t)+\dot{y}_3(t) = -\theta_2$ and $\dot{y}_2(t) + \dot{y}_3(t) = -\theta_3$. Thus, $\theta_2 = \theta_3 \leq 0$, and consequently $\theta_2 - R_{23}\theta_3 = 0$ and $\theta_3 - R_{32}\theta_2 = 0$. By Lemma 2,

$$\theta_1 - R_{13}\theta_3 < 0, \tag{4.5}$$

$$\theta_1 - R_{12}\theta_2 < 0. \tag{4.6}$$

Now $\dot{f}(t) = h_1\dot{z}_1(t)$, where

$$\dot{z}_1(t) = \theta_1 + R_{12}\dot{y}_2(t) + R_{13}\dot{y}_3(t).$$

When $R_{12} \leq R_{13}$, one has

$$\begin{aligned} \dot{z}_1(t) &= \theta_1 + (R_{12} - R_{13})\dot{y}_2(t) + R_{13}(\dot{y}_2(t) + \dot{y}_3(t)) \\ &= \theta_1 - R_{13}\theta_3 + (R_{12} - R_{13})\dot{y}_2(t) \\ &\leq \theta_1 - R_{13}\theta_3 < 0, \end{aligned}$$

where the last inequality follows from (4.5). When $R_{13} \leq R_{12}$,

$$\begin{aligned} \dot{z}_1(t) &= \theta_1 + R_{12}(\dot{y}_2(t) + \dot{y}_3(t)) + (R_{13} - R_{12})\dot{y}_3(t) \\ &= \theta_1 - R_{12}\theta_2 + (R_{13} - R_{12})\dot{y}_3(t) \\ &\leq \theta_1 - R_{12}\theta_2 < 0, \end{aligned}$$

where the last inequality follows from (4.6).

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