Newsvendor Networks and Assemble-to-Order Inventory Systems

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Talk Outline

• The Assemble-to-Order (ATO) inventory system
• The inventory control problem
• A related stochastic program (SP): The ‘Newsvendor Network’
• A relaxed SP
• SP solution for W model & translation into control policy
• Long lead time limits
• SP solution for M model & translation into control policy
The Assemble-to-Order (ATO) Inventory System

- \( m \) products assembled from \( n \) components
- Product \( i \) requires \( a_{ij} \) units of component \( j \)
- Stochastic demand for products: lead time demand \( D_i \)
- Identical component procurement lead time \( L \) (suppliers are uncapacitated)
- Assembly time is negligible
- Only component inventories are kept
- Backlogging: unit backlog cost \( b_i \)
- Unit inventory holding cost \( h_j \)
- Continuous or periodic review (compound Poisson or i.i.d. demand)

**Goal:** Find replenishment & allocation policies to minimize the long run average expected cost
The (Continuous Review) Demand Process

Let

\[ D_i(t) = \text{demand for product } i \text{ during } [0, t], \ 1 \leq i \leq m, \]

and \( D(t) \equiv (D_1(t), \ldots, D_m(t)) \).

Assume that \( \{D(t), t \geq 0\} \) is a compound Poisson process.

Let

\[ \Delta_i \equiv \mathbb{E}[(D_i(1)], \ 1 \leq i \leq m, \quad \Delta \equiv (\Delta_1, \ldots, \Delta_m), \]

and

\[ \sigma_{ij}^2 \equiv \mathbb{E}[(D_i(1) - \Delta_i)(D_j(1) - \Delta_j)]. \]

Assume that

\[ 0 < \Delta_i < \infty \text{ and } 0 < \sigma_{ii}^2 < \infty, \ 1 \leq i \leq m. \]
The (Continuous Review) Inventory Control Problem

- We want to choose a replenishment policy $\gamma$ and an allocation policy $p$ to minimize

$$C^{\gamma,p} = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \left( \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \right) dt \right]$$

where $B_i(t)$ is the product $i$ backlog at time $t$ and $I_j(t)$ is the component $j$ inventory at time $t$.

- Feasible policy:
  1. $B_i(t) \geq 0$
  2. $I_j(t) \geq 0$
  3. Decisions cannot be based on future demand information
The Stochastic Programming Based Approach

1. Introduce a (two stage) stochastic (linear) program (with complete recourse) whose solution provides a lower bound on the achievable cost in the inventory control system

2. Solve the stochastic program (SP)

3. Translate the SP solution into a control policy for the inventory system

4. Prove asymptotic optimality

This is similar to the approach introduced in Harrison (1988) and used in many papers since then
A Related Stochastic Program (The ‘Newsvendor Network’)

Let $y = (y_1, \ldots, y_n)$. Choose $y \geq 0$ to minimize $C_s(y) \equiv \mathbb{E}[\varphi(y; D)] + \sum_{j=1}^{n} h_j y_j$, where

$$
\varphi(y; D) = \min_{z \geq 0} \left\{ \sum_{i=1}^{m} b_i (D_i - z_i) - \sum_{j=1}^{n} h_j \sum_{i=1}^{m} a_{ij} z_i \left| z_i \leq D_i, 1 \leq i \leq m, \sum_{i=1}^{m} a_{ij} z_i \leq y_j, 1 \leq j \leq n \right. \right\}
$$

$$
= b \cdot D - \max_{z \geq 0} \left\{ c \cdot z \left| z \leq D, zA \leq y \right. \right\},
$$

$c_i = b_i + \sum_{j=1}^{n} a_{ij} h_j$, and $D_i$ represents the demand for product $i$ over a lead time (so that $D_i \overset{d}{=} D_i(L)$).

Let $C^*_s = \min_{y \geq 0} C_s(y)$.

- This stochastic program ‘relaxes’ the allocation decisions ($z$): They are made at the end of the lead time, as opposed to each demand/replenishment arrival time. But it is not a relaxation of the inventory control problem because it assumes initial conditions $B=0$ and $I=0$. 
References on Newsvendor Networks

• 1 Period ATO models
  ▪ Baker, Magazine and Nuttle (1986)
  ▪ Gerchak, Magazine and Gamble (1988)
  ▪ Song and Zipkin (2003)

• Capacity Investment/Production/Inventory
  ▪ Harrison & Van Mieghem (1999)
  ▪ Van Mieghem and Rudi (2002)

• Call Center Staffing/Routing
  ▪ Harrison and Zeevi (2005)
  ▪ Bassamboo, Harrison and Zeevi (2006)
The Relaxed Stochastic Program

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$. Choose $y \geq 0$ and $\alpha \geq 0$ to minimize $C_s(y, \alpha) \equiv E[\varphi(y, \alpha; D)] + h \cdot y + b \cdot \alpha$, where

$$\varphi(y, \alpha; D) = b \cdot D - \max_{z \geq 0} \left\{ c \cdot z \mid z \leq D + \alpha, zA \leq y \right\}.$$

Let

$$C^*_s \equiv \inf_{y \geq 0, \alpha \geq 0} C_s(y, \alpha).$$

**Theorem (Dogru, Reiman, Wang, 2009).** Let $(\gamma, p)$ be any feasible policy, and $C^{\gamma, p}$ be the corresponding cost. If the demand is an i.i.d. sequence for the periodic review case, or a compound Poisson process in the continuous review case then

$$C^*_s \leq C^{\gamma, p}.$$
Solution of SP for W Model and Translation into Control Policy

• The newsvendor network for the W Model can be solved exactly. (There is no need for sampling.)

• The Relaxed SP can be transformed and solved exactly. (For all parameter values in our experiments, the solutions of the newsvendor network and relaxed SPs for the W model coincide.)

• For replenishment, follow a base-stock policy, where the base-stock levels are \((y_0^*, y_1^*, y_2^*)\), the solution of the newsvendor network.

• For allocation, follow a priority policy, which is motivated by the solution of the recourse LP.
Base-Stock Policies

Let $R_j(t) = \text{total replenishment orders for component } j \text{ placed up to time } t$ and $R_j(t) = R(t) - R(t - L)$: lead time replenishment orders. The 'inventory position' of component $j$ is defined as

$$Y_j(t) = I_j(t) - \sum_{i=1}^{m} a_{ij} B_i(t) + R_j(t), \ 1 \leq j \leq n.$$ 

A base-stock policy with base-stock levels $y = (y_1, \ldots, y_n)$ places replenishment orders to keep

$$Y_j(t) = y_j, \ 1 \leq j \leq n.$$ 

After an initial 'start-up' order, this is 'order-for-order replenishment', so that

$$R_j(t) = \sum_{i=1}^{m} a_{ij} [D_i(t) - D_i(t - L)].$$
The Solution of the Recourse LP for the W System

Here $c_1 = b_1 + h_0 + h_1$ and $c_2 = b_2 + h_0 + h_2$.

Given $y \geq 0$ the recourse LP for the W system is

$$\max_{z \geq 0} \left\{ c_1 z_1 + c_2 z_2 \mid z_i \leq D_i, \ y_i, i = 1, 2, \ z_1 + z_2 \leq y_0 \right\}.$$ 

Assume (without loss of generality) that $c_1 \geq c_2$.

The recourse LP has solution

$$z_1^* = D_1 \land y_1, \quad z_2^* = D_2 \land y_2 \land (y_0 - z_1^*).$$

This motivates priority to product 1 in allocation:

- A demand is served as long as all required components are available.
- If both products have backlogs due to the lack of the common component, when a replenishment arrives, then all product 1 backlogs are cleared first (as long as there are enough unique components) before serving product 2 demand.
The SP-Based Solution is Sometimes Optimal for W Model

• Symmetric cost case: \( c_1 = h_0 + h_1 + b_1 = c_2 = h_0 + h_2 + b_2 \)
  - All ‘myopic’ policies (maintaining \( I_i^* I_0^* B_i = 0, \ i=1,2 \)) achieve the same total allocation \( (z_1 + z_2) \) as the associated recourse LP.
  - With \( c_1 = c_2 \), \( C_s^* = C_s^* \) and all allocations with the same total have the same cost.

• SP solution has \( y_0^* = y_1^* + y_2^* \)
  - In this case the supply chain decomposes into 2 separate supply chains: There is no advantage to component commonality here, even if the common component costs the same as the components that it is replacing.
  - Numerical examples show that this is not a rare occurrence, especially if \( h_1 \) and/or \( h_2 \) are large relative to \( h_0 \)
Asymptotic Optimality?

For $0 < L < \infty$, let $C_s^*(L)$ denote the optimal objective of the relaxed SP with demand $D(L)$.

Let $C_{SP}^{(L)}$ denote the expected long-run average cost of the inventory system using the SP based policy in the $W$ system with lead time $L$.

**Conjecture.** \[ \frac{C_{SP}^{(L)}}{C_s^*(L)} \to 1 \quad \text{as} \quad L \to \infty. \]

This is borne out numerically.
The Component Shortage Process

Let

\[ Q_j(t) = \sum_{i=1}^{m} a_{ij} B_i(t) - I_j(t) : \text{component } j \text{ 'shortage'} \]

Recall that, under a base-stock policy,

\[ I_j(t) - \sum_{i=1}^{m} a_{ij} B_i(t) + R_j(t) = y_j, \quad 1 \leq j \leq n, \quad t \geq 0, \]

and \( R_j = \sum_{i=1}^{m} a_{ij} [D_i(t) - D_i(t - L)] \).

Thus

\[ Q_j(t) = \sum_{i=1}^{m} a_{ij} [D_i(t) - D_i(t - L)] - y_j. \]

This process is not controllable.
Target Backlog Levels

Given $Q(t) = Q$, we can look for feasible values of $B(t)$ and $I(t)$ that minimize the total inventory/backlog cost:

$$\min_{B, I \geq 0} \left\{ \sum_{i=1}^{m} b_i B_i + \sum_{j=1}^{n} h_j I_j \mid \sum_{i=1}^{m} a_{ij} B_i - I_j = Q_j \right\}$$

$$= \min_{B \geq 0} \left\{ \sum_{i=1}^{m} c_i B_i \mid \sum_{i=1}^{m} a_{ij} B_i \geq Q_j \right\} - \sum_{j=1}^{n} h_j Q_j$$

This is a translation of the recourse LP of the relaxed SP.
Let $B^*(Q)$ denote an optimal solution.
If for all $t \geq 0$ we can control the system so that $B(t) = B^*(Q(t))$ then the inventory system will achieve the lower bound.
Functional Central Limit Theorem

Let
\[ \hat{D}^{(L)}(t) = \frac{D(Lt) - LT \Delta}{\sqrt{L}}, \quad L > 0, \ t \geq 0, \ \text{and} \ \hat{Q}^{(L)}(t) = \frac{Q(Lt)}{\sqrt{L}}. \]

By Donsker’s (Functional Central Limit) Theorem,
\[ \hat{D}^{(L)} \xrightarrow{d} \hat{D}, \]
where \( \hat{D} \) is a driftless Brownian motion with covariance matrix \( \{\sigma^2_{ij}, 1 \leq i, j \leq m\} \).

This motivates representing \( y(L) \) and \( z(L) \) as
\[ y(L) = L \Delta A + \sqrt{L} \hat{y}(L) \quad \text{and} \quad z(L) = L \Delta + \sqrt{L} \hat{z}(L). \]

If \( \hat{y}(L) \to \hat{y} \) as \( L \to \infty \), then
\[ \hat{Q}^{(L)} \xrightarrow{d} \hat{Q} \quad \text{as} \quad L \to \infty, \]
where
\[ \hat{Q}_{j}(t) = \sum_{j=1}^{n} [\hat{D}_{j}(t) - \hat{D}_{j}(t-1)] - \hat{y}_{j}. \]
State Space Collapse for W

Let

\[ \hat{B}^{(L)}(t) = \frac{B^{(L)}(Lt)}{\sqrt{L}} \quad \text{and} \quad \hat{I}^{(L)}(t) = \frac{I^{(L)}(Lt)}{\sqrt{L}}. \]

For the W system,

\[ \hat{Q}_0^{(L)}(t) = \hat{B}_1^{(L)}(t) + \hat{B}_2^{(L)}(t) - \hat{I}_0^{(L)}(t); \]

\[ \hat{Q}_1^{(L)}(t) = \hat{B}_1^{(L)}(t) - \hat{I}_1^{(L)}(t), \quad \hat{Q}_2^{(L)}(t) = \hat{B}_2^{(L)}(t) - \hat{I}_2^{(L)}(t). \]

The target backlog levels, given \( \hat{Q}^{(L)}(t) = \tilde{Q} \), are

\[ \tilde{B}_1 = \tilde{Q}_1^+ \quad \text{and} \quad \tilde{B}_2 = \tilde{Q}_2^+ \vee (\tilde{Q}_0 - \tilde{Q}_1^+)^+. \]

Suppose that \( \hat{B}_1^{(L)}(t) > \hat{Q}_1^+(t) \). Then \( \hat{I}_1^{(L)}(t) > 0 \).
With product 1 receiving priority in using component 0,

\[ \hat{B}_1^{(L)}(t + \frac{s}{\sqrt{L}}) - \hat{B}_1^{(L)}(t) \approx -\Delta_2 s. \]

So \( \hat{B}_1^{(L)} \) hits \( \hat{Q}_1^+ \) in \( O\left(\frac{1}{\sqrt{L}}\right) \) time. By Bramson(1998), this implies state-space collapse.
Solution of SP for M Model and Translation into Control Policy

- The newsvendor network for the M Model can be solved exactly. (There is no need for sampling.)
- The Relaxed SP for the M model can be transformed and solved exactly. (The solutions of the newsvendor network and relaxed SPs for the M model are sometimes different.)
- For replenishment, follow a base-stock policy, where the base-stock levels are \((y_1^*, y_2^*)\), the solution of the newsvendor network.
- For allocation, follow a priority policy, possibly with reservation.
  - The priority is motivated by the solution of the recourse LP
  - The need for reservation follows from a ‘fluid analysis’
Parameter Regions for $M$ System

Here $c_0 = b_0 + h_1 + h_2, c_1 = b_1 + h_1, c_2 = b_2 + h_2$.
Assuming (without loss of generality) that $c_1 \geq c_2$, there are 4 cost parameter regions:

\[
\begin{align*}
\text{(A)} & \quad c_1 + c_2 \leq c_0 \\
\text{(B)} & \quad c_1 \leq c_0 \leq c_1 + c_2 \\
\text{(C)} & \quad c_2 \leq c_0 \leq c_1 \\
\text{(D)} & \quad c_0 \leq c_2
\end{align*}
\]

- The recourse solution in region A motivates priority to product 0 in allocation
  - Priority alone does not yield asymptotically optimal performance
  - Minimal reservation: hold 1 unit of component 2 in reserve for product 0. Fluid analysis (and numerical results) suggest state-space collapse

- The recourse solution in region B motivates a state-dependent policy:
  - If all products have backlogs, clear product 1 and 2 backlogs before product 0
  - If only products 0 and 1 have backlogs, clear product 0 backlog first
  - If only products 0 and 2 have backlogs, clear product 0 backlog first