Brownian network models of ramp metering

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Outline

• Rate control in communication networks (relatively well understood)

• Philosophy: optimization vs fairness

• Ramp metering (very preliminary)
Network structure

\[ J \quad - \quad \text{set of resources} \]
\[ R \quad - \quad \text{set of routes} \]
\[ A_{jr} = 1 \quad - \quad \text{if resource } j \text{ is on route } r \]
\[ A_{jr} = 0 \quad - \quad \text{otherwise} \]
Rate allocation

\( n_r \) - number of flows on route \( r \)
\( x_r \) - rate of each flow on route \( r \)

Given the vector \( n = (n_r, r \in R) \)
how are the rates \( x = (x_r, r \in R) \)
chosen?
Optimization formulation

Suppose \( x = x(n) \) is chosen to

\[
\text{maximize} \quad \sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}
\]

subject to \( \sum_r A_{jr} n_r x_r \leq C_j \quad j \in J \)

\( x_r \geq 0 \quad r \in R \)

(weighted \( \alpha \)-fair allocations, Mo and Walrand 2000)

\( 0 < \alpha < \infty \) (replace \( \frac{x_r^{1-\alpha}}{1-\alpha} \) by \( \log(x_r) \) if \( \alpha = 1 \) )
Solution

\[ x_r = \left( \frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \quad r \in R \]

where

\[ \sum_r A_{jr} n_r x_r \leq C_j \quad j \in J; \quad x_r \geq 0 \quad r \in R \]

\[ p_j(n) \geq 0 \quad j \in J \]

\[ p_j(n) \left( C_j - \sum_r A_{jr} n_r x_r \right) \geq 0 \quad j \in J \]

\[ p_j(n) \quad - \text{shadow price (Lagrange multiplier)} \] for the resource \( j \) capacity constraint
Examples of $\alpha$-fair allocations

$$\text{maximize} \quad \sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$$

$$\text{subject to} \quad \sum_r A_{jr} n_r x_r \leq C_j \quad j \in J$$

$$x_r \geq 0 \quad r \in R$$

$$x_r = \left( \frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \quad r \in R$$

$\alpha \to 0$ $(w = 1)$ - maximum flow
$\alpha \to 1$ $(w = 1)$ - proportionally fair
$\alpha = 2$ $(w_r = 1/T_r^2)$ - TCP fair
$\alpha \to \infty$ $(w = 1)$ - max-min fair
Example

\begin{align*}
\text{max-min fairness:} & \quad \alpha \to \infty \\
\text{proportional fairness:} & \quad \alpha = 1 \\
\text{maximum flow:} & \quad \alpha \to 0
\end{align*}

\[ n_r = 1, \ w_r = 1 \quad r \in R, \]
\[ C_j = 1 \quad j \in J \]
Source: CAIDA - Young Hyun, Bradley Huffaker (displayed at MOMA)
Flow level model

Define a Markov process \( n(t) = (n_r(t), r \in R) \)
with transition rates

\[
\begin{align*}
n_r &\rightarrow n_r + 1 \quad \text{at rate} \quad \nu_r \quad \quad r \in R \\
n_r &\rightarrow n_r - 1 \quad \text{at rate} \quad n_r x_r(n) \mu_r \quad r \in R
\end{align*}
\]

- Poisson arrivals, exponentially distributed file sizes

Roberts and Massoulié 1998
Stability

Let
\[ \rho_r = \frac{\nu_r}{\mu_r} \quad r \in R \]

If
\[ \sum_r A_{jr} \rho_r < C_j \quad j \in J \]

then the Markov chain
\[ n(t) = (n_r(t), r \in R) \]
is positive recurrent

De Veciana, Lee & Konstantopoulos 1999;
Bonald & Massoulié 2001
Heavy traffic: balanced fluid model

The following are equivalent:

- $n$ is an invariant state
- there exists a non-negative vector $p$ with

$$n_r = \frac{\nu_r}{\mu_r} \sum_j A_{jr} p_j \quad r \in R$$

Thus the set of invariant states forms a $J$ dimensional subspace, parameterized by $p$. 

$\alpha = 1$
Example

\[ \mu_r = 1, \quad r \in R \]

Each bounding face corresponds to a resource not working at full capacity.

Entrainment: congestion at some resources may prevent other resources from working at their full capacity.
Williams (1987) determined sufficient conditions, in terms of the reflection angles and covariance matrix, for a SRBM in a polyhedral domain to have a product form invariant distribution – a skew symmetry condition.
Local traffic condition

Assume the matrix $A$ contains the columns of the unit matrix amongst its columns:

\[
A = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & 1 & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & 1 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & 0 & 1 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & 0 & 0 & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

i.e. each resource has some local traffic -
Product form under proportional fairness

\[ \alpha = 1, \ w_r = 1, \ r \in R \]

Under the stationary distribution for the reflected Brownian motion, the (scaled) components of \( p \) are independent and exponentially distributed. The corresponding approximation for \( n \) is

\[ n_r \approx \rho_r \sum_j A_{jr} p_j \quad r \in R \]

where

\[ p_j \sim \text{Exp}(C_j - \sum_r A_{jr} \rho_r) \quad j \in J \]

Dual random variables are independent and exponential

Kang, K, Lee and Williams 2009
What we've learned about highway congestion

Data, modelling and inference in road traffic networks
R.J. Gibbens and Y. Saatci
Phil. Trans. R. Soc. A366 (2008), 1907-1919.

Figure 2. The relationship between the speed and flow of vehicles observed on the morning of Wednesday, 14 July 2004 using the M25 midway between junctions 11 and 12 in the clockwise direction. In the free-flow regime, flow rapidly increases with only a modest decline in speeds. Above a critical occupancy of vehicles there is a marked drop in speed with little, if any, improvement in flow which is then followed by a severe collapse into a congested regime where both flow and speed are highly variable and attain very low levels. Finally, the situation recovers with a return to higher flows and an improvement in speeds.
A linear network

\[ m_i(t) = m_i(0) + e_i(t) - \int_0^t \Lambda_i(m(s)) \, ds, \quad t \geq 0 \]
Suppose the metering rates can be chosen to be any vector \( \Lambda = \Lambda(m) \) satisfying

\[
\sum_i A_{ji} \Lambda_i \leq C_j, \quad j \in J
\]

\[
\Lambda_i \geq 0, \quad i \in I
\]

\[
\Lambda_i = 0, \quad m_i = 0
\]

and such that

\[
m_i(t) = m_i(0) + e_i(t) - \int_0^t \Lambda_i(m(s))ds \geq 0, \quad t \geq 0
\]
Optimal policy?

For each of \( i = I, I-1, \ldots, 1 \) in turn choose

\[ \int_0^t \Lambda_i(m(s))ds \geq 0 \]

to be maximal, subject to the constraints.

This policy minimizes

\[ \sum_i m_i(t) \]

for all times \( t \)
Proportionally fair metering

Suppose \( \Lambda(m) = (\Lambda_i(m), i \in I) \) is chosen to

maximize \( \sum_{i} m_i \log \Lambda_i \)

subject to \( \sum_{i} A_{ji} \Lambda_i \leq C_j, \quad j \in J \)

\( \Lambda_i \geq 0, \quad i \in I \)

\( \Lambda_i = 0, \quad m_i = 0 \)
Proportionally fair metering

$$\Lambda_i(m) = \frac{m_i}{\sum_j p_j A_{ji}}, \quad i \in I$$

where

$$\Lambda_i \geq 0, \quad i \in I$$

$$\sum_i A_{ji} \Lambda_i \leq C_j, \quad j \in J$$

$$p_j \geq 0, \quad j \in J$$

$$p_j \left( C_j - \sum_i A_{ji} \Lambda_i \right) \geq 0, \quad j \in J$$

$p_j$ - shadow price (Lagrange multiplier) for the resource $j$ capacity constraint
Brownian network model

Suppose that \((e_i(t), t \geq 0)\) is a Brownian motion, starting from the origin, with drift \(\rho_i\) and variance \(\rho_i \sigma^2\). Let

\[
X_j(t) = \sum_i A_{ji} e_i(t) - C_j t
\]

Then \(X(t) = (X_j(t), j \in J)\) is a \(J\)-dimensional Brownian motion starting from the origin with drift \(-\theta = A\rho - C\) and variance \(\Gamma = \sigma^2 A[\rho] A'\).
Brownian network model

Let \( W = A[\rho]A'R_+^J \)
and \( W^J = \{ A[\rho]A' : q \in R^J, \quad q_j = 0 \} \).

Define \( W(t) \) by the following relationships:

(i) \( W(t) = X(t) + U(t) \) for all \( t \geq 0 \)

(ii) \( W \) has continuous paths, \( W(t) \in W \)

(iii) for each \( j \in J, U_j \) is a one-dimensional process such that

(a) \( U_j \) is continuous and non-decreasing, with \( U_j(0) = 0 \),

(b) \( U_j(t) = \int_0^t I\{W(s) \in W^j\}dU_j(s) \) for all \( t \geq 0 \).
Brownian network model

If $\theta_j > 0, j \in J$, then there is a unique stationary distribution $W$ under which the components of

$$Q = (A[\rho]A')^{-1}W$$

are independent, and $Q_j$ is exponentially distributed with parameter

$$\frac{\sigma^2}{2} \theta_j, \quad j \in J$$

and queue sizes are given by

$$M = [\rho]A'Q$$
Delays

Let \( D_i(m) = \frac{m_i}{\Lambda_i(m)} \)
- the time it would take to process the work in queue \( i \) at the current metered rate. Then

\[
D_i(M) = \sum_j Q_j A_{ji}
\]
A tree network
A tree network

\[ Q_1 + Q_4 + Q_5 \]
Route choices
Route choices
Route choices

\[ Q_1 + Q_2 + Q_3 + Q_4 \]
Route choices

\[ Q_2 \sim \frac{\sigma^2}{2} \exp(C_1 + C_2 - \rho_1 - \rho_2) \]

\[ Q_1 + Q_2 + Q_3 + Q_4 \]
Brownian network model

- As Mike has cogently argued, for many applications it may be *easier* to describe the workload arrival process in terms of the mean and variance of a Brownian motion.
- If relevant time periods long enough, negative increments less likely.
- The Brownian model exposes structure in a way that more detailed models (e.g. MDP models) do not.