Positive Recurrence of Reflecting
Brownian Motion in 3 Dimensions

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(Abbreviated) Definition of semimartingale reflecting Brownian motion (SRBM) -

Process $Z(t)$, $t \geq 0$, on the nonnegative orthant on $\mathbb{R}_d^+$ with $d$-dim SRBM $\mathcal{d}$-dim BM w/ drift vector $\Theta$, covariance matrix $\Gamma$ (nondegenerate)

(1) $Z(t) = X_t + RY_t(t)$, continuous, nondecreasing w/ $Y_i(t) = 0 \forall i$; $Y_i(t)$ only increases where $Z_i(t) = 0$

reflection matrix - determines direction of reflection on boundary

$X$, $Y$ adapted wrt filtered space

- $Z$ behaves like BM on $\mathbb{R}_d^+$ and reflects on $\partial \mathbb{R}_d^+$ according to $R$. 
Existence & uniqueness of $Z$ (in law) for given $(\theta, \Gamma, R)$ iff $R$ is completely $S$ (Reiman-Williams, Taylor-Williams).

Here, we will scale $R^d$ so that $R_{ii} = 1$ for $i=1, \ldots, d$.

**Question:** When is $Z$ positive recurrent?

Elementary requirement:

1. $R$ is nonsingular with $R^{-1}\theta < 0$ (El Kharrouchi et al. (2000))

**Some background:**

- For all $d$, (2) is necessary & sufficient when $R$ is an $M$-matrix (Harrison-Williams (1987)).

- For $d=2$, positive recurrent iff $R$ is a $P$-matrix (2) holds (K et al. (2000)).
  
  *Each principal submatrix of $R$ has positive determinant*

- For $d \geq 3$, (2) is not sufficient (even when $R$ is a $P$-matrix) (K et al. (2000)).

*Will discuss their example later*

Here, will answer question when $d=3$.

For $d \geq 4$ - unknown.
Fluid paths: basics

Wish to reduce problem of positive recurrence to a deterministic problem.

Fluid path \((y,z)\) satisfies deterministic analog of (1):

\[
\begin{align*}
\text{same } & \theta, \mathcal{R} \text{ as in (1)} \\
(3) \quad z(t) &= y(0) + \theta t + \mathcal{R}y(t), \\
\mathcal{R} &\subset \mathbb{R}^d \\
( y,z ) &\text{ is continuous, nondecreasing} \\
&\text{w/ } y(t) \downarrow_0 \forall i; \ y^i(t) \text{ only increases where } z^i(t) = 0 \\
&\text{associated w/ } (\theta, \mathcal{R})
\end{align*}
\]

vocabulary:

A fluid path (FP) \((y,z)\) is attracted to the origin if \(z(t) \to 0\) as \(t \to \infty\).

A FP \((y,z)\) is divergent if \(|z(t)| \to \infty\) as \(t \to \infty\).

Sufficient condition for positive recurrence

Theorem (Dupuis-Williams (1994)). Let \(Z\) be a \(d\)-dimensional SRBM w/ data \((\theta, \mathcal{R})\). If every FP associated w/ \((\theta, \mathcal{R})\) is attracted to the origin, then \(Z\) is positive recurrent.
Linear fluid paths

A FP \((y, z)\) of (3) is linear if \( y(t) = vt \), \( z(t) = vt \) for some \( u, v \geq 0 \). \( \text{LFP} \)

For a LFP, (3) and its side conditions reduce to

\[ (4) \quad v = \Theta + Ru, \quad u \cdot v = 0, \quad u, v \geq 0. \]

(\( \text{LCP} \))

An LFP is stable if \( v = 0 \).
An LFP is divergent if \( v \neq 0 \).

Lemma 1. Suppose (2) holds and set \( u^* = -R^{-1}\Theta \). Then \((u^*, 0)\) is a stable LFP and all other LFPs are divergent.
Example of SRBM in $d=3$ that is not positive recurrent:


(5) $\Theta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

with

$Z(0) = (0, 0, k)$, $k > 0$.

Evolution of FP -

diagram 1

- FP is divergent.
- Using various estimates, K. et al. (2000) showed corresponding SRBM transient.

for any nonsingular $\Gamma$

Can generalize (5) in a natural way so $z(t)$ moves piecewise linearly among the $z_0, z_2, z_3$ axes and diverges as above; denote this set of $(\Theta, R)$ by $C_1$.

$C_2 =$ corresponding $(\Theta, R)$ in clockwise direction,

$C = C_1 \cup C_2$.

Set

$\beta(\Theta, R) =$ scaling factor upon 1st return to $z_3$ axis.

For $(\Theta, R) \in C$, SRBM is transient if $\beta(\Theta, R) > 1$ and $R$ is a $P$-matrix (K. et al. 2000).
Summary of results in 3 dimensions

Stability result for FP

Theorem 2 (El Kharroubi et al. (2002)).

Suppose (2) and either
(a) \((\Theta, R) \in C\) and \(\beta(\Theta, R) < 1\)
or
(b) \((\Theta, R) \notin C\) and (4) has a unique solution.

Then all FPs associated with \((\Theta, R)\) are attracted to the origin.

By DW (1994), the corresponding SRBM is positive recurrent.

Note: proof is not entirely rigorous;
alternative proof given in Dai-Harrison (2009).

Want to know behavior in converse direction.
Given by following results:
For \((\Theta, R) \in C\):

**Theorem 3** (B-Dai-Harrison (2009)). If \((\Theta, R) \in C\) and \(\beta(\Theta, R) \geq 1\), then the SRBM is not positive recurrent.

For \((\Theta, R) \notin C\), it suffices to consider the case where there is a divergent LFP.

Because of Theorem 2(b) and Lemma 1.

**Theorem 4** (BDH (2009)). If there exists a divergent LFP, then the SRBM is not positive recurrent.

So, the conditions (a) and (b) in Theorem 2 give necessary and sufficient conditions for positive recurrence.

The proof of Theorem 3 is conceptual and not difficult.

The proof of Theorem 4 breaks into 5 cases, depending on the number of nonzero components of \(u\) and \(v\) for the LFP \((u, v)\). A fair amount of work; some cases easier than others. Need to examine the asymptotic behavior of related BMs including hitting times at different boundaries.
Summary of proof of Theorem 3

For concreteness, use the Bernard–El Kharroubi example, with
\[(S') \Theta = (E), \quad \mathcal{R} = \left( \begin{smallmatrix} 1 & 3 \\ 3 & 0 \\ 3 & 1 \end{smallmatrix} \right), \quad \Pi = I.\]

Also, assume \(Z_1(0) = 0, \ Z_2(0) = 0, \ Z_3(0) > 0.\)

(Argument holds for \((E, R) \in C.\))

In the spirit of diagram 1, set:

\[\gamma_1 = \inf \{ t > 0 : Z_3(t) = 0 \};\]
\[\gamma_2 = \inf \{ t > \gamma_1 : Z_1(t) = 0 \};\]
\[\gamma_3 = \inf \{ t > \gamma_2 : Z_2(t) = 0 \};\]

... when \(Z(t)\) first hits \(z_1z_2\) plane
Can check:

\[ \text{standard BMs} \]

\[ \begin{align*}
\text{(7a) } Z_2(t) &= \frac{1}{2} \left[ B_2(t) - B_2(t_0) \right] - t + Y_2(t), \quad 0 \leq t \leq t_1, \\
\text{(7b) } Z_3(t) &= [B_3(t) - B_3(t_1)] - (t-t_1) + [Y_3(t) - Y_3(t_1)], \quad t_1 \leq t \leq t_2, \\
\text{(7c) } Z_1(t) &= [B_1(t) - B_1(t_2)] - (t-t_2) + [Y_1(t) - Y_1(t_2)], \quad t_2 \leq t \leq t_3,
\end{align*} \]

Rewriting (7a):

\[ (8) \quad Y_2(t) = t + Z_2(t) - B_2(t) \quad \text{on } 0 \leq t \leq t_1. \]

On the other hand, note that for

\[ u = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \text{one has} \]

\[ u' \mathcal{R} e' = 1, \quad u' e = 1. \]

We wish to define a "distance" from

\[ Z(t) \] to \( 0 \). Set

\[ \mathcal{E}(t) = u' Z(t). \]

will construct a martingale

from this & compare w/ 1-d BM (which is not positive recurrent)
For \( t \geq 0 \),

\[
\begin{align*}
\varepsilon(t) - \varepsilon(0) &= u' B(t) - u' e^t + u' R Y(t) \\
\geq & \ u' B(t) - t + e' Y(t).
\end{align*}
\]

On \( t \leq r_1 \), this is, by (8),

\[
= u' B(t) - t + Y_1(t) + [t + Z_2(t) - B_2(t)]
\]
\[
= u' B(t) - B_2(t) + Y_1(t) + Z_2(t).
\]

Define \( M(t) \), \( A(t) \) analogously on \( (T_i, T_{i+1}] \).

Then

\[
\varepsilon(t) - \varepsilon(0) \geq M(t) + A(t) \text{ for all } t \geq 0.
\]

\( M(t) \) is a continuous martingale with quadratic variation

\[
\langle M, M \rangle(t) - \langle M, M \rangle(s) \leq \gamma (t-s) \text{ for } s \leq t,
\]

\( A(t) \) is continuous with \( A(t) \geq 0 \) for all \( t \).
For given \( \varepsilon \in (0, \varepsilon(0)) \), set

\[
\sigma = \inf \{ t > 0 : \varepsilon(t) + M(t) = \varepsilon^3 \},
\]
\[
T = \inf \{ t > 0 : \varepsilon(t) = \varepsilon^3 \}.
\]

\( \sigma \leq T \)

\[M(t) + \varepsilon(t) \quad \text{(which is } \leq \varepsilon(t))\]

A standard argument shows that

\[E[\sigma] = \infty. \quad \text{same reasoning as for BM}\]

Hence

\[E[T] = \infty.\]

This implies \( Z(t) \) is not positive recurrent.