On the Asymptotic Optimality of the Randomized Linear Program for Network Revenue Management

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Abstract

For network revenue management problems, it is known that the bid prices computed through the so-called deterministic linear program are asymptotically optimal as the capacities on the flight legs and the expected numbers of product requests increase linearly with the same rate. In this paper, we show that the same asymptotic optimality result holds for the bid prices computed through the so-called randomized linear program. We computationally investigate how the performance of the randomized linear program changes with different problem parameters and with the number of samples. The hope is that our asymptotic optimality result and computational experiments will raise awareness for the randomized linear program, which has yet not been popular in the research community or industry.
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The concept of bid prices forms a practical tool for constructing good policies for network revenue management problems. The fundamental idea is to associate a bid price with each flight leg that captures the opportunity cost of a unit of capacity. A request for a specific itinerary-fare-class combination is accepted only when the revenue from the requested itinerary-fare-class exceeds the sum of the bid prices associated with the flight legs in the requested itinerary-fare-class; see Williamson (1992), Talluri and van Ryzin (1998) and Talluri and van Ryzin (2004).

Bid prices are traditionally computed by solving a deterministic linear program, which uses the expected numbers of itinerary-fare-class requests that are to arrive until the time of departure. This deterministic linear program dates back to Simpson (1989), and since then, there have been numerous successful attempts to improve it. In this paper, we focus on one of these attempts, namely the randomized linear program proposed by Talluri and van Ryzin (1999). Rather than the expected numbers of itinerary-fare-class requests, the randomized linear program uses actual samples of itinerary-fare-class requests. This linear program is solved for many samples and the results corresponding to different samples are averaged to obtain bid prices. Implementing the randomized linear program requires a small amount of extra work over the deterministic linear program, but computational experiments indicate that the bid prices obtained by the randomized linear program can perform better than the ones obtained by the deterministic linear program.

Nevertheless, despite its relative ease of implementation and potentially superior performance, the randomized linear program has yet not been popular in the research community or industry. For example, numerous solution methods for network revenue management problems have appeared in the recent literature, but very few of them are compared with the randomized linear program; see, for example, de Boer, Freling and Piersma (2002), Bertsimas and Popescu (2003) and Adelman (2006). As far as we are aware, comparisons with the randomized linear program are given only in Talluri and van Ryzin (1999) and Talluri and van Ryzin (2004). In our opinion, one reason for this lack of attention is that there has not been much computational work on the randomized linear program. It is not clear how the performance of the randomized linear program changes with different problem parameters and how many samples are needed to obtain good bid prices. Talluri and van Ryzin (1999) also mention in their original paper that the performance of the randomized linear program warrants further investigation. Furthermore, Talluri and van Ryzin (1998) provide theoretical support for the deterministic linear program by showing that the bid prices obtained by the deterministic linear program are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary-fare-class requests increase linearly with the same rate. A comparable result is not available for the randomized linear program. Our main goal in this paper is to fill these two gaps. We show that the bid prices obtained by the randomized linear program are asymptotically optimal. We conduct extensive computational experiments to investigate how the performance of the randomized linear program changes with different problem parameters and with the number of samples.

After its introduction by Simpson (1989), the idea of using a deterministic linear program to compute bid prices was studied more thoroughly by Williamson (1992). Talluri and van Ryzin (1998) provide a careful treatment of the policies that are based on bid prices and prove the asymptotic optimality result mentioned above. They also provide counterexamples indicating that the policies that are based on bid prices can be suboptimal. Talluri and van Ryzin (1999) introduce the randomized linear program that we focus on in this paper. Simulation studies of de Boer et al. (2002) carefully compare several approaches for solving network revenue management problems. Bertsimas and Popescu (2003) propose enhancements on the deterministic linear program to better capture the total opportunity cost of the leg capacities consumed by an itinerary-fare-class request. They also show how to extend the deterministic linear program to incorporate cancellations. Their approach appears to obtain better control policies when compared with the deterministic linear program, but it requires solving one linear program for each itinerary-fare-class and the number of itinerary-fare-class combinations can easily get large. Adelman (2006) computes bid prices by formulating the Markov decision process that characterizes the network revenue management problem as a linear program. The numbers of constraints and decision variables in this linear program are exponential in the number of flight legs and the author uses approximations of the value functions to make this linear program tractable. Topaloglu (2006) relaxes certain constraints in the dynamic programming formulation of the network revenue management problem, in which case the problem decomposes by the flight legs and it is possible to compute bid prices by concentrating on one flight leg at a time.

Besides using bid prices, another popular strategy for network revenue management problems is to use protection levels. Roughly speaking, this strategy associates a protection level with each itinerary-fare-class combination on each flight leg. A request for a specific itinerary-fare-class is accepted only when the remaining leg capacities exceed the protection levels associated with the requested itinerary-fare-class combination on all of the flight legs. For practical tractability, the itinerary-fare-classes are usually bucketed into a small number of virtual classes and one only associates a protection level with each virtual class on each flight leg; see Talluri and van Ryzin (2004). There are different approaches for computing protection levels. For example, displacement adjusted virtual nesting uses the dynamic programming formulations of a series of revenue management problems that take place over a single flight leg; see Talluri and van Ryzin (2004). Bertsimas and de Boer (2005) and van Ryzin and Vulcano (2006) propose using stochastic approximation methods to compute protection levels. Finally, we emphasize that other strategies for computing bid prices and protection levels exist in the literature and we refer the reader to Talluri and van Ryzin (2004) for a comprehensive review.

We make the following research contributions in this paper. 1) We show that the bid prices obtained by the randomized linear program are asymptotically optimal. A key step is to abandon the traditional discrete-time model, which assumes that the numbers of requests for different itinerary-fare-classes have a multinomial distribution, and to switch to a continuous-time model, which assumes that the arrivals of requests for different itinerary-fare-classes are characterized by independent Poisson processes. This enables us to use Chebyshev's inequality to bound certain probabilities. 2) While showing the asymptotic optimality of the bid prices obtained by the randomized linear program, we also show the asymptotic optimality of the bid prices obtained by the deterministic linear program under assumptions that are different from those in Talluri and van Ryzin (1998). In particular, Talluri and van Ryzin (1998) smooth the problem by assuming that the revenues from accepting the itinerary-fare-class requests are random variables with continuous distributions, whereas we work with deterministic revenues. 3) Through extensive computational experiments, we investigate how the performance of the randomized linear program changes with different problem parameters and with the number of samples.

The rest of the paper is organized as follows. In Section 1, we derive the Hamilton-Jacobi-Bellman equation for the network revenue management problem. Section 2 describes the deterministic and randomized linear programs in detail. Section 3 shows the asymptotic optimality of the bid prices obtained by the randomized linear program. We present computational experiments in Section 4.

1 Problem Formulation

We have a set of flight legs that can be used to satisfy the itinerary-fare-class requests that arrive randomly over time. Whenever a request arrives, we have to decide whether to accept or reject it. An accepted request generates a revenue and consumes the capacities on the relevant flight legs. A rejected request simply leaves the system.

The problem takes place over the time interval $[0, \tau]$. All flight legs depart at time τ. The set of flight legs is $\mathcal{L}$ and the set of itinerary-fare-class combinations is $\mathcal{J}$. If we accept a request for itinerary-fare-class combination j, then we make a revenue of f_j and consume a_{ij} units of capacity on flight leg i. If itinerary-fare-class combination j does not use flight leg i, then we have $a_{ij} = 0$. The capacity on flight leg i is c_i. The requests for itinerary-fare-class combination j arrive according to a nonstationary Poisson process with the intensity function $\lambda_j(\cdot)$. We assume that the arrivals of requests for different itinerary-fare-class combinations are independent. Throughout the paper, we refer to an itinerary-fare-class combination simply as a product.

We let x_{it} be the remaining capacity on flight leg i at time t so that $x_t = \{x_{it} : i \in \mathcal{L}\}$ gives the state of the system. We capture the decisions by $u = \{u_j : j \in \mathcal{J}\}$, where u_j takes value 1 if we accept a request for product j, and 0 otherwise. In this case, the set of feasible decisions at time t becomes

$$\mathcal{U}(x_t) = \big\{u \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, u_j \le x_{it} \;\text{ for all } i \in \mathcal{L},\; j \in \mathcal{J}\big\}.$$

The Hamilton-Jacobi-Bellman equation for the problem can be written by appealing to the following argument. Over a small time interval of length Δ around time t, the probability that we have a request for product j is $\Delta\, \lambda_j(t) + o(\Delta)$, where $o(\cdot)$ stands for a function $h(\cdot)$ that satisfies $\lim_{\Delta \to 0} h(\Delta)/\Delta = 0$. The probability that we do not have a product request is $1 - \Delta \sum_{j \in \mathcal{J}} \lambda_j(t) + o(\Delta)$. Therefore, the optimal policy in discrete time can be found by solving the optimality equation

$$V_t(x_t) = \Big[1 - \Delta \sum_{j \in \mathcal{J}} \lambda_j(t)\Big]\, V_{t+\Delta}(x_t) + \max_{u \in \mathcal{U}(x_t)} \Bigg\{ \sum_{j \in \mathcal{J}} \Delta\, \lambda_j(t) \Big[f_j\, u_j + V_{t+\Delta}(x_t - u_j \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i)\Big] \Bigg\} + o(\Delta),$$

where e_i is the $|\mathcal{L}|$-dimensional unit vector with a 1 in the element corresponding to $i \in \mathcal{L}$. Subtracting $V_{t+\Delta}(x_t)$, dividing by Δ and taking the limit as Δ approaches 0 in the expression above, we obtain

$$\frac{\partial V_t(x_t)}{\partial t} = \sum_{j \in \mathcal{J}} \lambda_j(t)\, V_t(x_t) - \max_{u \in \mathcal{U}(x_t)} \Bigg\{ \sum_{j \in \mathcal{J}} \lambda_j(t) \Big[f_j\, u_j + V_t(x_t - u_j \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i)\Big] \Bigg\} \tag{1}$$

with the boundary conditions that $V_\tau(\cdot) = 0$ and $V_\cdot(0) = 0$. The derivation of (1) is somewhat heuristic because we do not justify the exchange of the limit and the maximization operator on the right side. One can show that these operations are valid by referring to the general theory, but we prefer to omit the details since we never directly work with the Hamilton-Jacobi-Bellman equation in (1).

2 Heuristic Strategies for Network Revenue Management

Due to the well-known curse of dimensionality, finding the optimal policy through the Hamilton-Jacobi-Bellman equation in (1) is impossible for problems of practical significance. An alternative solution method is to use a deterministic linear program. Letting y_j be the number of requests for product j that we plan to accept, this linear program has the form

$$\max \quad \sum_{j \in \mathcal{J}} f_j \, y_j \tag{2}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_{ij} \, y_j \le c_i \qquad \text{for all } i \in \mathcal{L} \tag{3}$$

$$0 \le y_j \le \int_0^\tau \lambda_j(t) \, dt \qquad \text{for all } j \in \mathcal{J}. \tag{4}$$

Constraints (3) ensure that the product requests that we plan to accept do not violate the leg capacities, whereas constraints (4) ensure that we do not plan to accept more product requests than the expected numbers of product requests. Letting $\Lambda_j = \int_0^\tau \lambda_j(t) \, dt$, we use $L(c, \Lambda)$ to denote the optimal objective value of the problem above as a function of $c = \{c_i : i \in \mathcal{L}\}$ and $\Lambda = \{\Lambda_j : j \in \mathcal{J}\}$. We also let $y(c, \Lambda) = \{y_j(c, \Lambda) : j \in \mathcal{J}\}$ be the optimal solution and $\mu(c, \Lambda) = \{\mu_i(c, \Lambda) : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (3) in problem (2)-(4).

Throughout the paper, we use numerous linear programs that differ from problem (2)-(4) only in the right sides of constraints (3) and (4). Whenever the right side of constraints (3) is replaced with $\chi = \{\chi_i : i \in \mathcal{L}\}$ and the right side of constraints (4) is replaced with $\ell = \{\ell_j : j \in \mathcal{J}\}$, we continue using $y(\chi, \ell) = \{y_j(\chi, \ell) : j \in \mathcal{J}\}$ to denote the optimal solution and $\mu(\chi, \ell) = \{\mu_i(\chi, \ell) : i \in \mathcal{L}\}$ to denote the optimal values of the dual variables associated with constraints (3).

There are two uses of problem (2)-(4). First, this problem can be used to decide whether to accept or reject a product request. The fundamental idea is to estimate the opportunity cost of a unit of capacity on flight leg i by $\mu_i(c, \Lambda)$. These opportunity costs are referred to as the bid prices in network revenue management vocabulary. If the revenue from a product request exceeds the sum of the bid prices of the flight legs in the requested product, then we accept the requested product subject to the capacity availability. More specifically, if we have $f_j > \sum_{i \in \mathcal{L}} a_{ij} \, \mu_i(c, \Lambda)$ and there is enough capacity, then we accept a request for product j. If, on the other hand, we have $f_j = \sum_{i \in \mathcal{L}} a_{ij} \, \mu_i(c, \Lambda)$ and there is enough capacity, then we break the tie by accepting a request for product j with probability $y_j(c, \Lambda) / \Lambda_j$. Finally, if we have $f_j < \sum_{i \in \mathcal{L}} a_{ij} \, \mu_i(c, \Lambda)$, then we reject a request for product j. We refer to this decision rule as the deterministic bid price heuristic. It is important to emphasize that our understanding in Sections 2 and 3 is that this heuristic computes the bid prices and the product acceptance probabilities once at the beginning of the planning horizon and uses these quantities throughout the whole planning horizon.

However, our computational experiments in Section 4 follow the traditional practice and periodically recompute the bid prices. The randomized decision rule that we use to break the tie when we have $f_j = \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$ becomes instrumental in showing the final asymptotic optimality result.

Second, it is easy to see that the optimal objective value of problem (2)-(4) provides an upper bound on the maximum expected revenue. Specifically, if we let N_j be the number of requests for product j over the whole planning horizon, then $L(c, N)$ is an upper bound on the performance of any nonanticipatory policy when the numbers of requests for the products are given by $N = \{N_j : j \in \mathcal{J}\}$. In other words, $L(c, N)$ is the optimal objective value of the perfect hindsight problem and we have $V_0(c) \le \mathbb{E}\{L(c, N)\}$. Jensen's inequality and the fact that $L(c, \Lambda)$ is a concave function of Λ yield

$$V_0(c) \le \mathbb{E}\{L(c, N)\} \le L(c, \mathbb{E}\{N\}) = L(c, \Lambda). \tag{5}$$

This information can be useful to assess the performance of a suboptimal decision rule such as the one described in the previous paragraph.

Another solution method for network revenue management problems is to use a randomized variant of problem (2)-(4). Recalling that the random variable N_j gives the number of requests for product j over the whole planning horizon, this randomized linear program has the form

$$\max \quad \sum_{j \in \mathcal{J}} f_j\, y_j \tag{6}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_{ij}\, y_j \le c_i \quad \text{for all } i \in \mathcal{L} \tag{7}$$

$$0 \le y_j \le N_j \qquad \text{for all } j \in \mathcal{J}. \tag{8}$$

In this case, letting $\hat{N}^1, \ldots, \hat{N}^K$ be K independent samples of $N = \{N_j : j \in \mathcal{J}\}$ and

$$\bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K) = \frac{1}{K} \sum_{k=1}^{K} \mu_i(c, \hat{N}^k),$$

we use $\bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K)$ as the bid price of flight leg i. More specifically, if we have $f_j > \sum_{i \in \mathcal{L}} a_{ij} \times \bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K)$ and there is enough capacity, then we accept a request for product j. If, on the other hand, we have $f_j = \sum_{i \in \mathcal{L}} a_{ij}\, \bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K)$ and there is enough capacity, then we accept a request for product j with probability $y_j(c, \Lambda)/\Lambda_j$. Finally, if we have $f_j < \sum_{i \in \mathcal{L}} a_{ij}\, \bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K)$, then we reject a request for product j. We refer to this decision rule as the randomized bid price heuristic. Similar to the deterministic bid price heuristic, our understanding in Sections 2 and 3 is that the randomized bid price heuristic computes the bid prices and the product acceptance probabilities once at the beginning of the planning horizon and uses these quantities throughout the whole planning horizon. We emphasize that the bid prices used by the randomized bid price heuristic depend on K independent samples of N. Therefore, these bid prices are actually random variables.

3 Asymptotic Analysis of the Randomized Linear Program

In this section, we consider a family of network revenue management problems $\{P^\alpha : \alpha \in \mathbb{Z}_+\}$ parameterized by the scalar parameter α. Problem P^α takes place over the time interval $[0, \alpha\tau]$. In this problem, the capacity on flight leg i is αc_i and the requests for product j arrive according to a nonstationary Poisson process with the intensity function $\lambda_j^\alpha(\cdot)$, where we have $\lambda_j^\alpha(t) = \lambda_j(t/\alpha)$. Therefore, the problem that we formulated in Section 1 is P^1. Since we have

$$\int_0^{\alpha\tau} \lambda_j^\alpha(t)\, dt = \int_0^{\alpha\tau} \lambda_j^1(t/\alpha)\, dt = \alpha \int_0^{\tau} \lambda_j^1(s)\, ds,$$

the leg capacities and the expected numbers of product requests in problem P^α are α times larger than those in problem P^1. We let $V_0^\alpha(\alpha c)$ be the maximum expected revenue for problem P^α. This quantity can be obtained by replacing τ with $\alpha\tau$, c_i with αc_i, $\lambda_j(\cdot)$ with $\lambda_j^\alpha(\cdot)$ in (1).

It is easy to see that we have $\mu_i(\alpha c, \alpha\Lambda) = \mu_i(c, \Lambda)$ and $y_j(\alpha c, \alpha\Lambda) = \alpha\, y_j(c, \Lambda)$. Therefore, the deterministic bid price heuristic for problem P^α uses the same bid prices and product acceptance probabilities as the deterministic bid price heuristic for problem P^1. On the other hand, letting N_j^α be the number of requests for product j over the whole planning horizon in problem P^α and $\hat{N}^{\alpha 1}, \ldots, \hat{N}^{\alpha K}$ be K independent samples of $N^\alpha = \{N_j^\alpha : j \in \mathcal{J}\}$, the randomized bid price heuristic for problem P^α uses $\bar{\mu}_i(\alpha c, \hat{N}^{\alpha 1}, \ldots, \hat{N}^{\alpha K})$ as the bid price of flight leg i, which can be different from the bid price used by the randomized bid price heuristic for problem P^1.

As a function of the bid prices $\bar{\mu}(\alpha c, \hat{N}^{\alpha 1}, \ldots, \hat{N}^{\alpha K}) = \{\bar{\mu}_i(\alpha c, \hat{N}^{\alpha 1}, \ldots, \hat{N}^{\alpha K}) : i \in \mathcal{L}\}$, we let $B_0^\alpha(\alpha c \,|\, \bar{\mu}(\alpha c, \hat{N}^{\alpha 1}, \ldots, \hat{N}^{\alpha K}))$ be the expected revenue obtained by the randomized bid price heuristic for problem P^α. In this section, we show that

$$\lim_{\alpha \to \infty} \frac{\mathbb{E}\{B_0^\alpha(\alpha c \,|\, \bar{\mu}(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\}}{V_0^\alpha(\alpha c)} = 1, \tag{9}$$

where $N^{\alpha 1}, \ldots, N^{\alpha K}$ are K independent random variables having the same distribution as N^α and the expectation above involves the random variables $N^{\alpha 1}, \ldots, N^{\alpha K}$. Therefore, the expected performance of the randomized bid price heuristic becomes asymptotically optimal as the capacities on the flight legs and the expected numbers of product requests increase linearly with the same rate. Throughout the rest of the paper, we assume that problem (2)-(4) has a unique optimal dual solution. Although this is somewhat limiting, since the parameters of problem (2)-(4) are deterministic, one can check this assumption easily in practice, and when violated, force this assumption by perturbing the right sides of the constraints by infinitesimal random amounts. At the end of this section, we explain how one can show our results under the assumption that problem (2)-(4) has a unique optimal primal solution. One can also check this assumption easily in practice, and when violated, force this assumption by perturbing the objective function coefficients by infinitesimal random amounts.

The next lemma is key to proving the final result.

Lemma 1 *There exists $\epsilon > 0$ such that*

$$\mathbb{P}\{\bar{\mu}(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) = \mu(c, \Lambda)\} \geq \prod_{j \in \mathcal{J}} \left[1 - \frac{\Lambda_j}{\alpha\, \epsilon}\right]^K.$$

Proof Associating the dual variables $\{\mu_i : i \in \mathcal{L}\}$ and $\{\eta_j : j \in \mathcal{J}\}$ with constraints (3) and (4), the dual of problem (2)-(4) can be written as

$$\min \quad \sum_{i\in\mathcal{L}} c_i\,\mu_i + \sum_{j\in\mathcal{J}} \Lambda_j\,\eta_j \tag{10}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i + \eta_j \geq f_j \qquad \text{for all } j \in \mathcal{J} \tag{11}$$

$$\mu_i, \eta_j \geq 0 \qquad \text{for all } i \in \mathcal{L},\ j \in \mathcal{J}. \tag{12}$$

This problem is assumed to have a unique optimal solution. Therefore, there exists $\epsilon > 0$ such that if we perturb the objective function coefficients in problem (10)-(12) by up to $\sqrt{\epsilon}$, then the optimal solution to problem (10)-(12) does not change and is still unique. On the other hand, $\mu(\alpha c, N^\alpha)$ can be obtained by solving the problem

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{L}} c_i\,\mu_i + \sum_{j\in\mathcal{J}} \frac{N_j^\alpha}{\alpha}\,\eta_j \\ \text{subject to} \quad & (11), (12). \end{aligned}$$

Consequently, if we have $\Lambda_j - \sqrt{\epsilon} \leq N_j^\alpha/\alpha \leq \Lambda_j + \sqrt{\epsilon}$ for all $j \in \mathcal{J}$, then the optimal solution to the problem above is unique and $\mu(\alpha c, N^\alpha) = \mu(c, \Lambda)$ holds.

The random variable N_j^α has a Poisson distribution with mean $\alpha\,\Lambda_j$, which implies that

$$\mathbb{P}\{|N_j^\alpha - \alpha\,\Lambda_j| \leq \alpha\,\sqrt{\epsilon}\} \geq 1 - \frac{\alpha\,\Lambda_j}{\alpha^2\,\epsilon}$$

by Chebyshev's inequality; see Ross (1993). The arrival processes for requests for different products are independent. Therefore, we have

$$\mathbb{P}\{\mu(\alpha c, N^\alpha) = \mu(c, \Lambda)\} \geq \mathbb{P}\{|N_j^\alpha/\alpha - \Lambda_j| \leq \sqrt{\epsilon} \text{ for all } j \in \mathcal{J}\} \geq \prod_{j\in\mathcal{J}} \left[1 - \frac{\Lambda_j}{\alpha\,\epsilon}\right].$$

If $\mu(\alpha c, N^{\alpha k}) = \mu(c, \Lambda)$ holds for all $k = 1, \ldots, K$, then we have $\bar{\mu}(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) = \mu(c, \Lambda)$. In this case, the result follows by noting that the random variables $N^{\alpha 1}, \ldots, N^{\alpha K}$ are independent. □

The next lemma gives a characterization of the optimal primal-dual solution to problem (2)-(4).

Lemma 2 *Letting* $\mathbf{1}(\cdot)$ *be the indicator function, we have*

$$L(c, \Lambda) = \sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c, \Lambda))\,\Lambda_j + \displaystyle\sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j = \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c, \Lambda))\,y_j(c, \Lambda) \tag{13}$$

$$c_i \geq \sum_{j\in\mathcal{J}} a_{ij}\,\mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c, \Lambda))\,\Lambda_j + \displaystyle\sum_{j\in\mathcal{J}} a_{ij}\,\mathbf{1}(f_j = \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c, \Lambda))\,y_j(c, \Lambda). \tag{14}$$

Proof Letting $\eta(c, \Lambda) = \{\eta_j(c, \Lambda) : j \in \mathcal{J}\}$ be the optimal values of the dual variables associated with constraints (4), some of the complementary slackness conditions for problem (2)-(4) are

$$\big[\Lambda_j - y_j(c, \Lambda)\big]\,\eta_j(c, \Lambda) = 0 \tag{15}$$

$$\Big[\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c, \Lambda) + \eta_j(c, \Lambda) - f_j\Big]\,y_j(c, \Lambda) = 0. \tag{16}$$

If we have $f_j > \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then constraints (11) imply that $\eta_j(c, \Lambda) > 0$ and we obtain $y_j(c, \Lambda) = \Lambda_j$ by (15). On the other hand, if we have $f_j < \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then nonnegativity of $\eta_j(c, \Lambda)$ implies that $\sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda) + \eta_j(c, \Lambda) - f_j > 0$ and we obtain $y_j(c, \Lambda) = 0$ by (16). Using these observations in the objective function of problem (2)-(4) shows that (13) holds, whereas using these observations in constraints (3) shows that (14) holds. □

We let $D_0^\alpha(\alpha c)$ be the expected revenue obtained by the deterministic bid price heuristic for problem P^α. In the next proposition, we show that

$$\lim_{\alpha \to \infty} \frac{D_0^\alpha(\alpha c)}{V_0^\alpha(\alpha c)} = 1.$$

Therefore, the expected performance of the deterministic bid price heuristic becomes asymptotically optimal as the capacities on the flight legs and the expected numbers of product requests increase linearly with the same rate. Our proof follows the proof of Theorem 1 in Talluri and van Ryzin (1998), but we do not smooth the problem by assuming that the revenues from accepting the product requests are random variables with continuous distributions. Instead, we directly work with the optimality conditions of a nonsmooth problem.

Proposition 3 *We have* $\lim_{\alpha \to \infty} D_0^\alpha(\alpha c) / V_0^\alpha(\alpha c) = 1$.

Proof Using (5) on problem P^α and the fact that $V_0^\alpha(\alpha c)$ is the maximum expected profit over the whole planning horizon, we have $D_0^\alpha(\alpha c) / [\alpha\, L(c, \Lambda)] = D_0^\alpha(\alpha c) /\, L(\alpha c, \alpha \Lambda) \leq D_0^\alpha(\alpha c) / V_0^\alpha(\alpha c) \leq 1$. Therefore, it is enough to show that $\lim_{\alpha \to \infty} D_0^\alpha(\alpha c) / [\alpha\, L(c, \Lambda)] \geq 1$.

By the discussion at the beginning of this section, the deterministic bid price heuristic for problem P^α uses the bid prices $\{\mu_i(c, \Lambda) : i \in \mathcal{L}\}$ and the product acceptance probabilities $\{y_j(c, \Lambda)/\Lambda_j : j \in \mathcal{J}\}$. Letting $f_\phi = \max_{j \in \mathcal{J}} f_j$, we consider a variant of the deterministic bid price heuristic for a variant of problem P^α. In the new heuristic, if we have $f_j > \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then we accept a request for product j and collect a revenue of f_j, irrespective of the capacity availability. Similarly, if we have $f_j = \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then we accept a request for product j with probability $y_j(c, \Lambda_j)/\Lambda_j$ and collect a revenue of f_j, irrespective of the capacity availability. Finally, if we have $f_j < \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then we reject a request for product j. In the new heuristic, however, we incur a cost of f_ϕ for each unit of leg capacity sold in excess of the leg capacities $\{\alpha c_i : i \in \mathcal{L}\}$.

Following the proof of Theorem 1 in Talluri and van Ryzin (1998), one can see that the new heuristic incurs a catastrophic cost whenever it accepts a product request that violates the capacity availability. Furthermore, accepting this product request leaves the system with even less capacity. Therefore, the expected revenue obtained by the new heuristic is smaller than the expected revenue obtained by the deterministic bid price heuristic. That is, letting $H_0^\alpha(\alpha c)$ be the expected revenue obtained by the new heuristic for problem P^α, we have $H_0^\alpha(\alpha c) \leq D_0^\alpha(\alpha c)$.

We let A_j be the number of requests for product j accepted by the new heuristic. If we have $f_j > \sum_{i \in \mathcal{L}} a_{ij}\, \mu_i(c, \Lambda)$, then A_j has a Poisson distribution with mean $\alpha \Lambda_j$, whereas if we have $f_j =$

$\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda)$, then A_j has a Poisson distribution with mean $\alpha\Lambda_j\,[y_j(c,\Lambda)/\Lambda_j]$. Therefore, we have

$$\begin{aligned} H_0^\alpha(\alpha c) &= \sum_{j\in\mathcal{J}} f_j\,\mathbb{E}\big\{A_j\big\} - f_\phi \sum_{i\in\mathcal{L}} \mathbb{E}\big\{\big[\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j - \alpha c_i\big]^+\big\} \\ &= \sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,\alpha\,\Lambda_j \\ &\quad + \sum_{j\in\mathcal{J}} f_j\,\mathbf{1}(f_j = \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,\alpha\,y_j(c,\Lambda) - f_\phi \displaystyle\sum_{i\in\mathcal{L}} \mathbb{E}\big\{\big[\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j - \alpha c_i\big]^+\big\} \\ &= \alpha\,L(c,\Lambda) - f_\phi \sum_{i\in\mathcal{L}} \mathbb{E}\big\{\big[\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j - \alpha c_i\big]^+\big\} \le D_0^\alpha(\alpha c), \end{aligned} \tag{17}$$

where the third equality follows from (13), the inequality follows from the fact that $H_0^\alpha(\alpha c) \le D_0^\alpha(\alpha c)$ and we use $[\cdot]^+ = \max\{\cdot, 0\}$.

Since A_j has a Poisson distribution, we have

$$Var\big\{A_j\big\} = \mathbb{E}\big\{A_j\big\} = \mathbf{1}(f_j > \textstyle\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,\alpha\,\Lambda_j + \mathbf{1}(f_j = \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,\alpha\,y_j(c,\Lambda),$$

in which case (14) implies that $\mathbb{E}\big\{\sum_{j\in\mathcal{J}} a_{ij}\,A_j\big\} \le \alpha c_i$. For a random variable Z with finite first two moments, Talluri and van Ryzin (1998) show that $\mathbb{E}\big\{\big[Z - z\big]^+\big\} \le \sqrt{Var\big\{Z\big\}}/2$ holds for all $z \ge \mathbb{E}\big\{Z\big\}$. Therefore, letting $\Omega_j = \mathbf{1}(f_j > \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,\Lambda_j + \mathbf{1}(f_j = \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda))\,y_j(c,\Lambda)$ and using the fact that the arrival processes for requests for different products are independent, we obtain

$$\mathbb{E}\big\{\big[\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j - \alpha c_i\big]^+\big\} \le \dfrac{1}{2}\sqrt{Var\big\{\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j\big\}} = \dfrac{1}{2}\sqrt{\displaystyle\sum_{j\in\mathcal{J}} a_{ij}^2\,Var\big\{A_j\big\}} = \dfrac{1}{2}\sqrt{\displaystyle\sum_{j\in\mathcal{J}} a_{ij}^2\,\alpha\,\Omega_j}.$$

Using this relationship in (17), we obtain

$$\alpha\,L(c,\Lambda) - \frac{f_\phi}{2}\sum_{i\in\mathcal{L}}\sqrt{\sum_{j\in\mathcal{J}} a_{ij}^2\,\alpha\,\Omega_j} \le \alpha\,L(c,\Lambda) - f_\phi \sum_{i\in\mathcal{L}} \mathbb{E}\big\{\big[\textstyle\sum_{j\in\mathcal{J}} a_{ij}\,A_j - \alpha c_i\big]^+\big\} \le D_0^\alpha(\alpha c).$$

We obtain $\lim_{\alpha\to\infty} D_0^\alpha(\alpha c)/[\alpha\,L(c,\Lambda)] \ge 1$ by dividing the expression above by $\alpha\,L(c,\Lambda)$ and taking the limit. □

Although Proposition 3 is not the final result that we are interested in, this result, by itself, shows the asymptotic optimality of the bid prices obtained by solving problem (2)-(4) under assumptions that are different from those in Talluri and van Ryzin (1998). Talluri and van Ryzin (1998) smooth the problem by assuming that $\big\{f_j : j \in \mathcal{J}\big\}$ are random variables with continuous distributions. Proposition 3 does not require this assumption. Instead, the deterministic bid price heuristic uses a randomized decision rule to accept or reject a request for product j when we have $f_j = \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda)$, but it uses a deterministic decision rule to accept or reject a request for product j when we have $f_j \ne \sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(c,\Lambda)$. We feel that using a randomized decision rule, rather than assuming that $\big\{f_j : j \in \mathcal{J}\big\}$ are random variables, may be more appropriate in certain cases. We also emphasize that Lemma 2 and Proposition 3 continue to hold when the optimal dual solution to problem (2)-(4) is not unique.

The next proposition shows that (9) holds.

Proposition 4 *We have* $\lim_{\alpha\to\infty} \mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\} / V_0^\alpha(\alpha c) = 1$.

Proof We show that $\lim_{\alpha\to\infty} \mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\} / D_0^\alpha(\alpha c) \geq 1$ and the result follows from Proposition 3. The only difference between the deterministic and randomized bid price heuristics is in the bid prices they use, which implies that $\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K})) \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) = \mu(c,\Lambda)\big\} = D_0^\alpha(\alpha c)$. Furthermore, the randomized bid price heuristic obtains a nonnegative revenue. Therefore, Lemma 1 implies that

$$
\begin{aligned}
&E\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\} \\
&\qquad = \mathbb{P}\big\{\bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) = \mu(c,\Lambda)\big\}\, D_0^\alpha(\alpha c) \\
&\qquad\qquad + \big[1 - \mathbb{P}\big\{\bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) = \mu(c,\Lambda)\big\}\big] \\
&\qquad\qquad\qquad \times \mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K})) \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) \neq \mu(c,\Lambda)\big\} \\
&\qquad \geq \prod_{j\in\mathcal{J}} \left[1 - \frac{\Lambda_j}{\alpha\,\epsilon}\right]^K D_0^\alpha(\alpha c).
\end{aligned}
$$

The result follows by dividing the expression above by $D_0^\alpha(\alpha c)$ and taking the limit. □

For three test problems, Table 1 shows the ratio $\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\} / [\alpha\, L(c,\Lambda)]$ for different values of α. We approximate all expectations through simulation. Using the fact that $V_0^\alpha(\alpha c) \leq L(\alpha c, \alpha\Lambda) = \alpha\, L(c,\Lambda)$, we have

$$
\frac{\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\}}{\alpha\, L(c,\Lambda)} \leq \frac{\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\}}{V_0^\alpha(\alpha c)} \leq 1.
$$

Table 1 indicates that the first ratio above gets very close to 1 as α increases. This implies that $\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\}$ gets very close to $V_0^\alpha(\alpha c)$ as α increases.

It is possible to show that Proposition 4 holds under the assumption that problem (2)-(4) has a unique optimal primal solution, but not necessarily a unique optimal dual solution. To illustrate the idea, we let $\mathcal{M}(c,\Lambda)$ be the set of possible optimal values of the dual variables associated with constraints (3) in problem (2)-(4). Since a convex combination of the optimal dual solutions to problem (2)-(4) is another optimal dual solution, Lemma 1 can be modified to show that there exists $\epsilon > 0$ such that

$$
\mathbb{P}\big\{\bar\mu(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}) \in \mathcal{M}(c,\Lambda)\big\} \geq \prod_{j\in\mathcal{J}} \left[1 - \frac{\Lambda_j}{\alpha\,\epsilon}\right]^K .
$$

Since the unique optimal primal solution $\big\{y_j(c,\Lambda) : j \in \mathcal{J}\big\}$ to problem (2)-(4), together with any optimal dual solution, satisfies the complementary slackness conditions in (15) and (16), Lemma 2 and Propositions 3 and 4 go through with the interpretation that $\mu(c,\Lambda)$ is an element of the set $\mathcal{M}(c,\Lambda)$.

4 Computational Experiments

In this section, we compare the randomized bid price heuristic with other solution methods. The computational experiments in Talluri and van Ryzin (1999) show that the randomized bid price heuristic can provide a small but noticeable improvement over the deterministic bid price heuristic. Our goal is to investigate how different problem parameters contribute to the performance gap and how the choice of K affects the performance of the randomized bid price heuristic.

4.1 Benchmark Strategies and Experimental Setup

We compare the performances of three solution methods.

Randomized Bid Price Heuristic (RBP). This is the solution method that we focus on in this paper, but our implementation recomputes the bid prices whenever we make a decision. In particular, we let $N_j(t,\tau)$ be the number of requests for product j over the time interval $[t,\tau]$. Given the remaining leg capacities $\{x_{it} : i \in \mathcal{L}\}$ at time t, we replace the right side of constraints (7) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (8) with a sample of $\{N_j(t,\tau) : j \in \mathcal{J}\}$, and solve problem (6)-(8). We repeat this K times. The mean of the optimal values of the dual variables associated with constraints (7) give the bid prices at time t.

Deterministic Bid Price Heuristic (DBP). Similar to RBP, our implementation of DBP recomputes the bid prices whenever we make a decision. In particular, given the remaining leg capacities $\{x_{it} : i \in \mathcal{L}\}$ at time t, we replace the right side of constraints (3) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (4) with $\{\int_t^\tau \lambda_j(s)\,ds : j \in \mathcal{J}\}$, and solve problem (2)-(4). The optimal values of the dual variables associated with constraints (3) give the bid prices at time t.

We note that if problem (6)-(8) or (2)-(4) has multiple optimal dual solutions, then we simply use the optimal dual solution provided by the linear programming package that we use. This appears to be the standard approach in practice; see Talluri and van Ryzin (2004).

Deterministic Bid Price Heuristic with Finite Differences (FD). This solution method tries to capture the total opportunity cost of the leg capacities consumed by a product request more accurately; see Bertsimas and Popescu (2003). Letting $\Lambda_j(t,\tau) = \int_t^\tau \lambda_j(s)\,ds$ and $\Lambda(t,\tau) = \{\Lambda_j(t,\tau) : j \in \mathcal{J}\}$, we first compute $L(x_t, \Lambda(t,\tau))$ by replacing the right side of constraints (3) with $\{x_{it} : i \in \mathcal{L}\}$ and the right side of constraints (4) with $\{\Lambda_j(t,\tau) : j \in \mathcal{J}\}$ in problem (2)-(4). We then replace the right side of constraints (3) with $\{x_{it} - a_{ij} : i \in \mathcal{L}\}$ and resolve problem (2)-(4) to obtain the optimal objective value $L_j^-(x_t, \Lambda(t,\tau))$. We follow the decision rule for DBP, but use $L(x_t, \Lambda(t,\tau)) - L_j^-(x_t, \Lambda(t,\tau))$ instead of $\sum_{i\in\mathcal{L}} a_{ij}\,\mu_i(x_t, \Lambda(t,\tau))$.

In our test problems, we consider two types of airline networks serving N spokes out of a single hub. In the first airline network, there are two flights associated with each spoke. One of these flights is out of the hub and the other one is to the hub. There is a high-fare and a low-fare product associated with each possible origin-destination pair. The revenue of a high-fare product is κ times larger than that of a low-fare product. Therefore, the first airline network involves $2N$ flight legs and $2N(N+1)$ products, $4N$ of which include one flight leg and $2N(N-1)$ of which include two flight legs. Figure 1.a shows the first airline network for the case where $N = 8$. In the second airline network, there is one flight associated with each spoke. For the first half of the spokes, this flight is to the hub and for the second half of the spokes, this flight is from the hub. Similar to the first airline network, there is a high-fare and a low-fare product associated with each possible origin-destination pair. Therefore, the second airline network involves N flight legs and $N(N/2+2)$ products, $2N$ of which include one flight leg and $2(N/2)^2$ of which include two flight legs. Figure 1.b shows the second airline network for the case where $N = 8$. For both airline networks, the arrival rates for the high-fare products increase over time, whereas the arrival rates for the low-fare products decrease over time.

Since the expected demand for the capacity on flight leg i is $\sum_{j \in \mathcal{J}} a_{ij} \Lambda_j$, we measure the tightness of the leg capacities by $\psi = \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} a_{ij} \Lambda_j / \sum_{i \in \mathcal{L}} c_i$. This measure of congestion is referred to as the demand factor in network revenue management vocabulary. We label our test problems by $(N, \psi, \kappa) \in \{4, 8, 12\} \times \{0.8, 0.9, 1.0, 1.2, 1.6\} \times \{2, 4, 8\}$. Unless stated otherwise, we use $K = 50$ for RBP.

4.2 Computational Results

Tables 2 and 3 respectively show the results for the first and second airline network. The second and third columns in these tables give $\mathbb{E}\{L(c, N)\}$ and $L(c, \Lambda)$, which are respectively the expected value of the optimal objective value of problem (6)-(8) and the optimal objective value of problem (2)-(4). By (5), both $\mathbb{E}\{L(c, N)\}$ and $L(c, \Lambda)$ give upper bounds on the maximum expected revenue. Since it is impossible to compute the expectation, we provide a confidence interval for $\mathbb{E}\{L(c, N)\}$ at 99% significance level by using 10,000 samples. The fourth, fifth and sixth columns give the expected revenues obtained by RBP, DBP and FD. We estimate these expected revenues by simulating the performances of the three solution methods for multiple demand trajectories. The seventh column gives the percent gap between the expected revenues obtained by RBP and DBP. The eighth column includes a "✓" if RBP performs better than DBP, a "×" if DBP performs better than RBP and a "○" if there does not exist a statistically significant difference between the performances of RBP and DBP at 95% significance level. We use common random numbers when simulating the performances of the three solution methods for multiple demand trajectories and use the pairwise t-test to make comparisons; see Law and Kelton (2000). The ninth and tenth columns compare the performances of RBP and FD. Their interpretations are similar to those of the seventh and eighth columns. The last three columns give the seat utilizations obtained by the three solution methods. This performance measure is referred to as the load factor in network revenue management vocabulary and is computed as the ratio of the total number of seats sold on all flight legs to the total available capacity on all flight legs. Similar to the demand factor, the load factor is a measure of congestion, but its value depends on the solution method used, whereas the value of the demand factor depends only on the problem data.

The results in Tables 2 and 3 indicate that RBP performs noticeably better than DBP and at least comparable to FD. The performance gap between RBP and DBP becomes more pronounced for test problems with tight leg capacities and large fare differences between high-fare and low-fare products. The tightness of the leg capacities also appears in some of the experiments in Talluri and van Ryzin (1999) as a factor that impacts the performance gap between RBP and DBP; see Figures 1-4 in Talluri and van Ryzin (1999). In 67 out of 90 test problems, the performance of RBP is significantly better than that of DBP, whereas in only 1 out of 90 test problems, the performance of DBP is significantly better than that of RBP. Similarly, in 41 out of 90 test problems, the performance of RBP is significantly better than that of FD, whereas in only 7 out of 90 test problems, the performance of FD is significantly better than that of RBP. An interesting observation is that RBP almost always obtains lower seat utilization than does DBP. Therefore, the superior performance of RBP seems to be not due to accepting more requests, but due to accepting more requests for high-fare products. We also note that FD obtains lower seat utilization than does RBP whenever it performs significantly better than RBP. Finally, we observe that the seat utilization obtained by RBP tends to decrease as the fare difference between high-fare and low-fare products increases. A possible explanation for this behavior is that the bid prices obtained by RBP allocate more seats for high-fare products as the fare difference between high-fare and low-fare products increases. Since the requests for high-fare products tend to arrive later in the planning horizon, this results in a larger number of unutilized seats.

For the first airline network, Figure 2 shows how the relative performances of the three solution methods change with different problem parameters. The thick data series in this figure show the ratios of the expected revenues obtained by RBP and DBP, whereas the thin data series show the ratios of the expected revenues obtained by RBP and FD. A block of five consecutive test problems in the horizontal axis share the same problem characteristics other than the tightness of the leg capacities. Consequently, the saw tooth pattern of the thick data series indicates that tight leg capacities cause the performance gap between RBP and DBP to grow. For test problems with extremely tight leg capacities and large fare differences between high-fare and low-fare products, the expected revenues obtained by RBP are up to 8% better than those obtained by DBP. The thin data series indicate that FD may perform better than RBP for some test problems, but the performance gap is not large except for one or two cases. For test problems with relatively loose leg capacities and small fare differences between high-fare and low-fare products, the performances of the three solution methods are somewhat comparable. Lastly, the dashed data series in Figure 2 show $\mathbb{E}\big\{L(c,N)\big\}/L(c,\Lambda)$ and indicate that RBP provides up to 6% tighter upper bounds on the maximum expected revenue than does DBP. Similar observations hold for the second airline network.

Table 4 shows the CPU seconds required to simulate the performances of RBP, DBP and FD for one demand trajectory on a Pentium IV PC running Windows XP with 2.4 GHz of CPU and 1 GB of RAM. These CPU seconds include the time required to sample the product requests, solve the linear programs and make the acceptance or rejection decisions over the whole planning horizon. Since the structure of the airline network and the number of spokes are the primary factors that affect the CPU seconds, we only show the average CPU seconds over different test problems. When solving problem (6)-(8) for a particular sample of product requests, we use the optimal basis from a previous sample as a starting basis and this noticeably boosts the performance of RBP. However, the CPU seconds for RBP are still, as expected, significantly higher than those for DBP and FD.

The two charts in Figure 3 show how the performance of RBP changes with the number of samples for 12 typical test problems. The value of K ranges in the set $\big\{1,2,3,5,10,25,50,100,250\big\}$ and each data series in Figure 3 show the expected revenues obtained by RBP for a particular test problem for different values of K. The figure indicates that the benefit from using more than 50 samples is marginal. Our findings are in agreement with those of Talluri and van Ryzin (1999) where the authors suggest using $K = 30$. Although we do not report the details here, we note that RBP obtains better expected revenues than does DBP for almost all of our test problems by using 25 or fewer samples.

To conclude, RBP emerges as a robust solution method that performs better than both DBP and FD on a majority of our test problems. Implementing RBP is only slightly more involved than implementing DBP and using a relatively small number of samples is enough for RBP to obtain good solutions. Furthermore, by (5), RBP always provides a tighter upper bound on the maximum expected revenue than does DBP. Finally, RBP preserves the bid price structure. In particular, RBP characterizes a decision rule by the parameters $\{\bar{\mu}_i(c, \hat{N}^1, \ldots, \hat{N}^K) : i \in \mathcal{L}\}$, whereas FD characterizes a decision rule by the parameters $\{L(c, \Lambda) - L_j^-(c, \Lambda) : j \in \mathcal{J}\}$. In most settings, $|\mathcal{J}|$ is much larger than $|\mathcal{L}|$.

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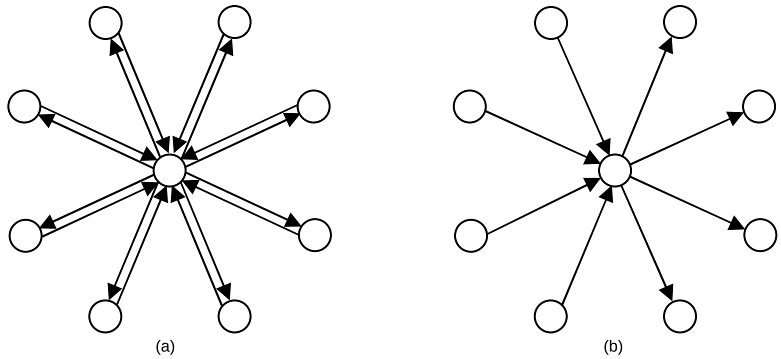

Figure 1: The airline networks that we use in our computational experiments for the case where $N = 8$.

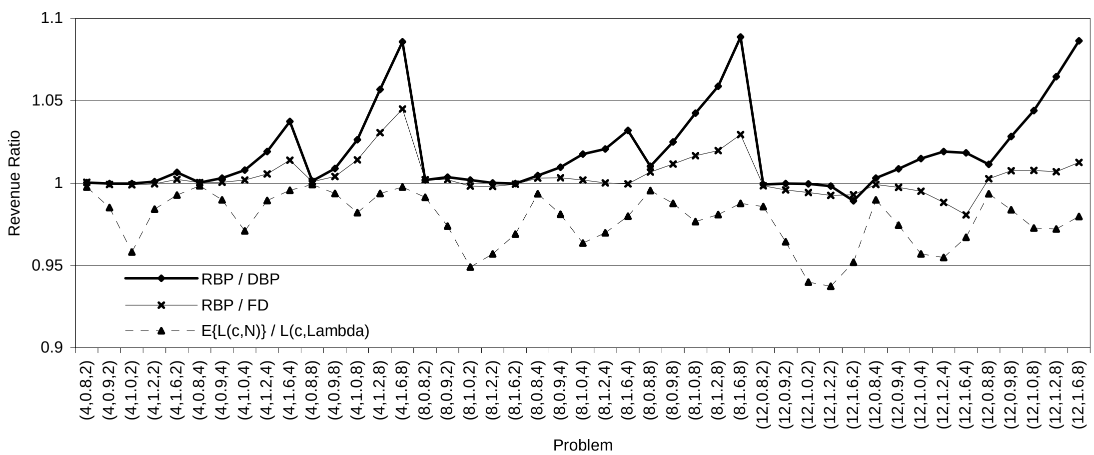

Figure 2: Comparison of the performances of RBP, DBP and FD along with the upper bounds obtained by RBP and DBP for the first airline network.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 4$	$\alpha = 8$	$\alpha = 16$	$\alpha = 32$
Problem 1	0.931	0.956	0.970	0.979	0.984	0.990
Problem 2	0.924	0.952	0.969	0.978	0.985	0.991
Problem 3	0.918	0.947	0.968	0.977	0.984	0.989

Table 1: The ratio $\mathbb{E}\big\{B_0^\alpha(\alpha c \,|\, \bar{\mu}(\alpha c, N^{\alpha 1}, \ldots, N^{\alpha K}))\big\}/[\alpha\, L(c, \Lambda)]$ as a function of α.

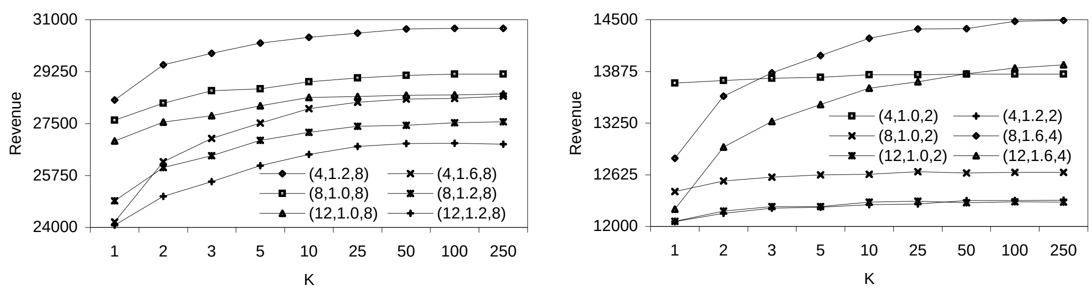

Figure 3: Performance of RBP with different values for K on 12 typical test problems that take place over the first airline network.

	Upper Bounds		Expected Revenues			RBP vs. DBP		RBP vs. FD		Seat Utilization		
Problem	$\mathbb{E}\{L(c,\Lambda)\}$	$L(c,\Lambda)$	RBP	DBP	FD	% Diff.	Sig.	% Diff.	Sig.	RBP	DBP	FD
$(4,0.8,2)$	$15,004 \mp 13$	15,042	14,887	14,884	14,881	0.03	$\circ$	0.04	$\circ$	0.80	0.80	0.79
$(4,0.9,2)$	$14,819 \mp 12$	15,042	14,559	14,563	14,572	-0.03	$\circ$	-0.09	$\circ$	0.88	0.88	0.87
$(4,1.0,2)$	$14,382 \mp 11$	15,011	13,840	13,844	13,854	-0.03	$\circ$	-0.10	$\circ$	0.92	0.92	0.92
$(4,1.2,2)$	$13,151 \mp 9$	13,362	12,299	12,288	12,304	0.09	$\circ$	-0.04	$\circ$	0.95	0.96	0.94
$(4,1.6,2)$	$10,930 \mp 8$	11,010	10,043	9,977	10,020	0.66	$\checkmark$	0.23	$\circ$	0.95	0.96	0.93
$(4,0.8,4)$	$21,525 \mp 26$	21,562	21,302	21,295	21,296	0.03	$\circ$	0.03	$\circ$	0.79	0.80	0.79
$(4,0.9,4)$	$21,340 \mp 26$	21,562	20,862	20,802	20,854	0.29	$\checkmark$	0.04	$\circ$	0.87	0.88	0.87
$(4,1.0,4)$	$20,904 \mp 25$	21,531	19,936	19,780	19,899	0.78	$\checkmark$	0.18	$\circ$	0.92	0.92	0.91
$(4,1.2,4)$	$19,672 \mp 23$	19,882	18,217	17,873	18,116	1.89	$\checkmark$	0.55	$\checkmark$	0.94	0.95	0.94
$(4,1.6,4)$	$17,452 \mp 22$	17,530	15,920	15,345	15,701	3.62	$\checkmark$	1.38	$\checkmark$	0.92	0.95	0.93
$(4,0.8,8)$	$34,569 \mp 56$	34,602	34,159	34,118	34,130	0.12	$\checkmark$	0.08	$\checkmark$	0.79	0.80	0.79
$(4,0.9,8)$	$34,383 \mp 55$	34,602	33,570	33,275	33,438	0.88	$\checkmark$	0.39	$\checkmark$	0.87	0.88	0.87
$(4,1.0,8)$	$33,947 \mp 54$	34,571	32,463	31,632	32,013	2.56	$\checkmark$	1.39	$\checkmark$	0.91	0.92	0.91
$(4,1.2,8)$	$32,715 \mp 53$	32,922	30,671	29,022	29,759	5.38	$\checkmark$	2.97	$\checkmark$	0.92	0.95	0.93
$(4,1.6,8)$	$30,494 \mp 52$	30,570	28,323	26,086	27,105	7.90	$\checkmark$	4.30	$\checkmark$	0.90	0.95	0.93
$(8,0.8,2)$	$14,092 \mp 14$	14,216	13,814	13,789	13,786	0.19	$\checkmark$	0.20	$\checkmark$	0.78	0.78	0.78
$(8,0.9,2)$	$13,844 \mp 13$	14,216	13,334	13,287	13,307	0.36	$\checkmark$	0.20	$\circ$	0.84	0.84	0.84
$(8,1.0,2)$	$13,439 \mp 12$	14,161	12,641	12,618	12,664	0.19	$\circ$	-0.18	$\circ$	0.88	0.88	0.87
$(8,1.2,2)$	$12,502 \mp 11$	13,064	11,441	11,439	11,464	0.02	$\circ$	-0.20	$\circ$	0.91	0.91	0.88
$(8,1.6,2)$	$10,641 \mp 9$	10,982	9,457	9,460	9,463	-0.03	$\circ$	-0.06	$\circ$	0.93	0.92	0.88
$(8,0.8,4)$	$19,976 \mp 26$	20,107	19,469	19,379	19,410	0.46	$\checkmark$	0.30	$\checkmark$	0.78	0.78	0.78
$(8,0.9,4)$	$19,727 \mp 26$	20,107	18,808	18,630	18,748	0.94	$\checkmark$	0.32	$\circ$	0.83	0.84	0.83
$(8,1.0,4)$	$19,321 \mp 25$	20,052	18,005	17,694	17,972	1.73	$\checkmark$	0.19	$\circ$	0.87	0.88	0.86
$(8,1.2,4)$	$18,378 \mp 24$	18,952	16,600	16,264	16,598	2.02	$\checkmark$	0.01	$\circ$	0.89	0.91	0.88
$(8,1.6,4)$	$16,495 \mp 22$	16,833	14,420	13,974	14,425	3.09	$\checkmark$	-0.04	$\circ$	0.89	0.92	0.88
$(8,0.8,8)$	$31,745 \mp 54$	31,889	30,876	30,564	30,671	1.01	$\checkmark$	0.66	$\checkmark$	0.77	0.78	0.78
$(8,0.9,8)$	$31,494 \mp 54$	31,889	30,065	29,333	29,718	2.43	$\checkmark$	1.15	$\checkmark$	0.83	0.84	0.83
$(8,1.0,8)$	$31,086 \mp 53$	31,835	29,099	27,913	28,624	4.08	$\checkmark$	1.63	$\checkmark$	0.86	0.88	0.86
$(8,1.2,8)$	$30,142 \mp 52$	30,727	27,477	25,950	26,945	5.56	$\checkmark$	1.94	$\checkmark$	0.87	0.91	0.88
$(8,1.6,8)$	$28,255 \mp 51$	28,608	25,202	23,146	24,479	8.16	$\checkmark$	2.87	$\checkmark$	0.85	0.92	0.87
$(12,0.8,2)$	$14,020 \mp 14$	14,223	13,592	13,604	13,613	-0.09	$\circ$	-0.16	$\circ$	0.76	0.76	0.76
$(12,0.9,2)$	$13,716 \mp 14$	14,223	12,949	12,953	13,002	-0.03	$\circ$	-0.42	$\times$	0.82	0.82	0.81
$(12,1.0,2)$	$13,325 \mp 13$	14,177	12,302	12,307	12,373	-0.04	$\circ$	-0.58	$\times$	0.85	0.85	0.82
$(12,1.2,2)$	$12,368 \mp 12$	13,194	11,046	11,067	11,129	-0.19	$\circ$	-0.75	$\times$	0.88	0.88	0.84
$(12,1.6,2)$	$10,622 \mp 10$	11,159	9,124	9,223	9,190	-1.09	$\times$	-0.72	$\times$	0.90	0.90	0.84
$(12,0.8,4)$	$19,959 \mp 27$	20,165	19,209	19,151	19,227	0.30	$\checkmark$	-0.09	$\circ$	0.76	0.76	0.76
$(12,0.9,4)$	$19,650 \mp 26$	20,165	18,323	18,167	18,370	0.85	$\checkmark$	-0.26	$\circ$	0.81	0.82	0.80
$(12,1.0,4)$	$19,251 \mp 26$	20,119	17,511	17,254	17,599	1.46	$\checkmark$	-0.50	$\times$	0.84	0.85	0.83
$(12,1.2,4)$	$18,268 \mp 25$	19,132	16,031	15,731	16,221	1.87	$\checkmark$	-1.19	$\times$	0.86	0.88	0.84
$(12,1.6,4)$	$16,458 \mp 23$	17,020	13,825	13,575	14,098	1.80	$\checkmark$	-1.98	$\times$	0.86	0.88	0.83
$(12,0.8,8)$	$31,840 \mp 55$	32,050	30,612	30,271	30,534	1.12	$\checkmark$	0.26	$\circ$	0.76	0.76	0.76
$(12,0.9,8)$	$31,531 \mp 54$	32,050	29,453	28,641	29,235	2.76	$\checkmark$	0.74	$\checkmark$	0.80	0.82	0.80
$(12,1.0,8)$	$31,131 \mp 54$	32,004	28,458	27,259	28,243	4.21	$\checkmark$	0.76	$\checkmark$	0.82	0.85	0.82
$(12,1.2,8)$	$30,146 \mp 53$	31,014	26,789	25,161	26,603	6.08	$\checkmark$	0.69	$\checkmark$	0.84	0.87	0.84
$(12,1.6,8)$	$28,317 \mp 52$	28,902	24,441	22,499	24,140	7.95	$\checkmark$	1.23	$\checkmark$	0.82	0.88	0.82

Table 2: Performances of RBP, DBP and FD for the first airline network.

	Upper Bounds		Expected Revenues			RBP vs. DBP		RBP vs. FD		Seat Utilization		
Problem	$\mathbb{E}\{L(c,\Lambda)\}$	$L(c,\Lambda)$	RBP	DBP	FD	% Diff.	Sig.	% Diff.	Sig.	RBP	DBP	FD
$(4, 0.8, 2)$	$14,927 \mp 10$	14,931	14,929	14,930	14,930	-0.01	○	-0.01	○	0.80	0.80	0.80
$(4, 0.9, 2)$	$14,870 \mp 10$	14,931	14,844	14,843	14,847	0.01	○	-0.02	○	0.89	0.89	0.89
$(4, 1.0, 2)$	$14,538 \mp 9$	14,925	14,335	14,321	14,321	0.10	○	0.10	○	0.95	0.95	0.95
$(4, 1.2, 2)$	$13,115 \mp 7$	13,203	12,707	12,655	12,682	0.41	✓	0.19	✓	0.97	0.97	0.96
$(4, 1.6, 2)$	$10,852 \mp 6$	10,888	10,430	10,331	10,365	0.95	✓	0.62	✓	0.95	0.96	0.95
$(4, 0.8, 4)$	$20,981 \mp 21$	20,985	20,956	20,956	20,956	0.00	○	0.00	○	0.80	0.80	0.80
$(4, 0.9, 4)$	$20,924 \mp 21$	20,985	20,817	20,800	20,822	0.09	✓	-0.02	○	0.89	0.89	0.89
$(4, 1.0, 4)$	$20,592 \mp 20$	20,979	20,169	20,064	20,127	0.52	✓	0.20	✓	0.95	0.95	0.95
$(4, 1.2, 4)$	$19,169 \mp 19$	19,257	18,458	18,114	18,284	1.86	✓	0.94	✓	0.95	0.97	0.96
$(4, 1.6, 4)$	$16,906 \mp 18$	16,942	16,176	15,702	15,898	2.93	✓	1.72	✓	0.94	0.96	0.95
$(4, 0.8, 8)$	$33,090 \mp 45$	33,093	33,002	32,997	32,996	0.02	○	0.02	○	0.80	0.80	0.80
$(4, 0.9, 8)$	$33,033 \mp 45$	33,093	32,796	32,711	32,770	0.26	✓	0.08	○	0.89	0.89	0.89
$(4, 1.0, 8)$	$32,701 \mp 45$	33,086	31,996	31,548	31,740	1.40	✓	0.80	✓	0.94	0.95	0.95
$(4, 1.2, 8)$	$31,278 \mp 43$	31,365	30,244	29,014	29,491	4.07	✓	2.49	✓	0.94	0.97	0.96
$(4, 1.6, 8)$	$29,015 \mp 43$	29,050	27,945	26,430	26,967	5.42	✓	3.50	✓	0.93	0.96	0.95
$(8, 0.8, 2)$	$13,759 \mp 15$	13,775	13,688	13,681	13,690	0.05	○	-0.01	○	0.79	0.79	0.80
$(8, 0.9, 2)$	$13,652 \mp 15$	13,775	13,494	13,464	13,467	0.22	✓	0.20	✓	0.87	0.87	0.87
$(8, 1.0, 2)$	$13,380 \mp 14$	13,752	13,047	12,947	12,984	0.77	✓	0.48	✓	0.90	0.92	0.91
$(8, 1.2, 2)$	$12,601 \mp 13$	12,818	12,081	11,933	12,030	1.23	✓	0.43	✓	0.94	0.95	0.93
$(8, 1.6, 2)$	$10,767 \mp 9$	10,908	10,040	9,948	10,005	0.92	✓	0.35	○	0.95	0.96	0.94
$(8, 0.8, 4)$	$19,635 \mp 28$	19,651	19,526	19,504	19,522	0.11	✓	0.02	○	0.80	0.80	0.80
$(8, 0.9, 4)$	$19,528 \mp 28$	19,651	19,248	19,156	19,196	0.48	✓	0.27	✓	0.87	0.87	0.87
$(8, 1.0, 4)$	$19,255 \mp 28$	19,628	18,701	18,419	18,556	1.51	✓	0.78	✓	0.90	0.92	0.91
$(8, 1.2, 4)$	$18,467 \mp 26$	18,695	17,580	17,158	17,452	2.40	✓	0.73	✓	0.92	0.94	0.93
$(8, 1.6, 4)$	$16,614 \mp 24$	16,776	15,386	14,893	15,277	3.20	✓	0.70	✓	0.93	0.95	0.93
$(8, 0.8, 8)$	$31,387 \mp 57$	31,405	31,209	31,150	31,191	0.19	✓	0.06	○	0.79	0.80	0.80
$(8, 0.9, 8)$	$31,280 \mp 57$	31,405	30,835	30,556	30,678	0.91	✓	0.51	✓	0.86	0.87	0.87
$(8, 1.0, 8)$	$31,005 \mp 57$	31,382	30,127	29,357	29,738	2.56	✓	1.29	✓	0.89	0.92	0.91
$(8, 1.2, 8)$	$30,206 \mp 56$	30,449	28,885	27,657	28,359	4.25	✓	1.82	✓	0.91	0.94	0.93
$(8, 1.6, 8)$	$28,346 \mp 54$	28,510	26,638	24,871	25,862	6.63	✓	2.91	✓	0.90	0.95	0.93
$(12, 0.8, 2)$	$13,834 \mp 15$	13,899	13,688	13,679	13,684	0.06	○	0.03	○	0.79	0.79	0.79
$(12, 0.9, 2)$	$13,629 \mp 14$	13,899	13,264	13,248	13,251	0.12	○	0.10	○	0.86	0.86	0.85
$(12, 1.0, 2)$	$13,265 \mp 13$	13,858	12,641	12,631	12,641	0.07	○	0.00	○	0.90	0.90	0.89
$(12, 1.2, 2)$	$12,364 \mp 11$	12,782	11,510	11,470	11,515	0.35	✓	-0.04	○	0.92	0.92	0.90
$(12, 1.6, 2)$	$10,483 \mp 9$	10,688	9,500	9,496	9,479	0.05	○	0.23	○	0.93	0.93	0.90
$(12, 0.8, 4)$	$19,589 \mp 28$	19,658	19,270	19,248	19,278	0.11	✓	-0.04	○	0.79	0.80	0.79
$(12, 0.9, 4)$	$19,383 \mp 27$	19,658	18,670	18,605	18,652	0.35	✓	0.10	○	0.86	0.86	0.85
$(12, 1.0, 4)$	$19,015 \mp 26$	19,617	17,891	17,753	17,860	0.77	✓	0.18	○	0.89	0.90	0.88
$(12, 1.2, 4)$	$18,099 \mp 25$	18,540	16,541	16,305	16,573	1.42	✓	-0.20	○	0.90	0.92	0.89
$(12, 1.6, 4)$	$16,162 \mp 23$	16,367	14,346	14,092	14,398	1.77	✓	-0.36	○	0.90	0.92	0.89
$(12, 0.8, 8)$	$31,102 \mp 57$	31,175	30,510	30,400	30,478	0.36	✓	0.11	○	0.79	0.79	0.79
$(12, 0.9, 8)$	$30,895 \mp 56$	31,175	29,693	29,350	29,504	1.16	✓	0.64	✓	0.85	0.86	0.85
$(12, 1.0, 8)$	$30,527 \mp 56$	31,134	28,762	28,048	28,394	2.48	✓	1.28	✓	0.88	0.90	0.88
$(12, 1.2, 8)$	$29,611 \mp 54$	30,058	27,203	26,083	26,877	4.12	✓	1.20	✓	0.88	0.92	0.89
$(12, 1.6, 8)$	$27,673 \mp 53$	27,885	24,917	23,420	24,389	6.01	✓	2.12	✓	0.87	0.92	0.89

Table 3: Performances of RBP, DBP and FD for the second airline network.

	First Airline Network			Second Airline Network		
	$N = 4$	$N = 8$	$N = 12$	$N = 4$	$N = 8$	$N = 12$
RBP	8.17	18.38	30.51	5.79	8.61	14.87
DBP	0.21	0.48	1.05	0.12	0.16	0.28
FD	0.44	1.16	2.17	0.25	0.39	0.59

Table 4: CPU seconds for RBP, DBP and FD.