A Unified Framework to Impose Market Share Constraints for Selected Product Classes: Randomized and Deterministic Assortments under the Multinomial Logit Model

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September 29, 2024

We consider assortment optimization problems with market share constraints. Each product has a fixed revenue. The products are partitioned into product classes. Each product class has a market share threshold. If a product class is represented in the offered assortment, meaning that we offer at least one product within the product class, then the total purchase probability of the products offered in the class should be above the market share threshold of the product class. Customers choose among all offered products according to the multinomial logit model. The goal is to maximize the expected revenue while satisfying the market share constraints. Our work is motivated by the fact that focusing only on maximizing the expected revenue often results in offering many products each commanding small demand quantities, which causes frustration for online retailers due to operational and managerial burden. Imposing the market share constraints only for the product classes represented in the offered assortment brings unique dynamics that have not been explored previously. We consider two variants. In the randomized variant, we randomize the offered assortment. In the deterministic variant, we offer a single assortment. The randomized variant is NP-hard, whereas the deterministic variant is NP-hard to approximate within a factor of $\frac{1}{2}$. Our main technical contributions are a fully polynomial-time approximation scheme for the randomized variant, an approximation scheme for the deterministic variant that yields a $(1-\epsilon)$-approximate solution while violating the market share constraints with a $(1+\epsilon)$-factor in running time that is polynomial in $\frac{1}{\epsilon}$, as well as a $\frac{1}{2}$-approximation algorithm for the deterministic variant that yields a solution satisfying the market share constraints exactly in running time that is pseudo-polynomial in the input size. To give these approximation schemes, we give a unified approximation framework that applies to both variants and leverage this approximate framework. Our computational experiments on a dataset from an online electronics retailer indicate that imposing our market share constraints is a natural way to avoid products commanding excessively small market shares.

1. Introduction

Using discrete choice models has steadily become a common approach for capturing the demand in revenue management. Discrete choice models allow us to capture the fact that customers choose and substitute among the available products, so if one product is not available for purchase, then a portion of the customers interested in this product substitute to another one. Given that we can model the customer choice and substitution process through a discrete choice model, a natural question is to choose an assortment of products to offer to maximize the expected revenue. There is significant literature indicating that building on discrete choice models to compute assortments of products to offer to customers yields significant increases in expected revenues; see Vulcano et al. (2010), Cao et al. (2022), Jagabathula et al. (2024). Even so, offering assortments that exclusively focus on maximizing the expected revenue creates undesirable side effects. Assortments that maximize the expected revenue often include a long list of so-called tail products, each commanding a small demand quantity. From the perspective of the retailer, tail products create unwarranted managerial burden in terms of, for example, placing procurement orders and maintaining relationships with vendors. Also, small demand volume often implies large demand variability, which results in large safety stocks. From the perspective of the customer, an excessive array of choices distracts customers during their choice process. One approach to avoid tail products is to limit the number of products in the offered assortments, but products commanding small demand quantities can still creep into the assortment under cardinality constraints. Thus, there is significant interest in constructing assortments with an explicit constraint on the market share of each product. Moreover, it is costly for the online retailers to form and maintain relationships with different sellers, to learn customs requirements for different origin countries and to build physical receiving infrastructure for different shipment modes. Thus, when choosing the assortments of products to offer to customers, it is important to ensure that groups of products coming from a certain seller, origin country or shipment mode command a large demand quantity. In this case, it becomes necessary to impose market share constraints on groups of products.

In this paper, we consider assortment optimization problems with market share constraints. There is a fixed revenue associated with each product. The products are partitioned into product classes. Each product class has a market share threshold. If we utilize a product class in the offered assortment, meaning that we offer a product within the product class, then the total purchase probability of the products offered in the class should be above the market share threshold of the product class. If we do not utilize a product class, then we do not need to satisfy its market share constraint. Customers choose among all offered products according to the multinomial logit model. Our goal is to pick an assortment of products to offer so as to maximize the expected revenue from a customer, while satisfying the market share constraints. If each product class includes a single product, then our problem imposes a market share constraint on each product separately, so that the purchase probability of a product, if offered, must be above a given threshold. We consider two variants. In the randomized variant, we pick a distribution over the universe of assortments and offer a random assortment sampled according to this distribution. The expected purchase probability of each product class needs to satisfy the market share threshold. In the deterministic variant, we pick a single assortment to offer.

Main Contributions: Our technical contributions are characterizing the complexity of the problem, bounding the value from randomization and giving approximation algorithms.

Model Formulation and Complexity. We formulate a new class of assortment optimization problems with market share constraints on product classes, ensuring that if the offered assortment includes any product in a particular product class, then the total purchase probability of the product class must be above a given market share threshold. We do not impose a market share constraint on the product class if the product class is not represented in the offered assortment. Each product class may include a single product, in which case, the purchase probability of a product, if offered, must exceed a given market share threshold. The novel feature of our market share constraints is that they *become active only when* a product in a particular product class is utilized. Although our market share constraints are quite natural, enforcing market share constraints only for the product classes represented in the offered assortment brings combinatorial aspects that were not explored before, to our knowledge. We show that the randomized variant is NP-hard, whereas the deterministic variant is NP-hard to approximate within a factor of $\frac{1}{2}$.

Value of Randomization. We show that if we randomize the offered assortment instead of offering a deterministic assortment, then we can increase the expected revenue arbitrarily. This result is, perhaps, not too surprising considering a single product class with a high revenue product H and a low revenue product L. Offering only product H by itself may not be enough to satisfy the market share constraint for the product class. Offering products H and L together may hurt the expected revenue. However, offering products H and L together with probability ϵ and offering product H by itself with probability $1-\epsilon$ may satisfy the market share constraint without hurting the expected revenue too much. Surprisingly, however, we show that if each product class includes a single product, imposing a market share constraint on each offered product separately, then a randomized assortment can improve the expected revenue by at most a tight factor of two.

Approximation Framework. We develop an overarching approximation framework that applies to both the randomized and deterministic variants. Our approximation framework uses the sales-based formulation of the assortment optimization problem under the multinomial logit model. The sales-based formulation uses the purchase probabilities of the products as decision variables, but it has combinatorial aspects because we impose the market share constraints only for the product classes in which we offer a product. Thus, it is not straightforward to solve the sales-based formulation directly. We give a meta-theorem to show that our approximation framework yields an approximate solution to the randomized or deterministic variants, as long as we can execute its steps with tunable parameters. Our approximation algorithms for both variants are based on finding ways to execute our approximation framework with specific values for the tunable parameters.

Approximation Schemes. As our algorithmic contributions, we use our approximation framework to give a fully polynomial-time approximation scheme (FPTAS) for the randomized variant, an approximation scheme for the deterministic variant to get a $(1-\epsilon)$-approximate solution while violating the market share constraints with a $(1+\epsilon)$-factor in running time that is polynomial in

$\frac{1}{\epsilon}$, as well as a $\frac{1}{2}$-approximation algorithm for the deterministic variant to get a solution satisfying the market share constraints exactly in running time that is pseudo-polynomial in the input size.

Related Literature: Several variants of the assortment optimization problem under the multinomial logit model can be solved in polynomial time. In particular, Talluri and van Ryzin (2004) focus on the case without any constraints on the offered assortment. Rusmevichientong et al. (2010) impose a constraint on the number of offered products. Sumida et al. (2021) work with constraints that can be encoded using a totally unimodular matrix. Gallego et al. (2015) give the sales-based formulation. Cao et al. (2023) work with a mixture of independent demand and multinomial logit models. Gao et al. (2021) focus on the case, where the offered assortment is revealed gradually in multiple stages and the customer choses within the offered assortment so far or decides to move on to the next stage. Moving to the cases that yield NP-hard optimization problems, Feldman and Topaloglu (2018) give an FPTAS when there are nested consideration sets and each customer chooses among the products in her consideration set. Bront et al. (2009) and Desir et al. (2022) study the problem under a mixture of multinomial logit models, where customers of different types choose according to different multinomial logit models.

Market share or display constraints recently started to appear in the literature, though in a rather different sense than ours. Chen et al. (2023) focus on fairness considerations, where products similar to each other need to garner similar market shares. Barre et al. (2023) study an assortment problem with multiple display opportunities, where the number times a product is offered should be above a given display threshold. Lu et al. (2023) focus on a randomized version of the same problem, where the expected number of times a product is offered should be above a given display threshold. In all of these papers, however, the market share or display constraints are hard constraints imposed on all products. There is no flexibility to sidestep the market share or display constraint for a product when the product is not offered. In contrast, we impose the market share constraints only for a product class that is represented in the offered assortment. If a product class is not represented in the offered assortment, then we do not need to impose its market share constraint. Such market share constraints are often more natural for the online retail setting and they bring additional combinatorial aspects that were surprisingly unexplored, to our knowledge.

Organization: In Section 2, we formulate our problem and characterize its complexity. In Section 3, we characterize the value of randomization. In Section 4, we give our approximation framework. In Section 5, we give an FPTAS for the randomized variant. In Section 6, we give an approximate scheme for the deterministic variant with controlled violations of the market share constraints. In Section 7, we give a $\frac{1}{2}$-approximation for the deterministic variant in pseudo-polynomial time. In Section 8, we give computational experiments. In Section 9, we conclude.

2. Problem Formulation and Complexity

We index the set of product classes as $\mathcal{K} = \{1, \ldots, K\}$. A product class may correspond to the products offered by a particular vendor, procured through a particular shipment mode or coming from a particular origin country. The set of products in class k is $\mathcal{N}^k = \{1, \ldots, n^k\}$. We capture the set of all products by $\mathcal{N} = \cup_{k \in \mathcal{K}} \mathcal{N}^k$. All products are distinct from each other. The total number of products is $n = \sum_{k \in \mathcal{K}} n^k$. The revenue of product i is $r_i > 0$. Customers choose among the offered products according to the multinomial logit model. The preference weight of product i is $v_i > 0$. We normalize the preference weight of the no-purchase option to one. Letting $\mathbf{1}(\cdot)$ be the indicator function, if we offer the assortment of products $S \subseteq \mathcal{N}$, then a customer chooses product i with probability $\phi_i(S) = \mathbf{1}(i \in S)\, v_i / (1 + \sum_{\ell \in S} v_\ell)$. If $i \notin S$, then $\phi_i(S) = 0$, so the purchase probability $\phi_i(S)$ is defined for all $i \in \mathcal{N}$. The minimum market share for product class k is $\beta^k > 0$. If we offer any product in class k, then the total purchase probability of the products in class k has to be at least β^k. So, if we utilize a particular vendor, shipment mode or country of origin, then the expected demand experienced by this seller, shipment mode and origin country satisfies a threshold.

Our goal is to find an assortment to offer to maximize the expected revenue from a customer while satisfying the market share constraints. We consider two variants of our problem. In the randomized variant, we can randomize the assortment offered to each customer. In the deterministic variant, we offer the same assortment to all customers. The randomized variant may be suitable for online retail where it is easy to offer different assortments to different customers, whereas the deterministic variant may be suitable for physical stores, where it is difficult to offer different assortments to different customers. We use the vector of decision variables $\boldsymbol{h} = (h(S) : S \subseteq \mathcal{N})$, where $h(S)$ is the probability that we offer assortment S to a customer. We need to have $\sum_{S \subseteq \mathcal{N}} h(S) = 1$ so that the total probability of offering an assortment is one. In the randomized variant, we have $h(S) \in [0, 1]$ so that we can randomize the assortment offered to each customer. In the deterministic variant, we have $h(S) \in \{0, 1\}$ so that we offer a deterministic assortment. To capture both variants, we impose the constraint $\boldsymbol{h} \in \mathcal{H}$ on our decision variables. If $\mathcal{H} = [0, 1]^{2^n}$, then we get the randomized variant, whereas if $\mathcal{H} = \{0, 1\}^{2^n}$, then we get the deterministic variant. We want to solve

$$\textsf{opt} \;=\; \max_{\boldsymbol{h} \in \mathcal{H}} \left\{ \sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} r_i \, \phi_i(S)\, h(S) \,:\, \sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}^k} \phi_i(S)\, h(S) \,\in\, \{0\} \cup [\beta^k, 1] \quad \forall\, k \in \mathcal{K}, \; \sum_{S \subseteq \mathcal{N}} h(S) = 1 \right\}. \qquad \textsf{(Market-Share)}$$

In the objective function, we account for the expected revenue from products in all classes. The left side of the first constraint is the total purchase probability of the products in class k. Therefore, the total purchase probability of the products in class k should either be zero, which is the case where we do not use any products in class k, or should be at least β^k, which is the case where we use at least one product in class k. The second constraint, as discussed in the previous paragraph, ensures that we offer an assortment with probability one. Because of the first constraint in the Market-Share problem, even when we offer a randomized assortment so that $\mathcal{H} = [0,1]^{2^n}$, the set of feasible solutions is not convex. Nevertheless, the set of feasible solutions is closed and bounded irrespective of we offer a randomized or a deterministic assortment, so an optimal solution exists. We can give an alternative formulation of the Market-Share problem by introducing the additional decision variables $\boldsymbol{y} = (y^k : k \in \mathcal{K}) \in \{0,1\}^K$, where $y^k = 1$ if and only if we offer a product in class k. In this case, we replace the first constraint with the two constraints $\sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}^k} \phi_i(S)\, h(S) \geq \beta^k\, y^k$ and $\sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}^k} \phi_i(S)\, h(S) \leq y^k$, where the first constraint ensures that if we offer a product in class k, then the total purchase probability of the products in this class has to be at least β^k, whereas the second constraint ensures that if we do not offer a product in class k, then we cannot use any assortment that includes a product in class k. While the latter formulation follows a standard mixed integer programming framework, the way we formulate the Market-Share problem will be adequate for our purpose and it avoids an extra set of decision variables.

An important special case of the Market-Share problem occurs when each product class includes a single product, so that $|\mathcal{N}^k| = 1$ for all $k \in \mathcal{K}$. If each product class includes a single product, then the first constraint in the Market-Share problem ensures that the purchase probability of each product, if offered, must satisfy a certain threshold. In this case, we have two options for each product. If we offer the product, then the purchase probability of the product must exceed a certain threshold, but if we cannot ensure that the purchase probability of the product exceeds this threshold, then we cannot offer the product. In this way, we try to offer assortments that command a significant amount of expected demand for each product in the offered assortment, so we avoid assortments that include tail products with small expected demands. If each product class includes a single product, then we index both the products classes and products with $\mathcal{N}$. We have a market share threshold for each product. Using β^i to denote the market share threshold for product i, the first constraint in the Market-Share problem takes the form $\sum_{S \subseteq \mathcal{N}} \phi_i(S)\, h(S) \in \{0\} \cup [\beta^i, 1]$ for all $i \in \mathcal{N}$. Other aspects of the problem remain unchanged. Throughout the paper, we mainly focus on the case where each product class can include multiple products. We can obtain stronger results when each product class includes a single product. We give these results for the singleton product classes when appropriate, but defer the details to the appendix.

In the next theorem, we use a reduction from the partition problem to characterize the complexity of the randomized and deterministic variants, establishing that the deterministic variant is significantly more difficult than the randomized one from computational complexity perspective. We defer the proof of the theorem to Appendix A.

Theorem 2.1 (Computational Complexity) *The randomized variant of the* Market-Share *problem is NP-hard. Furthermore, unless $P = NP$, there does not exist an α-approximation algorithm for the deterministic variant of the* Market-Share *problem for any $\alpha \in (\frac{1}{2}, 1]$.*

The partition problem is often used in the complexity proofs for various assortment optimization problems, but our proof uniquely uses the partition problem to establish inapproximability.

Sales-Based Formulation:

Motivated by our complexity result, we focus on developing approximation algorithms for the randomized and deterministic variants. One of the tools that we use is the so-called sales-based reformulation of the Market-Share problem. This reformulation is called sales-based because its decision variables correspond to the purchase probabilities of different products, rather than the offer probabilities of different assortments. We use the decision variable x_i to capture the probability that a customer purchases product i and x_0 to capture the probability that a customer leaves without a purchase. In the sales-based reformulation of the Market-Share problem, we need the constraint $x_i \in [0, v_i\, x_0]$ for the randomized variant, whereas we need the constraint $x_i \in \{0, v_i\, x_0\}$ for the deterministic variant. To capture both variants, we use the constraint $x_i \in \mathcal{H}_i(x_0)$, where we have $\mathcal{H}_i(x_0) = [0, v_i\, x_0]$ and $\mathcal{H}_i(x_0) = \{0, v_i\, x_0\}$, respectively, in the randomized and deterministic variants. Using the vector of decision variables $\boldsymbol{x} = (x_i : i \in \mathcal{N})$, we consider the problem

$$\overline{\mathsf{opt}} = \max_{(x_0, \boldsymbol{x}) \in [0,1]^{1+n}} \left\{ \sum_{i \in \mathcal{N}} r_i\, x_i : \sum_{i \in \mathcal{N}^k} x_i \in \{0\} \cup [\beta^k, 1] \quad \forall k \in \mathcal{K}, \qquad x_0 + \sum_{i \in \mathcal{N}} x_i \le 1, \right.$$
$$\left. x_i \in \mathcal{H}_i(x_0) \quad \forall i \in \mathcal{N} \right\}. \qquad \text{(Sales-Based)}$$

In the objective function, recalling x_i corresponds to the purchase probability of product i, we account for the expected revenue from all products in all classes. The first constraint ensures that the total purchase probability of all products in class k is either zero or at least β^k. The second constraint ensures that a customer either purchases a product or leaves without a purchase. We express this constraint as an inequality constraint, but it is not difficult to see that there exists an optimal solution that satisfies this constraint as equality. If $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is an optimal solution to the Sales-Based problem with $\sum_{i \in \mathcal{N}} \widehat{x}_i + \widehat{x}_0 < 1$, then letting $\widetilde{x}_0 = \frac{\widehat{x}_0}{\sum_{j \in \mathcal{N}} \widehat{x}_j + \widehat{x}_0}$ and $\widetilde{x}_i = \frac{\widehat{x}_i}{\sum_{j \in \mathcal{N}} \widehat{x}_j + \widehat{x}_0}$ for all $i \in \mathcal{N}$, the solution $(\widetilde{x}_0, \widetilde{\boldsymbol{x}})$ is feasible to problem Sales-Based, satisfies the second constraint as equality and provides an objective value that is at least as large as that of $(\widehat{x}_0, \widehat{\boldsymbol{x}})$. Noting the definition of $\mathcal{H}_i(x_0)$ in the previous paragraph, the preference weights of the products in the multinomial logit model appear only in the third constraint. Sales-based reformulations of assortment optimization problems go back to Gallego et al. (2015), where the authors give a linear program to solve the assortment optimization problem under the multinomial logit model.

The decision variables in the Sales-Based problem correspond to the probability of purchase for different products, whereas the decision variables in the Market-Share problem correspond to the probability of offering different assortments. However, Topaloglu (2013) shows that we can use an optimal solution to the sales-based formulation of an unconstrained assortment optimization problem to obtain an optimal solution to the original assortment optimization problem. We give an approximate version of this result for our constrained assortment optimization problem. In particular, we show that we can use a near-optimal and near-feasible solution to the Sales-Based problem to get a near-optimal and near-feasible solution to the Market-Share problem. In the rest of this section, we give this result. This results holds *irrespective* of whether we focus on the randomized or deterministic variant. Consider a solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ to the Sales-Based problem with $\widehat{x}_i \in \mathcal{H}_i(\widehat{x}_0)$ for all $i \in \mathcal{N}$. We index the products such that $\frac{\widehat{x}_1}{v_1} \geq \frac{\widehat{x}_2}{v_2} \geq \ldots \geq \frac{\widehat{x}_n}{v_n}$. By the definition of $\mathcal{H}_i(\widehat{x}_0)$ for both the randomized and deterministic variants, we have $\widehat{x}_0 \geq \frac{\widehat{x}_i}{v_i}$ for all $i \in \mathcal{N}$. Therefore, we have $\widehat{x}_0 \geq \frac{\widehat{x}_1}{v_1} \geq \frac{\widehat{x}_2}{v_2} \geq \ldots \geq \frac{\widehat{x}_n}{v_n}$. Using the sets $S_i = \{1, \ldots, i\}$ for all $i \in \mathcal{N}$ with $S_0 = \varnothing$, corresponding to the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$, we define the solution $\widehat{\boldsymbol{h}} = (\widehat{h}(S) : S \subseteq \mathcal{N})$ to the Market-Share problem as follows. We set $\widehat{h}(S) = 0$ for all $S \notin \{S_0, S_1, \ldots, S_n\}$. Setting $v_0 = 1$ and $\widehat{x}_{n+1} = 0$ for notational uniformity, for all $S \in \{S_0, S_1, \ldots, S_n\}$, we define $\widehat{h}(S)$ as

$$\widehat{h}(S_i) = \frac{\left(\dfrac{\widehat{x}_i}{v_i} - \dfrac{\widehat{x}_{i+1}}{v_{i+1}}\right)\left(1 + \displaystyle\sum_{j \in S_i} v_j\right)}{\widehat{x}_0 + \displaystyle\sum_{j \in \mathcal{N}} \widehat{x}_j} \quad \forall\, i = 0, 1, \ldots, n. \tag{1}$$

By the next theorem, if the solution $\widehat{\boldsymbol{x}}$ is near-optimal and near-feasible to the Sales-Based problem, then the solution $\widehat{\boldsymbol{h}}$ is near-optimal and near-feasible to the Market-Share problem.

Theorem 2.2 (Sales-Based to Market-Share) *For some $\gamma_1, \gamma_2 \in [0,1)$, if the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ satisfies $\sum_{i \in \mathcal{N}^k} \widehat{x}_i \in \{0\} \cup [\beta^k, \infty)$ for all $k \in \mathcal{K}$, $\widehat{x}_0 + \sum_{i \in \mathcal{N}} \widehat{x}_i \leq \frac{1}{1-\gamma_1}$, $\widehat{x}_i \in \mathcal{H}_i(\widehat{x}_0)$ for all $i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} r_i\, \widehat{x}_i \geq (1-\gamma_2)\,\overline{\mathsf{opt}}$, then the solution $\widehat{\boldsymbol{h}}$ in (1) satisfies*

$$\widehat{\boldsymbol{h}} \in \mathcal{H}, \qquad \sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}^k} \phi_i(S)\, \widehat{h}(S) \in \{0\} \cup [(1-\gamma_1)\,\beta^k, 1] \quad \forall\, k \in \mathcal{K}, \qquad \sum_{S \subseteq \mathcal{N}} \widehat{h}(S) = 1,$$
$$\sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} r_i\, \phi_i(S)\, \widehat{h}(S) \geq (1-\gamma_1)\,(1-\gamma_2)\,\mathsf{opt}.$$

The proof is in Appendix B. By a simple corollary in the same appendix, given as Corollary B.1, the Market-Share and Sales-Based problems have the same optimal objective value, so $\mathsf{opt} = \overline{\mathsf{opt}}$.

3. Value of Randomized Assortments

The optimal expected revenue in the randomized variant is at least as large as that in the deterministic variant. In this section, we demonstrate that the randomized variant can improve the optimal expected revenue in the deterministic variant by an arbitrarily large factor. However, if each product class includes a single product, then we show that the randomized variant can improve the optimal expected revenue by at most a factor of two and this bound is tight. Throughout this section, we use $\mathsf{opt_R}$ and $\mathsf{opt_D}$ to, respectively, denote the optimal objective values of the randomized and deterministic variants of the Market-Share problem. We use $R(S) = \sum_{i \in S} r_i \, \phi_i(S)$ to denote the expected revenue from assortment S. Under arbitrary product classes, we demonstrate that the ratio $\frac{\mathsf{opt_R}}{\mathsf{opt_D}}$ can be arbitrarily large. Under singleton product classes, we show that we always have $\frac{\mathsf{opt_R}}{\mathsf{opt_D}} \leq 2$ and there are problem instances such that the ratio $\frac{\mathsf{opt_R}}{\mathsf{opt_D}}$ is arbitrarily close to two.

Arbitrary Product Classes:

To see that the randomized variant can improve the optimal expected revenue in the deterministic variant by an arbitrarily large factor, consider the following problem instance. We have one product class that includes two products. For some $k \geq 2$, the revenues and preference weights of the products are given by $r_1 = k$, $r_2 = 1$, $v_1 = 1$ and $v_2 = k$. The market share threshold for the single product class is $\beta^1 = \frac{9}{16}$. For this problem instance, the purchase probabilities of the products under different assortments are $\phi_1(\{1\}) = \frac{1}{2}$, $\phi_2(\{2\}) = \frac{k}{1+k}$, $\phi_1(\{1,2\}) = \frac{1}{2+k}$ and $\phi_2(\{1,2\}) = \frac{k}{2+k}$. Therefore, we have $\phi_2(\{2\}) = \frac{k}{1+k} \geq \frac{2}{3}$ and $\phi_1(\{1,2\}) + \phi_2(\{1,2\}) = \frac{1+k}{2+k} \geq \frac{3}{4}$, where we use the fact that $k \geq 2$. On the other hand, the expected revenues under different assortments are $R(\{1\}) = \frac{k}{2}$, $R(\{2\}) = \frac{k}{1+k}$ and $R(\{1,2\}) = \frac{2k}{2+k}$. Using the fact that $k \geq 2$ once more, we have $\frac{k}{2} \geq \frac{2k}{2+k} \geq \frac{k}{1+k}$, in which case, we obtain $R(\{1\}) \geq R(\{1,2\}) \geq R(\{2\})$ as well.

Consider the value of $\mathsf{opt_D}$. Because $\phi_1(\{1\}) < \frac{9}{16}$, $\phi_2(\{2\}) \geq \frac{9}{16}$ and $\phi_1(\{1,2\}) + \phi_2(\{1,2\}) \geq \frac{9}{16}$ by the discussion in the previous paragraph, offering the assortment $\{1\}$ by itself does not satisfy the market share constraint, but offering either of the assortments $\{2\}$ or $\{1,2\}$ by itself does, so $\mathsf{opt_D} = \max\{R(\{2\}), R(\{1,2\})\} = R(\{1,2\}) = \frac{2k}{2+k}$. Consider the value of $\mathsf{opt_R}$. If we offer the assortments $\{1\}$ and $\{1,2\}$, respectively, with probabilities $\frac{3}{4}$ and $\frac{1}{4}$, then the total purchase probability of the products is $\frac{3}{4}\,\phi_1(\{1\}) + \frac{1}{4}(\phi_1(\{1,2\}) + \phi_2(\{1,2\})) \geq \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{3}{4} = \frac{9}{16} = \beta^1$, so this solution satisfies the market share constraint for the product class under a randomized assortment. This solution provides an expected revenue of $\frac{3}{4}R(\{1\}) + \frac{1}{4}\,R(\{1,2\}) = \frac{3}{4} \times \frac{k}{2} + \frac{1}{4} \times \frac{2k}{2+k} \geq \frac{3k}{8}$. Thus, we have $\mathsf{opt_R} \geq \frac{3k}{8}$. In this case, we get $\frac{\mathsf{opt_R}}{\mathsf{opt_D}} \geq \frac{3k/8}{2k/(1+k)} = \frac{3}{16}\,(k+1)$. Therefore, it follows that if we choose k arbitrarily large, then the fraction $\frac{\mathsf{opt_R}}{\mathsf{opt_D}}$ becomes arbitrarily large.

In our problem instance, if we choose k arbitrarily large, then $R(\{1\})$ becomes arbitrarily large, but $R(\{2\})$ and $R(\{1,2\})$ are always at most two. However, if we offer the assortment $\{1\}$ by itself, then we cannot satisfy the market share constraint. Using a randomized assortment allows us to take advantage of the large expected revenue from the assortment $\{1\}$.

Singleton Product Classes:

Turning to the case with one product in each class, we show that $\frac{\mathsf{opt_R}}{\mathsf{opt_D}} \leq 2$. We use the Sales-Based problem to show this result. As discussed in the previous section, if we set $\mathcal{H}_i(x_0) = [0, v_i\, x_0]$, then the optimal objective value of the Sales-Based problem corresponds to the optimal expected revenue for the randomized variant, whereas if we set $\mathcal{H}_i(x_0) = \{0, v_i\, x_0\}$, then the optimal objective value of the Sales-Based problem corresponds to the optimal expected revenue for the deterministic variant. Because we have one product in each class, we index the product classes and products both with $\mathcal{N}$. In this case, we have a market share threshold for each product. Letting β^i be the market share threshold for product i, noting that there is one product in each class, the first constraint in the Sales-Based problem becomes $x_i \in \{0\} \cup [\beta^i, 1]$ for all $i \in \mathcal{N}$. Letting $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem for the randomized variant, we define the set of products $\mathcal{P} = \{i \in \mathcal{N} : \widehat{x}_i \geq \beta^i\}$, including the products with non-zero market share in the optimal solution. If we solve the Sales-Based problem for the randomized variant by focusing only on these products and fixing the decision variable x_0 at $\widehat{x}_0$, then the optimal objective value does not change. In the next theorem, we build on this observation to show the upper bound of two.

Theorem 3.1 (Value of Randomized Assortments) *If we have one product in each class so that $|\mathcal{N}^k| = 1$ for all $k \in \mathcal{K}$, then we have $2\,\mathsf{opt_D} \geq \mathsf{opt_R}$.*

Proof: Letting $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem for the randomized variant, by the discussion just before the theorem, if we solve the Sales-Based problem for the randomized variant by focusing only on the products in $\mathcal{P}$ and fixing the decision variable x_0 at $\widehat{x}_0$, then the optimal objective value does not change. So, dropping the first constraint in Sales-Based problem as well, using the vector of decision variables $\boldsymbol{x}^{\mathcal{P}} = (x_i : i \in \mathcal{P})$, we can upper bound $\mathsf{opt_R}$ by the optimal objective value of the problem $\zeta = \max_{\boldsymbol{x}^{\mathcal{P}} \in \mathbb{R}_+^{|\mathcal{P}|}} \{\sum_{i \in \mathcal{P}} r_i\, x_i : \sum_{i \in \mathcal{P}} x_i \leq 1 - \widehat{x}_0,\ x_i \leq v_i\, \widehat{x}_0 \ \ \forall\, i \in \mathcal{P}\}$, so we have $\zeta \geq \mathsf{opt_R}$. In the last constraint, we write $x_i \in \mathcal{H}_i(\widehat{x}_0)$ explicitly as $x_i \leq v_i\, \widehat{x}_0$. Noting that the last problem is a knapsack problem, there exists an optimal solution to this problem with at most one strictly positive decision variable that does not attain its upper bound. In other words, letting $\widetilde{\boldsymbol{x}}^{\mathcal{P}} = (\widetilde{x}_i : i \in \mathcal{P})$ be such an optimal solution, the exists a set of products $\mathcal{Q} \subseteq \mathcal{P}$ and $j \in \mathcal{P} \setminus \mathcal{Q}$ such that $\widetilde{x}_i^{\mathcal{P}} = v_i\, \widehat{x}_0$ for all $i \in \mathcal{Q}$, $\widetilde{x}_j^{\mathcal{P}} \in [0, v_j\, \widehat{x}_0]$ and $\widetilde{x}_i^{\mathcal{P}} = 0$ for all $i \in \mathcal{P} \setminus (\mathcal{Q} \cup \{j\})$. Thus, the optimal objective value of the knapsack problem satisfies $\zeta = \sum_{i \in \mathcal{Q}} r_i\, v_i\, \widehat{x}_0 + r_j\, \widetilde{x}_j^{\mathcal{P}}$. We construct two feasible solutions to the Sales-Based problem for the deterministic variant. The objective value provided by the first solution will be $\sum_{i \in \mathcal{Q}} r_i\, v_i\, \widehat{x}_0$, whereas the objective value provided by the second solution will be at least $r_j\,\widetilde{x}_j^{\mathcal{P}}$, so we have $\mathsf{opt}_\mathsf{D} \geq \sum_{i\in\mathcal{Q}} r_i\,v_i\,\widehat{x}_0$ and $\mathsf{opt}_\mathsf{D} \geq r_j\,\widetilde{x}_j^{\mathcal{P}}$, in which case, as desired, we obtain $2\,\mathsf{opt}_\mathsf{D} \geq \sum_{i\in\mathcal{Q}} r_i\,v_i\,\widehat{x}_0 + r_j\,\widetilde{x}_j^{\mathcal{P}} = \zeta \geq \mathsf{opt}_\mathsf{R}$.

In the first solution $(\widehat{x}_0, \overline{\boldsymbol{x}})$, we set $\overline{x}_i = v_i\,\widehat{x}_0$ for all $i \in Q$ and $\overline{x}_i = 0$ for all $i \in \mathcal{N} \setminus \mathcal{Q}$. For all $i \in \mathcal{Q}$, we have $\overline{x}_i = v_i\,\widehat{x}_0 \geq \widehat{x}_i \geq \beta^i$, where the first inequality holds because $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem for the randomized variant and the second inequality holds by the definition of $\mathcal{P}$ and the fact that $Q \subseteq \mathcal{P}$. Thus, $(\widehat{x}_0, \overline{\boldsymbol{x}})$ satisfies the first constraint in the Sales-Based problem for the deterministic variant. We have $\sum_{i\in\mathcal{N}} \overline{x}_i = \sum_{i\in\mathcal{Q}} v_i\,\widehat{x}_0 = \sum_{i\in\mathcal{Q}} \widetilde{x}_i^{\mathcal{P}} \leq 1 - \widehat{x}_0$, where the second equality holds because $\widetilde{x}_i^{\mathcal{P}} = v_i\,\widehat{x}_0$ for all $i \in \mathcal{Q}$ and the inequality holds because $\widetilde{\boldsymbol{x}}^{\mathcal{P}}$ is a feasible solution to the earlier knapsack problem. Thus, $(\widehat{x}_0, \overline{\boldsymbol{x}})$ satisfies the second constraint in the Sales-Based problem for the deterministic variant. Lastly, $\overline{x}_i \in \{0, v_i\,\widehat{x}_0\}$ for all $i \in \mathcal{N}$, so $(\widehat{x}_0, \overline{\boldsymbol{x}})$ satisfies the third constraint in the Sales-Based problem for the deterministic variant.

In the second solution $(\frac{1}{1+v_j}, \overline{\boldsymbol{x}})$, we set $\overline{x}_j = \frac{v_j}{1+v_j}$ and $\overline{x}_i = 0$ for all $i \in \mathcal{N} \setminus \{j\}$. Because $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem for the randomized variant, by the second and third constraints in this problem, we have $\widehat{x}_j \leq 1 - \widehat{x}_0 \leq 1 - \frac{\widehat{x}_j}{v_j}$, in which case, focusing on the first and last terms in this chain of inequalities yields $\widehat{x}_j \leq \frac{v_j}{1+v_j}$. By the definition of $\mathcal{P}$, we also have $\widehat{x}_j \geq \beta^j$, so $\overline{x}_j = \frac{v_j}{1+v_j} \geq \beta^j$. Thus, $(\frac{1}{1+v_j}, \overline{\boldsymbol{x}})$ satisfies the first constraint in the Sales-Based problem for the deterministic variant. It is immediate that $(\frac{1}{1+v_j}, \overline{\boldsymbol{x}})$ satisfies the second and third constraints in the Sales-Based problem for the deterministic variant. Noting that $\widetilde{\boldsymbol{x}}^{\mathcal{P}}$ is feasible to the earlier knapsack problem, we have $\widetilde{x}_j^{\mathcal{P}} \leq 1 - \widehat{x}_0 \leq 1 - \frac{\widetilde{x}_j^{\mathcal{P}}}{v_j}$, which yields $\widetilde{x}_j^{\mathcal{P}} \leq \frac{v_j}{1+v_j}$. In this case, the objective value provided by $(\frac{1}{1+v_j}, \overline{\boldsymbol{x}})$ to the Sales-Based problem for the deterministic variant is $r_j\,\frac{v_j}{1+v_j} \geq r_j\,\widetilde{x}_j^{\mathcal{P}}$. ■

Thus, if each product class includes a single product, then switching from deterministic assortments to randomized assortments can at most double the optimal expected revenue.

In the remainder of this section, we give a problem instance with singleton product classes such that the ratio $\frac{\mathsf{opt}_\mathsf{R}}{\mathsf{opt}_\mathsf{D}}$ is arbitrarily close to two. We have two product classes and two products, each class including one of the products. For some $k \geq 2$, the revenues and preference weights of the products are given by $r_1 = r_2 = 1$ and $v_1 = v_2 = 1/k$. The market share thresholds for the two product classes are $\beta^1 = \frac{k-1}{k}\,\frac{1}{2+k} + \frac{1}{k}\,\frac{1}{1+k}$ and $\beta^2 = \frac{1}{k\,(k+2)}$. The market share of the first product class is a strict convex combination of $\frac{1}{2+k}$ and $\frac{1}{1+k}$, which implies that $\frac{1}{2+k} < \beta^1 < \frac{1}{1+k}$. For this problem instance, the purchase probabilities of the products under different assortments are $\phi_1(\{1\}) = \frac{1}{1+k}$, $\phi_2(\{2\}) = \frac{1}{1+k}$ and $\phi_1(\{1,2\}) = \phi_2(\{1,2\}) = \frac{1}{2+k}$. On the other hand, the expected revenues under different assortments are $R(\{1\}) = \frac{1}{1+k}$, $R(\{2\}) = \frac{1}{1+k}$ and $R(\{1,2\}) = \frac{2}{2+k}$.

Consider the value of opt_D. Because $\beta^1 > \frac{1}{2+k}$, if we offer the assortment $\{1,2\}$, then the total purchase probability of the products in the first product class is $\phi_1(\{1,2\}) = \frac{1}{2+k} < \beta^1$, which implies that we cannot offer the assortment $\{1,2\}$ by itself. Using the fact that $\beta^1 < \frac{1}{1+k}$, because $\phi_1(\{1\}) = \frac{1}{1+k} > \beta^1$ and $\phi_2(\{2\}) = \frac{1}{1+k} \geq \frac{1}{k\,(k+2)} = \beta^2$, if we offer either of the assortments $\{1\}$ or $\{2\}$ by itself, then we satisfy the market share constraints, so $\mathsf{opt_D} = \max\{R(\{1\}), R(\{2\})\} = \frac{1}{1+k}$. Consider the value of $\mathsf{opt_R}$. Offering the assortments $\{1\}$ and $\{1,2\}$, respectively, with probabilities $\frac{1}{k}$ and $\frac{k-1}{k}$, the purchase probabilities of the products in each class are $\frac{1}{k}\phi_1(\{1\}) + \frac{k-1}{k}\,\phi_1(\{1,2\}) = \frac{1}{k}\,\frac{1}{1+k} + \frac{k-1}{k}\,\frac{1}{2+k} = \beta^1$ and $\frac{k-1}{k}\,\phi_2(\{1,2\}) = \frac{k-1}{k}\,\frac{1}{2+k} \geq \beta^2$, so this solution satisfies the market share constraints. This solution provides an expected revenue of $\frac{1}{k}\,R(\{1\}) + \frac{k-1}{k}\,R(\{1,2\}) = \frac{1}{k}\,\frac{1}{1+k} + \frac{k-1}{k}\,\frac{2}{2+k}$, so $\mathsf{opt_R} \geq \frac{1}{k}\,\frac{1}{1+k} + \frac{k-1}{k}\,\frac{2}{2+k} \geq \frac{1}{k}\,\frac{1}{2+k} + \frac{k-1}{k}\,\frac{2}{2+k} = \frac{2k-1}{k\,(2+k)}$. Thus, we get $\frac{\mathsf{opt_R}}{\mathsf{opt_D}} \geq \frac{(2k-1)/(k(2+k))}{1/(1+k)}$. The last fraction converges to two as k gets arbitrarily large.

Randomized assortments may not be feasible to implement in all settings, but the two problem instances in this section demonstrate that their benefits can be significant.

4. Approximation Framework

We give a unified approximation framework that applies to both the randomized and deterministic variants. The approximation framework has some gaps in it. We fill these gaps using different approaches for the randomized and deterministic variants. Letting $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem, $\widehat{x}_0$ is the optimal no-purchase probability, whereas $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$ is the optimal market share of product class k. The idea behind our approximation framework is that if we know the values of $\widehat{x}_0$ and $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$ for all $k \in \mathcal{K}$, then the Sales-Based problem decomposes by the product classes. In particular, letting $\widehat{z}^k = \sum_{i\in\mathcal{N}^k} \widehat{x}_i$ and using the vector of decision variables $\boldsymbol{x}^k = (x_i : i \in \mathcal{N}^k)$, if we know the values of $\widehat{x}_0$ and $\widehat{z}^k$, then we can recover an optimal solution to the Sales-Based problem for product class k by solving the problem

$$\max_{\boldsymbol{x}^k \in [0,1]^{n^k}} \left\{ \sum_{i\in\mathcal{N}^k} r_i\, x_i \, : \, \sum_{i\in\mathcal{N}^k} x_i = \widehat{z}^k, \quad x_i \in \mathcal{H}_i(\widehat{x}_0) \;\; \forall\, i \in \mathcal{N}^k \right\}. \tag{2}$$

In the problem above, we will use $\mathcal{H}_i(\widehat{x}_0) = [0, v_i\,\widehat{x}_0]$ when we are interested in approximating the randomized variant, whereas we will use $\mathcal{H}_i(\widehat{x}_0) = \{0, v_i\,\widehat{x}_0\}$ when we are interested in approximating the deterministic variant. Because we do not know the optimal no-purchase probability $\widehat{x}_0$ and market share $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$ for product class k, we cannot immediately use problem (2). Letting $v_{\max} = \max_{i\in\mathcal{N}} v_i$, it is straightforward to check that the value of $\widehat{x}_0$ in a feasible solution to the Sales-Based problem is in the interval $[\frac{1}{1+n\,v_{\max}}, 1]$. Also, a non-zero value of $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$ in a feasible solution to the Sales-Based problem is in the interval $[\beta^k, 1]$. Therefore, we construct two collections of grid points, each of which covering the intervals $[\frac{1}{1+n\,v_{\max}}, 1]$ and $[\beta^k, 1]$. In this case our approximation framework proceeds as follows. In the first step, we guess the optimal no-purchase probability $\widehat{x}^0$ and market share $\sum_{i\in\mathcal{N}^k}\widehat{x}_i$ for product class k to be a pair of points in these two sets of grid points. For each product class k, we solve a variant of problem (2) to generate a candidate solution for each guess of the no-purchase probability and market share of the product class. In the second step, fixing a guess for the no-purchase probability, we solve a multiple-choice knapsack problem to choose a candidate solution from the different product classes to maximize the expected revenue. Thus, we stitch together a solution for the Sales-Based problem by choosing one candidate solution for each product class. In the third step, we find the guess of the no-purchase probability that provides the largest expected revenue. In the fourth step, lastly, we use this guess to construct an approximate solution to the Sales-Based problem. Below is our approximation framework. Before we execute our approximation framework, we construct a collection of grid points $\mathsf{Grid}_0=\{\overline{x}_0(p):p=1,\dots,L_0\}$, as well as a collection of grid points $\mathsf{Grid}^k=\{\overline{z}^k(q):q=1,\dots,L^k\}$ for each $k\in\mathcal{K}$, where the two collections serve as the guesses for the no-purchase probability and market share of product class k. We also fix the precision parameters $\epsilon_1,\epsilon_2,\epsilon_3\in[0,1]$.

Approximation Framework:

Step 1. (Generate Candidate Solutions) For each $k\in\mathcal{K}$, $p=1,\dots,L_0$ and $q=1,\dots,L^k$, construct a candidate solution vector $\overline{\boldsymbol{x}}^k(p,q)=(\overline{x}_i(p,q):i\in\mathcal{N}^k)$ for product class k such that $\sum_{i\in\mathcal{N}^k}\overline{x}_i(p,q)\in\{0\}\cup[\beta^k,\frac{1}{1-\epsilon_1}\overline{z}^k(q)]$ and $\overline{x}_i(p,q)\in\mathcal{H}_i(\overline{x}_0(p))$ for all $i\in\mathcal{N}^k$.

Step 2. (Stitch Candidates) The expected revenue from the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ is $\overline{r}^k(p,q)=\sum_{i\in\mathcal{N}^k}r_i\,\overline{x}_i(p,q)$. For each $p=1,\dots,L_0$, using the vector of decision variables $\boldsymbol{u}(p)=(u^k(p,q):k\in\mathcal{K},\ q=1,\dots,L^k)$, pick a candidate solution for each product class by solving

$$Z(p)=\max_{\boldsymbol{u}(p)\in\{0,1\}^{L^1+\dots+L^K}}\left\{\sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k}\overline{r}^k(p,q)\ u^k(p,q)\ :\ \sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k}\overline{z}^k(q)\,u^k(p,q)+\overline{x}_0(p)\le\frac{1}{1-\epsilon_2}\right.$$
$$\left.\sum_{q=1}^{L^k}u^k(p,q)=1\quad\forall\,k\in\mathcal{K}\right\}.\qquad(\mathsf{Knapsack})$$

Step 3. (Find Best No-Purchase) Using $\widehat{\boldsymbol{u}}(p)=(\widehat{u}^k(p,q):k\in\mathcal{K},\ q=1,\dots,L^k)$ to denote a $(1-\epsilon_3)$-approximate solution to the Knapsack problem above with the corresponding objective value $\widehat{Z}(p)=\sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k}\overline{r}^k(p,q)\,\widehat{u}^k(p,q)$, set $\widehat{p}=\arg\max_{p=1,\dots,L_0}\widehat{Z}(p)$.

Step 4. (Build Solution) Letting $\widehat{\boldsymbol{u}}(p)$ be as in the previous step, considering the Knapsack problem with $p=\widehat{p}$, for each $k\in\mathcal{K}$, choose the unique grid point $\widehat{q}^k=1,\dots,L^k$ with $\widehat{u}^k(\widehat{p},\widehat{q}^k)=1$. Return the solution $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ such that $\widehat{x}_0=\overline{x}_0(\widehat{p})$ and $\widehat{x}_i=\overline{x}_i^k(\widehat{p},\widehat{q}^k)$ for all $i\in\mathcal{N}^k$ and $k\in\mathcal{K}$.

Our collection of grid points will be a standard exponential grid of the form $\{(1+\rho)^\ell:\ell\in\mathbb{Z}\}$ for some precision parameter ρ. In Step 1, for each guess of the no-purchase probability $\overline{x}_0(q)$ and market share $\overline{z}^k(q)$ of product class k, we construct a candidate solution for product class k. We use a variant of problem (2) for this purpose, but our specific approach will depend on whether we are interested in approximating the randomized or deterministic variant. This candidate solution may violate the guess of the market share by a tolerance of $\frac{1}{1-\epsilon_1}$. In this way, for each product class k and guess of the no-purchase probability $\overline{x}_0(p)$, we generate a sequence of candidate solutions for product class k, which are given by $\overline{\boldsymbol{x}}^k(p,q)$ for $q=1,\ldots,L^k$. Because our candidate solutions satisfy $\overline{x}_i^k(p,q) \in \mathcal{H}_i(\overline{x}_0(p))$, setting $\mathcal{H}_i(x_0)=[0,v_i\,x_0]$ will allow us to use the approximation framework for the randomized variant, whereas setting $\mathcal{H}_i(x_0)=\{0,v_i\,x_0\}$ will allow us to use the approximation framework for the deterministic variant. In Step 2, fixing the no-purchase probability at the guess $\overline{x}_0(p)$, we pick exactly one candidate solution from each product class to maximize the expected revenue while making sure that the sum of the market shares of the picked solutions and the no-purchase probability exceeds one by at most a tolerance of $\frac{1}{1-\epsilon_2}$. The Knapsack problem is a multiple-choice knapsack problem, which is NP-hard, but it admits an FPTAS. In Step 3, for fixed no-purchase probability guess $\overline{x}_0(p)$, we compute a $(1-\epsilon_3)$-approximate solution to the Knapsack problem. For fixed no-purchase probability guess $\overline{x}_0(p)$, this approximate solution specifies which candidate solution we should use for each product class. We find the no-purchase probability guess that provides the largest expected revenue. In Step 4, for the no-purchase probability guess that provides the largest expected revenue, we use the approximate solution to the Knapsack problem to decide which candidate solution to pick for each product class.

In the next theorem, we give a performance guarantee for our approximation framework. Recall that $Z(p)$ denotes the optimal objective value of the Knapsack problem.

Theorem 4.1 (Approximation Framework) *Setting* $\alpha=\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}$, *letting* $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ *be the solution from the approximation framework, if we obtain the solution* $\widehat{\boldsymbol{h}}=(\widehat{h}(S):S\subseteq\mathcal{N})$ *to the* Market-Share *problem by applying the transformation in* (1) *on* $(\widehat{x}_0,\widehat{\boldsymbol{x}})$, *then this solution provides an expected revenue of at least* $\alpha\,(1-\epsilon_1)\,(1-\epsilon_2)\,(1-\epsilon_3)\,\mathsf{opt}$ *and satisfies*

$$\boldsymbol{h}\in\mathcal{H},\qquad \sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\in\{0\}\cup[(1-\epsilon_1)\,(1-\epsilon_2)\,\beta^k,1]\quad\forall\,k\in\mathcal{K},\qquad \sum_{S\subseteq\mathcal{N}}\widehat{h}(S)=1.$$

Proof: We verify that the solution $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ provided by our approximation framework satisfies the assumptions of Theorem 2.2 with $1-\gamma_1=(1-\epsilon_1)(1-\epsilon_2)$ and $1-\gamma_2=\alpha\,(1-\epsilon_3)$. In this case, the desired result follows from Theorem 2.2. Because the solution $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is obtained by stitching together the candidate solutions, by Step 1, we have $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\in\{0\}\cup[\beta^k,\infty)$ for all $k\in\mathcal{K}$ and $\widehat{x}_i\in\mathcal{H}_i(\widehat{x}_0)$ for all $i\in\mathcal{N}$, verifying the first and third assumptions in Theorem 2.2. Noting the choice of $\widehat{p}$ in Step 3 and $\widehat{q}^k$ is Step 4, by Step 1, we also have $\sum_{i\in\mathcal{N}^k}\widehat{x}_i=\sum_{i\in\mathcal{N}^k}\overline{x}_i^k(\widehat{p},\widehat{q}^k)\le\frac{1}{1-\epsilon_1}\,\overline{z}^k(\widehat{q}^k)$ for all $k\in\mathcal{K}$. In this case, using the first constraint in the Knapsack problem in Step 2, we get $\widehat{x}_0+$

$\sum_{i\in\mathcal{N}} \widehat{x}_i = \overline{x}_0(\widehat{p}) + \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}^k} \overline{x}_i^k(\widehat{p},\widehat{q}^k) \leq \frac{1}{1-\epsilon_1}\,\overline{x}_0(\widehat{p}) + \sum_{k\in\mathcal{K}} \frac{1}{1-\epsilon_1}\,\overline{z}^k(\widehat{q}^k) \leq \frac{1}{(1-\epsilon_1)\,(1-\epsilon_2)}$, verifying the second assumption in Theorem 2.2 with $1-\gamma_1 = (1-\epsilon_1)\,(1-\epsilon_2)$. Let $\widetilde{p} = \arg\max_{p=1,\ldots,L_0} Z(p)$, so by the definition of α, we have $Z(\widetilde{p}) = \alpha\,\mathsf{opt}$. By Step 3, $\widehat{u}(\widetilde{p})$ is a $(1-\epsilon_3)$-approximate solution to the Knapsack problem when we solve this problem with $p = \widetilde{p}$, which implies that $\sum_{k\in\mathcal{K}} \sum_{q=1}^{L^k} \overline{r}^k(\widetilde{p},q)\,\widehat{u}^k(\widetilde{p},q) \geq (1-\epsilon_3)\,Z(\widetilde{p})$. By the construction of the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ in Step 4, we have $\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i = \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i^k(\widehat{p},\widehat{q}^k) = \sum_{k\in\mathcal{K}} \overline{r}^k(\widehat{p},\widehat{q}^k) = \sum_{k\in\mathcal{K}} \sum_{q=1}^{L^k} \overline{r}^k(\widehat{p},q)\,\widehat{u}^k(\widehat{p},q)$, where the second equality uses the definition of $\overline{r}^k(p,q)$. In this case, we obtain

$$\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i \;=\; \sum_{k\in\mathcal{K}} \sum_{q=1}^{L^k} \overline{r}^k(\widehat{p},q)\,\widehat{u}^k(\widehat{p},q) \;\geq\; \sum_{k\in\mathcal{K}} \sum_{q=1}^{L^k} \overline{r}^k(\widetilde{p},q)\,\widehat{u}^k(\widetilde{p},q) \;\geq\; (1-\epsilon_3)\,Z(\widetilde{p}) \;=\; \alpha\,(1-\epsilon_3)\,\mathsf{opt},$$

where the first inequality uses the fact that we have $\widehat{p} = \arg\max_{p=1,\ldots,L_0} \widehat{Z}(p)$ in Step 3, verifying the fourth assumption in Theorem 2.2 with $1-\gamma_2 = \alpha\,(1-\epsilon_3)$. ■

We use our approximation framework to construct approximation algorithms for both the randomized and deterministic variants. To do so, we need to fill three gaps. First, we need to construct the grid points that serve as our guesses for the no-purchase probability and market shares of the different product classes. Second, for each product class, we need to construct the candidate solutions as in Step 1 for each guess of the no-purchase probability and market share of the product class. Third, we need to verify that the ratio $\alpha = \max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}$ that we use in Theorem 4.1 is large, based on our choice of the grid points and our approach for constructing the candidate solutions. Multiple-choice knapsack admits an FPTAS; see Bansal and Venkaiah (2004). We use this FPTAS to solve the Knapsack problem in Step 2 approximately. In the next section, we use our approximation framework to develop an FPTAS for the randomized variant.

5. Approximation Scheme for Randomized Variant

In this section, considering the randomized variant of the Market-Share problem, for any $\epsilon \in (0,1)$, setting $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_3 = \epsilon$ in our approximation framework, as well as setting $\alpha = 1-\epsilon$ in Theorem 4.1, we will execute our approximation framework in running time that is polynomial in input size and $\frac{1}{\epsilon}$. We specify our collection of grid points that serve as guesses of the no-purchase probability and market shares of different product classes in the Sales-Based problem. As discussed right after the formulation of the Sales-Based problem, we can replace the second constraint in this problem with an equality constraint. In this case, because we have $\mathcal{H}_i(x_0) = [0, v_i\,x_0]$ for the randomized variant, noting the constraint $x_i \in \mathcal{H}_i(x_0)$ in the Sales-Based problem, even when the decision variable x_i takes its largest possible value of $v_i\,x_0$, by the second constraint in the Sales-Based problem, we have $x_0 + \sum_{i\in\mathcal{N}} v_i\,x_0 = 1$. So, the smallest possible value of x_0 in a feasible solution is $\frac{1}{1+\sum_{i\in\mathcal{N}} v_i}$. Thus, the grid points that we use for our guesses for the no-purchase probability covers the interval $[\frac{1}{1+\sum_{i\in\mathcal{N}} v_i}, 1]$. Furthermore, if we have $x_0 < \frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i}$, then even when the decision variable x_i takes its largest value of $v_i x_0$ for all $i \in \mathcal{N}^k$, we get $\sum_{i\in\mathcal{N}^k} x_i = \sum_{i\in\mathcal{N}^k} v_i x_0 < \sum_{i\in\mathcal{N}^k} v_i \frac{\beta^k}{\sum_{j\in\mathcal{N}^k} v_j} = \beta^k$, so we cannot satisfy the market share constraint for product class k. Therefore, the value of $\frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i}$ for the no-purchase probability is critical to decide whether we can satisfy the market share for product class k. Letting $\overline{\mathcal{K}} = \left\{k \in \mathcal{K} : \frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i} > \frac{1}{1+\sum_{i\in\mathcal{N}} v_i}\right\}$, we add the point $\frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i}$ for each product class $k \in \overline{\mathcal{K}}$ into the grid. On the other hand, the market share for product class k takes values in the interval $[\beta^k, 1]$, so the grid points that we use for our guesses for the market share of product class k covers the interval $[\beta^k, 1]$. We add the points zero and β^k into the grid. In this case, letting $\rho > 0$ be a precision parameter, using $\lceil \cdot \rceil$ to denote the round up function, as our guesses for the no-purchase probability and market share of product class k, we use the grid points

$$\mathsf{Grid}_0 = \left\{ \frac{1}{(1+\rho)^\ell} : \ell = 0, \ldots, \left\lceil \frac{\log(1+\sum_{i\in\mathcal{N}} v_i)}{\log(1+\rho)} \right\rceil \right\} \bigcup \left\{ \frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i} : k \in \overline{\mathcal{K}} \right\} \tag{3}$$

$$\mathsf{Grid}^k = \left\{ \frac{1}{(1+\rho)^\ell} : \ell = 0, \ldots, \left\lceil \frac{\log(1/\beta^k)}{\log(1+\rho)} \right\rceil \right\} \bigcup \left\{ 0, \beta^k \right\}.$$

Successive non-zero grid points differ by at most a factor of $1+\rho$. Recall that we denote the grid points succinctly as $\mathsf{Grid}_0 = \{\overline{x}_0(p) : p = 1, \ldots, L_0\}$ and $\mathsf{Grid}^k = \{\overline{z}^k(q) : q = 1, \ldots, L^k\}$.

Generating Candidate Solutions:

In Step 1 of our approximation framework, for each product class k, to construct the candidate solution corresponding to the grid points $\overline{x}_0(p)$ and $\overline{z}^k(q)$, we solve the problem

$$\max_{\boldsymbol{x}^k \in [0,1]^{n^k}} \left\{ \sum_{i\in\mathcal{N}^k} r_i x_i \ : \ \sum_{i\in\mathcal{N}^k} x_i \le \overline{z}^k(q), \quad x_i \le v_i \overline{x}_0(p) \ \ \forall i \in \mathcal{N}^k \right\}. \tag{4}$$

In our approximation framework, for product class k, we used $\overline{\boldsymbol{x}}^k(p,q) = (\overline{x}_i(p,q) : i \in \mathcal{N}^k)$ to denote the candidate solution corresponding to the grid points $\overline{x}_0(p)$ and $\overline{z}^k(q)$. By the discussion in the previous paragraph, if $\overline{x}_0(p) < \frac{\beta^k}{\sum_{i\in\mathcal{N}} v_i}$, then we cannot satisfy the market share for product class k, so we set the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ as $\overline{\boldsymbol{x}}^k(p,q) = \mathbf{0} \in \mathbb{R}_+^{n^k}$. If $\overline{z}^k(q) < \beta^k$, then we cannot satisfy the market share constraint for product class k either, so we also set the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ as $\overline{\boldsymbol{x}}^k(p,q) = \mathbf{0} \in \mathbb{R}_+^{n^k}$. Lastly, if $\overline{x}_0(p) \ge \frac{\beta^k}{\sum_{i\in\mathcal{N}} v_i}$ and $\overline{z}^k(q) \ge \beta^k$, then we set the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ as an optimal solution to problem (4). We verify that constructing the candidate solutions in this way satisfies the requirements in Step 1. Problem (4) is a continuous knapsack problem, where the capacity of the knapsack is $\overline{z}^k(q)$ and the upper bound for item k is $v_i \overline{x}_0(p)$. When $\overline{x}_0(p) \ge \frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i}$ and $\overline{z}^k(q) \ge \beta^k$, the sum of the upper bounds for all items satisfies $\sum_{i\in\mathcal{N}^k} v_i \overline{x}_0(p) \ge \beta^k$ and the capacity of the knapsack is also at least β^k, so an optimal solution to (4) consumes at least β^k units of capacity. Thus, we have $\sum_{i\in\mathcal{N}^k} \overline{x}_i(p,q) \in [\beta^k, \infty)$. For the other two cases earlier in this paragraph, $\sum_{i\in\mathcal{N}^k}\overline{x}_i(p,q)=0$. Thus, we have $\sum_{i\in\mathcal{N}^k}\overline{x}_i(p,q)\in\{0\}\cup[\beta^k,\overline{z}^k(q)]$. By the second constraint in (4), we have $\overline{x}_i(p,q)\in\mathcal{H}_i(\overline{x}_0(p))$ for all $i\in\mathcal{N}^k$. So, the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ satisfies the requirements of Step 1 of our approximation framework with $\epsilon_1=0$.

Degradation in the Objective:

Recalling that $Z(p)$ is the optimal objective value of the Knapsack problem, we verify that if we set $\epsilon_2=0$ in this problem, then we have $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}\geq 1/(1+\rho)$, where ρ is the precision parameter in the construction of our collection of grid points. This result will allow us to use Theorem 4.1 with $\epsilon_2=0$ and $\alpha=\frac{1}{1+\rho}$ in the theorem to get an approximate solution to the randomized variant through our approximation framework. To show that $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}\geq 1/(1+\rho)$, letting $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem for the randomized variant, we choose the grid point $\overline{x}_0(\widehat{p})\in\mathsf{Grid}_0$ that is closest to but no larger than $\widehat{x}_0$. Thus, we choose $\widehat{p}=1,\ldots,L_0$ such that $\overline{x}_0(\widehat{p})\leq\widehat{x}_0\leq(1+\rho)\,\overline{x}_0(\widehat{p})$. Similarly, we choose the grid point $\overline{z}^k(\widehat{q}^k)\in\mathsf{Grid}^k$ such that $\overline{z}^k(\widehat{q}^k)\leq\sum_{i\in\mathcal{N}^k}\widehat{x}_i\leq(1+\rho)\,\overline{z}^k(\widehat{q}^k)$. Considering the Knapsack problem with $p=\widehat{p}$, we define the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})=(\widehat{u}^k(\widehat{p},q):q=1,\ldots,L^k)$ to this problem by setting $\widehat{u}^k(\widehat{p},\widehat{q}^k)=1$ and $\widehat{u}^k(\widehat{p},q)=0$ for all $q\neq\widehat{q}^k$. In the next lemma, we use this solution to show that $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}\geq 1/(1+\rho)$, when we consider the Knapsack problem with $\epsilon_2=0$.

Lemma 5.1 (Degradation for Randomized Variant) *Letting $Z(p)$ be the optimal objective value of the* Knapsack *problem with $\epsilon_2=0$, we have $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt}\geq 1/(1+\rho)$.*

Proof: Letting $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem, we define the grid points $\overline{x}_0(\widehat{p})$ and $\overline{z}^k(\widehat{q}^k)$, as well as the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$, as right before the lemma. We claim that $\sum_{i\in\mathcal{N}^l} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k)\geq\frac{1}{1+\rho}\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$ for all $k\in\mathcal{K}$. To see the claim, consider the case $\widehat{x}_0\geq\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$ and $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\geq\beta^k$. By the definition of $\overline{\mathcal{K}}$ used in (3), either $\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$ is included in Grid_0 or the smallest point in Grid_0 exceeds $\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$. In either case, because $\overline{x}_0(\widehat{p})$ is a grid point that is closest to but no larger than $\widehat{x}_0$, having $\widehat{x}_0\geq\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$ implies that $\overline{x}_0(\widehat{p})\geq\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$. Similarly, because β^k is included in Grid^k and $\overline{z}^k(\widehat{q}^k)$ is a grid point that is closest to but no larger than $\sum_{i\in\mathcal{N}^k}\widehat{x}_i$, having $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\geq\beta^k$ implies that $\overline{z}^k(\widehat{q}^k)\geq\beta^k$. In this case, by the discussion right after (4), we choose the candidate solution $\overline{\boldsymbol{x}}^k(\widehat{p},\widehat{q}^k)$ as an optimal solution to (4) with $(p,q)=(\widehat{p},\widehat{q}^k)$. By the definition of $\widehat{p}$, we have $\frac{1}{1+\rho}\sum_{i\in\mathcal{N}^k}\widehat{x}_i\leq\overline{z}^k(\widehat{q}^k)$. Since $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem, $\frac{1}{1+\rho}\widehat{x}_i\leq v_i\,\frac{1}{1+\rho}\,\widehat{x}_0\leq v_i\,\overline{x}_0(\widehat{p})$. Thus, $\frac{1}{1+\rho}\,\widehat{\boldsymbol{x}}^k$ with $\widehat{\boldsymbol{x}}^k=(\widehat{x}_i:i\in\mathcal{N}^k)$ is feasible to (4) at $(p,q)=(\widehat{p},\widehat{q}^k)$, but $\overline{\boldsymbol{x}}_i^k(\widehat{p},\widehat{q}^k)$ is optimal to the same problem, so we get $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k)\geq\frac{1}{1+\rho}\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$.

By the discussion in the previous paragraph, we have $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k)\geq\frac{1}{1+\rho}\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i$ as long as $\widehat{x}_0\geq\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$ and $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\geq\beta^k$. To check the other cases, if $\widehat{x}_0<\frac{\beta^k}{\sum_{i\in\mathcal{N}^k}v_i}$, then we get $\sum_{i\in\mathcal{N}^k}\widehat{x}_i \le \widehat{x}_0 \sum_{i\in\mathcal{N}^k} v_i < \beta^k$, where the first inequality holds because the solution $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem, in which case, by the first constraint in this problem, we get $\sum_{i\in\mathcal{N}^k}\widehat{x}_i = 0$. Because $\overline{z}^k(\widehat{q}^k) \le \sum_{i\in\mathcal{N}^k}\widehat{x}_i$, we obtain $\overline{z}^k(\widehat{q}^k)=0$, so by the discussion that follows (4), we choose the candidate solution $\overline{\boldsymbol{x}}^k(\widehat{p},\widehat{q}^k)$ as $\overline{\boldsymbol{x}}^k(\widehat{p},\widehat{q}^k)=\mathbf{0}$. Therefore, if $\widehat{x}_0 < \frac{\beta^k}{\sum_{i\in\mathcal{N}^k} v_i}$, then we have $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) = 0 = \sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$. If $\sum_{i\in\mathcal{N}^k}\widehat{x}_i < \beta^k$, then a similar argument also yields $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) = 0 = \sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$. Thus, $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \ge \frac{1}{1+\rho}\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$ for all $k\in\mathcal{K}$. Furthermore, note that the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$ defined just before the lemma is feasible to the Knapsack problem with $p=\widehat{p}$ and $\epsilon_2=0$. In particular, by the definition of $\widehat{\boldsymbol{u}}^k(\widehat{p})$, we have $\sum_{q=1}^{L^k}\widehat{u}^k(\widehat{p},q)=1$, checking the second constraint in the Knapsack problem. To check the first constraint, we have $\overline{x}_0(\widehat{p}) + \sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k}\overline{z}^k(q)\,\widehat{u}^k(\widehat{p},q) = \overline{x}_0(\widehat{p}) + \sum_{k\in\mathcal{K}}\overline{z}^k(\widehat{q}^k) \le \widehat{x}_0 + \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{N}^k}\widehat{x}_i \le 1$, where the first inequality uses the definition of $\widehat{p}$ and $\widehat{q}^k$, whereas the second inequality holds because $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem. In this case, we obtain the chain of inequalities

$$\max_{p=1,\ldots,L_0} Z(p) \ge Z(\widehat{p}) \overset{(a)}{\ge} \sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k}\overline{r}^k(\widehat{p},q)\,\widehat{u}^k(\widehat{p},q) \overset{(b)}{=} \sum_{k\in\mathcal{K}}\overline{r}^k(\widehat{p},\widehat{q}^k) \overset{(c)}{=} \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \ge \frac{1}{1+\rho}\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i,$$

where (a) holds because $\widehat{\boldsymbol{u}}^k(\widehat{p})$ is feasible to the Knapsack problem with $p=\widehat{p}$, (b) uses the definition of $\widehat{\boldsymbol{u}}^k(\widehat{p})$ and (c) uses the definition of $\overline{r}^k(p,q)$. The result follows as we have $\mathsf{opt} = \sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i$. ■

In the next theorem, we use Lemma 5.1 together with Theorem 4.1 to give an FPTAS for the randomized variant of the Market-Share problem. We set $v_{\max} = \max_{i\in\mathcal{N}} v_i$ and $\beta_{\min} = \min_{k\in\mathcal{K}}\beta^k$.

Theorem 5.2 (FPTAS for Randomized Variant) *Considering the randomized variant of the* Market-Share *problem, for any $\epsilon\in(0,1)$, we can obtain a $(1-\epsilon)$-approximate solution in a running time of $O\Big(\frac{1}{\epsilon^2}K^2 n\Big(K+\frac{1}{\epsilon}\log(1+nv_{\max})\Big)\log(\frac{1}{\beta_{\min}}) + n\log n\Big)$ operations.*

Proof: Given $\epsilon\in(0,1)$, we set the precision parameter for our grid as $\rho=\frac{\epsilon}{2}$. By the discussion right after (4), the candidate solutions that we construct satisfy the requirements of Step 1 with $\epsilon_1=0$. Multiple-choice knapsack problem admits an FPTAS. Considering the Knapsack problem with $\epsilon_2=0$ in Step 2, we use this FTPAS for to obtain a $(1-\epsilon/2)$-approximate solution, so we set $\epsilon_3=\frac{\epsilon}{2}$ in Step 3. By Lemma 5.1, we have $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt} \ge \frac{1}{1+\rho} = \frac{1}{1+\epsilon/2} \ge 1-\frac{\epsilon}{2}$. Thus, we can use Theorem 4.1 with $\alpha=1-\frac{\epsilon}{2}$, $\epsilon_1=\epsilon_2=0$ and $\epsilon_3=\frac{\epsilon}{2}$. We have $\alpha\,(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3) = (1-\frac{\epsilon}{2})^2 \ge 1-\epsilon$, so the solution from our approximation framework yields an expected revenue of at least $(1-\epsilon)\,\mathsf{opt}$. Because $(1-\epsilon_1)\,(1-\epsilon_2)=1$, this solution satisfies the market share constraints. The number of grid points and the running time to obtain a $(1-\epsilon)$-approximate solution to the Knapsack problem are polynomial in input size and $\frac{1}{\epsilon}$. We can solve the continuous knapsack problem in (4) in polynomial time in input size. Thus, the running time of our approximation framework is polynomial in input size and $\frac{1}{\epsilon}$. We account for the precise running time in Appendix C. ■

6. Controlled Market Share Violations for Deterministic Variant

Turning our attention to the deterministic variant of the Market-Share problem, we give an algorithm to obtain a solution to the deterministic variant that provides an expected revenue of at least $(1-\epsilon)\,\mathsf{opt}$, while possibly violating the market share constraints with a factor of at most $1+\epsilon$ in running time that is polynomial in input size and $\frac{1}{\epsilon}$. The construction of this algorithm uses our approximation framework. The algorithm that we give in this section provides a solution that may violate the market share constraints by amounts that we control with our choice of ϵ. Small violations of market share constraints may be acceptable, making this algorithm useful in its own right. More interestingly, perhaps, we build on this algorithm to give another algorithm in the next section that provides a solution to the deterministic variant with an expected revenue of at least $\frac{1}{2}\mathsf{opt}$ while satisfying the market share constraints exactly. This algorithm is akin to a constant-factor approximation algorithm, but its running time is pseudo-polynomial.

For the deterministic variant, we continue using the same grid points given in (3). We explain the construction of the candidate solutions in Step 1 of our approximation framework.

Generating Candidate Solutions:

In Step 1 of our approximation framework, for each product class k, to construct the candidate solution corresponding to the grid points $\overline{x}_0(p)$ and $\overline{z}^k(q)$, we solve the problem

$$\max_{\boldsymbol{x}^k \in [0,1]^{n^k}} \left\{ \sum_{i\in\mathcal{N}^k} r_i\, x_i \;:\; \sum_{i\in\mathcal{N}^k} x_i \in \{0\}\cup[\beta^k,\infty), \quad \sum_{i\in\mathcal{N}^k} x_i \le (1+\rho)\,\overline{z}^k(q), \right.$$
$$\left. x_i \in \{0, v_i\,\overline{x}_0(p)\} \quad \forall\, i\in\mathcal{N}^k \right\}. \tag{5}$$

In the problem above, the parameter ρ corresponds to the precision parameter in our grid points. Letting $\overline{\boldsymbol{x}}^k(p,q) = (\overline{x}_i(p,q) : i\in\mathcal{N}^k)$ be an optimal solution to problem (5), noting that we have $\mathcal{H}_i(x_0) = \{0, v_i\, x_0\}$ for the deterministic variant, we have $\sum_{i\in\mathcal{N}^k} \overline{x}_i(p,q) \in \{0\}\cup[\beta^k, (1+\rho)\,\overline{z}^k(q)]$ and $x_i \in \mathcal{H}_i(\overline{x}_0(q))$ for all $i\in\mathcal{N}^k$ by the constraints in (5). Thus, the solution $\overline{\boldsymbol{x}}^k(p,q)$ satisfies the requirements in Step 1 of our approximation framework with $\epsilon_1 = 1-\frac{1}{1+\rho}$. However, it is difficult to solve problem (5) as above. In problem (5), we have both upper and lower bound constrains on the quantity $\sum_{i\in\mathcal{N}^k} x_i$. Also, the decision variable x_i takes either the value 0 or $v_i\,\overline{x}_0(p)$. Due to these considerations, it is difficult to find even an approximate solution to this problem that satisfies both the upper and lower bound constraints. Instead, we find an approximate solution to problem (5) that satisfies the lower bound constraint exactly, while allowing controlled violations of the upper bound constraint. We use a dynamic program for this purpose as described next.

In (5), we maximize the revenue contribution, while the total market share of the products in class k, if non-zero, is lower bounded by β^k and upper bounded by $(1+\rho)\,\overline{z}^k(q)$. We consider maximizing the market share, while the revenue contribution is lower bounded by some value γ and the market share is upper bounded by $(1+\rho)\,\overline{z}^k(q)$, yielding the problem

$$\max_{\boldsymbol{y}^k \in \{0,1\}^{n^k}} \left\{ \sum_{i \in \mathcal{N}^k} v_i \,\overline{x}_0(p)\, y_i \;:\; \sum_{i \in \mathcal{N}^k} r_i\, v_i\, \overline{x}_0(p)\, y_i \geq \gamma, \quad \sum_{i \in \mathcal{N}^k} v_i\, \overline{x}_0(p)\, y_i \leq (1+\rho)\, \overline{z}^k(q) \right\}. \tag{6}$$

In the problem above, recalling that x_i takes one of the values 0 or $v_i\,\overline{x}_0(p)$ in (5), the decision variables are $\boldsymbol{y}^k = (y_i : i \in \mathcal{N}^k) \in \{0,1\}^{n^k}$, where $y_i = 1$ if and only if the market share of product i takes the value $v_i\,\overline{x}_0(p)$. In the objective function, we maximize the market share of the offered products in class k. Interpreting γ as a guess of the optimal objective value of problem (5), the first constraint ensures that the revenue contribution is at least as large as the guess, whereas the second constraint ensures that the market share of the products in class k is at most $(1+\rho)\,\overline{z}^k(q)$. To come up with possible guesses for the optimal objective value of problem (5), we use a collection of grid points. In particular, letting $v_{\min} = \min_{i\in\mathcal{N}} v_i$, $r_{\max} = \max_{i\in\mathcal{N}} r_i$ and $r_{\min} = \min_{i\in\mathcal{N}} r_i$, trivial upper and lower bounds on the non-zero optimal objective value of problem (5) are, respectively, given by $n\, r_{\max}\, v_{\max}\, \overline{x}_0(p)$ and $r_{\min}\, v_{\min}\, \overline{x}_0(p)$. Using the same precision parameter ρ that we used in the collection of grid points in (3) and letting $\lfloor \cdot \rfloor$ be the round down function, to cover the interval $[r_{\min}\, v_{\min}\, \overline{x}_0(q), n\, r_{\max}\, v_{\max}\, \overline{x}_0(p)]$, we define the collection of grid points

$$\overline{\mathsf{Grid}} = \left\{ (1+\rho)^\ell : \ell = \left\lfloor \frac{\log(r_{\min}\, v_{\min}\, \overline{x}_0(p))}{\log(1+\rho)} \right\rfloor, \ldots, \left\lceil \frac{\log(n\, r_{\max}\, v_{\max}\, \overline{x}_0(p))}{\log(1+\rho)} \right\rceil \right\}.$$

To construct our candidate solutions, we will obtain an approximate solution to problem (6) for the guesses γ taking values in the collection of grid points above.

Problem (6) is a two-dimensional knapsack problem, where the first constraint is a cover constraint and the second constraint is a packing constraint. We can formulate this problem as a dynamic program with a two-dimensional state variable, where each component of the state variable keeps track of the capacity consumption of the knapsack in each dimension. However, if we follow this approach, then the number of possible values for the state variable in the dynamic program still increases exponentially with the number of products, making the dynamic program computationally intractable. To get around this difficulty, we use a discretization approach proposed by Desir et al. (2022). In particular, we multiply both sides of the first constraint in problem (6) by $\frac{n}{\gamma\,\rho}$ to write this constraint equivalently as $\sum_{i\in\mathcal{N}^k} \frac{n}{\rho\,\gamma}\, r_i\, v_i\, \overline{x}_0(p)\, y_i \geq \frac{n}{\rho}$, in which case, a relaxed version of the first constraint in problem (6) is given by $\sum_{i\in\mathcal{N}^k} \lceil \frac{n}{\rho\,\gamma}\, r_i\, v_i\, \overline{x}_0(p) \rceil\, y_i \geq \lfloor \frac{n}{\rho} \rfloor$. Switching the roles of rounding up and down, using a similar approach, we obtain a relaxed version of the second constraint in problem (6), which is given by $\sum_{i\in\mathcal{N}^k} \lfloor \frac{n}{\rho\,(1+\rho)\,\overline{z}^k(q)}\, v_i\, \overline{x}_0(p) \rfloor\, y_i \geq \lceil \frac{n}{\rho} \rceil$.

Setting $\sigma_i(p,\gamma) = \lceil \frac{n}{\rho\,\gamma}\, r_i\, v_i\, \overline{x}_0(p) \rceil$ and $\delta_i(p,q) = \lfloor \frac{n}{\rho\,(1+\rho)\,\overline{z}^k(q)}\, v_i\, \overline{x}_0(p) \rfloor$, a relaxed version of problem (6) is $\max_{\boldsymbol{y}^k\in\{0,1\}^{n^k}} \{ \sum_{i\in\mathcal{N}^k} v_i\, \overline{x}_0(p)\, y_i : \sum_{i\in\mathcal{N}^k} \sigma_i(p,\gamma)\, y_i \geq \lfloor \frac{n}{\rho} \rfloor,\ \sum_{i\in\mathcal{N}^k} \delta_i(p,q)\, y_i \leq \lceil \frac{n}{\rho} \rceil \}$. There are two useful properties of the relaxed version. First, the constraint coefficients $\sigma_i(p,\gamma)$ and $\delta_i(p,q)$ in the relaxed version take integer values. Thus, if we solve the relaxed version through a dynamic program, then the state variable, which keeps track of the capacity consumptions of the knapsack in the two dimensions, takes integer values. Second, the values of the right sides of both constrains is $O(n/\rho)$. Thus, as we solve the relaxed version through a dynamic program, it is enough to consider $O(n/\rho)$ possible values for each of the two dimensions of the state variable. Because of these two properties, we can solve the relaxed version of problem (6) in running time that is polynomial in n and ρ. When ρ is small, we expect the relaxed version to be a good approximation to problem (6). In particular, intuitively speaking, if ρ is small, then $\frac{n}{\rho\,\gamma}\, r_i\, v_i\, \overline{x}_0(p)$, $\frac{n}{\rho\,(1+\rho)\,\overline{z}^k(q)}\, v_i\, \overline{x}_0(p)$ and $\frac{n}{\rho}$ are large, so rounding these quantities up and down does not change their value significantly. By solving the relaxed problem, we obtain an approximate solution to problem (6) that satisfies the constraints in this problem with some violation. In this case, by obtaining such an approximate solution to problem (6) for all guesses of the optimal revenue contribution in $\overline{\mathsf{Grid}}$, we, in turn, obtain an approximate solution to problem (5) that satisfies the constraints in this problem with some violation. In the next lemma, we make this result precise. We defer the proof of the lemma to Appendix D. The proof follows by accounting for the loss incurred by using the relaxed version to solve problem (6) and using the guesses of the optimal objective value of (5) over a grid. In the lemma, recall that ρ refers to the precision parameter in the collection of our grid points.

Lemma 6.1 (Candidates for Deterministic Variant) *We can obtain a solution $\widehat{\boldsymbol{x}}^k$ to problem* (5) *that provides at least* $(1-3\,\rho)$*-fraction of the optimal objective value of this problem and satisfies the first and third constraints exactly, while satisfying the second constraint as* $\sum_{i\in\mathcal{N}^k}\widehat{x}_i \le (1+2\rho)\,(1+\rho)\,\overline{z}^k(q)$, *in a running time of* $O\Big(\frac{n^3}{\rho^3}\log(\frac{n\, r_{\max}\, v_{\max}}{r_{\min}\, v_{\min}})\Big)$ *operations.*

We implicitly assume that $\rho \le \frac{1}{3}$. Otherwise, the result holds trivially. By Lemma 6.1, the candidate solutions that we obtain by using approximate solutions to problem (5) as in the lemma above satisfy $\sum_{i\in\mathcal{N}^k}\overline{x}_i(p,q) \in \{0\}\cup[\beta^k, (1+2\rho)(1+\rho)\,\overline{z}^k(q)]$ and $\overline{x}_i(p,q)\in\mathcal{H}_i(\overline{x}_0(p))$ for all $i\in\mathcal{N}^k$. Thus, the candidate solutions satisfy the requirements of Step 1 of our approximation framework with $\epsilon_1 = 1-\frac{1}{(1+2\rho)(1+\rho)}$. Note that the discussion so far in this section holds for $\overline{z}^k(q)\neq 0$, because we construct the relaxed version of problem (6) by dividing both sides of the second constraint in (6) by $\overline{z}^k(q)$. When $\overline{z}^k(q)=0$, which corresponds to one of the points in Grid^k, the trivial optimal solution to problem (5) is obtained by setting $\boldsymbol{x}^k=\mathbf{0}\in\mathbb{R}_+^{n^k}$. Thus, for each $k\in\mathcal{K}$, we proceed with the understanding that $\overline{\boldsymbol{x}}^k(p,q)=\mathbf{0}\in\mathbb{R}_+^{n^k}$ is one of the candidate solutions.

Degradation in the Objective:

We verify that the collection of candidate solutions that we generate is rich enough that we can use the Knapsack problem to approximate the deterministic variant. In particular, we consider the Knapsack problem with $\epsilon_2 = 1 - \frac{1}{1+\rho}$, where ρ is the precision parameter that we use when constructing our grid points. Using $Z(p)$ to denote the optimal objective value of the Knapsack problem with $\epsilon_2 = 1 - \frac{1}{1+\rho}$, we will show that $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt} \geq 1 - 3\rho$. In this case, we will be able to use Theorem 4.1 with $\epsilon_2 = 1 - \frac{1}{1+\rho}$ and $\alpha = 1 - 3\rho$ in the theorem to obtain an approximate solution to the deterministic variant with controlled market share violations through our approximation framework. To show this result, we use $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ to denote an optimal solution to the Sales-Based problem for the deterministic variant. We choose the grid point $\overline{x}_0(\widehat{p}) \in \mathsf{Grid}_0$ that is closest to but no smaller than $\widehat{x}_0$. In other words, we choose $\widehat{p} = 1, \ldots, L_0$ such that $\widehat{x}_0 \leq \overline{x}_0(\widehat{p}) \leq (1+\rho)\,\widehat{x}_0$. Similarly, for each $k \in \mathcal{K}$, we choose the grid point $\overline{z}^k(\widehat{q}^k) \in \mathsf{Grid}^k$ that is closest to but no smaller than $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$. In other words, we choose $\widehat{q}^k = 1, \ldots, L^k$ such that $\sum_{i\in\mathcal{N}^k} \widehat{x}_i \leq \overline{z}^k(\widehat{q}^k) \leq (1+\rho) \sum_{i\in\mathcal{N}^k} \widehat{x}_i$. Considering the Knapsack problem with $p = \widehat{p}$ and $\epsilon_2 = 1 - \frac{1}{1+\rho}$, we define the solution $\widehat{\boldsymbol{u}}^k(\widehat{p}) = (\widehat{u}^k(\widehat{p}, q) : q = 1, \ldots, L^k)$ to this problem by setting $\widehat{u}^k(\widehat{p}, \widehat{q}^k) = 1$ and $\widehat{u}^k(\widehat{p}, q) = 0$ for all $q \neq \widehat{q}^k$. In the next lemma, we use this solution to show that $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt} \geq 1 - 3\rho$ when we consider the Knapsack problem with $\epsilon_2 = 1 - \frac{1}{1+\rho}$.

Lemma 6.2 (Degradation for Deterministic Variant) *Letting $Z(p)$ be the optimal objective value of the* Knapsack *problem with $\epsilon_2 = 1 - \frac{1}{1+\rho}$, we have $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt} \geq 1 - 3\,\rho$.*

Proof: Letting $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem for the deterministic variant, we define the grid points $\overline{x}_0(\widehat{p})$ and $\overline{z}^k(\widehat{q}^k)$, as well as the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$, as right before the lemma. For problem (5), we define the solution $\widetilde{\boldsymbol{x}}^k = (\widetilde{x}_i : i \in \mathcal{N}^k)$ as $\widetilde{x}_i = v_i\, \overline{x}_0(\widehat{p})\, \mathbf{1}(\widehat{x}_i > 0)$ for all $i \in \mathcal{N}^k$. We claim that the solution $\widetilde{\boldsymbol{x}}^k$ is feasible to problem (5) with $(p,q) = (\widehat{p}, \widehat{q}^k)$. If $\sum_{i\in\mathcal{N}^k} \widehat{x}_i > 0$, then we have $\sum_{i\in\mathcal{N}^k} \widetilde{x}_i = \sum_{i\in\mathcal{N}^k} v_i\, \overline{x}_0(\widehat{p})\, \mathbf{1}(\widehat{x}_i > 0) \geq \sum_{i\in\mathcal{N}^k} v_i\, \widehat{x}_0\, \mathbf{1}(\widehat{x}_i > 0) = \sum_{i\in\mathcal{N}^k} \widehat{x}_i \geq \beta^k$, where the first inequality uses the definition of the grid point $\widehat{p}$, the second equality holds because $\widehat{x}_i \in \{0, v_i\, \widehat{x}_0\}$ by the third constraint in the Sales-Based problem and the second inequality holds by the first constraint in the Sales-Based problem and noting that $\sum_{i\in\mathcal{N}^k} \widehat{x}_i > 0$. If $\sum_{i\in\mathcal{N}^k} \widehat{x}_i = 0$, then $\sum_{i\in\mathcal{N}^k} \widetilde{x}_i = 0$ by the definition of $\widetilde{\boldsymbol{x}}^k$. Thus, the solution $\widetilde{\boldsymbol{x}}^k$ satisfies the first constraint in (5). Similarly, we have $\sum_{i\in\mathcal{N}^k} \widetilde{x}_i = \sum_{i\in\mathcal{N}^k} v_i\, \overline{x}_0(\widehat{p})\, \mathbf{1}(\widehat{x}_i > 0) \leq (1+\rho) \sum_{i\in\mathcal{N}^k} v_i\, \widehat{x}_0\, \mathbf{1}(\widehat{x}_i > 0) = (1+\rho) \sum_{i\in\mathcal{N}^k} \widehat{x}_i \leq (1+\rho)\overline{z}^k(\widehat{q}^k)$, where the first and second inequalities use the definitions of the grid points $\widehat{p}$ and $\widehat{q}^k$. Thus, the solution $\widetilde{\boldsymbol{x}}^k$ satisfies the second constraint in (5). By the definition of $\widetilde{\boldsymbol{x}}^k$, we have $\widetilde{x}_i \in \{0, v_i\, \overline{x}_0(\widehat{p})\}$, so the solution $\widetilde{\boldsymbol{x}}^k$ satisfies the third constraint in (5). Thus, the claim holds and the solution $\widetilde{\boldsymbol{x}}^k$ is feasible to problem (5) with $(p,q) = (\widehat{p}, \widehat{q}^k)$.

By the claim in the previous paragraph, using $\zeta^k(p,q)$ to denote the optimal objective value of problem (5), we obtain $\zeta^k(\widehat{p}, \widehat{q}^k) \geq \sum_{i\in\mathcal{N}^k} r_i\, \widetilde{x}_i \geq \sum_{i\in\mathcal{N}^k} r_i\, v_i\, \widehat{x}_0\, \mathbf{1}(\widehat{x}_i > 0) = \sum_{i\in\mathcal{N}^k} r_i\, \widehat{x}_i$, where the second inequality holds by the definition of $\widetilde{\boldsymbol{x}}^k$, as well as noting that $\overline{x}_0(\widehat{p}) \geq \widehat{x}_0$, whereas the equality holds because $\widehat{x}_i \in \{0, v_i\,\widehat{x}_0\}$ by the last constraint in the Sales-Based problem. On the other hand, by Lemma 6.1, the candidate solution that we generate corresponding to the grid points $(\widehat{p}, \widehat{q}^k)$ satisfies $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \geq (1-3\,\rho)\,\zeta^k(\widehat{p},\widehat{q}^k)$, so by the chain of inequalities at the beginning of this paragraph, we get $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \geq (1-3\,\rho)\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$. Lastly, the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$ is feasible to Knapsack problem with $p=\widehat{p}$ and $\epsilon_2 = 1-\frac{1}{1+\rho}$. In particular, by the definition of the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$, $\sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k} \overline{z}^k(q)\,\widehat{u}^k(\widehat{p},q) + \overline{x}_0(\widehat{p}) = \overline{z}^k(\widehat{q}^k) + \overline{x}_0(\widehat{p}) \leq (1+\rho)\sum_{i\in\mathcal{N}^k}\widehat{x}_i + (1+\rho)\,\widehat{x}_0 \leq 1+\rho = \frac{1}{1-\epsilon_2}$, where the first in equality uses the definitions of the grid points $\widehat{p}$ and $\widehat{q}^k$, whereas the second inequality holds because $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem. Thus, the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$ satisfies the first constraint in the Knapsack problem. The second constraint in the Knapsack problem holds immediately by the definition of $\widehat{\boldsymbol{u}}^k(\widehat{p})$. Because the solution $\widehat{\boldsymbol{u}}^k(\widehat{p})$ is feasible to the Knapsack problem with $p=\widehat{p}$, we obtain

$$\max_{p=1,\ldots,L_0} Z(p) \geq Z(\widehat{p}) \geq \sum_{k\in\mathcal{K}}\sum_{q=1}^{L^k} \overline{r}^k(\widehat{p},q)\;\widehat{u}^k(\widehat{p},q) \overset{(a)}{=} \sum_{k\in\mathcal{K}} \overline{r}^k(\widehat{p},\widehat{q}^k) \overset{(b)}{=} \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \overset{(c)}{\geq} (1-3\rho)\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i,$$

where (a) uses the definition of $\widehat{\boldsymbol{u}}^k(\widehat{p})$ and (b) is by the definition of $\overline{r}^k(p,q)$ and (c) holds because $\sum_{i\in\mathcal{N}^k} r_i\,\overline{x}_i(\widehat{p},\widehat{q}^k) \geq (1-3\,\rho)\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$. The result follows because we have $\mathsf{opt} = \sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i$. ■

In the next theorem, we use the lemma above together with Theorem 4.1 to obtain an approximate solution to the deterministic variant with controlled market share violations.

Theorem 6.3 (Controlled Market Share Violations) *Considering the deterministic variant of the* Market-Share *problem, for any $\epsilon\in(0,1)$, we can obtain a solution with an expected revenue of at least $(1-\epsilon)\,\mathsf{opt}$, while providing a market share of zero or at least $(1-\epsilon)\,\beta^k$ for each product class k, in a running time of $O\Big(\frac{1}{\epsilon^4}K^2n^3\log(\frac{n\,r_{max}\,v_{max}}{r_{\min}\,v_{\min}})\,\Big(K+\frac{1}{\epsilon}\log(1+nv_{\max})\Big)\log(\frac{1}{\beta_{\min}})\Big)$.*

Proof: Given $\epsilon\in(0,1)$, we set the precision parameter for our grid as $\rho=\frac{\epsilon}{8}$. By the discussion right after Lemma 6.1, the candidate solutions that we construct satisfy the requirements of Step 1 with $\epsilon_1 = 1-\frac{1}{(1+2\rho)(1+\rho)} = 1-\frac{1}{(1+\frac{\epsilon}{4})(1+\frac{\epsilon}{8})}$. Multiple-choice knapsack problem has an FPTAS. For the Knapsack problem with $\epsilon_2 = 1-\frac{1}{1+\rho}$, we use this FTPAS to get a $(1-\epsilon/8)$-approximate solution, so $\epsilon_3=\frac{\epsilon}{8}$ in Step 3. By Lemma 6.2, we have $\max_{p=1,\ldots,L_0} Z(p)/\mathsf{opt} \geq 1-3\rho = 1-\frac{3\epsilon}{8}$. Thus, we can use Theorem 4.1 with $\alpha = 1-\frac{3\epsilon}{8}$, $\epsilon_1 = 1-\frac{1}{(1+\frac{\epsilon}{4})(1+\frac{\epsilon}{8})}$, $\epsilon_2 = 1-\frac{1}{1+\frac{\epsilon}{8}}$ and $\epsilon_3=\frac{\epsilon}{8}$. We have $\alpha\,(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3) = (1-\frac{3\epsilon}{8})\frac{1}{(1+\frac{\epsilon}{4})(1+\frac{\epsilon}{8})}\,\frac{1}{1+\frac{\epsilon}{8}}(1-\frac{\epsilon}{8}) \geq (1-\frac{3\epsilon}{8})(1-\frac{\epsilon}{4})\,(1-\frac{\epsilon}{8})^3 \geq 1-\epsilon$, so we get a solution with an expected revenue of at least $(1-\epsilon)\,\mathsf{opt}$. Because $(1-\epsilon_1)\,(1-\epsilon_2) = \frac{1}{(1+\frac{\epsilon}{4})(1+\frac{\epsilon}{8})}\,\frac{1}{1+\frac{\epsilon}{8}} \geq 1-\frac{\epsilon}{2}$, this solution provides a market share of at least $(1-\frac{\epsilon}{2})\,\beta^k$ or zero for product class k. The number of grid points, as well as the running time to generate our candidate solutions as in Lemma 6.1, is polynomial in input size and $\frac{1}{\epsilon}$. Thus, the running time of our approximation framework is polynomial in input size and $\frac{1}{\epsilon}$. We account for the precise running time in Appendix E. ■

7. Constant-Factor in Pseudo-Polynomial Time for Deterministic Variant

Considering the deterministic variant of the Market-Share problem, we give an algorithm that provides a solution with an expected revenue of at least $\frac{1}{2}\mathsf{opt}$, while satisfying all of the market share constraints exactly. In this sense, this algorithm is akin to a constant-factor approximation algorithm, but its running time is pseudo-polynomial in the input size. Throughout this section, we proceed with the understanding that we offer products in at least two product classes in an optimal solution to the deterministic variant. In Appendix F, to cover the case where we offer products in exactly one product class in an optimal solution to the deterministic variant, we give an FPTAS for the deterministic variant when there is one product class. Thus, if we offer products in only one product class in an optimal solution to the deterministic variant, then we can focus on each product class one by one, ignoring the products in other classes, execute the FPTAS and pick the solution with the largest expected revenue. The running time of this FPTAS is dominated by the algorithm that we give in this section. Noting the constraint $\boldsymbol{h} \in \{0,1\}^n$ in the deterministic variant of the Market-Share problem, using the fact that the choice probability of product i from assortment S is given by $\phi_i(S) = v_i/(1+\sum_{j\in S} v_j)$, we express the deterministic variant as

$$\mathsf{opt} \;=\; \max_{S\subseteq\mathcal{N}} \left\{ \sum_{k\in\mathcal{K}} \frac{\sum_{i\in S\cap\mathcal{N}^k} r_i\, v_i}{1+\sum_{i\in S} v_i} \;:\; \frac{\sum_{i\in S\cap\mathcal{N}^k} v_i}{1+\sum_{i\in S} v_i} \in \{0\}\cup[\beta^k,1] \right\}. \tag{7}$$

In (7), a customer chooses product i within assortment S with probability $v_i/(1+\sum_{j\in S} v_j)$, so the expected revenue from the products in class k is $\sum_{i\in S\cap\mathcal{N}^k} r_i\, v_i/(1+\sum_{j\in S} v_j)$ and the total purchase probability of the products in class k is $\sum_{i\in S\cap\mathcal{N}^k} v_i/(1+\sum_{i\in S} v_i)$. We outline our approach to get a solution with an expected revenue of at least $\frac{1}{2}\mathsf{opt}$, while satisfying all market share constraints exactly. We focus on the product class that provides the smallest expected revenue in an optimal solution. Calling this product class the weak one, we stop offering the products in the weak class. Noting that we offer products in at least two product classes, not offering the products in the weak class loses an expected revenue of at most $\frac{1}{2}\mathsf{opt}$. Once we stop offering the products in the weak class, the market share of all other product classes increases. Considering the non-weak class k that has non-zero market share in the optimal solution, we argue that if we stop offering the weak class, then the market share of product class k is at least $\frac{\beta^k}{1-\beta_{\min}}$. Thus, focusing on a problem instance with the market share threshold of class k being $\frac{\beta^k}{1-\beta_{\min}}$, we use the algorithm for controlled market share violations with a market share violation budget of $1+\beta_{\min}$. Noting that $\frac{\beta^k}{(1-\beta_{\min})\,(1+\beta_{\min})} \geq \beta^k$, the latter solution satisfies all market share constraints without any violation.

To formalize the outline given in the previous paragraph, using $\widehat{S}$ to denote an optimal solution to problem (7), set $\widehat{x}_i = v_i/(1+\sum_{j\in\widehat{S}} v_j)$ for all $i\in\widehat{S}$ and $\widehat{x}_i = 0$ for all $i\in\mathcal{N}\setminus\widehat{S}$, in which case, $\widehat{x}_i$

corresponds to the purchase probability of product $i \in \widehat{S}$ under the optimal solution. The optimal objective value of problem (7) is given by $\mathsf{opt} = \sum_{k\in\mathcal{K}} \sum_{i\in\widehat{S}\cap\mathcal{N}^k} r_i\,\widehat{x}_i$, whereas the contribution of product class k to the optimal total expected revenue is $\sum_{i\in\widehat{S}\cap\mathcal{N}^k} r_i\,\widehat{x}_i$. Define the weak product class as $\lambda = \arg\min_{k\in\mathcal{K}} \{\sum_{i\in\widehat{S}\cap\mathcal{N}^k} r_i\,\widehat{x}_i : |\widehat{S}\cap\mathcal{N}^k| > 0\}$, which is the product class making the smallest non-zero expected revenue contribution. Dropping the products in the weak class from the optimal solution, consider the solution $\overline{S} = \widehat{S}\setminus\mathcal{N}^\lambda$ for problem (7). We proceed with the understanding that the optimal solution offers products in at least two product classes, so $\overline{S}$ is non-empty. Set $\overline{x}_i = v_i/(1+\sum_{j\in\overline{S}} v_j)$ for all $i\in\overline{S}$ and $\overline{x}_i = 0$ for all $i\in\mathcal{N}\setminus\overline{S}$, so $\overline{x}_i$ corresponds to the purchase probability of product i under the solution $\overline{S}$. Because the solution $\overline{S}$ does not offer any products in the weak class, the expected revenue from the solution $\overline{S}$ is $\sum_{k\in\mathcal{K}\setminus\{\lambda\}} \sum_{i\in\overline{S}\cap\mathcal{N}^k} r_i\,\overline{x}_i$. We lower bound the purchase probability of each product in the solution $\overline{S}$ with the corresponding quantity in the solution $\widehat{S}$. Because the optimal solution offers products in the weak class, the market share of the weak product class in the optimal solution is non-zero, so by the constraint in (7), we have $\sum_{i\in\widehat{S}\cap\mathcal{N}^\lambda} v_i/(1+\sum_{i\in\widehat{S}} v_i) \ge \beta^\lambda \ge \beta_{\min}$. In this case, for all $i\in\overline{S}\subseteq\widehat{S}$, we can lower bound the purchase probability of product i under the solution $\overline{S}$ as

$$\overline{x}_i \;=\; \frac{v_i}{1+\sum_{j\in\overline{S}} v_j} \;=\; \frac{v_i}{1+\sum_{j\in\widehat{S}\setminus\mathcal{N}^\lambda} v_j} \;=\; \frac{1}{1-\dfrac{\sum_{j\in\widehat{S}\cap\mathcal{N}^\lambda} v_j}{1+\sum_{j\in\widehat{S}} v_j}}\,\frac{v_i}{1+\sum_{j\in\widehat{S}} v_j} \;\ge\; \frac{1}{1-\beta_{\min}}\,\widehat{x}_i. \tag{8}$$

Considering the expected revenue from the solution $\overline{S}$, the weak product class is the one with the smallest non-zero expected revenue under the optimal solution $\widehat{S}$ and the optimal solution offers products in at least two product classes, so we have $\sum_{i\in\widehat{S}\cap\mathcal{N}^\lambda} r_i\,\widehat{x}_i \le \frac{1}{2}\mathsf{opt}$. Focusing on the other product classes, we get $\sum_{k\in\mathcal{K}\setminus\{\lambda\}} \sum_{i\in\widehat{S}\cap\mathcal{N}^k} r_i\,\widehat{x}_i \ge \frac{1}{2}\,\mathsf{opt}$. In this case, the expected revenue from the solution $\overline{S}$ satisfies $\sum_{k\in\mathcal{K}\setminus\{\lambda\}} \sum_{i\in\overline{S}\cap\mathcal{N}^k} r_i\,\overline{x}_i \ge \frac{1}{1-\beta_{\min}} \sum_{k\in\mathcal{K}\setminus\{\lambda\}} \sum_{i\in\overline{S}\cap\mathcal{N}^k} r_i\,\widehat{x}_i \ge \frac{1}{2(1-\beta_{\min})}\,\mathsf{opt}$, where the first inequality uses (8) and the second inequality holds because the definition of $\overline{S}$ implies that $\overline{S}\cap\mathcal{N}^k = \widehat{S}\cap\mathcal{N}^k$ for all $k\ne\lambda$. Considering the market share of each product class under the solution $\overline{S}$, we do not offer any products in the weak class under the solution $\overline{S}$, so the market share of the weak product class is zero. The products offered by the solutions $\overline{S}$ and $\widehat{S}$ in all product classes other than the weak class are identical. Therefore, if the market share of product class k under solution $\widehat{S}$ is zero, then the market share of product class k under the solution $\overline{S}$ is also zero. If the market share of non-weak product class k under solution $\widehat{S}$ exceeds β^k, then the market share of non-weak product class k under the solution $\overline{S}$ is $\sum_{i\in\overline{S}\cap\mathcal{N}^k} \overline{x}_i \ge \frac{1}{1-\beta_{\min}} \sum_{i\in\overline{S}\cap\mathcal{N}^k} \widehat{x}_i = \frac{1}{1-\beta_{\min}} \sum_{i\in\widehat{S}\cap\mathcal{N}^k} \widehat{x}_i \ge \frac{\beta^k}{1-\beta_{\min}}$, where the first inequality is by (8) and the equality holds as $\overline{S}\cap\mathcal{N}^k = \widehat{S}\cap\mathcal{N}^k$ for all $k\ne\lambda$. Thus, noting that $\sum_{i\in\overline{S}} \overline{x}_i \le 1$, we get $\sum_{i\in\overline{S}\cap\mathcal{N}^k} \overline{x}_i \in \{0\}\cup[\frac{\beta^k}{1-\beta_{\min}}, 1]$ for all $k\in\mathcal{K}$.

By the discussion in the previous paragraph, the solution $\overline{S}$ provides an expected revenue of at least $\frac{1}{2(1-\beta_{\min})}\,\mathsf{opt}$ and provides a market share of zero or at least $\frac{\beta^k}{1-\beta_{\min}}$ for each product class k. We consider a perturbed problem instance without changing the set of products, set of product classes and preference weights, but setting the market share of product class k as $\overline{\beta}^k = \frac{\beta^k}{1-\beta_{\min}}$. Letting $\overline{\mathsf{opt}}$ be the optimal objective value of the perturbed problem, noting that the solution $\overline{S}$ is feasible to the perturbed problem, we have $\overline{\mathsf{opt}} \geq \frac{1}{2(1-\beta_{\min})}\,\mathsf{opt}$. In this case, we can use Theorem 6.3 with $\epsilon = \beta_{\min}$ to find an approximate solution to the perturbed problem that provides an expected revenue of at least $(1-\beta_{\min})\,\overline{\mathsf{opt}}$ and satisfies the market share constraints within a factor of $1-\beta_{\min}$. Because $(1-\beta_{\min})\,\overline{\mathsf{opt}} \geq \frac{1}{2}\mathsf{opt}$ and $(1-\beta_{\min})\,\overline{\beta}^k = \beta^k$, this solution provides an expected revenue of at least $\frac{1}{2}\mathsf{opt}$ and provides a market share of at least β^k for each product class k with a non-zero market share. To cover the case where the optimal solution may offer products in only one class, we can focus on each product class one by one, use the FPTAS with one product class in Appendix F to obtain a $\frac{1}{2}$-approximate solution. Among all the solutions obtained by using Theorem 6.3 and the FPTAS, we pick the one that provides the largest expected revenue. Considering the running time in Theorem 6.3 with $\epsilon = \beta_{\min}$, along with noting that the running time of the FPTAS is dominated by the running time in Theorem 6.3, we obtain the following result.

Theorem 7.1 (Constant-Factor for Deterministic Variant) *Considering the deterministic variant of the* Market-Share *problem, we can obtain a* $\frac{1}{2}$*-approximate solution in a running time of* $O\Big(\frac{1}{\beta_{\min}^4} K^2 n^3 \log(\frac{n\, r_{\max}\, v_{\max}}{r_{\min}\, v_{\min}}) \Big(K + \frac{1}{\beta_{\min}} \log(1 + n v_{\max})\Big) \log(\frac{1}{\beta_{\min}})\Big)$ *operations.*

In the next section, we give computational experiments to check the effectiveness of our approximation schemes and to understand the impact of imposing market share constraints.

8. Computational Experiments

We use synthetically generated datasets to test our approximation schemes. We use a dataset from an electronics seller to investigate the impact of imposing market share constraints.

8.1 Performance of the Approximation Schemes

We give our experimental setup followed by our computational results. We focus on both the randomized and deterministic variants, allowing us to check the benefit from randomization.

Experimental Setup: We synthetically generate a large number of test problems. For each test problem, we use our approximation schemes to obtain solutions to the randomized and deterministic variants. We also develop efficient approaches to compute upper bounds on the optimal objective values of the two variants. By comparing the expected revenue that we obtain for the two variants with the upper bounds on the optimal objective values for their respective problems, we check the optimality gaps of the solutions that we obtain for the randomized and deterministic variants. Furthermore, by comparing the expected revenues provided by the solutions for the randomized and deterministic variants, we check the benefit from using randomized solutions. We use the following approach to generate our test problems. In all of our test problems, we have $K = 10$ product classes. We sample the number of products in each class from the uniform distribution over $\{5, \ldots, 10\}$. To introduce some variety in our product classes in terms of product revenues and preference weights, we proceed with the understanding that some product classes include products with high revenues and low preference weights, whereas some others include products with low revenues and high preference weights. In particular, half of the product classes belong to each of the two product categories. If a product class belongs to the first category, then we sample the revenue and preference weights of the products in this product class, respectively, from the uniform distributions over $[40, 50]$ and $[1, 5]$. On the other hand, if a product class belongs to the second category, then we sample the revenue and preference weights of the products in this product class, respectively, from the uniform distributions over $[1, 5]$ and $[40, 50]$. Once we generate all preference weights, we normalize the preference weights by $\alpha = \frac{p_0}{1-p_0} \sum_{i \in \mathcal{N}} v_i$, where p_0 is a parameter that we vary. In this way, if we offer all products, then a customer leaves without a purchase with probability $1/(1 + \frac{1}{\alpha} \sum_{i \in \mathcal{N}} v_i) = 1/(1 + \frac{1-p_0}{p_0}) = p_0$, so the parameter p_0 controls the propensity of the customers to leave without a purchase.

To come up with the market share thresholds, we find the assortment that maximizes the expected revenue without market share constraints as $S^* = \arg\max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i(S)\, r_i$. Thus, if we set the market share of each product class k as $\underline{\beta}^k = \sum_{i \in \mathcal{N}^k} \phi_i(S^*)$, then the assortment S^* is still feasible to the Market-Share problem. If we only offer the products in class k, then we attain a market share of $\overline{\beta}^k = \sum_{i \in \mathcal{N}^k} \phi_i(\mathcal{N}^k)$, which is the largest attainable market share for product class k. We set the market share threshold of product class k as a convex combination of $\underline{\beta}^k$ and $\overline{\beta}^k$, so $\beta^k = \underline{\beta}^k + \gamma\, (\overline{\beta}^k - \underline{\beta}^k)$, where γ is another parameter that we vary. We vary $p_0 \in \{0.1, 0.4\}$ and $\gamma \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ to obtain 10 parameter configurations. For each parameter configuration, we use the approach discussed so far to generate 100 test problems. When using the FPTAS for the randomized variant, noting the proof of Theorem 5.2, we choose the precision parameter for the grid as $\rho = \frac{\epsilon}{2}$ to obtain a $(1 - \frac{\epsilon}{2})^2$-approximate solution. We set $\epsilon = 0.4$. When using the pseudo-polynomial time constant-factor approximation algorithm for the deterministic variant, noting the discussion right before Theorem 7.1, we need to set the precision parameter for the grid as $\rho = \frac{\beta_{\min}}{2}$. For some test problems, this choice results in too small precision parameters, yielding excessive running times. Instead, we run the approximation algorithm with $\rho = \frac{1}{2}\,(0.05 + 0.05 \times k)$ for $k = 0, \ldots, 9$ and return the solution with the largest expected revenue.

Computational Results: We give our results in Table 1. On the left side of the table, we give the parameter configuration (p_0, γ). In Appendix G, we give efficient approaches to compute

Param.	Rand. Var.		Det. Var.		Rand. vs Det.		
(p_0,γ)	Avg.	95-th	Avg	95-th	Avg.	5-th	95-th
(0.1, 0.1)	1.17	2.22	1.41	2.43	11.00	8.27	14.19
(0.1, 0.2)	1.38	2.74	4.76	8.68	7.00	2.31	10.91
(0.1, 0.3)	1.64	3.62	3.90	6.12	5.80	2.45	9.57
(0.1, 0.4)	2.42	4.59	6.09	9.16	6.15	2.52	9.64
(0.1, 0.5)	2.38	5.70	4.40	8.70	4.04	0.06	10.11
Avg.	1.80	3.77	4.11	7.02	6.80	3.12	10.88

Param.	Rand. Var.		Det. Var.		Rand. vs Det.		
(p_0,γ)	Avg.	95-th	Avg	95-th	Avg.	5-th	95-th
(0.4, 0.1)	0.83	2.67	2.01	4.63	3.70	0.70	8.12
(0.4, 0.2)	1.17	3.10	2.86	6.07	6.65	3.05	10.28
(0.4, 0.3)	1.31	3.21	2.47	6.18	5.43	1.58	10.01
(0.4, 0.4)	1.92	3.96	3.09	6.82	4.88	1.02	9.71
(0.4, 0.5)	2.62	5.69	4.17	8.69	4.92	0.00	10.19
Avg.	1.72	3.85	3.15	6.79	5.56	1.63	10.00

Table 1 Computational results for the synthetically generated datasets.

upper bounds on the optimal objective values of the randomized and deterministic variants. Considering the randomized variant, the first two columns give the average and 95-th percentile of the gaps between the expected revenue from the solution obtained by our approximation scheme and the upper bound on the optimal expected revenue, where these statistics are over the 100 test problems in a parameter configuration. In particular, for test problem k, letting Rand^k be the expected revenue from the solution obtained by our approximation scheme for the randomized variant and UB^k be the upper bound, these columns give the average and 95-th percentile of $\{100 \times \frac{\mathsf{UB}^k - \mathsf{Rand}^k}{\mathsf{UB}^k} : k = 1, \ldots, 100\}$. The next two columns give the same statistics by focusing on the deterministic variant. The last three columns give the average, 5-th percentile and 95-th percentile of the gaps between the expected revenues obtained by our approximation schemes for the randomized and deterministic variants. Thus, letting Det^k be the expected revenue of the solution obtained by our approximation scheme for the deterministic variant, these columns give the average, 5-th percentile and 95-th percentile of $\{100 \times \frac{\mathsf{Rand}^k - \mathsf{Det}^k}{\mathsf{Rand}^k} : k = 1, \ldots, 100\}$.

Our results indicate that our approximation schemes for both the randomized and deterministic variants perform quite well. The average optimality gap of the solutions obtained by our approximation schemes is, respectively, 1.76% and 3.63% for the randomized and deterministic variants. These figures are upper bounds on the optimality gaps, as we compare the performance of our approximation schemes with an upper bound on the optimal expected revenue, rather than the optimal expected revenue itself. The optimality gaps tend to get larger as the parameter γ gets larger. As the parameter γ gets larger, the market share thresholds get larger. Thus, as the market share thresholds get larger, the optimality gaps of the solutions tend to get larger, but it is difficult to pinpoint whether this trend holds because the solutions get poorer or the upper bounds get looser. The benefit from using a randomized solution, as opposed to a deterministic one, can be significant. On average, we can improve the expected revenues by 6.18% by switching from a deterministic solution to a randomized one. There are test problems for which the expected revenue gap between the randomized and deterministic solutions can exceed 14.19%.

In our computational experiments, we used Python 3.9 on MacOS with 2.4 GHz Intel Core i9 CPU and 16 GB RAM. The running time per test problem ranges from 0.15 to 2.8 seconds for the randomized variant and from 2.21 to 40.78 seconds for the deterministic variant.

8.2 Impact of the Market Share Constraints

We use a dataset from an online electronics retailer to investigate the impact of market share constraints on the expected revenues. The dataset is publicly available at Kaggle (2021). Our goal is to understand tradeoffs between expected revenues and market share constraints.

Experimental Setup: The dataset focuses on the sale transactions from an online electronics retailer in the period of January 5, 2020 to November 21, 2020. Products come from different product *categories*, such as smart phones, laptops, tablets and headphones. Each product has a *brand*, such as Apple, Samsung, LG and Beko. There can be multiple products with the same brand, such as multiple smart phones with the brand Apple. Each row in the dataset corresponds to a different sale transaction, providing the unique product code, brand and category of the purchased product, date of the transaction and price of the product. We split the dataset by the product categories so that we focus on the sale transactions from each product category separately. To keep the dataset rich, we only keep the product categories that have at least 10,000 sale transactions, 50 products and 10 brands. Considering the sale transactions in each product category, if a product has fewer than 20 sales, then we remove all sale transactions for that product from the dataset. In this way, we end up with 18 product categories and 50,379 sale transactions on average from each product category. Using the unique product code to identify the products in the dataset, based on the sale transactions for each product category separately, we fit a multinomial logit model to capture the choice process of the customers within a product category as follows.

The dataset does not provide the assortment of products available during the time of each sale transaction. To infer the assortment available during the time of each sale transaction, we proceed with the assumption that all products for which we observe a sale in the dataset within a one week window around the sale transaction was in the available assortment. The same approach is used by Jagabathula et al. (2024) for inferring the available assortments. The dataset does not provide the customers that leave without making a purchase either. We proceed with the assumption that 10% of the customers leave without a purchase and add no-purchase sale transactions into the dataset accordingly. Noting that we already inferred the assortment available during the time of each sale transaction, we randomly sample 10% of the assortments. For each sampled assortment, we add a sale transaction, where the available assortment corresponds to the sampled one, but the customer leaves without a purchasing. We borrow the idea of generating the no-purchase transaction records in this way from Gao et al. (2021). Different values for the fraction of customers without a purchase did not charge our results qualitatively. By using the approach discussed so far, for each product category, we end up with the data $\{(S_t, i_t) : t = 1, \ldots, T\}$, where T is the number of sale transactions in the product category including those that end up with a no-purchase, S_t is the assortment of products available during sale transaction t and i_t is the product, if any, purchased. We set $i_t = \varnothing$ when the sale transaction t ends up with a no-purchase. Using this data, we use maximum likelihood estimation to fit a multinomial logit model to estimate the preference weights of the products in the category, yielding $\{v_i : i \in \mathcal{N}\}$. The price of a product varies minimally in the dataset. We use the average price of each product to come up with the revenues $\{r_i : i \in \mathcal{N}\}$.

We focus on the randomized variant. We treat all products with the same brand as one product class and set the market share threshold for all product classes the same. We investigate the loss in the expected revenue when we impose the market share constraints. For example, there are four product classes in the air conditioner product category, corresponding to the brands Samsung, Beko, LG and AVA. If we compute the assortment of products that maximizes the expected revenue without any market share constraints, then the market share of the four product classes come out to be 0.219, 0.207, 0.188 and 0.053. If we impose a market share threshold of 0.16 on all four product classes and compute the assortment that maximizes the expected revenue under the market share constraint, then the market shares of the four product classes come out to be 0.205, 0.164, 0.160 and 0.160. Imposing the market share constraints ensures that different product classes enjoy more balanced market shares. The cost of imposing the market share constraints is that the gap between the expected revenues obtained by the unconstrained and constrained assortments is 2%.

In our computational experiments, for each product category, we find the market share constraint we should impose so that the expected revenue gap between the constrained and unconstrained solutions is 1%, 2% or 3%. We compare the market shares of the product classes before and after imposing the market share constraints. To pursue this line of investigation, we proceed as follows. For each product category, using the multinomial logit model estimated for this product category, we compute the optimal solution to the randomized variant without any market share constraints. Furthermore, we find the market share constraint to impose in the randomized variant such that optimal expected revenues between the unconstrained and constrained versions of the randomized variant differ by 1%. Because we impose the same market share threshold for all product classes, we use bisection search for this purpose. We compare the market shares of the product classes with and without market share constraints. We repeat the same process when the targeted expected revenue drop from the unconstrained to constrained solutions is 2% and 3%, as opposed to 1%.

Computational Results: We give our results in Table 2. On the left side of the table, we list the product categories. There are three panels in the table. The first panel focuses on the case where imposing the market share constraints reduces the expected revenue by 1%. We refer to the product classes in which we offer products in both the constrained and unconstrained solutions as the stable product classes. We compare the market shares of the stable product classes in the constrained

Product Category	1% Loss in Exp. Rev. Unconst. Min	Max	Change in Const. Min	Max	2% Loss in Exp. Rev. Unconst. Min	Max	Change in Const. Min	Max	3% Loss in Exp. Rev. Unconst. Min	Max	Change in Const. Min	Max
Air Conditioner	5.25	21.92	2.65	0.99	5.25	21.92	3.05	0.93	5.25	21.92	3.30	0.79
Clock	9.24	69.86	1.47	0.94	9.24	69.86	1.61	0.89	9.24	69.86	1.75	0.84
Cooler	6.80	25.69	1.09	1.07	6.80	25.69	1.42	1.05	6.80	25.69	1.65	1.07
Hard Disk	10.84	33.08	1.52	0.94	10.84	33.08	1.87	0.89	10.84	33.08	2.09	0.68
Headphone	1.28	49.89	3.22	0.96	2.61	49.89	1.98	0.98	2.61	49.89	2.36	0.95
Iron	12.59	16.02	1.17	0.96	12.59	16.02	1.24	0.98	12.59	16.02	1.30	1.02
Laptop	8.88	26.96	2.07	0.68	8.88	26.96	2.12	0.70	8.88	26.96	2.12	0.70
Microwave	1.72	33.24	2.13	0.87	1.72	33.24	2.32	0.69	1.72	33.24	2.38	0.63
Oven	7.81	27.99	1.36	0.79	7.81	27.99	1.62	0.71	21.03	27.99	1.29	0.97
Processor	5.69	35.87	1.25	1.03	5.69	35.87	1.77	1.00	11.18	35.87	1.14	1.08
Router	7.99	33.33	1.71	0.94	7.99	33.33	2.01	0.91	7.99	33.33	2.23	0.77
Smart Phone	15.40	51.34	1.35	0.94	15.40	51.34	1.61	0.83	15.40	51.34	1.77	0.81
Tablet	30.82	31.70	1.10	1.07	30.82	31.70	1.14	1.11	30.82	31.70	1.17	1.14
Telephone	3.18	21.09	1.13	1.09	3.62	21.09	1.13	1.05	3.62	21.09	1.19	1.06
Television	9.57	32.02	1.47	0.90	9.57	32.02	1.69	0.86	9.57	32.02	1.89	0.84
Vacuum	2.46	20.22	1.20	0.96	3.52	20.22	1.20	1.04	3.52	20.22	1.47	1.01
Washer	8.68	42.55	1.68	0.97	8.68	42.55	1.95	0.80	8.68	42.55	2.13	0.77
Water Heater	3.74	67.93	2.51	0.94	3.74	67.93	3.55	0.85	8.57	67.93	3.30	0.83
Avg.			1.67	0.95			1.85	0.90			1.92	0.89

Table 2 Computational results for the online electronics retailer dataset.

and unconstrained solutions. The first and second columns show the minimum and maximum market shares over the stable product categories in the unconstrained solution. That is, letting $\widehat{\boldsymbol{h}}_{\mathsf{U}}$ and $\widehat{\boldsymbol{h}}_{\mathsf{C}}$, respectively, be optimal solutions to the unconstrained and constrained problems, we set $\mathcal{K}_{\mathsf{U}} = \{k \in \mathcal{K} : \sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{U}}(S) > 0\}$ and $\mathcal{K}_{\mathsf{C}} = \{k \in \mathcal{K} : \sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{C}}(S) > 0\}$ to capture the active product classes in the unconstrained and constrained solutions. The stable product classes are $\mathcal{K}_{\mathsf{U}} \cap \mathcal{K}_{\mathsf{C}}$. Thus, the first two columns give the minimum and maximum of $\{\sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{U}}(S) : k \in \mathcal{K}_{\mathsf{U}} \cap \mathcal{K}_{\mathsf{C}}\}$, yielding the minimum and maximum market share of a stable product class in the unconstrained solution. The third and fourth columns, respectively, give by how much the minimum and maximum market shares change from the unconstrained solution to the constrained solution. Thus, letting $k_{\min}$ and $k_{\max}$, respectively, be the product classes that attain the minimum and maximum of $\{\sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{U}}(S) : k \in \mathcal{K}_{\mathsf{U}} \cap \mathcal{K}_{\mathsf{C}}\}$, the last two columns give $\frac{\sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{C}}(S)}{\sum_{i\in\mathcal{N}^k} \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}_{\mathsf{U}}(S)}$ for $k = k_{\min}$ and $k = k_{\max}$, checking how much the smallest and largest market shares change from the unconstrained to the constrained solution. The second and third panels have the same layout as that of the first, but they focus on the the cases with expected revenue drops of 2% and 3% from the unconstrained to the constrained solution.

Our results indicate that we can get significant bumps in the minimum market shares without significantly affecting the maximum market shares. For the air conditioners, for example, if we allow a 1% decrease in the expected revenue, then we can increase the minimum market share over the product classes from 5.25% to $5.25 \times 2.65 = 13.91\%$, while decreasing the maximum market share from 21.92% to $21.92 \times 0.99 = 21.70\%$, ending with a significantly more balanced market

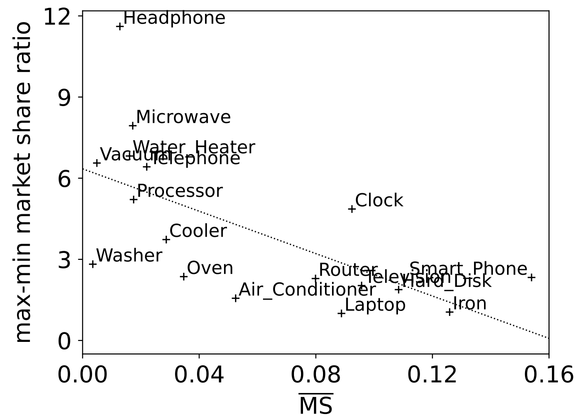

Figure 1 The pairs $\left(\overline{\mathsf{MS}}, \frac{\max_{k \in \mathsf{K}_\mathsf{C}} \mathsf{MS}_\mathsf{C}^k}{\min_{k \in \mathsf{K}_\mathsf{C}} \mathsf{MS}_\mathsf{C}^k}\right)$ for different product categories, where we note that the tablet product category has coordinates $(0.31, 1.01)$, which is outside the chart but fits the trend.

shares across the product classes. Similarly, for the laptops, we can increase the minimum market share over the product classes from 8.88% to $8.88 \times 2.07 = 18.38\%$, while decreasing the maximum market share from 26.96% to $26.96 \times 0.68 = 18.33\%$. Not surprisingly, this observation does not hold for all product categories, but our model is useful to balance favorable expected revenues with assortments that command more balanced demand volumes across the product classes.

A natural question is when we can expect imposing market share constraints to result in more balanced demand volumes across the product classes in a product category. Recalling the definitions of $\widehat{\boldsymbol{h}}_\mathsf{U}$, $\widehat{\boldsymbol{h}}_\mathsf{C}$, $\mathcal{K}_\mathsf{U}$ and $\mathcal{K}_\mathsf{C}$ in this section, we, respectively, use $\mathsf{MS}_\mathsf{U}^k = \sum_{i \in \mathcal{N}^k} \sum_{S \subseteq \mathcal{N}} \phi_i(S)\, \widehat{h}_\mathsf{U}(S)$ and $\mathsf{Rev}_\mathsf{U}^k = \sum_{i \in \mathcal{N}^k} \sum_{S \subseteq \mathcal{N}} r_i\, \phi_i(S)\, \widehat{h}_\mathsf{U}(S)$ to capture the market share and expected revenue from product class k in the unconstrained solution. We compute the market share for product class k in the constrained solution, denoted by MS_C^k, similarly, but using the solution $\widehat{\boldsymbol{h}}_\mathsf{C}$. Indexing the product classes in $\mathcal{K}_\mathsf{U}$ such that $\mathsf{MS}_\mathsf{U}^1 \le \mathsf{MS}_\mathsf{U}^2 \le \ldots \le \mathsf{MS}_\mathsf{U}^L$ with $L = |\mathcal{K}_\mathsf{U}|$, our intuitive expectation is that if we impose market share constraints, then we stop offering products in product classes in increasing order of $1, \ldots, L$, dropping the product classes with smaller market shares first. In this case, letting opt_U be the optimal expected revenue for the unconstrained problem, we heuristically use $\overline{\mathsf{MS}} = \min_{k \in \mathcal{K}} \left\{ \mathsf{MS}_\mathsf{U}^k : \sum_{\ell=k}^L \mathsf{Rev}_\mathsf{U}^\ell \ge 0.99 \times \mathsf{opt}_\mathsf{U} \right\}$ as a proxy for the smallest market share for the product class still represented in the constrained solution when we allow 1% decrease in the optimal expected revenue. One conjecture is that if $\overline{\mathsf{MS}}$ is larger, then we obtain more balanced demand volumes after we impose the market share constraints, because the smallest market share for the product class still represented in the constrained solution is larger. In Figure 1, each data point corresponds to a product category. The horizontal axis gives $\overline{\mathsf{MS}}$, whereas the vertical axis gives $\frac{\max_{k \in \mathsf{K}_\mathsf{C}} \mathsf{MS}_\mathsf{C}^k}{\min_{k \in \mathsf{K}_\mathsf{C}} \mathsf{MS}_\mathsf{C}^k}$. Smaller values of the last ratio indicates that the product classes have similar market shares in the constrained solution, pointing out more balanced demand volumes across the product classes. The figure indicates that we get more balanced demand volumes when the value of $\overline{\mathsf{MS}}$ is larger, as expected. Thus, inspecting the unconstrained solution can hint at when we can expect to obtain more balanced demand volumes through market share constraints.

9. Conclusions

We studied assortment optimization problems under the multinomial logit model with market share constraints. The novel aspect of our class of problems is we impose the constraints only for the product classes represented in the offered assortment. This feature brings unique combinatorial aspects that, to our knowledge, have not been explored previously. We give three research directions. In Appendix H, we give an FPTAS for the deterministic variant when each product class includes a single product. For arbitrary product classes, the running time for our $\frac{1}{2}$-approximation algorithm for the deterministic variant is pseudo-polynomial, but our complexity results do not rule out polynomial running time. Despite our best efforts, we were not able to get a constant-factor for the deterministic variant in polynomial running time, which we leave open. Also, we can consider other constraints that become active when the offered assortment satisfies certain conditions. Lastly, it is useful to investigate similar market share constraints under other choice models.

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A Unified Framework to Impose Market Share Constraints for Selected Product Classes: Randomized and Deterministic Assortments under the Multinomial Logit Model

Appendix A: Proof of Theorem 2.1

We give a proof for Theorem 2.1. We use two auxiliary lemmas with proofs deferred to the end of the section. We consider a generalized version of the subset sum problem, which we refer to as the fixed-proportion subset sum problem. In this problem, we have n items with weights $a_1, \ldots, a_n$ with $\sum_{i=1}^n a_i = B$, as well as relatively prime natural numbers p and q such that $\frac{1}{2} \leq \frac{p}{q} < 1$. The fixed-proportion subset sum problem asks whether there exists a subset of items $Q \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in Q} a_i = \frac{p}{q} B$. We have the next lemma regarding the complexity of this problem.

Lemma A.1 (Complexity of Fixed-Proportion Subset Sum) *The fixed-proportion subset sum problem is NP-complete.*

In the next lemma, we use purely algebraic manipulations to check that a sequence of inequalities is satisfied by a certain collection of rational numbers.

Lemma A.2 (Instance) *For fixed $\alpha \in (\frac{1}{2}, 1]$, setting $\epsilon = \frac{1}{2}(\frac{1}{2} + \alpha)$, $\delta = \frac{3}{2} - \epsilon$, $w = \frac{\epsilon - \frac{1}{2}}{1-\epsilon}$, $V = \frac{1}{2} w$ and $r = \frac{V}{\delta\, w} \frac{\delta\, w + (1-\epsilon)\,(1+V)}{\epsilon\,(1+V)}$, we have $\epsilon \in (\frac{1}{2}, \alpha)$, $\delta \in (\frac{1}{2}, 1)$ and $\alpha \, \frac{V + r\,\delta\, w}{1 + V + \delta\, w} > \max\{\frac{V}{1+V}, \frac{r\, w}{1+w}\}$.*

Below is the prof of Theorem 2.1. We focus on the complexity of the randomized variant, followed by the complexity of the deterministic variant.

<u>Complexity of the Randomized Variant:</u>

We consider the feasibility version of the randomized variant, where we have an instance of the randomized variant and a threshold T. We ask the question of whether there exists a feasible assortment with an expected revenue of T or more. Given an instance of the fixed-proportion subset sum problem with the items weights $a_1, \ldots, a_n$, $\sum_{i=1}^n a_i = B$ and $\frac{p}{q} = \frac{1}{2}$, we show that there exists a subset of items $Q \subseteq \{1, \ldots, n\}$ that satisfies $\sum_{i \in Q} a_i = \frac{1}{2} B$ if and only if there exists a feasible assortment $S \subseteq \mathcal{N}$ that provides an expected revenue of T or more. In the instance of the fixed-proportion subset sum problem, we normalize the weights by B so that $B = 1$. From this instance, we construct an instance of the randomized variant by setting $\mathcal{N} = \mathcal{K} = \{1, \ldots, n\}$, $\mathcal{N}^k = \{k\}$, $r_i = 1$, $v_i = 2\, a_i$ for all $i \in \mathcal{N}$, $\beta^k = a_k$ for all $k \in \mathcal{K}$ and $T = \frac{1}{2}$. In this instance, each product belongs to a different class. First, assuming that there exists a subset of items $Q \subseteq \{1, \ldots, n\}$ with $\sum_{i \in Q} a_i = \frac{1}{2}$, we show that there exists a feasible solution for the randomized variant that provides an expected revenue of $\frac{1}{2}$. Define the solution $\widehat{\boldsymbol{h}}$ to the randomized variant of the Market-Share problem as $\widehat{h}(Q) = 1$ and $\widehat{h}(S) = 0$ for all $S \neq Q$. Checking the market share for each class, for all $i \in Q$, we have

$\sum_{S\subseteq\mathcal{N}}\phi_i(S)\,\widehat{h}(S)=\frac{2a_i}{1+\sum_{i\in Q}2a_i}=a_i=\beta^i$, where we use the fact that $\sum_{i\in Q}a_i=\frac{1}{2}$. For all $i\notin Q$, there is no assortment offered in the solution $\widehat{\boldsymbol{h}}$ that includes product i, so $\sum_{S\subseteq\mathcal{N}}\phi_i(S)\,\widehat{h}(S)=0$. Thus, the solution $\widehat{\boldsymbol{h}}$ is feasible to the randomized variant. The expected revenue from the solution $\widehat{\boldsymbol{h}}$ is given by $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}}\phi_i(S)\,\widehat{h}(S)=\sum_{i\in Q}\frac{2a_i}{1+\sum_{j\in Q}2a_j}=\frac{1}{2}\geq T$. Second, assuming that there exists a feasible solution $\widehat{\boldsymbol{h}}$ for the randomized variant that provides an expected revenue of $\frac{1}{2}$ or more, we show that there exists a subset of items $Q\subseteq\{1,\dots,n\}$ with $\sum_{i\in Q}a_i=\frac{1}{2}$. Define the subset of items $\widehat{Q}=\{i\in\mathcal{N}:\sum_{S\subseteq\mathcal{N}}\widehat{h}(S)\,\mathbf{1}(i\in S)>0\}$. The set $\widehat{Q}$ includes the products offered in some assortment in the optimal solution to the randomized variant, so these products must satisfy the market share constraints, which is to say that we have $\sum_{S\subseteq\mathcal{N}}\mathbf{1}(i\in S)\frac{2a_i}{1+\sum_{j\in S}2a_j}\widehat{h}(S)\geq a_i$ for all $i\in\widehat{Q}$. Thus, for all $i\in\widehat{Q}$, we get $\sum_{S\subseteq\mathcal{N}}\mathbf{1}(i\in S)\frac{1}{1+\sum_{j\in S}2a_j}\widehat{h}(S)\geq\frac{1}{2}$. In this case, for all $i\in\widehat{Q}$, we have

$$\sum_{S\subseteq\mathcal{N}}\sum_{j\in\mathcal{N}}r_j\,\phi_j(S)\,\widehat{h}(S)\overset{(a)}{=}\sum_{S\subseteq\mathcal{N}}\frac{\sum_{j\in S}2a_j}{1+\sum_{j\in S}2a_j}\,\widehat{h}(S)$$
$$\overset{(b)}{=}1-\sum_{S\subseteq\mathcal{N}}\frac{1}{1+\sum_{j\in S}2a_j}\,\widehat{h}(S)\;\leq\;1-\sum_{S\subseteq\mathcal{N}}\mathbf{1}(i\in S)\frac{1}{1+\sum_{j\in S}2a_j}\widehat{h}(S)\;\leq\;\frac{1}{2},$$

where (a) uses the fact that $r_i=1$ and $v_i=2a_i$ along with the form of the choice probability under the multinomial logit model, whereas (b) holds because $\sum_{S\subseteq\mathcal{N}}\widehat{h}(S)=1$ and $\frac{x}{1+x}=1-\frac{1}{1+x}$.

The expression on the left side above is the expected revenue from the solution $\widehat{\boldsymbol{h}}$, which is at least $\frac{1}{2}$, so all of the inequalities above hold as equalities. Therefore, for all $i\in\widehat{Q}$, we have $\sum_{S\subseteq\mathcal{N}}\frac{1}{1+\sum_{j\in S}2a_j}\,\widehat{h}(S)=\sum_{S\subseteq\mathcal{N}}\mathbf{1}(i\in S)\frac{1}{1+\sum_{j\in S}2a_j}\widehat{h}(S)$ and the quantities in both sides of the equality is equal to $\frac{1}{2}$. For the last equality hold, we must have $\mathbf{1}(i\in S)\,\widehat{h}(S)=\widehat{h}(S)$ for all $i\in\widehat{Q}$ and $S\subseteq\mathcal{N}$. Considering any $\widehat{S}\subseteq\mathcal{N}$ with $\widehat{h}(\widehat{S})>0$, we claim that $\widehat{S}=\widehat{Q}$. In particular, if $i\in\widehat{Q}$, then we have $\mathbf{1}(i\in\widehat{S})\,\widehat{h}(\widehat{S})=\widehat{h}(\widehat{S})$ by the last equality, but because $\widehat{h}(\widehat{S})>0$, we get $\mathbf{1}(i\in\widehat{S})=1$. Therefore, having $i\in\widehat{Q}$ implies $i\in\widehat{S}$. If $i\in\widehat{S}$, then $\widehat{h}(\widehat{S})\,\mathbf{1}(i\in\widehat{S})>0$, so $\sum_{S\subseteq\mathcal{N}}\widehat{h}(S)\,\mathbf{1}(i\in S)>0$, in which case, by the definition of $\widehat{Q}$, we have $i\in\widehat{Q}$. Therefore, having $i\in\widehat{S}$ implies $i\in\widehat{Q}$, establishing the claim. By the claim, $\widehat{h}(\widehat{Q})=1$ and $\widehat{h}(S)=0$ for all $S\neq\widehat{Q}$. In this case, because $\sum_{S\subseteq\mathcal{N}}\frac{1}{1+\sum_{j\in S}2a_j}\,\widehat{h}(S)=\frac{1}{2}$, we get $\frac{1}{1+\sum_{j\in\widehat{Q}}2a_j}=\frac{1}{2}$, which is equivalent to $\sum_{j\in\widehat{Q}}a_j=\frac{1}{2}$. ■

Complexity of the Deterministic Variant:

For fixed $\alpha\in(\frac{1}{2},1]$, we construct a problem instance with two product classes that does not admit an α-approximation algorithm. In particular, for fixed α, letting ϵ, δ, w, V and r be as in Lemma A.2, we show that if such an α-approximation existed, then we would be able to solve any instance of the fixed-proportion subset sum problem with $B=w$ and $\frac{p}{q}=\delta$. Considering an instance of the fixed-proportion subset sum problem with the weights $a_1,\dots,a_n$ with $\sum_{i=1}^n a_i=B$, we would like to know whether there exists a set $Q\subseteq\{1,\dots,n\}$ such that $\sum_{i\in Q}a_i=\delta B$. Corresponding to the instance of the fixed-proportion subset sum problem, we define an instance of the deterministic variant as follows. The set of products is $\mathcal{N}=\{1,\dots,n+1\}$. There are two product classes with $\mathcal{N}^1=\{1,\dots,n\}$ and $\mathcal{N}^2=\{n+1\}$. The revenues and preference weights of the products are given by $r_i=r$ and $v_i=a_i$ for all $i\in\mathcal{N}^1$, whereas $r_{n+1}=1$ and $v_{n+1}=V$. The market share thresholds are $\beta^1=\frac{\delta w}{1+V+\delta w}$ and $\beta^2=\frac{V}{1+V+\delta w}$. Assume that the α-approximation algorithm returns the solution $\widehat{\boldsymbol{h}}=(\widehat{h}(S):S\subseteq\mathcal{N})\in\{0,1\}^{2^{n+1}}$. Noting that we focus on the deterministic variant, we have $\widehat{h}(\widehat{S})=1$ and $\widehat{h}(S)=0$ for all $S\neq\widehat{S}$. We claim that there exists a set $Q\subseteq\{1,\dots,n\}$ such that $\sum_{i\in Q}a_i=\delta w$ if and only if $\widehat{S}\cap\mathcal{N}^1\neq\varnothing$ and $\widehat{S}\cap\mathcal{N}^2\neq\varnothing$. In this case, to see whether there exists a set $Q\subseteq\{1,\dots,n\}$ such that $\sum_{i\in Q}a_i=\delta w$, we can execute the α-approximation algorithm and check whether the output satisfies $\widehat{S}\cap\mathcal{N}^1\neq\varnothing$ and $\widehat{S}\cap\mathcal{N}^2\neq\varnothing$. We proceed to showing the claim.

First, assume that $\widehat{S}\cap\mathcal{N}^1\neq\varnothing$ and $\widehat{S}\cap\mathcal{N}^2\neq\varnothing$. In this case, we have $\widehat{S}=\widehat{S}^1\cap\{n+1\}$ for some $\widehat{S}^1\subseteq\mathcal{N}^1$. Because the solution $\widehat{\boldsymbol{h}}$ is feasible to the deterministic variant, the market share for the first product class must be satisfied, so $\sum_{i\in\mathcal{N}^1}\phi_i(\widehat{S})=\frac{\sum_{i\in\widehat{S}^1}a_i}{1+\sum_{i\in\widehat{S}^1}a_i+V}\geq\beta^1=\frac{\delta w}{1+\delta w+V}$. Because $\frac{x}{1+x}$ is increasing in x, we get $\sum_{i\in\widehat{S}^1}a_i\geq\delta w$. Similarly, we have $\sum_{i\in\mathcal{N}^2}\phi_i(\widehat{S})=\frac{V}{1+\sum_{i\in\widehat{S}^1}a_i+V}\geq\beta^2=\frac{V}{1+\delta w+V}$, so $\sum_{i\in\widehat{S}^1}a_i\leq\delta w$. Therefore, we must have $\sum_{i\in\widehat{S}^1}a_i=\delta w$. Second, assume that there exists a set $Q\subseteq\{1,\dots,n\}$ such that $\sum_{i\in Q}a_i=\delta w$. Consider any solution to the deterministic variant $\boldsymbol{h}^1=(h^1(S):S\subseteq\mathcal{N})$ with $h^1(S^1)=1$ for some $S^1\subseteq\mathcal{N}^1$ and $h^1(S)=0$ for all $S\neq S^1$. The expected revenue of this solution is $\sum_{i\in\mathcal{N}^1}r_i\,\phi_i(S_1)=\frac{r\sum_{i\in S^1}a_i}{1+\sum_{i\in S^1}a_i}\leq\frac{rw}{1+w}$, where the inequality uses $\sum_{i\in S^1}a_i\leq\sum_{i=1}^n a_i=B=w$. The last inequality is attained at $S^1=\mathcal{N}^1$. Also, if we set $S^1=\mathcal{N}^1$, then the market share of the first product class is $\sum_{i\in\mathcal{N}^1}\phi_i(\mathcal{N}^1)=\sum_{i\in\mathcal{N}^1}\frac{a_i}{1+\sum_{i\in\mathcal{N}^1}a_i}=\frac{w}{1+w}\geq\beta^1$. Thus, if we only offer the products in the first class, then we can get an expected revenue of at most $\frac{rw}{1+w}$. Consider any solution to the deterministic variant $\boldsymbol{h}^2=(h^2(S):S\subseteq\mathcal{N})$ with $h^2(\{n+1\})=1$ and $h^2(S)=0$ for all $S\neq\{n+1\}$. The expected revenue of this solution is $r_{n+1}\,\phi_{n+1}(\{n+1\})=\frac{V}{1+V}$. The market share of the second product class is $\phi_{n+1}(\{n+1\})=\frac{V}{1+V}\geq\beta^2$. Thus, if we offer only the product in the second class, then we can obtain an expected revenue of at most $\frac{V}{1+V}$.

To complete the proof, consider a solution to the deterministic variant $\boldsymbol{h}^0=(h^0(S):S\subseteq\mathcal{N})$ such that $h^0(Q\cup\{n+1\})=1$ and $h^0(S)=0$ for all $S\neq Q\cup\{n+1\}$. Under this solution, the market share of the first product class is $\sum_{i\in\mathcal{N}^1}\phi_i(Q\cup\{n+1\})=\frac{\sum_{i\in Q}a_i}{1+\sum_{i\in Q}a_i+V}=\frac{\delta w}{1+\delta w+V}=\beta^1$, whereas the market share of the second product class is $\phi_{n+1}(Q\cup\{n+1\})=\frac{V}{1+\sum_{i\in Q}a_i+V}=\frac{V}{1+\delta w+V}=\beta^2$. Thus, the solution $\boldsymbol{h}^0$ is feasible to the deterministic variant. The expected revenue from this solution is $\sum_{i\in\mathcal{N}}r_i\,\phi_i(Q\cup\{n+1\})=\frac{r\delta w+V}{1+\delta w+V}$. Therefore, we have $\mathsf{opt}\geq\frac{r\delta w+V}{1+\delta w+V}$, so by Lemma A.2, we get $\alpha\,\mathsf{opt}\geq\alpha\frac{r\delta w+V}{1+\delta w+V}>\max\{\frac{V}{1+V},\frac{r\,w}{1+w}\}$. Noting that $\max\{\frac{V}{1+V},\frac{r\,w}{1+w}\}$ is the expected revenue obtained by offering products in only one of the two product classes, any α-approximate solution must offer products in both product classes. Therefore, we must have $\widehat{S}\cap\mathcal{N}^1\neq\varnothing$ and $\widehat{S}\cap\mathcal{N}^2\neq\varnothing$. ∎

We give a proof for Lemma A.1. We use a reduction from the standard subset sum problem, which is NP-complete; see Garey and Johnson (1979). In the standard subset sum problem, we are given a set of positive numbers $\{\alpha_1,\ldots,\alpha_n\}$ and a target value k. We ask whether there exists a subset $Q\subseteq\{1,\ldots,n\}$ such that $\sum_{i\in Q}\alpha_i=k$. Below is the proof of Lemma A.1.

Proof of Lemma A.1:

The fixed-proportion subset sum problem with $\frac{p}{q}=\frac{1}{2}$ corresponds to the partition problem, which is already known to be NP-complete; see Garey and Johnson (1979). Thus, we assume that $\frac{p}{q}>\frac{1}{2}$. We use a reduction from the standard subset sum problem. We are given an instance of the standard subset sum problem with the set of positive numbers $\{\alpha_1,\ldots,\alpha_n\}$ and target value k with $k\leq\sum_{i=1}^n\alpha_i$. Fixing relatively prime natural numbers p and q such that $\frac{1}{2}<\frac{p}{q}<1$ and $B>0$, corresponding to the instance of the standard subset sum problem, we construct an instance of the fixed-proportion subset sum problem as follows. Setting $A=\sum_{i=1}^n\alpha_i$, $a_i=\frac{B}{2qA}\,\alpha_i$, $b=\frac{B}{2qA}(2pA-k)$ and $c=\frac{B}{2qA}((2(q-p)-1)A+k)$, there are $n+2$ items with weights $\{a_1,\ldots,a_n,b,c\}$. Noting that $q>p\geq 1$ with $p,q\in\mathbb{Z}_+$, both b and c are non-negative. By the definitions of b and c, we have $\sum_{i=1}^n a_i+b+c=B$. It is enough to verify that there exists $Q\subseteq\{1,\ldots,n\}$ such that $\sum_{i\in Q}\alpha_i=k$ if and only if there exists $S\subseteq\{a_1,\ldots,a_n,b,c\}$ such that $\sum_{\ell\in S}\ell=\frac{p}{q}B$. First, assume that there exists $Q\subseteq\{1,\ldots,n\}$ such that $\sum_{i\in Q}\alpha_i=k$. In this case, we have $b+\sum_{i\in Q}a_i=\frac{B}{2qA}\,(2pA-k)+\frac{B}{2qA}\,k=\frac{p}{q}B$. Therefore, setting $S=\{a_i:i\in Q\}\cup\{b\}$ suffices. Second, assume that there exists $S\subseteq\{a_1,\ldots,a_n,b,c\}$ such that $\sum_{\ell\in S}\ell=\frac{p}{q}B$. We cannot have both b and c in S because $b+c=\frac{B}{2qA}\,(2q-1)A>\frac{p}{q}\,B$, where we use the fact that $2q-1\geq 2p+1>2p$. Furthermore, we cannot have c without b in S, because we have $c+\sum_{i=1}^n a_i=\frac{B}{2qA}\,(2(q-p)A+k)\leq\frac{B}{2qA}\,(2(q-p)+1)A<\frac{p}{q}\,B$, where the last inequality holds by $q+1\leq 2p$, so $2q-2p+1<2p$. Therefore, we must have $S=\{b\}\cup\{a_i:i\in X\}$ for some $X\subseteq\{1,\ldots,n\}$. Because $b=\frac{B}{2qA}\,(2pA-k)$, having $\sum_{\ell\in S}\ell=\frac{p}{q}\,B$ is equivalent to having $\frac{B}{2qA}\,(2pA-k)+\sum_{i\in X}a_i=\frac{p}{q}\,B$, which implies that $\frac{2qA}{B}\,\sum_{i\in X}a_i=k$. Therefore, we obtain $\sum_{i\in X}\alpha_i=k$, so setting $Q=X$ suffices. ∎

To close this section, we give a proof for Lemma A.2. The proof of this lemma follows from a purely algebraic argument.

Proof of Lemma A.2:

Noting that $\alpha\in(\frac{1}{2},1]$ and ϵ is the average of α and $\frac{1}{2}$, we get $\epsilon\in(\frac{1}{2},\alpha)\subseteq(\frac{1}{2},1)$, in which case, the definition of δ also implies that $\delta\in(\frac{1}{2},1)$. We will verify the two identities $\frac{V}{1+V}=\epsilon\frac{r\delta w+V}{1+V+\delta w}$ and $\frac{rw}{1+w}=\frac{V}{1+V}$. In this case, we obtain $\alpha\frac{r\delta w+V}{1+V+\delta w}>\epsilon\frac{r\delta w+V}{1+V+\delta w}=\frac{V}{1+V}=\frac{rw}{1+w}$, in which case, we obtain $\alpha\,\frac{V+r\,\delta\,w}{1+V+\delta\,w}>\max\{\frac{V}{1+V},\frac{r\,w}{1+w}\}$. To see that the first identity holds, by the definition of r, we have $r=\frac{V}{\delta\,w}\left(\frac{1+V+\delta w}{\epsilon(1+V)}-1\right)$, which is equivalent to $r\delta w=\frac{V(1+V+\delta w)}{\epsilon(1+V)}-V$. Arranging the terms in the last equality, we obtain $\frac{r\delta w+V}{1+V+\delta w}=\frac{V}{\epsilon(1+V)}$, in which case, we obtain $\frac{V}{1+V}=\epsilon\frac{r\delta w+V}{1+V+\delta w}$. Thus, the first identity holds. To see that the second identity holds, using the definition of V, we express this identity as $\frac{rw}{1+w}=\frac{w}{2+w}$, so it is enough to verify that $r=\frac{1+w}{2+w}$. Noting the definition of r, we have

$r = \frac{\delta w+(1-\epsilon)\,(1+w/2)}{2\delta\epsilon\,(1+w/2)} = \frac{1-\epsilon+(\delta+\frac{1}{2}(1-\epsilon))w}{\delta\epsilon(2+w)} = \frac{1}{2+w}\,\frac{1}{\delta\epsilon}\,(1-\epsilon+(\delta+\frac{1}{2}(1-\epsilon))w)$. Thus, if we can show that $\frac{1}{\delta\epsilon}\,(1-\epsilon+(\delta+\frac{1}{2}(1-\epsilon))w) = 1+w$, then $r = \frac{1+w}{2+w}$, so the second identity follows. We have

$$\begin{aligned}\frac{1}{\delta\epsilon}\left(1-\epsilon+\left(\delta+\frac{1}{2}(1-\epsilon)\right)w\right) &= \frac{1-\epsilon+(2-\frac{3}{2}\epsilon)\,w}{\epsilon\,(\frac{3}{2}-\epsilon)} = \frac{1-\epsilon+(2-\frac{3}{2}\epsilon)\,\frac{\epsilon-\frac{1}{2}}{1-\epsilon}}{\epsilon\,(\frac{3}{2}-\epsilon)} \\ &= \frac{(1-\epsilon)^2+(2-\frac{3}{2}\epsilon)\,(\epsilon-\frac{1}{2})}{\epsilon\,(\frac{3}{2}-\epsilon)\,(1-\epsilon)} = \frac{\frac{1}{2}\,\epsilon\,(\frac{3}{2}-\epsilon)}{\epsilon\,(\frac{3}{2}-\epsilon)\,(1-\epsilon)} = \frac{1}{2(1-\epsilon)} = 1+w,\end{aligned}$$

where the first equality uses the definition of δ, the second equality uses the definition of w and the last equality, once again, uses the definition of w. ■

Appendix B: Proof of Theorem 2.2

We give a proof for Theorem 2.2. Following the proof, we give a simple corollary to the theorem to show that the Market-Share and Sales-Based problems have the same optimal objective value.

Proof of Theorem 2.2:

We verify the four relationships in the theorem. First, we verify that $\sum_{S\subseteq\mathcal{N}} \widehat{h}(S) = 1$. Letting $B = \widehat{x}_0 + \sum_{j\in\mathcal{N}} \widehat{x}_j$, noting that $\widehat{h}(S) = 0$ for $S \neq S_0, S_1, \ldots, S_n$, using (1), we have

$$\begin{aligned}\sum_{S\subseteq\mathcal{N}} \widehat{h}(S) &= \frac{1}{B}\sum_{i=0}^{n}\left(\frac{\widehat{x}_i}{v_i}-\frac{\widehat{x}_{i+1}}{v_{i+1}}\right)\left(1+\sum_{j\in S_i} v_j\right) \overset{(a)}{=} \frac{1}{B}\sum_{i=0}^{n}\left(\frac{\widehat{x}_i}{v_i}-\frac{\widehat{x}_{i+1}}{v_{i+1}}\right) + \frac{1}{B}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{\widehat{x}_i}{v_i}-\frac{\widehat{x}_{i+1}}{v_{i+1}}\right)\sum_{j\in S_i} v_j \\ &\overset{(b)}{=} \frac{1}{B}\widehat{x}_0 + \frac{1}{B}\sum_{j=1}^{n} v_j \sum_{i=j}^{n}\left(\frac{\widehat{x}_i}{v_i}-\frac{\widehat{x}_{i+1}}{v_{i+1}}\right) \overset{(c)}{=} \frac{1}{B}\widehat{x}_0 + \frac{1}{B}\sum_{j=1}^{n} v_j\,\frac{\widehat{x}_j}{v_j} = \frac{1}{B}\left(\widehat{x}_0+\sum_{j=1}^{n}\widehat{x}_j\right) \overset{(d)}{=} 1, \qquad (9)\end{aligned}$$

where (a) uses $S_0 = 0$, so $\sum_{j\in S_0} v_j = 0$, (b) uses a telescoping sum and an interchange in the order sums, as well as the convention that $v_0 = 1$, (c) uses another telescoping sum and (d) follows from the definition of B. Second, we verify that $\widehat{\boldsymbol{h}} \in \mathcal{H}$. We have $\widehat{x}_i \in \mathcal{H}_i(\widehat{x}_0)$. For the deterministic variant of the Market-Share problem, by the definition of $\mathcal{H}_i(\widehat{x}_0)$, we have $\widehat{x}_i \in \{0, v_i\,\widehat{x}_0\}$ for all $i \in \mathcal{N}$, so $\frac{\widehat{x}_i}{v_i} \in \{0, \widehat{x}_0\}$. Therefore, we have $\widehat{x}_0 = \frac{\widehat{x}_1}{v_1} = \ldots = \frac{\widehat{x}_i}{v_i} > \frac{\widehat{x}_{i+1}}{v_{i+1}} = \ldots = \frac{\widehat{x}_n}{v_n} = 0$ for some $i = 0, \ldots, n$. In this case, by (1), we have $\widehat{h}(S_i) > 0$ and $\widehat{h}(S) = 0$ for all $S \neq S_i$. Thus, noting (9), we get $\sum_{S\subseteq\mathcal{N}} \widehat{h}(S) = \sum_{i=0}^{n} \widehat{h}(S_i) = 1$, so we must have $\widehat{h}(S_i) = 1$ and $\widehat{h}(S) = 0$ for all $S \neq S_i$. Therefore, we have $\widehat{\boldsymbol{h}} \in \{0,1\}^{2^n}$, which is equivalent to having $\widehat{\boldsymbol{h}} \in \mathcal{H}$. For the randomized variant of the Market-Share problem, by (1), we have $\widehat{h}(S) \geq 0$ for all $S \subseteq \mathcal{N}$. Thus, by (9), $\widehat{\boldsymbol{h}} \in [0,1]^{2^n}$, so $\widehat{\boldsymbol{h}} \in \mathcal{H}$. Third, we verify that $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k} \phi_i(S)\,\widehat{h}(S) \in \{0\} \cup [(1-\gamma_1)\,\beta^k, 1]$. We have

$$\sum_{S\subseteq\mathcal{N}} \phi_i(S)\,\widehat{h}(S) \overset{(e)}{=} \sum_{j=i}^{n} \phi_i(S_j)\,\widehat{h}(S_j) \overset{(f)}{=} \frac{1}{B}\sum_{j=i}^{n} \frac{v_i}{1+\sum_{\ell\in S_j} v_\ell}\left(\frac{\widehat{x}_j}{v_j}-\frac{\widehat{x}_{j+1}}{v_{j+1}}\right)\left(1+\sum_{\ell\in S_j} v_\ell\right) \overset{(g)}{=} \frac{\widehat{x}_i}{B}, \qquad (10)$$

where (e) holds by the fact that $\widehat{h}(S) = 0$ for all $S \notin \{S_0, S_1, \ldots, S_n\}$ and noting that product i is included in the sets $S_i, S_{i+1}, \ldots, S_n$, but not in the sets $S_0, S_1, \ldots, S_{i-1}$, (f) uses (1), as well as the form of the choice probability under the multinomial logit model and (g) uses a telescoping sum. By the chain of equalities above, we have $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)=\frac{1}{B}\sum_{i\in\mathcal{N}^k}\widehat{x}_i$.

Noting that $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\in\{0\}\cup[\beta^k,\infty)$, we get $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\in\{0\}\cup[\frac{1}{B}\,\beta^k,\infty)$. Because $\widehat{x}_0+\sum_{i\in\mathcal{N}}\widehat{x}_i\le\frac{1}{1-\gamma_1}$ by the assumption in the theorem, we get $\frac{1}{B}=1/(\widehat{x}_0+\sum_{i\in\mathcal{N}}\widehat{x}_i)\ge 1-\gamma_1$ as well, so $[\frac{1}{B}\,\beta^k,\infty)\subseteq[(1-\gamma_1)\,\beta^k,\infty)$. Therefore, having $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\in\{0\}\cup[\frac{1}{B}\,\beta^k,\infty)$ implies that $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\in\{0\}\cup[(1-\gamma_1)\,\beta^k,\infty)$. Lastly, using the fact that $\sum_{S\subseteq\mathcal{N}}\widehat{h}(S)=1$ by (9) and $\sum_{i\in\mathcal{N}}\phi_i(S)\le 1$, we have $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\le 1$, in which case, it follows that $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}^k}\phi_i(S)\,\widehat{h}(S)\in\{0\}\cup[(1-\gamma_1)\,\beta^k,1]$. Fourth, to finish the proof, we verify that $\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}}r_i\,\phi_i(S)\,\widehat{h}(S)\ge(1-\gamma_1)\,(1-\gamma_2)\,\mathsf{opt}$. Noting that opt and $\overline{\mathsf{opt}}$ are, respectively, the optimal objective values of the Market-Share and Sales-Based problems, we can show that $\overline{\mathsf{opt}}\ge\mathsf{opt}$. In particular, letting $\overline{\boldsymbol{h}}=(\overline{h}(S):S\subseteq\mathcal{N})$ be an optimal solution to the Market-Share problem, using $\phi_0(S)=1/(1+\sum_{i\in S}v_i)$ to denote the no-purchase probability within assortment S, we can define the solution $(\overline{x}_0,\overline{\boldsymbol{x}})$ to the Sales-Based problem as $\overline{x}_0=\sum_{S\subseteq\mathcal{N}}\phi_0(S)\,\overline{h}(S)$ and $\overline{x}_i=\sum_{S\subseteq\mathcal{N}}\phi_i(S)\,\overline{h}(S)$. It is simple to check that the solution $(\overline{x}_0,\overline{\boldsymbol{x}})$ is feasible to the Sales-Based problem and the solutions $\overline{\boldsymbol{h}}$ and $(\overline{x}_0,\overline{\boldsymbol{x}})$ provide the same objective values for their respective problems. Therefore, the optimal objective value of the Sales-Based problem is at least as large as that of the Market-Share problem. In this case, we obtain

$$\sum_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}}r_i\,\phi_i(S)\,\widehat{h}(S)\overset{(h)}{=}\frac{1}{B}\sum_{i\in\mathcal{N}}r_i\,\widehat{x}_i\overset{(i)}{\ge}(1-\gamma_1)\,(1-\gamma_2)\,\overline{\mathsf{opt}}\overset{(j)}{\ge}(1-\gamma_1)\,(1-\gamma_2)\,\mathsf{opt},$$

where (h) uses (10), (i) holds because we have $\frac{1}{B}\ge 1-\gamma_1$ by the discussion earlier in this paragraph and $\sum_{i\in\mathcal{N}}r_i\,\widehat{x}_i\ge(1-\gamma_2)\,\overline{\mathsf{opt}}$ by the assumption in the theorem and (j) is by $\overline{\mathsf{opt}}\ge\mathsf{opt}$. ■

In the next corollary, we use an argument similar to the one in the proof of Theorem 2.2 to show that the Market-Share and Sales-Based problems have the same optimal objective value.

Corollary B.1 (Equal Optimal Objectives) *Noting that* opt *and* $\overline{\mathsf{opt}}$ *are, respectively, the optimal objective values of the* Market-Share *and* Sales-Based *problems, we have* $\overline{\mathsf{opt}}=\mathsf{opt}$.

Proof: Let $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ be an optimal solution to the Sales-Based problem. Without loss of generality, we can assume that $\widehat{x}_0+\sum_{i\in\mathcal{N}}\widehat{x}_i=1$. In particular, setting $\delta=\widehat{x}_0+\sum_{i\in\mathcal{N}}\widehat{x}_i$, if $\delta<1$, then we construct the solution $(\widetilde{x}_0,\widetilde{\boldsymbol{x}})$ as $\widetilde{x}_0=\frac{1}{\delta}\,\widehat{x}_0$ and $\widetilde{x}_i=\frac{1}{\delta}\,\widehat{x}_i$ for all $i\in\mathcal{N}$. We claim that the solution $(\widetilde{x}_0,\widetilde{\boldsymbol{x}})$ is feasible to the Sales-Based problem. We have $\widetilde{x}_0+\sum_{i\in\mathcal{N}}\widetilde{x}_i=1$, verifying the second constraint. Because $(\widehat{x}_0,\widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem, we have $\sum_{i\in\mathcal{N}^k}\widehat{x}_i\in\{0\}\cup[\beta^k,1]$, so $\sum_{i\in\mathcal{N}^k}\widetilde{x}_i=\sum_{i\in\mathcal{N}^k}\frac{1}{\delta}\,\widehat{x}_i\in\{0\}\cup[\frac{1}{\delta}\beta^k,\frac{1}{\delta}]$. Noting that $\widetilde{x}_0+\sum_{i\in\mathcal{N}}\widetilde{x}_i=1$, we have $\sum_{i\in\mathcal{N}^k}\widetilde{x}_i\le 1$ as well. In this case, we get $\sum_{i\in\mathcal{N}^k}\widetilde{x}_i\in\{0\}\cup[\beta^k,1]$, verifying the first constraint. If we are interested in the deterministic variant, then having $\widehat{x}_i \in \mathcal{H}_i(\widehat{x}_0)$ is to say that $\widehat{x}_i \in \{0, v_i\, \widehat{x}_0\}$, which implies that $\frac{1}{\delta}\widehat{x}_i \in \{0, v_i\, \frac{1}{\delta}\, \widehat{x}_0\}$, so $\widetilde{x}_i \in \mathcal{H}_i(\widetilde{x}_0)$. Similarly, if we are interested in the randomized variant, then having $\widehat{x}_i \in \mathcal{H}_i(\widehat{x}_0)$ is to say that $\widehat{x}_i \in [0, v_i\, \widehat{x}_0]$, which implies that $\frac{1}{\delta}\widehat{x}_i \in [0, v_i\, \frac{1}{\delta}\, \widehat{x}_0]$, so $\widetilde{x}_i \in \mathcal{H}_i(\widetilde{x}_0)$. Therefore, we have $\widetilde{x}_i \in \mathcal{H}_i(\widetilde{x}_0)$, verifying the third constraint. Thus, the claim holds. In this case, because $\delta < 1$, we get $\sum_{i\in\mathcal{N}} r_i\, \widetilde{x}_i = \frac{1}{\delta} \sum_{i\in\mathcal{N}} r_i\, \widehat{x}_i \geq \sum_{i\in\mathcal{N}} r_i\, \widehat{x}_i$, so $(\widetilde{x}_0, \widetilde{\boldsymbol{x}}_0)$ is optimal for the Sales-Based problem. Therefore, we can indeed assume that $\widehat{x}_0 + \sum_{i\in\mathcal{N}} \widehat{x}_i = 1$.

First, we show that $\mathsf{opt} \geq \overline{\mathsf{opt}}$. Using the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$, we construct the solution $\widehat{\boldsymbol{h}} = (\widehat{h}(S) : S \subseteq \mathcal{N})$ to the Market-Share problem as in (1). By the same chain of equalities in (9), we have $\sum_{S\subseteq\mathcal{N}} \widehat{h}(S) = 1$, so the solution $\widehat{\boldsymbol{h}}$ satisfies the second constraint in the Market-Share problem. Noting that $\widehat{x}_0 + \sum_{i\in\mathcal{N}} \widehat{x}_i = 1$, the value of B in the chain of inequalities in (10) is one, so by the same chain of equalities in (10), we get $\sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}(S) = \widehat{x}_i$. In this case, we have $\sum_{S\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}^k} \phi_i(S)\, \widehat{h}(S) = \sum_{i\in\mathcal{N}^k} \widehat{x}_i \in \{0\} \cup [\beta^k, 1]$, where the last inclusion holds because $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem. Therefore, the solution $\widehat{\boldsymbol{h}}$ satisfies the first constraint in the Market-Share problem. Lastly, by the same argument that follows (9), we have $\widehat{\boldsymbol{h}} \in \{0,1\}^{2^n}$ for the deterministic variant, whereas $\widehat{\boldsymbol{h}} \in [0,1]^{2^n}$ for the randomized variant. Thus, the solution $\widehat{\boldsymbol{h}}$ is feasible to the Market-Share problem. Lastly, because $\sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \widehat{h}(S) = \widehat{x}_i$, we get $\sum_{S\subseteq\mathcal{N}} \sum_{i\in\mathcal{N}} r_i\, \phi_i(S)\, \widehat{h}(S) = \sum_{i\in\mathcal{N}} \widehat{x}_i = \overline{\mathsf{opt}}$. Therefore, the solution $\widehat{\boldsymbol{h}}$ is feasible to the Market-Share problem and provides an objective value of $\overline{\mathsf{opt}}$ for this problem, yielding $\mathsf{opt} \geq \overline{\mathsf{opt}}$.

Second, we show that $\overline{\mathsf{opt}} \geq \mathsf{opt}$. Let $\overline{\boldsymbol{h}} = (\overline{h}(S) : S \subseteq \mathcal{N})$ be an optimal solution to the Market-Share problem, providing the optimal objective value of opt. Using $\phi_0(S) = 1/(1 + \sum_{i\in S} v_i)$ to capture the no-purchase probability within assortment S, we define the solution $(\overline{x}_0, \overline{\boldsymbol{x}})$ to the Sales-Based problem as $\overline{x}_0 = \sum_{S\subseteq\mathcal{N}} \phi_0(S)\, \overline{h}(S)$ and $\overline{x}_i = \sum_{S\subseteq\mathcal{N}} \phi_i(S)\, \overline{h}(S)$. We can check that the solution $(\overline{x}_0, \overline{\boldsymbol{x}})$ is feasible to the Sales-Based problem and provides an objective value of opt. Therefore, the optimal objective value of the Sales-Based problem is at least opt, yielding $\overline{\mathsf{opt}} \geq \mathsf{opt}$. ∎

Appendix C: Proof of Theorem 5.2

We complete the proof of Theorem 5.2 by giving the running time for the FPTAS. Letting $V = \frac{\log(1+n\, v_{\max})}{\log(1+\rho)} + K + 1$, we have $|\mathsf{Grid}_0| \leq V$. Letting $B = \frac{\log(1/\beta_{\min})}{\log(1+\rho)} + 3$, we have $|\mathsf{Grid}^k| \leq B$. We account of the running time for each step of our approximation framework. In Step 1, after ordering the products according to their revenues once, for each $k \in \mathcal{K}$, $p = 1, \ldots, L_0$ and $q = 1, \ldots, L^k$, solving the problem (4) takes $O(n)$ operations. Thus, we can execute Step 1 in $O(KVBn)$ operations. In Step 2, for each $p = 1, \ldots, L_0$, we can use the FPTAS in Bansal and Venkaiah (2004) to obtain a $(1-\epsilon)$-approximate solution to the Knapsack problem in $O(K \sum_{k\in\mathcal{K}} L^k/\epsilon) = O(K^2 B/\epsilon)$ operations. Thus, we can execute Step 2 in $O(VK^2B/\epsilon)$ operations. The running times of Steps 3 and 4 are dominated by those of Steps 1 and 2. Lastly, we use (1) to transform the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ returned by our approximation framework into the probability of offering each assortment. This transformation requires sorting the values of $(\widehat{x}_i : i \in \mathcal{N})$, which can be done in $O(n \log n)$ operations. Thus, the running time for our approximation framework is $O(KVBn + K^2VB/\epsilon + n\log n)$, which we dominate by $O(\frac{1}{\epsilon} K^2 nVB + n \log n)$. We choose the precision parameter for our grid as $\rho = \frac{\epsilon}{2}$, so $V = O(\frac{\log(1+v_{\max})}{\epsilon} + K)$ and $B = O(\frac{\log(1/\beta_{\min})}{\epsilon})$. In this case, the desired running time follows by plugging these values for V and B into the expression $O(\frac{1}{\epsilon} K^2 nVB + n \log n)$.

Appendix D: Proof of Lemma 6.1

We give a proof for Lemma 6.1. Setting $\sigma_i(p,\gamma) = \lceil \frac{n}{\rho\,\gamma}\, r_i\, v_i\, \overline{x}_0(p) \rceil$ and $\delta_i(p,q) = \lfloor \frac{n}{\rho\,(1+\rho)\,\overline{z}^k(q)}\, v_i\, \overline{x}_0(p) \rfloor$, we formulated a relaxed version of problem (6), which is given by

$$\widehat{\zeta}(\gamma) \;=\; \max_{\boldsymbol{y}^k \in \{0,1\}^{n^k}} \left\{ \sum_{i \in \mathcal{N}^k} v_i\, \overline{x}_0(p)\, y_i : \sum_{i \in \mathcal{N}^k} \sigma_i(p,\gamma)\, y_i \geq \left\lfloor \frac{n}{\rho} \right\rfloor,\; \sum_{i \in \mathcal{N}^k} \delta_i(p,q)\, y_i \leq \left\lceil \frac{n}{\rho} \right\rceil \right\}. \tag{11}$$

Letting $\widehat{\gamma} = \max\{\gamma \in \overline{\mathsf{Grid}} : \widehat{\zeta}(\gamma) \geq \beta^k\}$, we use $\widehat{\boldsymbol{y}}^k = (\widehat{y}_i : i \in \mathcal{N}^k)$ to denote an optimal solution to problem (11) with $\gamma = \widehat{\gamma}$. If there does not exist any $\gamma \in \overline{\mathsf{Grid}}$ such that $\widehat{\zeta}(\gamma) \geq \beta^k$, then we set $\widehat{\boldsymbol{y}}^k = \mathbf{0} \in \mathbb{R}_+^{n^k}$. Lastly, we construct the solution $\widehat{\boldsymbol{x}}^k = (\widehat{x}_i : i \in \mathcal{N}^k)$ as $\widehat{x}_i = v_i\, \overline{x}_0(p)\, \widehat{y}_i$ for all $i \in \mathcal{N}^k$. We claim that the objective value of problem (5) at the solution $\widehat{\boldsymbol{x}}^k$ differs from the optimal objective value of this problem by at most a factor of $1 - 3\rho$. If the optimal objective value of problem (5) is zero, then there is nothing to prove, so we proceed with the understanding that the optimal objective value of problem (5) is strictly positive. Letting $\widehat{Z}$ be the optimal objective value of problem (5), the smallest strictly positive objective value for this problem is $r_{\min}\, v_{\min}\, \overline{x}_0(p)$. We choose the grid point $\widetilde{\gamma} \in \overline{\mathsf{Grid}}$ such that $\widetilde{\gamma} \leq \widehat{Z} \leq (1+\rho)\,\widetilde{\gamma}$. Letting $\widetilde{\boldsymbol{x}}^k = (\widetilde{x}_i : i \in \mathcal{N}^k)$ be an optimal solution to problem (5), we have $\sum_{i \in \mathcal{N}^k} r_i\, \widetilde{x}_i = \widehat{Z}$, $\sum_{i \in \mathcal{N}^k} \widetilde{x}_i \geq \beta^k$, $\sum_{i \in \mathcal{N}^k} \widetilde{x}_i \leq (1+\rho)\, \overline{z}^k(q)$ and $\widetilde{x}_i \in \{0, v_i\, \overline{x}_0(p)\}$ for all $i \in \mathcal{N}^k$, where the first inequality uses the fact that the optimal objective value of problem (5) is strictly positive. In this case, setting $\widetilde{y}_i = \widetilde{x}_i / (v_i\, \overline{x}_0(p))$ for all $i \in \mathcal{N}^k$, the solution $\widetilde{\boldsymbol{y}}^k = (\widetilde{y}_i : i \in \mathcal{N}^k)$ is feasible to problem (6) when we solve this problem with $\gamma = \widetilde{\gamma}$ and provides an objective value of at least β^k for problem (6). Therefore, the optimal objective value of problem (6) with $\gamma = \widetilde{\gamma}$ is at least β^k, but noting that problem (11) is a relaxation of problem (6), optimal objective value of problem (11) with $\gamma = \widetilde{\gamma}$ satisfies $\zeta(\widetilde{\gamma}) \geq \beta^k$. Thus, the solution $\widetilde{\gamma}$ is feasible to the problem $\max\{\gamma \in \overline{\mathsf{Grid}} : \widehat{\zeta}(\gamma) \geq \beta^k\}$ but $\widehat{\gamma}$ is optimal to the same problem, so $\widehat{\gamma} \geq \widetilde{\gamma}$. Also, by the definition of $\widetilde{\gamma}$, we have $\widetilde{\gamma} \geq \widehat{Z}/(1+\rho)$, so we obtain $\widehat{\gamma} \geq \widehat{Z}/(1+\rho)$.

The solution $\widehat{\boldsymbol{y}}^k$ is feasible to problem (11) with $\gamma = \widehat{\gamma}$. We use the second constraint in problem (11) to lower bound the expected revenue from the solution $\widehat{\boldsymbol{y}}^k$. In particular, noting the definition of $\sigma_i(p,\gamma)$, we have $\sum_{i\in\mathcal{N}^k}\left\lceil\frac{n}{\rho\,\widehat{\gamma}}\,r_i\,v_i\,\overline{x}_0(p)\right\rceil\widehat{y}_i \geq \left\lfloor\frac{n}{\rho}\right\rfloor$. Therefore, using the fact that $x\geq\lceil x\rceil-1$ and $\lfloor x\rfloor\geq x-1$, we obtain the chain of inequalities

$$\sum_{i\in\mathcal{N}^k}\frac{n}{\rho\,\widehat{\gamma}}\,r_i\,v_i\,\overline{x}_0(p)\,\widehat{y}_i \;\geq\; \sum_{i\in\mathcal{N}^k}\left(\left\lceil\frac{n}{\rho\,\widehat{\gamma}}\,r_i\,v_i\,\overline{x}_0(p)\right\rceil-1\right)\widehat{y}_i \;\overset{(a)}{\geq}\; \left\lfloor\frac{n}{\rho}\right\rfloor-\sum_{i\in\mathcal{N}^k}\widehat{y}_i \;\overset{(b)}{\geq}\; \frac{n}{\rho}-1-n, \tag{12}$$

where (a) holds because $\widehat{\boldsymbol{y}}$ is feasible to problem (11), whereas (b) holds by $\sum_{i\in\mathcal{N}^k}\widehat{y}_i\leq n$. Multiplying the chain of inequalities with $\frac{\rho}{n}\,\widehat{\gamma}$, as well as noting that $\widehat{x}_i=v_i\,\overline{x}_0(p)\,\widehat{y}_i$, we get $\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i \geq \frac{\rho\,\widehat{\gamma}}{n}(\frac{n}{\rho}-1-n)=\widehat{\gamma}\,(1-\frac{\rho}{n}-\rho)\geq\widehat{\gamma}\,(1-2\rho)$. Because $\widehat{\gamma}\geq\widehat{Z}/(1+\rho)$, we obtain $\sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i\geq(1-2\rho)\,\widehat{Z}/(1+\rho)\geq(1-3\rho)\,\widehat{Z}$. Therefore, the solution $\widehat{\boldsymbol{x}}^k$ provides at least $(1-3\rho)$-fraction of the optimal objective value of problem (5). By the definition of $\widehat{\gamma}$, we have $\widehat{\zeta}(\widehat{\gamma})\geq\beta^k$, but because $\widehat{\boldsymbol{y}}^k$ is optimal to (11) with $\gamma=\widehat{\gamma}$, we get $\sum_{i\in\mathcal{N}^k}\widehat{x}_i=\sum_{i\in\mathcal{N}^k}v_i\,\overline{x}_0(p)\,\widehat{y}_i=\widehat{\zeta}(\widehat{\gamma})$, which implies that the solution $\widehat{\boldsymbol{x}}^k$ satisfies the first constraint in (5). Because $\widehat{y}_i\in\{0,1\}$, we have $\widehat{x}_i=v_i\,\overline{x}_0(p)\,\widehat{y}_i\in\{0,v_i\,\overline{x}_0(p)\}$, so solution $\widehat{\boldsymbol{x}}^k$ satisfies the third constraint in (5). Using an argument analogous to the chain of inequalities in (12), but using the second constraint instead of the first constraint in problem (11), we can show that $\sum_{i\in\mathcal{N}^k}\frac{n}{\rho\,(1+\rho)\,\overline{z}^k(q)}\,v_i\,\overline{x}_0(p)\,\widehat{y}_i\leq\frac{n}{\rho}+1+n$, in which case, multiplying this inequality with $\frac{\rho}{n}\,(1+\rho)\,\overline{z}^k(q)$ and noting that $\widehat{x}_i=v_i\,\overline{x}_0(p)\,\widehat{y}_i$, we obtain $\sum_{i\in\mathcal{N}}^k\widehat{x}_i\leq\frac{\rho}{n}\,(1+\rho)\,\overline{z}^k(q)\,(\frac{n}{\rho}+1+n)\leq(1+\rho)\,\overline{z}^k(q)(1+\frac{n}{\rho}+\rho)\leq(1+2\rho)\,(1+\rho)\,\overline{z}^k(q)$, so the solution $\widehat{\boldsymbol{x}}^k$ satisfies the second constraint in (5) with a tolerance of $1+2\rho$.

To complete the proof, we give the running time to obtain the solution $\widehat{\boldsymbol{x}}^k$. Indexing the products in $\mathcal{N}^k$ as $\{1,\ldots,n^k\}$, we can solve problem (11) through the dynamic program

$$J_i(b,c)=\max_{y_i\in\{0,1\}}\Big\{v_i\,\overline{x}_0(p)\,y_i+J_{i+1}(b+\sigma_i(p,\gamma)\,y_i,c+\delta_i(p,q)\,y_i)\Big\}$$

with the boundary condition that $J_{n^k+1}(b,c)=0$ if $b\geq\left\lfloor\frac{n}{\rho}\right\rfloor$ and $c\leq\left\lceil\frac{n}{\rho}\right\rceil$, whereas $J_{n^k+1}(b,c)=-\infty$ otherwise. In the dynamic program above, the state variable keeps track of the accumulated value of the left side of the two constraints in (11) and the value function keeps track of the accumulated value of the objective function in (11). The boundary condition $J_{n^k+1}(c,b)$ is verified by comparing b and c with $\left\lfloor\frac{n}{\rho}\right\rfloor$ and $\left\lceil\frac{n}{\rho}\right\rceil$. Thus, during the course of the decision epochs, if the value of b exceeds $\left\lfloor\frac{n}{\rho}\right\rfloor$, then we do not need to keep the value of this state variable anymore, whereas if the value of c exceeds $\left\lceil\frac{n}{\rho}\right\rceil$, then we can immediately conclude that the value function takes the value negative infinity. Therefore, the number of state variables in $O((\frac{n}{\rho})^2)$. Because there are $O(n)$ decision epochs, we can solve the dynamic program in $O(\frac{n^3}{\rho^2})$ operations, providing an optimal solution to (11) and the optimal objective value $\zeta(\gamma)$ for fixed γ. The number of points in $\overline{\mathsf{Grid}}$ is $O\Big(\frac{1}{\rho}\log(\frac{n\,r_{\max}\,v_{\max}}{r_{\min}\,v_{\min}})\Big)$. For each $\gamma\in\overline{\mathsf{Grid}}$, compute $\zeta(\gamma)$ in $O(\frac{n^3}{\rho^2})$ operations, so we can solve the problem $\max\{\gamma\in\overline{\mathsf{Grid}}:\widehat{\zeta}(\gamma)\geq\beta^k\}$ in $O\Big(\frac{n^3}{\rho^3}\log(\frac{n\,r_{\max}\,v_{\max}}{r_{\min}\,v_{\min}})\Big)$. The number of operations for other work to obtain $\widehat{\boldsymbol{x}}^k$ is dominated by this running time. ■

Appendix E: Proof of Theorem 6.3

We complete the proof of Theorem 6.3 by giving the running time for our approximation scheme. Letting V and B as in Appendix C, we have $|\mathsf{Grid}_0| \le V$ and $|\mathsf{Grid}^k| \le B$. In Step 1, for each $k \in \mathcal{K}$, $p = 1, \dots, L_0$ and $q = 1, \dots, L^k$, the running time to obtain the candidate solution $\overline{\boldsymbol{x}}^k(p,q)$ is given in Lemma 6.1. Therefore, we can execute Step 1 in $O\Big(KVB \frac{n^3}{\rho^3} \log(\frac{n\, r_{max}\, v_{max}}{r_{\min}\, v_{\min}})\Big)$ operations. The running time to execute Step 2 is the same as the one in the proof of Theorem 5.2. Thus, we can execute Step 2 in $O(VK^2B/\epsilon)$ operations. The running times of Steps 3 and 4, as well as the running time to use (1) to transform the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ returned by our approximation framework into the probability of offering each assortment, are dominated by those of Steps 1 and 2. Thus, noting that we choose the precision parameter for our grid as $\rho = \frac{\epsilon}{2}$, the running time for our approximation framework is $O\Big(KVB \frac{n^3}{\epsilon^3} \log(\frac{n\, r_{\max}\, v_{\max}}{r_{\min}\, v_{\min}}) + \frac{1}{\epsilon} K^2 VB\Big)$, which we dominate by $O\Big(K^2VB \frac{n^3}{\epsilon^3} \log(\frac{n\, r_{\max}\, v_{\max}}{r_{\min}\, v_{\min}})\Big)$. We choose the precision parameter for our grid as $\rho = \frac{\epsilon}{2}$, so $V = O(\frac{\log(1+v_{\max})}{\epsilon} + K)$ and $B = O(\frac{\log(1/\beta_{\min})}{\epsilon})$. In this case, the desired running time follows by plugging these values for V and B into the expression $O\Big(K^2VB \frac{n^3}{\epsilon^3} \log(\frac{n\, r_{\max}\, v_{\max}}{r_{\min}\, v_{\min}})\Big)$.

Appendix F: Approximation Scheme with One Product Class

We give an FPTAS for the deterministic variant with one product class. We denote the products in the single class by $\mathcal{N}$. Considering the Sales-Based problem, we assume that $\sum_{i \in \mathcal{N}} x_i \in [\beta^1, 1]$ in an optimal solution, otherwise the optimal objective value of the Sales-Based problem is zero. We claim that we can replace the first two constraints in the Sales-Based problem with the single constraint $\sum_{i \in \mathcal{N}} x_i \in [\frac{\beta^1}{1-\beta^1}\, x_0, 1 - x_0]$. If the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem, then $\sum_{i \in \mathcal{N}} \widehat{x}_i \ge \beta^1$ and $\widehat{x}_0 + \sum_{i \in \mathcal{N}} \widehat{x}_i \le 1$, which yield $\sum_{i \in \mathcal{N}} \widehat{x}_i \ge \beta^1\, (\widehat{x}_0 + \sum_{i \in \mathcal{N}} \widehat{x}_i)$. Rearranging the terms in the last inequality, we get $\sum_{i \in \mathcal{N}} \widehat{x}_i \ge \frac{\beta^1}{1-\beta^1}\, \widehat{x}_0$, so $\sum_{i \in \mathcal{N}} \widehat{x}_i \in [\frac{\beta^1}{1-\beta^1}\, \widehat{x}_0, 1 - \widehat{x}_0]$. On the other hand, let $(\widetilde{x}_0, \widetilde{\boldsymbol{x}}_0)$ be an optimal solution to the Sales-Based problem after we replace the first two constraints with the single constraint $\sum_{i \in \mathcal{N}} x_i \in [\frac{\beta^1}{1-\beta^1}\, x_0, 1 - x_0]$. Thus, we have $\sum_{i \in \mathcal{N}} \widetilde{x}_i \ge \frac{\beta^1}{1-\beta^1}\, \widetilde{x}_0$, $\sum_{i \in \mathcal{N}} \widetilde{x}_i \le 1 - \widetilde{x}_0$ and $\widetilde{x}_i \in \{0, v_i\, \widetilde{x}_0\}$, using which we can check that $(\widehat{x}_0, \widehat{\boldsymbol{x}}_0)$ with $\widehat{x}_i = \frac{\widetilde{x}_i}{\widetilde{x}_0 + \sum_{j \in \mathcal{N}} \widetilde{x}_j}$ and $\widehat{x}_0 = \frac{\widetilde{x}_0}{\widetilde{x}_0 + \sum_{j \in \mathcal{N}} \widetilde{x}_j}$ is feasible to the Sales-Based problem. Because $\widetilde{x}_0 + \sum_{i \in \mathcal{N}} \widetilde{x}_i \le 1$, we have $\widehat{x}_i \ge \widetilde{x}_i$, so the objective value from $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is at least as large as that from $(\widetilde{x}_0, \widetilde{\boldsymbol{x}})$, establishing the claim. Letting Grid_0 be as in (3), for each $p = 1, \dots, L_0$, consider the problem

$$\zeta(p) \;=\; \max_{\boldsymbol{x} \in [0,1]^n} \left\{ \sum_{i \in \mathcal{N}} r_i\, x_i \;:\; \sum_{i \in \mathcal{N}} x_i \,\in\, \Big[\frac{\beta^1}{1-\beta^1}\, \overline{x}_0(p), 1 - \overline{x}_0(p)\Big],\; x_i \in \{0, v_i\, \overline{x}_0(p)\} \;\; \forall\, i \in \mathcal{N} \right\}. \quad (13)$$

In the problem above, recall that $\{\overline{x}_0(p) : p = 1, \dots, L_0\}$ corresponds to the set of points in Grid_0. To obtain an approximate solution to the deterministic variant of the Market-Share problem, we simply solve the problem above for each $p=1,\dots,L_0$ and return the solution that provides the largest expected revenue. Because of the second constraint in problem (13), it is not clear that we can solve this problem exactly. However, using precisely the same approach in the proof of Lemma 6.1, we can show that we can obtain an approximate solution to problem (13) that satisfies the first constraint with a given tolerance, while satisfying the second constraint exactly. We give this result in the next lemma. We omit the proof as it follows from an argument that closely reflects the proof of Lemma 6.1. In the next lemma, recall that ρ corresponds to the precision parameter in the construction of our grid points and $\zeta(p)$ is the optimal objective value of (13).

Lemma F.1 (Parameterized Market-Share) *For fixed $p=1,\dots,L_0$, we can obtain a solution $\widehat{\boldsymbol{x}}$ to problem* (13) *that provides an objective value of at least $(1-3\rho)\,\zeta(p)$ and satisfies the second constraint exactly, while satisfying the first constraint as $\sum_{i\in\mathcal{N}}\widehat{x}_i \in \Big[\frac{\beta^1}{1-\beta^1}\,\overline{x}_0(p), (1+2\,\rho)(1-\overline{x}_0(p))\Big]$, in a running time of $O\Big(\frac{n^3}{\rho^3}\log(\frac{n\,r_{\max}\,v_{\max}}{r_{\min}\,v_{\min}})\Big)$ operations.*

To obtain an approximate solution to the deterministic variant, for each $p=1,\dots,L_0$, we use the lemma above to get an approximate solution to problem (13). As a function of p, we denote this approximate solution as $\widehat{\boldsymbol{x}}(p)=(\widehat{x}_i(p): i\in\mathcal{N})$. In this case, letting $\widehat{p}=\arg\max_{p=1,\dots,L_0}\sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i(p)$ to capture the grid point that provides the largest expected revenue, we construct the solution $(\widetilde{x}_0,\widetilde{\boldsymbol{x}})$ as $\widetilde{x}_i=\widehat{x}_i(\widehat{p})/(\overline{x}_0(\widehat{p})+\sum_{j\in\mathcal{N}}\widehat{x}_j(\widehat{p}))$ for all $i\in\mathcal{N}$ and $\widetilde{x}_0=\overline{x}_0(\widehat{p})/(\overline{x}_0(\widehat{p})+\sum_{j\in\mathcal{N}}\widehat{x}_j(\widehat{p}))$, where we note that $\overline{x}_0(\widehat{p})$ is the grid point in Grid_0 with the index $\widehat{p}$. Finally, we use the transformation in (1) on the solution $(\widetilde{x}_0,\widetilde{\boldsymbol{x}})$ to obtain a solution to the deterministic variant of the Market-Share problem. In the next theorem, we give a performance guarantee for this solution.

Theorem F.2 (FPTAS for One Product Class) *Considering the deterministic variant of the* Market-Share *problem with one product class, for any $\epsilon\in(0,1)$, we can obtain a $(1-\epsilon)$-approximate solution in a running time of $O\Big(\frac{n^3}{\epsilon^4}\,\log(\frac{n\,r_{\max}\,v_{\max}}{r_{\min}\,v_{\min}})\,\log(1+nv_{\max})\Big)$ operations.*

Proof: We claim that the solution $(\widetilde{x}_0,\widetilde{\boldsymbol{x}})$ as given just before the theorem satisfies the assumptions of Theorem 2.2 with $\gamma_1=0$ and $\gamma_2=1-6\rho$. Let $(x_0^*,\boldsymbol{x}^*)$ be an optimal solution to the deterministic variant of the Sales-Based problem. For $\widetilde{p}=1,\dots,L_0$, we use $\overline{x}_0(\widetilde{p})$ to denote the grid point that is closest to but no larger than x_0^*, which is to say that $\overline{x}_0(\widetilde{p})\le x_0^*\le(1+\rho)\,\overline{x}_0(\widetilde{p})$. We construct a feasible solution to problem (13) when we solve this problem with $p=\widetilde{p}$. We have $\sum_{i\in\mathcal{N}} v_i\,\overline{x}_0(\widetilde{p})\,\mathbf{1}(x_i^*>0)\le\sum_{i\in\mathcal{N}} v_i\,x_0^*\,\mathbf{1}(x_i^*>0)=\sum_{i\in\mathcal{N}} x_i^*\le 1-x_0^*\le 1-\overline{x}_0(\widetilde{p})$, where the equality and the second inequality holds because $(x_0^*,\boldsymbol{x}^*)$ is feasible to the Sales-Based problem. Furthermore, we have $\sum_{i\in\mathcal{N}} x_i^*\ge\beta^1\ge\beta^1\,(x_0^*+\sum_{i\in\mathcal{N}} x_i^*)$, which yields $\frac{\beta^1}{1-\beta^1}\,x_0^*\le\sum_{i\in\mathcal{N}} x_i^*=\sum_{i\in\mathcal{N}} v_i\,x_0^*\,\mathbf{1}(x_i^*>0)$, where the last equality holds because $(x_0^*, \boldsymbol{x}^*)$ satisfies the last constraint in the Sales-Based problem. Multiplying the last chain of inequalities with $\frac{\overline{x}_0(\widetilde{p})}{x_0^*}$ yields $\sum_{i\in\mathcal{N}} v_i\, \overline{x}_0(\widetilde{p})\, \mathbf{1}(x_i^*>0) \geq \frac{\beta^1}{1-\beta^1}\, \overline{x}_0(\widetilde{p})$. Lastly, noting that $v_i\, \overline{x}_0(\widetilde{p})\, \mathbf{1}(x_i^*>0) \in \{0, v_i\, \overline{x}_0(\widetilde{p})\}$, by the discussion so far in this paragraph, the solution $(v_i\, \overline{x}_0(\widetilde{p})\, \mathbf{1}(x_i^*>0) : i\in\mathcal{N})$ is feasible to problem (13) with $p=\widetilde{p}$. Using Lemma F.1 for the grid point with the index $\widetilde{p}$, we can find a solution $\widehat{\boldsymbol{x}}(\widetilde{p}) = (\widehat{x}_i(\widetilde{p}) : i\in\mathcal{N})$ to problem (13) such that $\sum_{i\in\mathcal{N}} r_i\, \widehat{x}_i(\widetilde{p}) \geq (1-3\rho)\, \zeta(\widetilde{p})$ and $\sum_{i\in\mathcal{N}} \widehat{x}_i(\widetilde{p}) \leq (1+2\rho)\,(1-\overline{x}_0(\widetilde{p}))$. In this case, by the definition of $\widehat{p}$ just before the theorem, we get $\sum_{i\in\mathcal{N}} r_i\, \widehat{x}_i(\widehat{p}) \geq \sum_{i\in\mathcal{N}} r_i\, \widehat{x}_i(\widetilde{p}) \geq (1-3\rho)\, \zeta(\widetilde{p})$. Using $\sum_{i\in\mathcal{N}} \widehat{x}_i(\widehat{p}) \leq (1+2\rho)\,(1-\overline{x}_0(\widehat{p}))$, we get $\overline{x}_0(\widehat{p}) + \sum_{i\in\mathcal{N}} \widehat{x}_i(\widehat{p}) \leq (1+2\rho)\,(1-\overline{x}_0(\widehat{p})) + \overline{x}_0(\widehat{p}) \leq 1+2\rho$. By the last two chains of inequalities, for the solution $(\widetilde{x}_0, \widetilde{\boldsymbol{x}})$ just before the theorem, we get

$$\begin{aligned}\sum_{i\in\mathcal{N}} r_i\, \widetilde{x}_i \;&=\; \sum_{i\in\mathcal{N}} r_i\, \frac{\widehat{x}_i(\widehat{p})}{\overline{x}_0(\widehat{p}) + \sum_{j\in\mathcal{N}} \widehat{x}_j(\widehat{p})} \;\geq\; \frac{1-3\rho}{1+2\rho}\, \zeta(\widetilde{p}) \;\overset{(a)}{\geq}\; \frac{1-3\rho}{1+2\rho} \sum_{i\in\mathcal{N}} r_i\, v_i\, \overline{x}_0(\widetilde{p})\, \mathbf{1}(x_i^*>0) \\ &\overset{(b)}{\geq}\; \frac{1-3\rho}{(1+2\rho)(1+\rho)} \sum_{i\in\mathcal{N}} r_i\, v_i\, x_0^*\, \mathbf{1}(x_i^*>0) \;\overset{(c)}{=}\; \frac{1-3\rho}{(1+2\rho)(1+\rho)} \sum_{i\in\mathcal{N}} r_i\, x_i^* \;\geq\; (1-6\rho)\, \overline{\mathsf{opt}},\end{aligned}$$

where (a) holds because the solution $(v_i\, \overline{x}_0(\widetilde{p})\, \mathbf{1}(x_i^*>0) : i\in\mathcal{N})$ is feasible to problem (13) with $p=\widetilde{p}$, (b) holds by the choice of the grid point $\widetilde{p}$ and (c) holds because the third constraint in the Sales-Based problem implies that $x_i = v_i\, x_0\, \mathbf{1}(x_i>0)$ in any feasible solution. By the chain of inequalities, we can choose $\gamma_2 = 6\rho$ in Theorem 2.2. We have $\widetilde{x}_0 + \sum_{i\in\mathcal{N}} \widetilde{x}_i = 1$ by the definition of $(\widetilde{x}_0, \widetilde{\boldsymbol{x}})$, so we can choose $\gamma_1 = 0$ in Theorem 2.2. To verify the remaining assumptions in Theorem 2.2, because $\sum_{i\in\mathcal{N}} \widehat{x}_i(\widehat{p}) \geq \frac{\beta^1}{1-\beta^1} \overline{x}_0(\widehat{p})$, using the fact that $a/(1+a)$ is increasing in $a\geq 0$, we get $\sum_{i\in\mathcal{N}} \widetilde{x}_i = \sum_{i\in\mathcal{N}} \widehat{x}_i(\widehat{p}) / (\overline{x}_0(\widehat{p}) + \sum_{i\in\mathcal{N}} \widehat{x}_i(\widehat{p})) \geq \frac{\beta^1}{1-\beta^1} \overline{x}_0(\widehat{p}) / (\overline{x}_0(\widehat{p}) + \frac{\beta^1}{1-\beta^1} \overline{x}_0(\widehat{p})) = \beta^1$. Because $\widehat{x}_i(\widehat{p}) \in \{0, v_i\, \overline{x}_0(\widehat{p})\}$, we have $\widetilde{x}_i = \{0, v_i\, \widetilde{x}_0\}$ as well, so the solution $(\widetilde{x}, \widetilde{\boldsymbol{x}}_0)$ satisfies the assumptions of Theorem 2.2 with $\gamma_1 = 0$ and $\gamma_2 = 6\rho$. We choose $\rho = \frac{\epsilon}{6}$, so that $(1-\gamma_1)\,(1-\gamma_2) = 1-\epsilon$.

By Theorem 2.2, using (1) on the solution $(\widetilde{x}_0, \widetilde{\boldsymbol{x}})$ yields a $(1-\epsilon)$-approximate solution to the Market-Share problem. The running time is by Lemma F.1 and the number of points in Grid_0.

Appendix G: Upper Bounds on the Optimal Expected Revenue

Consider computing an upper bound on the optimal expected revenue in the randomized variant. Fix the points $\{\eta(t) : t=0,\ldots,L\}$ with $0=\eta(0)<\eta(1)<\ldots<\eta(L-1)<\eta(L)=1$. Any set of points will yield an upper bond, but a finer set will yield tighter upper bounds. For notational brevity, set $\sigma^k(t) = \beta^k + (1-\beta^k)\,\eta(t)$. For each $k\in\mathcal{K}$, $t=0,\ldots,L-1$ and $\ell=0,\ldots,L-1$, using the decision variables $\boldsymbol{x}^k = (x_i : i\ \in\mathcal{N}^k)$, we consider the continuous knapsack problem

$$\psi^k(t,\ell) = \max_{\boldsymbol{x}^k\in[0,1]^{n^k}} \left\{ \sum_{i\in\mathcal{N}^k} r_i\, x_i \,:\, \sigma^k(t) \leq \sum_{i\in\mathcal{N}^k} x_i \leq \sigma^k(t+1), \quad x_i \leq v_i\, \eta(\ell+1)\ \forall\, i\in\mathcal{N}^k \right\}. \qquad (14)$$

Because $\eta(0)=0$ and $\eta(L)=1$, we have $\sigma^k(0)=\beta^k$ and $\sigma^k(L)=1$, so the collection of intervals $\{[\sigma^k(t), \sigma^k(t+1)] : t=0,\ldots,L-1\}$ cover the interval $[\beta^k, 1]$. Noting that we have $\mathcal{H}_i(x_0) = [0, v_i\, x_0]$

for the randomized variant of the Market-Share problem, we interpret the knapsack problem above as approximate version of problem (2) when we guess the market share of product class k to take a value in the interval $[\sigma^k(t), \sigma^k(t+1)]$ and the no-purchase probability to be $\eta(\ell+1)$. The reason for using $\eta(\ell+1)$ rather than $\eta(\ell)$ for the guess of the no-purchase probability on the right side of the second constraint in the problem above will shortly be clear. To maximize the expected revenue in the randomized variant, we find the best combination of the market shares for each product class, as well as the no-purchase probability. In particular, noting that $\psi^k(t,\ell)$ is the optimal objective value of problem (14), using the vector of decision variables $\boldsymbol{u}(\ell) = (u^k(t,\ell) : t = 0, \ldots, L-1,\ k \in \mathcal{K})$, we solve the continuous multiple-choice knapsack problem

$$\Theta(\ell) = \max_{\boldsymbol{u}(\ell) \in [0,1]^{L\times K}} \left\{ \sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \psi^k(t,\ell)\, u^k(t,\ell) \ :\ \sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \sigma^k(t)\, u^k(t,\ell) + \eta(\ell) \le 1 \right.$$
$$\left. \sum_{t=0}^{L-1} u^k(t,\ell) \le 1 \quad \forall\, k \in \mathcal{K} \right\}. \tag{15}$$

In the next lemma, we show that we can use the problem above to get an upper bound on the optimal expected revenue for the randomized variant of the Market-Share problem.

Lemma G.1 (Upper Bound for Randomized Variant) *Using* opt *to denote the optimal expected revenue in the randomized variant, we have* $\max_{\ell=0,\ldots,L-1} \Theta(\ell) \ge \mathsf{opt}$.

Proof: Let $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ be an optimal solution to the randomized variant of the Sales-Based problem. Let $\widehat{\ell} \in \{0, \ldots, L-1\}$ be such that $\widehat{x}^0 \in [\eta(\widehat{\ell}), \eta(\widehat{\ell}+1)]$. It is enough to show that $\Theta(\widehat{\ell}) \ge \mathsf{opt}$. If $\widehat{x}_0 = 0$, then we have $\widehat{x}_i = 0$ for all $i \in \mathcal{N}$ by the last constraint in the Sales-Based problem. If $\widehat{x}_0 > 0$, then we have $\sum_{i\in\mathcal{N}^k} \widehat{x}_i \le \sum_{i\in\mathcal{N}} \widehat{x}_i \le 1 - \widehat{x}^0 < 1$ by the second constraint in the Sales-Based problem. Therefore, we have $\sum_{i\in\mathcal{N}^k} \widehat{x}_i < 1$ for all $k \in \mathcal{K}$. On the other hand, by the first constraint in the Sales-Based problem, if $\sum_{i\in\mathcal{N}^k} \widehat{x}_i > 0$, then $\sum_{i\in\mathcal{N}^k} \widehat{x}_i^k \in [\beta^k, 1]$, which implies that we have $\sigma^k(0) \le \sum_{i\in\mathcal{N}^k} \widehat{x}_i < \sigma^k(L)$. Considering problem (15) with $\ell = \widehat{\ell}$, we construct a solution $\widehat{\boldsymbol{u}}(\widehat{\ell}) = (\widehat{u}^k(t,\ell) : t = 0, \ldots, L-1,\ k \in \mathcal{K})$ to this problem as $\widehat{u}^k(t,\widehat{\ell}) = \mathbf{1}(\sum_{i\in\mathcal{N}^k} \widehat{x}_i \in [\sigma^k(t), \sigma^k(t+1)))$ for all $t = 0, \ldots, L-1$ and $k \in \mathcal{K}$. Noting that the intervals $\{[\sigma^k(t), \sigma^k(t+1)) : t = 0, \ldots, L-1\}$ are non-overlapping, by the definition of $\widehat{\boldsymbol{u}}(\widehat{\ell})$, we immediately obtain $\sum_{t=0}^{L-1} \widehat{u}^k(t,\widehat{\ell}) \le 1$, so the solution $\widehat{\boldsymbol{u}}(\widehat{\ell})$ satisfies the second constraint in (15). Furthermore, we have $\sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \sigma^k(t)\, \widehat{u}^k(t,\widehat{\ell}) = \sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \sigma^k(t)\, \mathbf{1}(\sum_{i\in\mathcal{N}^k} \widehat{x}_i \in [\sigma^k(t), \sigma^k(t+1))) \le \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}^k} \widehat{x}_i$, where the last inequality follows because if the interval $[\sigma^k(t), \sigma^k(t+1))$ includes $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$, then $\sigma^k(t)$ is upper bounded by $\sum_{i\in\mathcal{N}^k} \widehat{x}_i$. In this case, we get $\sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \sigma^k(t)\, \widehat{u}^k(t,\widehat{\ell}) + \eta(\widehat{\ell}) \le \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}^k} \widehat{x}_i + \widehat{x}_0 \le 1$, where the first equality uses the choice of $\widehat{\ell}$ and the second inequality holds because the solution $(\widehat{x}_0, \widehat{\boldsymbol{x}})$

is feasible to the Sales-Based problem. Thus, the solution $\widehat{\boldsymbol{u}}(\widehat{\ell})$ satisfies the second constraint in (15) with $\ell = \widehat{\ell}$. In this case, we get $\Theta(\widehat{\ell}) \geq \sum_{k\in\mathcal{K}} \sum_{t=0}^{L-1} \psi^k(t,\widehat{\ell})\,\mathbf{1}(\sum_{i\in\mathcal{N}^k} \widehat{x}_i \in [\sigma^k(t), \sigma^k(t+1)))$.

Let $\widehat{t}^k \in \{0,\ldots,L-1\}$ be such that $\sum_{i\in\mathcal{N}^k} \widehat{x}_i \in [\sigma^k(\widehat{t}^k), \sigma^k(\widehat{t}^k+1))$. We have $\widehat{x}_i \leq v_i\,\widehat{x}_0 \leq v_i\,\eta(\widehat{\ell}+1)$, where the first inequality holds because $(\widehat{x}_0, \widehat{\boldsymbol{x}})$ is feasible to the Sales-Based problem and the second inequality holds by our choice of $\widehat{\ell}$. So, the solution $\widehat{x}^k = (\widehat{x}_i : i \in \mathcal{N}^k)$ satisfies the second constraint in (14) with $(t,\ell) = (\widehat{t}^k, \widehat{\ell})$. By the definition of $\widehat{t}^k$, we have $\sigma^k(\widehat{t}^k) \leq \sum_{i\in\mathcal{N}^k} \widehat{x}_i \leq \sigma^k(\widehat{t}^k+1)$, so the solution $\widehat{x}^k = (\widehat{x}_i : i \in \mathcal{N}^k)$ satisfies the first constraint in (14) with $(t,\ell) = (\widehat{t}^k, \widehat{\ell})$. In this case, we get $\psi^k(\widehat{t}^k, \widehat{\ell}) \geq \sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i$. Using this inequality in the inequality at the end of the previous paragraph, noting the definition of $\widehat{t}^k$, we get $\Theta(\widehat{\ell}) \geq \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}^k} r_i\,\widehat{x}_i = \sum_{i\in\mathcal{N}} r_i\,\widehat{x}_i = \mathsf{opt}$. ■

By the lemma above, we can obtain an upper bound on the optimal expected revenue for the randomized variant of the Market-Share problem by solving a sequence of linear programs. We can use a similar outline to obtain an upper bound for the deterministic variant. For economy of space, we give an overview of our approach for the deterministic variant. We guess the no-purchase probability under the optimal assortment. If we offer the assortment S, then the no-purchase probability is $1/(1+\sum_{i\in S} v_i)$. Thus, if we guess the no-purchase probability to be η, then each offered product i contributes $v_i\,\eta$ to the market share of its product class and $r_i\,v_i\,\eta$ to the expected revenue over all product classes. In this case, we can formulate a multiple-choice knapsack problem similar to the one in (15) that chooses an assortment to offer in each product class such that we maximize the expected revenue over all product classes, while ensuring that the no-purchase probability is consistent with our guess. The number of decision variables in this problem increases exponentially with the number of products in each class, but linearly with the number of product classes. In particular, the number of decision variables is $\sum_{k\in\mathcal{K}} 2^{n^k}$. In our experimental setup, we have at most 10 products in each of 10 product classes, yielding about 10,000 decision variables. We solve a continuous relaxation of this multiple-choice knapsack problem to obtain an upper bound.

Appendix H: Approximation Scheme with Singleton Product Classes

We give an FPTAS for the deterministic variant when each product class is a singleton. We denote both the sets of products and product classes by $\mathcal{N}$. For each $i \in \mathcal{N}$, we have $\mathcal{N}^i = \{i\}$, so each product is a product class itself. We use β^i to denote the market share threshold of product i. We use the set of grid points Grid_0 in (3) for our guesses of the no-purchase probability. Because we focus on the deterministic variant, we offer a single assortment $\widehat{S}$, so if the no-purchase probability is x_0, then $x_0 = \frac{1}{1+\sum_{i\in\widehat{S}} v_j}$, in which case, the purchase probability of product $i \in \widehat{S}$ is $\frac{v_i}{1+\sum_{j\in\widehat{S}} v_j} = v_i\,x_0$. Thus, if we guess the no-purchase probability in an optimal solution to be x_0, then offering product i satisfies the market share of the product when $v_i\,x_0 \geq \beta^i$. For each $p = 1,\ldots,L_0$, we define the set $\mathcal{N}(p)=\{i\in\mathcal{N}: v_i\,\overline{x}_0(p)\geq\beta^i\}$, which is the set of products with attainable market shares when we guess the no-purchase probability as $\overline{x}_0(p)$. For fixed $p=1,\ldots,L_0$, consider the problem

$$\max_{S\subseteq\mathcal{N}(p)}\left\{\sum_{i\in S} r_i\, v_i\,\overline{x}_0(p)\;:\;\overline{x}_0(p)+\sum_{i\in S} v_i\,\overline{x}_0(p)\leq 1\right\}. \tag{16}$$

Problem (16) is a knapsack problem, so it admits an FPTAS that yields a $(1-\epsilon)$-approximate solution in a running time of $O(\frac{1}{\epsilon}\,n^3)$ operations; see Williamson and Shmoys (2011). In our FPTAS with singleton product classes, for each $p=1,\ldots,L_0$, we compute a $(\frac{1}{1+\rho})$-approximate solution to (16). Letting $\widehat{S}(p)$ be this approximate solution as a function of p, we compute the best grid point as $\widehat{p}=\arg\max_{p=1,\ldots,L_0}\sum_{i\in\mathcal{N}} r_i\,\phi_i(\widehat{S}(p))$. We define the solution $\widehat{\boldsymbol{h}}=(\widehat{h}(S):S\subseteq\mathcal{N})$ to the deterministic variant of the Market-Share problem as $\widehat{h}(S)=1$ if $S=\widehat{S}(\widehat{p})$ and $\widehat{h}(S)=0$ for all $S\neq\widehat{S}(\widehat{p})$. In other words, the solution $\widehat{\boldsymbol{h}}$ offers the assortment $\widehat{S}(\widehat{p})$ with probability one.

We will give a performance guarantee for the solution $\widehat{\boldsymbol{h}}$. In the next lemma, we show that the solution $\widehat{\boldsymbol{h}}$ is feasible to the Market-Share problem with singleton product classes.

Lemma H.1 (Feasibility of Generated Solution) *The solution $\widehat{\boldsymbol{h}}=(\widehat{h}(S):S\subseteq\mathcal{N})$ is feasible for the deterministic variant of the* Market-Share *problem with singleton product classes.*

Proof: Noting the definition of the solution $\widehat{S}(\widehat{p})$ right before the lemma, because this solution is feasible to problem (16) with $p=\widehat{p}$, we have $\overline{x}_0(\widehat{p})+\sum_{i\in\widehat{S}(\widehat{p})} v_i\,\overline{x}_0(\widehat{p})\leq 1$, so $\frac{1}{1+\sum_{i\in\widehat{S}(\widehat{p})} v_i}\geq\overline{x}_0(\widehat{p})$. In this case, we get $\sum_{S\subseteq\mathcal{N}}\phi_i(S)\,\widehat{h}(S)=\phi_i(\widehat{S}(\widehat{p}))=\frac{v_i}{1+\sum_{j\in\widehat{S}(\widehat{p})} v_j}\geq v_i\,\overline{x}_0(\widehat{p})\geq\beta^i$ for all $i\in\widehat{S}(\widehat{p})$, where the first equality holds because the solution $\widehat{\boldsymbol{h}}$ offers only the assortment $\widehat{S}(\widehat{p})$ and the last inequality holds by the definition of $\mathcal{N}(\widehat{p})$ and the fact that $\widehat{S}(\widehat{p})\subseteq\mathcal{N}(\widehat{p})$. On the other hand, for all $i\notin\widehat{S}(\widehat{p})$, we have $\sum_{S\subseteq\mathcal{N}}\phi_i(S)\,\widehat{h}(S)=\phi_i(\widehat{S}(\widehat{p}))=0$. By the definition of $\widehat{\boldsymbol{h}}$, we have $\sum_{S\subseteq\mathcal{N}}\widehat{h}(S)=\widehat{h}(\widehat{S}(\widehat{p}))=1$ as well. Once again, by the definition of $\widehat{\boldsymbol{h}}$, we have $\widehat{\boldsymbol{h}}\in\{0,1\}^{2^n}$. ■

In the next theorem, we show that the solution $\widehat{\boldsymbol{h}}$ provides a performance guarantee for the deterministic variant of the Market-Share problem with singleton product classes.

Theorem H.2 (FPTAS for Singleton Product Classes) *Considering the deterministic variant of the* Market-Share *problem with singleton product classes, for any $\epsilon\in(0,1)$, we can obtain a $(1-\epsilon)$-approximate solution in a running time of $O\Big(\frac{1}{\epsilon}n^4\Big(n+\frac{1}{\epsilon}\log(1+nv_{\max})\Big)\Big)$ operations.*

Proof: If $\frac{v_i}{1+\sum_{j\in\mathcal{N}} v_j}\geq\beta^i$, then we can satisfy the market share constraint for product i even when we offer product i along all other products and the purchase probability of product i takes its smallest possible value. Thus, it is enough to enforce the market share constraints for product i only when $\frac{v_i}{1+\sum_{j\in\mathcal{N}} v_j}<\beta^i$. If $\frac{v_i}{1+\sum_{j\in\mathcal{N}} v_j}\geq\beta^i$, then we simply set the market share threshold of product i to zero. Using $(x_0^*, \boldsymbol{x}^*)$ to denote an optimal solution to the Sales-Based problem for the deterministic variant, we define the assortment $S^* = \{i \in \mathcal{N} : x_i^* > 0\}$. Let $p^* = 1, \ldots, L_0$ be such that the grid point corresponding to p^* is the largest one that does not exceed x_0^*. In other words, we have $\overline{x}_0(p^*) \leq x_0^* \leq (1+\rho)\,\overline{x}_0(p^*)$. We claim that the solution S^* is feasible to problem (16) with $p = p^*$. We have $\overline{x}_0(p^*) + \sum_{i \in S^*} v_i\,\overline{x}_0(p^*) \leq x_0^* + \sum_{i \in S^*} v_i\,x_0^* = x_0^* + \sum_{i \in \mathcal{N}} x_i^* \leq 1$, where the fist inequality uses the definition of the grid point p^*, the equality is by the fact that we have $x_i^* \in \{0, v_i\,x_0^*\}$ and the second inequality holds because the solution $(x_0^*, \boldsymbol{x}^*)$ is feasible to the Sales-Based problem. Thus, the solution S^* satisfies the constraint in problem (16) with $p = p^*$.

To check that $S^* \subseteq \mathcal{N}(p^*)$, if $i \notin \overline{\mathcal{K}}$ and $i \in S^*$, then we have $\frac{v_i}{1+\sum_{j \in \mathcal{N}} v_j} \geq \beta^i$ by the definition of $\overline{\mathcal{K}}$ right before (3) and noting that $\mathcal{N}^k = \{k\}$, so we set β^i to zero by the discussion at the beginning of this paragraph. If, on the other hand, $i \in \overline{\mathcal{K}}$ and $i \in S^*$, then $\beta^i \leq x_i^* = v_i\,x_0^*$ by the first and third constraints in the Sales-Based problem, so $x_0^* \geq \frac{\beta^i}{v_i}$. Furthermore, noting that $i \in \overline{\mathcal{K}}$, by (3), $\frac{\beta^i}{v_i} \in \mathsf{Grid}_0$. In this case, because $\overline{x}_0(p^*)$ is the largest grid point that does not exceed x_0^*, using the fact that $x_0^* \geq \frac{\beta^i}{v_i}$ and $\frac{\beta^i}{v_i} \in \mathsf{Grid}_0$, we obtain $\overline{x}_0(p^*) \geq \frac{\beta^i}{v_i}$. Thus, if $i \in \overline{\mathcal{K}}$ and $i \in S^*$, then $\overline{x}_0(p^*) \geq \frac{\beta^i}{v_i}$, whereas if $i \notin \overline{\mathcal{K}}$ and $i \in S^*$, then $\beta^i = 0$. Thus, it follows that $v_i\,\overline{x}_0(p^*) \geq \beta^i$ for all $i \in S^*$, in which case, the definition of $\mathcal{N}(p^*)$ implies that $S^* \subseteq \mathcal{N}(p^*)$. Therefore, the claim holds.

The solution $\widehat{\boldsymbol{h}}$ offers the assortment $\widehat{S}(\widehat{p})$ with probability one, so $\sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} r_i\,\phi_i(S)\,\widehat{h}(S) = \sum_{i \in \mathcal{N}} r_i\,\phi_i(\widehat{S}(\widehat{p})) \geq \sum_{i \in \mathcal{N}} r_i\,\phi_i(\widehat{S}(p^*)) = \sum_{i \in \widehat{S}(p^*)} r_i\,v_i/(1 + \sum_{j \in \widehat{S}(p^*)} v_j)$, where the inequality uses the definition of $\widehat{p}$ right after (16). Because $\widehat{S}(p^*)$ is feasible to problem (16) with $p = p^*$, we get $\overline{x}_0(p^*) + \sum_{i \in \widehat{S}(p^*)} v_i\,\overline{x}_0(p^*) \leq 1$, so $\overline{x}_0(p^*) \leq 1/(1 + \sum_{i \in \widehat{S}(p^*)} v_i)$. In this case, we continue the last chain of inequalities as $\sum_{i \in \widehat{S}(p^*)} r_i\,v_i/(1 + \sum_{j \in \widehat{S}(p^*)} v_j) \geq \sum_{i \in \widehat{S}(p^*)} r_i\,v_i\,\overline{x}_0(p^*) \geq \frac{1}{1+\rho} \sum_{i \in S^*} r_i\,v_i\,\overline{x}_0(p^*) \geq \frac{1}{(1+\rho)^2} \sum_{i \in S^*} r_i\,v_i\,x_0^*$, where the second inequality holds because $\widehat{S}(p^*)$ is a $(\frac{1}{1+\rho})$-approximate and S^* is a feasible solution to (16) with $p = p^*$, whereas the third inequality uses the definition of p^*. Lastly, $\sum_{i \in S^*} r_i\,v_i\,x_0^* = \sum_{i \in \mathcal{N}} r_i\,x_i^* = \mathsf{opt}$, where we use the fact that $x_i^* \in \{0, v_i\,x_0^*\}$.

We get $\sum_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} r_i\,\phi_i(S)\,\widehat{h}(S) \geq \frac{1}{(1+\rho)^2}\,\mathsf{opt} \geq (1 - 2\rho)\,\mathsf{opt}$ by the discussion in the previous paragraph. By Lemma H.1, $\widehat{\boldsymbol{h}}$ is feasible to the deterministic variant of the Market-Share problem. Calculating the running time to get the solution $\widehat{\boldsymbol{h}}$ with $\rho = \frac{\epsilon}{2}$ yields the desired result. ∎