# A New Dynamic Programming Decomposition Method for the Network Revenue Management Problem with Customer Choice Behavior

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**Abstract**

In this paper, we propose a new dynamic programming decomposition method for the network revenue management problem with customer choice behavior. The fundamental idea behind our dynamic programming decomposition method is to allocate the revenue associated with an itinerary among the different flight legs and to solve a single-leg revenue management problem for each flight leg in the airline network. The novel aspect of our approach is that it chooses the revenue allocations by solving an auxiliary optimization problem that takes the probabilistic nature of the customer choices into consideration. We compare our approach with two standard benchmark methods. The first benchmark method uses a deterministic linear programming formulation. The second benchmark method is a dynamic programming decomposition idea that is similar to our approach, but it chooses the revenue allocations in an ad hoc manner. We establish that our approach provides an upper bound on the optimal total expected revenue, and this upper bound is tighter than the ones obtained by the two benchmark methods. Computational experiments indicate that our approach provides significant improvements over the performances of the benchmark methods.

Keywords: Network revenue management; customer choice; approximate dynamic programming.

An interesting feature of network revenue management systems is the customer choice behavior, where a customer arriving into the system observes the itineraries that are available for purchase and makes a choice among the itineraries that can satisfy its needs. Up until recently, this customer choice behavior has been largely ignored and many network revenue management models followed the assumption that each customer arrives into the system with the intention of purchasing a fixed itinerary. If this itinerary is available for purchase, then the customer purchases it. Otherwise, the customer leaves the system without purchasing anything. However, modeling the customer choice behavior is particularly important nowadays with the online sales channels offering easy access to a large variety of itineraries.

Incorporating the customer choice behavior into the network revenue management problem has recently started seeing attention in the literature. To address this issue, Liu and van Ryzin (2008) propose a deterministic linear program that is formulated under the assumption that the customer choices are deterministic and the itineraries can be sold in fractional amounts. This linear program includes one constraint for each flight leg and the right sides of these constraints are the remaining leg capacities. As a result, we can use the optimal values of the dual variables associated with these constraints to estimate the opportunity cost of a unit of capacity. As shown by Liu and van Ryzin (2008), this observation allows us to extend the traditional bid pricing and dynamic programming decomposition methods to deal with the customer choice behavior.

In this paper, we particularly focus on the dynamic programming decomposition method for the network revenue management problem with customer choice behavior. This method was proposed by Liu and van Ryzin (2008). Throughout the paper, we refer to this method as LvR decomposition, where the acronym stands for the initials of the authors. The fundamental idea behind LvR decomposition is to allocate the revenue associated with an itinerary among the different flight legs and to solve a single-leg revenue management problem for each flight leg in the airline network. In this case, we sum up the value functions obtained from the single-leg revenue management problems to construct value function approximations for the network revenue management problem. This approach performs quite well in practice, but it has some shortcomings when viewed from a theoretical standpoint. To begin with, LvR decomposition uses the opportunity costs obtained from the aforementioned deterministic linear program when allocating the revenue associated with an itinerary among the different flight legs. This deterministic linear program ignores the probabilistic nature of the customer choices and we ideally would like to have a less crude approach for allocating the revenue associated with an itinerary. Furthermore, approximating the value functions for the network revenue management problem by using the sum of the value functions obtained from the single-leg revenue management problems is entirely based on a heuristic argument. In particular, the value function approximations used by LvR decomposition are not necessarily upper or lower bounds on the exact value functions.

In this paper, we propose an alternative dynamic programming decomposition method to address some of the shortcomings of LvR decomposition. The novel aspect of our decomposition approach is that when allocating the revenue associated with an itinerary among the different flight legs, we view the revenue allocations as decision variables, and choose the revenue allocations by solving an auxiliary optimization problem. Throughout the paper, we refer to our dynamic programming decomposition method as AP decomposition, where the acronym stands for auxiliary problem. When compared with LvR decomposition, AP decomposition has several advantages. To begin with, unlike the deterministic linear program used by LvR decomposition, the auxiliary optimization problem used by AP decomposition takes the probabilistic nature of the customer choices into consideration. Furthermore, we can show that the value function approximations used by AP decomposition are upper bounds on the exact value functions. Another useful feature of AP decomposition becomes apparent when we want to compute upper bounds on the optimal total expected revenue. As shown by Liu and van Ryzin (2008), the optimal objective value of the deterministic linear program provides one such upper bound. Similarly, Zhang and Adelman (2009) show that it is also possible to obtain an upper bound on the optimal total expected revenue by building on LvR decomposition. It turns out that AP decomposition provides an upper bound on the optimal total expected revenue as well and the upper bound obtained by AP decomposition is provably tighter than the ones obtained by the deterministic linear program and LvR decomposition. Finally, our computational experiments indicate that the total expected revenues obtained by AP decomposition can provide significant improvements over those obtained by the deterministic linear program and LvR decomposition.

Incorporating the customer choice behavior into revenue management problems is an active area of research. There are several models that deal with a single flight leg or a number of parallel flight legs that operate between the same origin-destination pair. Belobaba and Weatherford (1996) propose extensions of the expected marginal seat revenue heuristic of Belobaba (1987) on a single flight leg to capture the possibility that a customer purchases a more expensive fare level when the cheaper fare level is not available. Talluri and van Ryzin (2004*b*) study the single-leg revenue management problem under the customer choice behavior and show that generalized protection level policies are optimal under reasonable conditions. Karaesmen and van Ryzin (2004) propose an overbooking model for dealing with multiple flight legs that serve as substitutes of each other, which is the case when we have parallel flight legs that operate between the same origin-destination pair. They develop a stochastic approximation method to compute overbooking limits that dictate by how much the number of accepted reservations should exceed the physically available seats. Zhang and Cooper (2005) also consider parallel flight legs that operate between the same origin-destination pair. They construct upper and lower bounds on the value functions when the customers are allowed to make a choice among the parallel flight legs. They ultimately use these bounds to construct tractable control policies.

There are also several models that take place over an airline network. The paper by Gallego, Iyengar, Phillips and Dubey (2004) is particularly noteworthy from the standpoint of incorporating the customer choice behavior into network revenue management models. The authors consider flexible itineraries that allow the customers to buy a reservation between a particular origin-destination pair in advance, but choose the specific itinerary just before the departure time. They analyze the benefit from offering flexible itineraries by formulating a deterministic linear program that approximates the optimal total expected revenue. This deterministic linear program captures the fact that the choices of the customers are driven not only by their personal preferences, but also by the availability of the different itineraries in the market. Liu and van Ryzin (2008) build on the deterministic linear program and extend the traditional bid pricing and dynamic programming decomposition methods to deal with the customer choice behavior. On one hand, the model developed by Liu and van Ryzin (2008) does not incorporate flexible products and it can be considered as a restriction of the one in Gallego et al. (2004). On the other hand, Liu and van Ryzin (2008) construct realistic dynamic control policies to decide which itineraries to make available to the customers over time. Topaloglu (2009) proposes a Lagrangian relaxation strategy to decompose the network revenue management problem by the flight legs. AP decomposition that we develop in this paper can be visualized as an extension of his strategy to deal with the customer choice behavior. However, it is not possible to derive our AP decomposition simply by using the Lagrangian relaxation ideas of Topaloglu (2009), since the probabilities that govern the evolution of the system in the presence of the customer choice behavior depend on what itineraries are offered to the customers. As a result, our derivations resort to entirely new arguments. Zhang and Adelman (2009) construct value function approximations by using the linear programming formulation of the Markov decision process that characterizes the network revenue management problem. The numbers of decision variables and constraints in this linear program increase exponentially with the number of flight legs and they use linear value function approximations to make this problem tractable. Kunnumkal and Topaloglu (2008) build on the work of Zhang and Adelman (2009) to construct linear approximations to the value functions, but their approach tends to be more computationally tractable. Finally, van Ryzin and Vulcano (2008) propose a stochastic approximation method to compute good protection levels in the presence of the customer choice behavior. The important aspect of their work is that it avoids parametric assumptions regarding the model that governs the customer choices.

We make the following research contributions in this paper. 1) We propose a new decomposition approach for the network revenue management problem with customer choice behavior. The fundamental idea behind our approach is to allocate the revenue associated with an itinerary among the different flight legs and solve a single-leg revenue management problem for each flight leg. The novel aspect of our approach is that we view the revenue allocations as decision variables and choose the revenue allocations by solving an auxiliary optimization problem that takes the probabilistic nature of the customer choices into consideration. 2) We show that our approach provides upper bounds on the value functions. This result naturally implies that our approach provides an upper bound on the optimal total expected revenue. 3) We show that the upper bound obtained by our approach is tighter than the ones obtained by the deterministic linear program and LvR decomposition. 4) Our computational experiments indicate that the total expected revenues obtained by our approach can significantly improve those that are obtained by the deterministic linear program and LvR decomposition.

The rest of the paper is organized as follows. In Section 1, we formulate the network revenue management problem with customer choice behavior as a dynamic program. In Section 2, we describe our decomposition approach and formulate an auxiliary optimization problem to choose the revenue allocations. In Section 3, we show that our decomposition approach provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the ones obtained by the deterministic linear program and LvR decomposition. In Section 4, we show that the objective function of the auxiliary optimization problem is convex and it is easy to make projections onto the feasible set of this problem. These results allow us to solve the auxiliary optimization problem by using subgradient search. In Section 5, we present computational experiments.

# 1 Problem Formulation

We have a set of flight legs to serve the customers who arrive over time with the intention of purchasing itineraries. At each time period, we need to decide which itineraries to offer to the customers. An arriving customer reviews the offered itineraries and purchases at most one of them according to a probability distribution defined over the set of offered itineraries. A sold itinerary generates a revenue and consumes the capacities on the relevant flight legs. We are interested in maximizing the total expected revenue over the decision horizon.

The set of flight legs in the airline network is $\mathcal{L}$ and the set of itineraries that we can offer to the customers is $\mathcal{J}$. An itinerary $j$ is characterized by the pair $(r_j, \mathcal{L}_j)$, where $r_j$ is the revenue that we generate by selling a ticket for itinerary $j$ and $\mathcal{L}_j$ is the set of connecting flight legs that are used by itinerary $j$. It is important to note that there can be multiple itineraries that use the same set of connecting flight legs and are offered at different revenues. This observation allows us to model multiple fare classes. Furthermore, there can be multiple itineraries that connect the same origin-destination pair through different sets of connecting flight legs, which is the case when the airline offers multiple routes to travel between the same origin-destination pair. In certain settings, an itinerary is referred to as an origin-destination pair and fare class combination, but we prefer not to use this terminology since it does not emphasize that an itinerary is characterized not only by its origin-destination pair and fare class, but also by its set of connecting flight legs. The initial capacity on flight leg $i$ is $c_i$. If a customer purchases itinerary $j$, then we generate a revenue of $r_j$ and consume $a_{ij}$ units of capacity on flight leg $i$. We note that $a_{ij}$ can take values that are greater than one when we are interested in modeling the possibility of group reservations. In the presence of group reservations, however, an itinerary $j$ is characterized by the triplet $(r_j, \mathcal{L}_j, s_j)$, where $r_j$ and $\mathcal{L}_j$ are as defined above and $s_j$ is the size of the group corresponding to itinerary $j$. Naturally, if itinerary $j$ does not use flight leg $i$, then we have $a_{ij} = 0$, whereas if $i \in \mathcal{L}_j$, then we have $a_{ij} > 0$.

The problem takes place over the finite decision horizon $\mathcal{T} = \{1, \ldots, \tau\}$ and all flight legs depart at time period $\tau + 1$. We assume that the time periods correspond to small intervals of time so that there is at most one customer arrival at each time period. The probability that there is a customer arrival at a time period is $\lambda$. We use the vector $u_t = \{u_{jt} : j \in \mathcal{J}\} \in \{0,1\}^{|\mathcal{J}|}$ to denote the set of itineraries that we offer to the customers at time period $t$ with the interpretation that $u_{jt} = 1$ if we offer itinerary $j$ at time period $t$ and $u_{jt} = 0$ otherwise. If the set of itineraries that we offer to the customers at time period $t$ is given by $u_t$, then an arriving customer purchases itinerary $j$ with probability $P_j(u_t)$. Naturally, if $u_{jt} = 0$, then we have $P_j(u_t) = 0$. We use $P_\phi(u_t) = 1 - \sum_{j \in \mathcal{J}} P_j(u_t)$ to denote the probability that a customer purchases nothing when the set of itineraries that we offer to the customers at time period $t$ is given by $u_t$. We assume that the customer arrivals at different time periods and the purchasing decisions of different customers are independent. As evident from our notation, we also assume that the customer arrival and purchase probabilities do not depend on the specific time periods. However, this assumption is only for notational brevity and it is straightforward to make these probabilities dependent on the time period.

We use $x_{it}$ to denote the remaining capacity on flight leg $i$ at time period $t$ so that the vector

$x_t = \{x_{it} : i \in \mathcal{L}\}$ captures the state of the leg capacities at time period $t$. Since we can offer an itinerary only if there is enough remaining capacity on all of the flight legs that are used by this itinerary, the set of itineraries that we can offer to the customers at time period $t$ is given by

$$\mathcal{U}(x_t) = \{u_t \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, u_{jt} \le x_{it} \quad \forall\, i \in \mathcal{L},\ j \in \mathcal{J}\}.$$

We can formulate the problem as a dynamic program by using $x_t$ as the state variable. Letting $C = \max_{i \in \mathcal{L}}\{c_i\}$ and $\mathcal{C} = \{0, \ldots, C\}$, we use $\mathcal{C}^{|\mathcal{L}|}$ as the state space. Using $e_i$ to denote the $|\mathcal{L}|$-dimensional unit vector with a one in the component corresponding to $i \in \mathcal{L}$, we can find the optimal policy by computing the value functions through the optimality equation

$$\begin{aligned} V_t(x_t) &= \max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} \lambda\, P_j(u_t) \Big[ r_j + V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) \Big] + \Big[ 1 - \lambda + \lambda\, P_\phi(u_t) \Big] V_{t+1}(x_t) \right\} \\ &= \max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} \lambda\, P_j(u_t) \Big[ r_j + V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) - V_{t+1}(x_t) \Big] \right\} + V_{t+1}(x_t), \qquad (1) \end{aligned}$$

where the second equality follows from the fact that $P_\phi(u_t) = 1 - \sum_{j \in \mathcal{J}} P_j(u_t)$. For notational brevity, we assume that $\lambda = 1$ throughout the rest of the paper. We note that this assumption is without loss of generality, since assuming $\lambda = 1$ is equivalent to letting $\tilde{P}_j(u_t) = \lambda\, P_j(u_t)$ and working with the probabilities $\{\tilde{P}_j(u_t) : j \in \mathcal{J}\}$ instead of $\{P_j(u_t) : j \in \mathcal{J}\}$.

Since the number of possible values for the state variable $x_t$ in the optimality equation in (1) grows exponentially with the number of flight legs, it is quite difficult to compute the value functions through this optimality equation. In the next two sections, we describe several approaches to approximate the value functions.

## 2 Decomposing the Network Revenue Management Problem

In this section, we present an approximate method that decomposes the optimality equation in (1) by the flight legs. This method corresponds to AP decomposition that we mention in the introduction section. To this end, we begin by allocating the revenue associated with an itinerary among the different flight legs. In particular, we let $\alpha_{ijt}$ be the portion of the revenue generated from selling itinerary $j$ at time period $t$ that is allocated to flight leg $i$. We do not specify yet how the revenue allocations are chosen, but they should clearly satisfy

$$\sum_{i \in \mathcal{L}} \alpha_{ijt} = r_j \qquad \forall\, j \in \mathcal{J},\ t \in \mathcal{T}. \qquad (2)$$

The revenue allocations $\{\alpha_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ immediately allow us to formulate a single-leg revenue management problem for each flight leg in the airline network. In the single-leg revenue management problem that takes place over flight leg $i$, if a customer purchases itinerary $j$ at time period $t$, then we generate a revenue of $\alpha_{ijt}$ and consume $a_{ij}$ units of capacity. We use $z_{it} = \{z_{ijt} : j \in \mathcal{J}\} \in \{0,1\}^{|\mathcal{J}|}$ to denote the set of itineraries that we offer to the customers at time period $t$ in the single-leg revenue management problem that takes place over flight leg $i$. Since we focus only on flight leg $i$, the set of itineraries that we can offer to the customers at time period $t$ is given by

$$\mathcal{U}_i(x_{it}) = \{z_{it} \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, z_{ijt} \le x_{it} \quad \forall\, j \in \mathcal{J}\}.$$

In this case, following the same argument that we use to obtain the optimality equation in (1) but focusing only on flight leg $i$, the single-leg revenue management problem that takes place over flight leg $i$ can be formulated as a dynamic program as

$$v_{it}(x_{it}\,|\,\alpha) = \max_{z_{it} \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} P_j(z_{it}) \Big[ \alpha_{ijt} + v_{i,t+1}(x_{it} - a_{ij}\,|\,\alpha) - v_{i,t+1}(x_{it}\,|\,\alpha) \Big] \right\} + v_{i,t+1}(x_{it}\,|\,\alpha), \quad (3)$$

where the argument $\alpha = \{\alpha_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ in the value functions emphasizes that the solution to the optimality equation above depends on the revenue allocations.

It is possible to show that we obtain upper bounds on the value functions for the network revenue management problem by using the approach outlined above. In other words, it is possible to show that we have $V_t(x_t) \le \sum_{i \in \mathcal{L}} v_{it}(x_{it}\,|\,\alpha)$ for all $x_t \in \mathcal{C}^{|\mathcal{L}|}$, $t \in \mathcal{T}$ as long as $\{\alpha_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ satisfy (2). Nevertheless, we can tighten this bound further by the following intuitive observation. We ideally would like to offer the same set of itineraries in the single-leg revenue management problems that take place over the different flight legs, as the network revenue management problem needs one set of itineraries at each time period to offer to the customers. However, the single-leg revenue management problems that take place over the different flight legs do not try to coordinate their decisions. For example, we may offer itineraries $\{j_1, j_2\}$ in the problem that takes place over flight leg $i_1$, but at the same time, offer itineraries $\{j_3, j_4\}$ in the problem that takes place over flight leg $i_2$. We propose including a penalty term in the optimality equation in (3) to penalize the discrepancies of the decisions in the single-leg revenue management problems for the different flight legs. In particular, we let $\beta = \{\beta_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ be penalty parameters that satisfy

$$\sum_{i \in \mathcal{L}} \beta_{ijt} = 0 \qquad \forall\, j \in \mathcal{J},\ t \in \mathcal{T} \quad (4)$$

and add the penalty term $\sum_{j \in \mathcal{J}} \beta_{ijt}\, z_{ijt}$ to the optimality equation in (3). Similar to the revenue allocations, we do not specify yet how the penalty parameters are chosen and the reason for imposing the condition in (4) will be clear shortly. With the addition of the penalty term, the single-leg revenue management problem that takes place over flight leg $i$ takes the form

$$v_{it}(x_{it}\,|\,\alpha,\beta) = \max_{z_{it} \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} P_j(z_{it}) \Big[ \alpha_{ijt} + v_{i,t+1}(x_{it} - a_{ij}\,|\,\alpha,\beta) - v_{i,t+1}(x_{it}\,|\,\alpha,\beta) \Big] + \sum_{j \in \mathcal{J}} \beta_{ijt}\, z_{ijt} \right\} + v_{i,t+1}(x_{it}\,|\,\alpha,\beta). \quad (5)$$

The next proposition shows that we obtain upper bounds on the value functions for the network revenue management problem by solving the optimality equation in (5). The proof of this proposition also gives an algebraic justification for imposing the condition in (4).

**Proposition 1** *If $(\alpha, \beta)$ satisfies the conditions in (2) and (4), then for all $x_t \in \mathcal{C}^{|\mathcal{L}|}$, $t \in \mathcal{T}$, we have $V_t(x_t) \le \sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha, \beta)$.*

**Proof** We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t+1$, we let $\hat{u}_t = \{\hat{u}_{jt} : j \in \mathcal{J}\}$ be the optimal solution to problem (1). Using (2) and (4), we obtain

$$\begin{aligned}
&\sum_{j \in \mathcal{J}} P_j(\hat{u}_t) \Big[ r_j + V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) - V_{t+1}(x_t) \Big] + V_{t+1}(x_t) \\
&= \sum_{j \in \mathcal{J}} P_j(\hat{u}_t) \Big[ \sum_{i \in \mathcal{L}} \alpha_{ijt} + V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i) \Big] + \sum_{j \in \mathcal{J}} \Big[ \sum_{i \in \mathcal{L}} \beta_{ijt} \Big] \hat{u}_{jt} + \Big[ 1 - \sum_{j \in \mathcal{J}} P_j(\hat{u}_t) \Big] V_{t+1}(x_t) \\
&\le \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} P_j(\hat{u}_t) \Big[ \alpha_{ijt} + v_{i,t+1}(x_{it} - a_{ij} \,|\, \alpha, \beta) \Big] + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \beta_{ijt}\, \hat{u}_{jt} + \sum_{i \in \mathcal{L}} \Big[ 1 - \sum_{j \in \mathcal{J}} P_j(\hat{u}_t) \Big] v_{i,t+1}(x_{it} \,|\, \alpha, \beta) \\
&\le \sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha, \beta),
\end{aligned}$$

where the first inequality follows from the induction assumption and the second inequality follows from the fact that $\hat{u}_t$ is a feasible but not necessarily an optimal solution to problem (5). The result follows by noting that the first expression in the chain of inequalities above is equal to $V_t(x_t)$. □

In addition to the algebraic justification in the proof of Proposition 1, the intuitive reason for imposing the condition in (4) on the penalty parameters is that if we make the same decision for itinerary $j$ in all of the single-leg revenue management problems (that is, we have $z_{ijt} = z_{ljt}$ for all $i, l \in \mathcal{L}$), then we have $\sum_{i \in \mathcal{L}} \beta_{ijt}\, z_{ijt} = 0$ so that the total penalty incurred for itinerary $j$ by all of the flight legs is equal to zero. We also emphasize that since the optimality equation in (3) is a special case of (5) that is obtained by setting the penalty parameters to zero, Proposition 1 shows that $V_t(x_t) \le \sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha)$ for all $x_t \in \mathcal{C}^{|\mathcal{L}|}$, $t \in \mathcal{T}$.

Since the initial leg capacities are $c = \{c_i : i \in \mathcal{L}\}$, the optimal total expected revenue for the network revenue management problem is $V_1(c)$. Proposition 1 implies that $\sum_{i \in \mathcal{L}} v_{i1}(c_i \,|\, \alpha, \beta)$ provides an upper bound on $V_1(c)$ as long as $(\alpha, \beta)$ satisfies (2) and (4). To obtain the tightest possible upper bound on the optimal total expected revenue, we propose solving the problem

$$\min_{(\alpha,\beta) \in \mathcal{P}} \left\{ \sum_{i \in \mathcal{L}} v_{i1}(c_i \,|\, \alpha, \beta) \right\}, \tag{6}$$

where the feasible set $\mathcal{P}$ is defined as

$$\begin{aligned}
\mathcal{P} = \Big\{ (\alpha, \beta) : &\sum_{i \in \mathcal{L}} \alpha_{ijt} = r_j \qquad \forall\, j \in \mathcal{J},\ t \in \mathcal{T} \\
&\sum_{i \in \mathcal{L}} \beta_{ijt} = 0 \qquad \forall\, j \in \mathcal{J},\ t \in \mathcal{T} \Big\}.
\end{aligned}$$

By Proposition 1 and the preceding discussion, the optimal objective value of problem (6) provides an upper bound on the optimal total expected revenue. We also note that problem (6) gives us a concrete method to choose the revenue allocations and the penalty parameters. We refer to problem (6) as the *auxiliary optimization problem*, and the objective function of problem (6) as the *dual function*. AP decomposition takes its name from this auxiliary optimization problem. In Section 4, we show that the dual function is convex. We also show how to compute subgradients of the dual function and how to carry out projections onto the feasible set $\mathcal{P}$. By using this information, problem (6) can be solved through subgradient search; see Wolsey (1998). Before doing so, however, we take a detour in Section 3 and compare the upper bound on the optimal total expected revenue that we obtain by solving problem (6) with the upper bounds obtained by using other solution methods in the literature.

# 3 Comparing the Upper Bounds

In this section, we first describe two alternative solution methods for the network revenue management problem with customer choice behavior, both of which provide upper bounds on the optimal total expected revenue. After this, we show that the upper bound obtained by our AP decomposition is tighter than the upper bounds obtained by these two solution methods.

## 3.1 Deterministic Linear Program

One alternative solution method involves formulating a linear program under the assumption that the customer choices are deterministic and the itineraries can be sold in fractional amounts. The decision variables in this linear program are $\{h_t(\mathcal{S}) : \mathcal{S} \subset \mathcal{J},\ t \in \mathcal{T}\}$ with the interpretation that $h_t(\mathcal{S})$ is the frequency with which we offer the set of itineraries $\mathcal{S}$ at time period $t$. In this case, the expected revenue that we generate at time period $t$ is $\sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, r_j\, h_t(\mathcal{S})$, where with slight notational abuse, we use $P_j(\mathcal{S})$ to denote the probability that a customer purchases itinerary $j$ when we offer the set of itineraries $\mathcal{S}$. Similarly, the expected consumption of the capacity on flight leg $i$ at time period $t$ is $\sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, a_{ij}\, h_t(\mathcal{S})$. Therefore, we can use the optimal objective value of the problem

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, r_j\, h_t(\mathcal{S}) \tag{7}$$

$$\text{subject to} \quad \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S})\, a_{ij}\, h_t(\mathcal{S}) \le c_i \qquad \forall\, i \in \mathcal{L} \tag{8}$$

$$\sum_{\mathcal{S} \subset \mathcal{J}} h_t(\mathcal{S}) = 1 \qquad \forall\, t \in \mathcal{T} \tag{9}$$

$$h_t(\mathcal{S}) \ge 0 \qquad \forall\, \mathcal{S} \subset \mathcal{J},\ t \in \mathcal{T} \tag{10}$$

as an approximation to the optimal total expected revenue; see Liu and van Ryzin (2008). The number of decision variables in problem (7)-(10) increases exponentially with the number of itineraries, but Liu and van Ryzin (2008) show that solving this problem by using column generation is relatively easy when the customer choices are governed by the multinomial logit model with disjoint consideration sets. We briefly review this choice model in Section 5.

There are two uses of problem (7)-(10). First, we can use this problem to decide which set of itineraries to offer. In particular, if we let $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (8), then $\hat{\pi}_i$ essentially captures the opportunity cost of a unit of capacity on flight leg $i$. In this case, we can approximate $V_t(x_t)$ with a linear function of the form $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\pi}_i \, x_{it}$. Plugging this approximation into the right side of the optimality equation in (1) and noting that $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} \, e_i) = \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\pi}_i$, we can solve the problem

$$\max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} P_j(u_t) \Big[ r_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\pi}_i \Big] \right\} + \sum_{i \in \mathcal{L}} \hat{\pi}_i \, x_{it} \tag{11}$$

to decide which set of itineraries to offer to the customers at time period $t$. This idea is used in Zhang and Adelman (2009) and it parallels the traditional bid pricing approach for the network revenue management problem without customer choice behavior; see Talluri and van Ryzin (2004*a*).

Second, letting $Z_{LP}$ be the optimal objective value of problem (7)-(10), Liu and van Ryzin (2008) show that we have $V_1(c) \le Z_{LP}$. Therefore, $Z_{LP}$ provides an upper bound on the optimal total expected revenue and this information can be useful when assessing the optimality gap of a suboptimal control policy. Later in this section, we show that the upper bound on $V_1(c)$ obtained by solving problem (6) is provably tighter than the one provided by $Z_{LP}$ and this is one of the advantages of our approach.

### 3.2 Dynamic Programming Decomposition Method of Liu and van Ryzin (2008)

A second solution method is the dynamic programming decomposition method proposed by Liu and van Ryzin (2008). This solution method corresponds to LvR decomposition that we mention in the introduction section. The starting point for LvR decomposition is a simple linear programming duality argument on problem (7)-(10). In particular, letting $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (8) in problem (7)-(10), we choose a fixed flight leg $i$ and relax constraints (8) for all other flight legs by associating the dual multipliers $\{\hat{\pi}_l : l \in \mathcal{L} \setminus \{i\}\}$. In this case, linear programming duality implies that the linear program

$$\begin{aligned} Z_{LP} = \max \quad & \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \Big[ r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l \Big] h_t(\mathcal{S}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l \\ \text{subject to} \quad & (9), (10) \\ & \sum_{t \in \mathcal{T}} \sum_{\mathcal{S} \subset \mathcal{J}} \sum_{j \in \mathcal{S}} P_j(\mathcal{S}) \, a_{ij} \, h_t(\mathcal{S}) \le c_i \end{aligned}$$

has the same optimal objective value as problem (7)-(10). Ignoring the constant term $\sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$ in the objective function and comparing the problem above with problem (7)-(10), it is easy to see that the problem above is the deterministic linear program for the single-leg revenue management problem that takes place over flight leg $i$ and associates the revenue $r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l$ with itinerary $j$. Therefore, $Z_{LP} - \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$ is an upper bound on the optimal total expected revenue in the single-leg revenue management problem that takes place over flight leg $i$.

On the other hand, we can obtain the optimal total expected revenue in the single-leg revenue management problem that takes place over flight leg $i$ by solving the optimality equation

$$\vartheta_{it}(x_{it}) = \max_{z_{it} \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} P_j(z_{it}) \Big[ r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l + \vartheta_{i,t+1}(x_{it} - a_{ij}) - \vartheta_{i,t+1}(x_{it}) \Big] \right\} + \vartheta_{i,t+1}(x_{it}). \tag{12}$$

The optimality equation above is similar to the one in (3), but instead of using the revenue allocations $\{\alpha_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$, the optimality equation above assumes that $r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj}\, \hat{\pi}_l$ is the portion of the revenue associated with itinerary $j$ that is allocated to flight leg $i$. Since the optimal total expected revenue in the single-leg revenue management problem that takes place over flight leg $i$ is given by $\vartheta_{i1}(c_i)$, we have $\vartheta_{i1}(c_i) \le Z_{LP} - \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l\, c_l$ by the discussion in the previous paragraph. Furthermore, it is possible to use an induction argument over the time periods to show that $\vartheta_{i1}(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l\, c_l$ provides an upper bound on the optimal total expected revenue in the original network revenue management problem; see Proposition 3 in Zhang and Adelman (2009). In other words, we have $V_1(c) \le \vartheta_{i1}(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l\, c_l$. Therefore, we obtain $V_1(c) \le \vartheta_{i1}(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l\, c_l \le Z_{LP}$, which implies that we can solve the optimality equation in (12) to obtain an upper bound on the optimal total expected revenue, and this upper bound is tighter than the one provided by $Z_{LP}$. Furthermore, since the last inequality is satisfied for all $i \in \mathcal{L}$, we can take the minimum over all $i \in \mathcal{L}$ and use

$$\min_{i \in \mathcal{L}} \left\{ \vartheta_{i1}(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l\, c_l \right\}$$

as an upper bound on the optimal total expected revenue.

Besides providing an upper bound on the optimal total expected revenue, another use of LvR decomposition is that we can use $\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ as an approximation to $V_t(x_t)$. Therefore, we can replace $\{V_t(x_t) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ in the right side of the optimality equation in (1) with the value function approximations $\{\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it}) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ and solve the problem

$$\max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} P_j(u_t) \Big[ r_j + \sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it} - a_{ij}) - \sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it}) \Big] \right\} + \sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it}) \qquad (13)$$

to decide which set of itineraries to offer to the customers at time period $t$.

### 3.3 Comparison of the Solution Methods

An important shortcoming of the deterministic linear program is that it assumes that the customer choices are deterministic, whereas both AP and LvR decomposition try to address the randomness in the customer choices by using dynamic programming formulations. Furthermore, the control policy that we derive from the deterministic linear program solves problem (11) to decide which set of itineraries to offer to the customers at time period $t$. As mentioned above, problem (11) is obtained by using $\sum_{i \in \mathcal{L}} \hat{\pi}_i\, x_{it}$ as an approximation to $V_t(x_t)$. The value function approximation $\sum_{i \in \mathcal{L}} \hat{\pi}_i\, x_{it}$ essentially assumes that the marginal value of an additional unit of capacity on flight leg $i$ is constant at $\hat{\pi}_i$ and does not depend on the time period or the remaining capacity on the flight leg. On the other hand, AP and LvR decomposition respectively use the value function approximations $\sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha, \beta)$ and $\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$, both of which capture the fact that the marginal value of an additional unit of capacity on a flight leg depends on the time period and the remaining capacity on the flight leg.

There are several advantages of AP decomposition when compared with LvR decomposition. Any approach that decomposes the network revenue management problem into a sequence of single-leg revenue management problems can suffer from two sources of approximation error. First, the revenue allocations associated with an itinerary in the different single-leg revenue management problems may not sum up to the original revenue associated with the itinerary. We call this potential source of error as the *revenue decoupling effect*. Second, the single-leg revenue management problems may not be able to coordinate their decisions by considering their impacts on each other. We call this potential source of error as the *capacity decoupling effect*. Noting the optimality equation in (12), LvR decomposition assumes that $r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l$ is the portion of the revenue associated with itinerary $j$ that is allocated to flight leg $i$. Since we do not necessarily have $\sum_{i \in \mathcal{L}} [r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l] = r_j$, the revenue allocations associated with itinerary $j$ in the different single-leg revenue management problems may not sum up to the original revenue associated with itinerary $j$. Therefore, LvR decomposition may suffer from the revenue decoupling effect. On the other hand, AP decomposition assumes that $\alpha_{ijt}$ is the portion of the revenue associated with itinerary $j$ that is allocated to flight leg $i$ at time period $t$ and the revenue allocations satisfy $\sum_{i \in \mathcal{L}} \alpha_{ijt} = r_j$ for all $j \in \mathcal{J}, \; t \in \mathcal{T}$. Therefore, the revenue decoupling effect is not an issue for AP decomposition. Furthermore, AP decomposition alleviates the capacity decoupling effect by using the penalty parameters $\{\beta_{ijt} : i \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}\}$ to penalize the discrepancies of the decisions in the different single-leg revenue management problems. In contrast, LvR decomposition has no such mechanism and it ignores the capacity decoupling effect. Due to the fact that AP decomposition is more competent than LvR decomposition in dealing with the revenue and capacity decoupling effects, we shortly show in Proposition 2 that the upper bound on the optimal total expected revenue obtained by AP decomposition is tighter than the one obtained by LvR decomposition.

Another important distinction between AP and LvR decomposition lies in the value function approximations used by these two methods. As mentioned above, LvR decomposition uses $\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ as an approximation to $V_t(x_t)$. Although $\vartheta_{it}(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt}$ is an upper bound on $V_t(x_t)$, $\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ is not necessarily an upper or a lower bound on $V_t(x_t)$. Using $\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ as an approximation to $V_t(x_t)$ is based on a heuristic argument and it is disconcerting that the value function approximations that are used by LvR decomposition are different from the upper bounds that it comes up with. On the other hand, AP decomposition uses $\sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha, \beta)$ as an approximation to $V_t(x_t)$ and Proposition 1 shows that $\sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \alpha, \beta)$ is indeed an upper bound on $V_t(x_t)$.

The next proposition compares the upper bounds on the optimal total expected revenue obtained by AP decomposition, the deterministic linear program and LvR decomposition.

**Proposition 2** *We have*

$$V_1(c) \leq \min_{(\alpha,\beta) \in \mathcal{P}} \left\{ \sum_{i \in \mathcal{L}} v_{i1}(c_i \,|\, \alpha, \beta) \right\} \leq \min_{i \in \mathcal{L}} \left\{ \vartheta_{i1}(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l \right\} \leq Z_{LP}.$$

**Proof** The first inequality follows from Proposition 1 and Zhang and Adelman (2009) show the third inequality. Therefore, we only focus on the second inequality here. We fix flight leg $i$ and let $\hat{\alpha}_{ijt} = r_j - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l$ for all $j \in \mathcal{J}, \; t \in \mathcal{T}$, $\hat{\alpha}_{ljt} = a_{lj} \, \hat{\pi}_l$ for all $l \in \mathcal{L} \setminus \{i\}, \; j \in \mathcal{J}, \; t \in \mathcal{T}$ and $\hat{\beta}_{ljt} = 0$ for all $l \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}$. We clearly have $(\hat{\alpha}, \hat{\beta}) \in \mathcal{P}$. Our proof shows that $\sum_{l \in \mathcal{L}} v_{lt}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \leq$

$\vartheta_{it}(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt}$ for all $x_t \in \mathcal{C}^{|\mathcal{L}|}$, $t \in \mathcal{T}$. The result then follows from the fact that the flight leg $i$ is arbitrary and $(\hat{\alpha}, \hat{\beta}) \in \mathcal{P}$.

First, it is easy to see that $v_{it}(x_{it} \,|\, \hat{\alpha}, \hat{\beta}) = \vartheta_{it}(x_{it})$ for all $x_{it} \in \mathcal{C}$, $t \in \mathcal{T}$. In particular, if we let $\alpha_{ijt} = \hat{\alpha}_{ijt}$ and $\beta_{ijt} = \hat{\beta}_{ijt}$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$ in (5), then the optimality equations in (5) and (12) become identical. Second, we show that $v_{lt}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \leq \hat{\pi}_l \, x_{lt}$ for all $x_{lt} \in \mathcal{C}$, $l \in \mathcal{L} \setminus \{i\}$, $t \in \mathcal{T}$ by using induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period $t$, we let $\hat{z}_{lt}$ be an optimal solution to problem (5) when we solve this problem for flight leg $l$ with $(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})$. We have

$$\begin{aligned}
v_{lt}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) &= \sum_{j \in \mathcal{J}} P_j(\hat{z}_{lt}) \Big[ a_{lj} \, \hat{\pi}_l + v_{l,t+1}(x_{lt} - a_{lj} \,|\, \hat{\alpha}, \hat{\beta}) - v_{l,t+1}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \Big] + v_{l,t+1}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \\
&= \sum_{j \in \mathcal{J}} P_j(\hat{z}_{lt}) \Big[ a_{lj} \, \hat{\pi}_l + v_{l,t+1}(x_{lt} - a_{lj} \,|\, \hat{\alpha}, \hat{\beta}) \Big] + \Big[ 1 - \sum_{j \in \mathcal{J}} P_j(\hat{z}_{lt}) \Big] v_{l,t+1}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \\
&\leq \sum_{j \in \mathcal{J}} P_j(\hat{z}_{lt}) \Big[ a_{lj} \, \hat{\pi}_l + \hat{\pi}_l \, [x_{lt} - a_{lj}] \Big] + \Big[ 1 - \sum_{j \in \mathcal{J}} P_j(\hat{z}_{lt}) \Big] \hat{\pi}_l \, x_{lt} = \hat{\pi}_l \, x_{lt},
\end{aligned}$$

where the inequality follows from the induction assumption and the fact that we have $0 \leq x_{lt} - a_{lj} \in \mathcal{C}$ whenever $P_j(\hat{z}_{lt}) > 0$. Therefore, we have $v_{it}(x_{it} \,|\, \hat{\alpha}, \hat{\beta}) = \vartheta_{it}(x_{it})$ for all $x_{it} \in \mathcal{C}$, $t \in \mathcal{T}$ and $v_{lt}(x_{lt} \,|\, \hat{\alpha}, \hat{\beta}) \leq \hat{\pi}_l \, x_{lt}$ for all $x_{lt} \in \mathcal{C}$, $l \in \mathcal{L} \setminus \{i\}$, $t \in \mathcal{T}$ and the result follows. □

## 4 Solving the Auxiliary Optimization Problem

In this section, we show that $v_{i1}(c_i \,|\, \alpha, \beta)$ is a convex function of $(\alpha, \beta)$ for all $i \in \mathcal{L}$. Since the dual function is of the form $\sum_{i \in \mathcal{L}} v_{i1}(c_i \,|\, \alpha, \beta)$, this result implies that the dual function is convex, in which case, we can solve problem (6) through subgradient search. To this end, we let $z_{it}(x_{it} \,|\, \alpha, \beta) = \{ z_{ijt}(x_{it} \,|\, \alpha, \beta) : j \in \mathcal{J} \}$ be an optimal solution to problem (5). We use the arguments $x_{it}$, $\alpha$ and $\beta$ to emphasize that the optimal solution to problem (5) depends on the remaining capacity on flight leg $i$, the revenue allocations and the penalty parameters. In this case, (5) can be written as

$$\begin{aligned}
v_{it}(x_{it} \,|\, \alpha, \beta) = \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \alpha_{ijt} &+ \sum_{k \in \mathcal{C}} \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \mathbf{1}(x_{it} - a_{ij} = k) \, v_{i,t+1}(k \,|\, \alpha, \beta) \\
&- \sum_{k \in \mathcal{C}} \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \mathbf{1}(x_{it} = k) \, v_{i,t+1}(k \,|\, \alpha, \beta) + \sum_{j \in \mathcal{J}} \beta_{ijt} \, z_{ijt}(x_{it} \,|\, \alpha, \beta) \\
&+ \sum_{k \in \mathcal{C}} \mathbf{1}(x_{it} = k) \, v_{i,t+1}(k \,|\, \alpha, \beta),
\end{aligned}$$

where $\mathbf{1}(\cdot)$ is the indicator function. Collecting the terms, we obtain

$$\begin{aligned}
v_{it}(x_{it} \,|\, \alpha, \beta) = \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \alpha_{ijt} &+ \sum_{j \in \mathcal{J}} \beta_{ijt} \, z_{ijt}(x_{it} \,|\, \alpha, \beta) \\
&+ \sum_{k \in \mathcal{C}} \Bigg\{ \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \mathbf{1}(x_{it} - a_{ij} = k) \\
&\qquad - \sum_{j \in \mathcal{J}} P_j(z_{it}(x_{it} \,|\, \alpha, \beta)) \, \mathbf{1}(x_{it} = k) + \mathbf{1}(x_{it} = k) \Bigg\} v_{i,t+1}(k \,|\, \alpha, \beta). \qquad (14)
\end{aligned}$$

Although we make use of (14) only algebraically, we note that we can interpret the expression in the curly brackets above as the probability that the remaining leg capacity at the next time period is $k$ given that the remaining leg capacity at the current time period is $x_{it}$ and the set of itineraries that we offer is given by $z_{it}(x_{it} \,|\, \alpha, \beta)$.

To write (14) in matrix notation, we define some vectors and matrices. We do not differentiate between column and row vectors since the difference will always be clear from the context. We let $v_{it}(\alpha, \beta)$ be the vector $\{v_{it}(x_{it} \,|\, \alpha, \beta) : x_{it} \in \mathcal{C}\}$, $\alpha_{it}$ be the vector $\{\alpha_{ijt} : j \in \mathcal{J}\}$ and $\beta_{it}$ be the vector $\{\beta_{ijt} : j \in \mathcal{J}\}$. We also let $P_{it}(\alpha, \beta)$ be the $|\mathcal{C}| \times |\mathcal{J}|$ dimensional matrix whose $(x_{it}, j)$th component is $P_j(z_{it}(x_{it} \,|\, \alpha, \beta))$, $Z_{it}(\alpha, \beta)$ be the $|\mathcal{C}| \times |\mathcal{J}|$ dimensional matrix whose $(x_{it}, j)$th component is $z_{ijt}(x_{it} \,|\, \alpha, \beta)$ and $Q_{it}(\alpha, \beta)$ be the $|\mathcal{C}| \times |\mathcal{C}|$ dimensional matrix whose $(x_{it}, k)$th component is the expression in the curly brackets in (14). With these definitions, we can write (14) in matrix notation as

$$v_{it}(\alpha, \beta) = P_{it}(\alpha, \beta)\, \alpha_{it} + Z_{it}(\alpha, \beta)\, \beta_{it} + Q_{it}(\alpha, \beta)\, v_{i,t+1}(\alpha, \beta). \tag{15}$$

The next proposition shows that $v_{i1}(c_i \,|\, \alpha, \beta)$ has a subgradient when visualized as a function of $(\alpha, \beta)$.

**Proposition 3** *If we let $(\alpha, \beta)$ and $(\hat{\alpha}, \hat{\beta})$ be two sets of revenue allocations and penalty parameters, then for all $i \in \mathcal{L}$, $t \in \mathcal{T}$, we have*

$$\begin{aligned} v_{it}(\alpha, \beta) - v_{it}(\hat{\alpha}, \hat{\beta}) \geq\ & P_{it}(\hat{\alpha}, \hat{\beta})\, [\alpha_{it} - \hat{\alpha}_{it}] + Z_{it}(\hat{\alpha}, \hat{\beta})\, [\beta_{it} - \hat{\beta}_{it}] \\ & + Q_{it}(\hat{\alpha}, \hat{\beta})\, P_{i,t+1}(\hat{\alpha}, \hat{\beta})\, [\alpha_{i,t+1} - \hat{\alpha}_{i,t+1}] + Q_{it}(\hat{\alpha}, \hat{\beta})\, Z_{i,t+1}(\hat{\alpha}, \hat{\beta})\, [\beta_{i,t+1} - \hat{\beta}_{i,t+1}] \\ & + \ldots + Q_{it}(\hat{\alpha}, \hat{\beta})\, Q_{i,t+1}(\hat{\alpha}, \hat{\beta}) \ldots Q_{i,\tau-1}(\hat{\alpha}, \hat{\beta})\, P_{i\tau}(\hat{\alpha}, \hat{\beta})\, [\alpha_{i\tau} - \hat{\alpha}_{i\tau}] \\ & + Q_{it}(\hat{\alpha}, \hat{\beta})\, Q_{i,t+1}(\hat{\alpha}, \hat{\beta}) \ldots Q_{i,\tau-1}(\hat{\alpha}, \hat{\beta})\, Z_{i\tau}(\hat{\alpha}, \hat{\beta})\, [\beta_{i\tau} - \hat{\beta}_{i\tau}]. \end{aligned}$$

**Proof** We show the result by induction over the time periods. It is easy to show the result for the last time period. We assume that the result holds for time period $t+1$ and show that it holds for time period $t$. Since $z_{it}(x_{it} \,|\, \hat{\alpha}, \hat{\beta})$ is not necessarily an optimal solution to problem (5) when the revenue allocations and the penalty parameters are $(\alpha, \beta)$, (14) (or its equivalent in matrix notation in (15)) implies that

$$\begin{aligned} v_{it}(\hat{\alpha}, \hat{\beta}) &= P_{it}(\hat{\alpha}, \hat{\beta})\, \hat{\alpha}_{it} + Z_{it}(\hat{\alpha}, \hat{\beta})\, \hat{\beta}_{it} + Q_{it}(\hat{\alpha}, \hat{\beta})\, v_{i,t+1}(\hat{\alpha}, \hat{\beta}) \\ v_{it}(\alpha, \beta) &\geq P_{it}(\hat{\alpha}, \hat{\beta})\, \alpha_{it} + Z_{it}(\hat{\alpha}, \hat{\beta})\, \beta_{it} + Q_{it}(\hat{\alpha}, \hat{\beta})\, v_{i,t+1}(\alpha, \beta). \end{aligned}$$

Subtracting the two expressions above, we obtain

$$v_{it}(\alpha, \beta) - v_{it}(\hat{\alpha}, \hat{\beta}) \geq P_{it}(\hat{\alpha}, \hat{\beta})\, [\alpha_{it} - \hat{\alpha}_{it}] + Z_{it}(\hat{\alpha}, \hat{\beta})\, [\beta_{it} - \hat{\beta}_{it}] + Q_{it}(\hat{\alpha}, \hat{\beta})\, [v_{i,t+1}(\alpha, \beta) - v_{i,t+1}(\hat{\alpha}, \hat{\beta})].$$

The result follows by using the induction assumption in the right side of the expression above and noting that all of the components of the matrix $Q_{it}(\hat{\alpha}, \hat{\beta})$ are positive. □

If we let $\Pi_{it}(\alpha, \beta) = Q_{i1}(\alpha, \beta)\, Q_{i2}(\alpha, \beta) \ldots Q_{i,t-1}(\alpha, \beta)\, P_{it}(\alpha, \beta)$ with $\Pi_{i1}(\alpha, \beta) = P_{i1}(\alpha, \beta)$ and $\Psi_{it}(\alpha, \beta) = Q_{i1}(\alpha, \beta)\, Q_{i2}(\alpha, \beta) \ldots Q_{i,t-1}(\alpha, \beta)\, Z_{it}(\alpha, \beta)$ with $\Psi_{i1}(\alpha, \beta) = Z_{i1}(\alpha, \beta)$, then Proposition 3 implies that

$$\begin{aligned} v_{i1}(\alpha, \beta) \geq\ & v_{i1}(\hat{\alpha}, \hat{\beta}) + \Pi_{i1}(\hat{\alpha}, \hat{\beta})\, [\alpha_{i1} - \hat{\alpha}_{i1}] + \Psi_{i1}(\hat{\alpha}, \hat{\beta})\, [\beta_{i1} - \hat{\beta}_{i1}] + \Pi_{i2}(\hat{\alpha}, \hat{\beta})\, [\alpha_{i2} - \hat{\alpha}_{i2}] \\ & + \Psi_{i2}(\hat{\alpha}, \hat{\beta})\, [\beta_{i2} - \hat{\beta}_{i2}] + \ldots + \Pi_{i\tau}(\hat{\alpha}, \hat{\beta})\, [\alpha_{i\tau} - \hat{\alpha}_{i\tau}] + \Psi_{i\tau}(\hat{\alpha}, \hat{\beta})\, [\beta_{i\tau} - \hat{\beta}_{i\tau}]. \end{aligned}$$

Letting $\epsilon_i$ be the $|\mathcal{C}|$-dimensional unit vector with a one in the $c_i$th component, if we multiply the inequality above by $\epsilon_i$ and note that $v_{i1}(c_i \,|\, \alpha, \beta) = \epsilon_i \, v_{i1}(\alpha, \beta)$, then we obtain

$$v_{i1}(c_i \,|\, \alpha, \beta) \geq \; v_{i1}(c_i \,|\, \hat{\alpha}, \hat{\beta}) + \sum_{t \in \mathcal{T}} \epsilon_i \, \Pi_{it}(\hat{\alpha}, \hat{\beta}) \, [\alpha_{it} - \hat{\alpha}_{it}] + \sum_{t \in \mathcal{T}} \epsilon_i \, \Psi_{it}(\hat{\alpha}, \hat{\beta}) \, [\beta_{it} - \hat{\beta}_{it}].$$

The last inequality shows that $v_{i1}(c_i \,|\, \alpha, \beta)$ has a subgradient when visualized as a function of $(\alpha, \beta)$, and hence, $v_{i1}(c_i \,|\, \alpha, \beta)$ is convex. The dual function, being a sum of convex functions, is also convex. The last inequality also shows how to obtain a subgradient of the dual function. Therefore, we can solve problem (6) by using subgradient search.

When solving problem (6) by using subgradient search, we may need to project the iterates onto the feasible set $\mathcal{P}$. Carrying out this projection turns out to be quite easy. To illustrate, we assume that we want to project the revenue allocations $p = \{p_{ijt} : i \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}\}$ and the penalty parameters $q = \{q_{ijt} : i \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}\}$ onto the feasible set $\mathcal{P}$. This requires solving the problem

$$\min \quad \frac{1}{2} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} [\alpha_{ijt} - p_{ijt}]^2 + \frac{1}{2} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} [\beta_{ijt} - q_{ijt}]^2 \tag{16}$$

$$\text{subject to} \quad (2), (4). \tag{17}$$

Associating the Lagrange multipliers $\mu = \{\mu_{jt} : j \in \mathcal{J}, \; t \in \mathcal{T}\}$ and $\sigma = \{\sigma_{jt} : j \in \mathcal{J}, \; t \in \mathcal{T}\}$ with the two sets of constraints, the Lagrange function is

$$L(\alpha, \beta, \mu, \sigma) = \frac{1}{2} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} [\alpha_{ijt} - p_{ijt}]^2 + \frac{1}{2} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} [\beta_{ijt} - q_{ijt}]^2 + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{L}} \Big[ \sum_{i \in \mathcal{L}} \alpha_{ijt} - r_j \Big] \mu_{jt} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{L}} \Big[ \sum_{i \in \mathcal{L}} \beta_{ijt} \Big] \sigma_{jt}.$$

If we minimize the Lagrange function over $(\alpha, \beta)$, then the first order conditions imply that the minimizer $(\hat{\alpha}, \hat{\beta})$ is given by $\hat{\alpha}_{ijt} = p_{ijt} - \mu_{jt}$ and $\hat{\beta}_{ijt} = q_{ijt} - \sigma_{jt}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$. Plugging $(\hat{\alpha}, \hat{\beta})$ into (2) and (4), it is easy to see that the Lagrange multipliers should satisfy $\mu_{jt} = \big[ \sum_{i \in \mathcal{L}} p_{ijt} - r_j \big] / |\mathcal{L}|$ and $\sigma_{jt} = \sum_{i \in \mathcal{L}} q_{ijt} / |\mathcal{L}|$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$. Therefore, if we let $\hat{\alpha}_{ijt} = p_{ijt} - \mu_{jt} = p_{ijt} - \big[ \sum_{i \in \mathcal{L}} p_{ijt} - r_j \big] / |\mathcal{L}|$ and $\hat{\beta}_{ijt} = q_{ijt} - \sigma_{jt} = q_{ijt} - \sum_{i \in \mathcal{L}} q_{ijt} / |\mathcal{L}|$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$, then $(\hat{\alpha}, \hat{\beta})$ is the optimal solution to problem (16)-(17).

## 5 Computational Experiments

In this section, we numerically compare the performances of AP decomposition, the deterministic linear program and LvR decomposition. We begin by describing the benchmark solution methods and the experimental setup. After this, we present our computational results.

### 5.1 Benchmark Solution Methods

We compare the performances of the following three solution methods.

**AP Decomposition** This solution method corresponds to AP decomposition that we describe in Section 2. In particular, AP decomposition solves problem (6) to obtain an optimal solution $(\hat{\alpha}, \hat{\beta})$ and decides which itineraries to offer at time period $t$ by using $\{\sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \hat{\alpha}, \hat{\beta}) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ as approximations to $\{V_t(x_t) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ in the optimality equation in (1). We use subgradient search to solve problem (6) and use a step size of the form $1,500/\sqrt{k}$ at iteration $k$. We note that our step size selection does not guarantee convergence to an optimal solution, but it consistently provided good solutions and stable performance for our test problems.

**Deterministic Linear Program** This solution method corresponds to the deterministic linear program that we describe in Section 3. We refer to this solution method as DLP. Our practical implementation of DLP divides the decision horizon into five equal segments and resolves problem (7)-(10) at the beginning of each segment by replacing the right side of constraints (8) with the current remaining leg capacities and the set of time periods $\mathcal{T}$ with the current set of remaining time periods. In this case, DLP plugs the optimal values of the dual variables associated with constraints (8) into problem (11) to decide which itineraries to offer at time period $t$.

**LvR Decomposition** This solution method corresponds to LvR decomposition that we describe in Section 3. In particular, LvR decomposition solves the optimality equation in (12) and decides which itineraries to offer at time period $t$ by using $\{\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it}) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ as approximations to $\{V_t(x_t) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ in the optimality equation in (1). Therefore, the only difference between AP and LvR decomposition lies in the value function approximations that they use.

## 5.2 Experimental Setup

In all of our test problems, we assume that the customer choices are governed by the multinomial logit model with disjoint consideration sets. For brevity, we refer to this choice model simply as the logit model. The logit model assumes that there are multiple customer types and the customers of different types are interested in disjoint sets of itineraries. We denote the set of customer types by $\mathcal{G}$. At each time period, a customer of type $g$ arrives with probability $\lambda_g$. The set of itineraries that a customer of type $g$ is interested in is $\mathcal{J}_g$. We assume that $\mathcal{J}_{g_1} \cap \mathcal{J}_{g_2} = \varnothing$ whenever $g_1 \neq g_2$ so that the customers of different types are interested in disjoint sets of itineraries. For example, a customer type may be associated with a particular origin-destination pair. In this case, $\mathcal{J}_g$ corresponds to the set of itineraries that connect the origin-destination pair associated with customer type $g$ and a customer of type $g$ makes a choice within the set of itineraries $\mathcal{J}_g$. We note that there can be different itineraries with different connecting flight legs and fare classes that connect the same origin-destination pair. To describe the choice process, the logit model associates the positive preference weights $\{w_j : j \in \mathcal{J}\}$ with the itineraries. If the set of itineraries that we offer to the customers at time period $t$ is given by $u_t = \{u_{jt} : j \in \mathcal{J}\}$ and a customer of type $g$ arrives at this time period, then this customer purchases itinerary $j \in \mathcal{J}_g$ with probability

$$\frac{w_j \, u_{jt}}{\sum_{\ell \in \mathcal{J}_g} w_\ell \, u_{\ell t} + w_g^0},$$

where $w_g^0$ is the preference weight for customer type $g$ associated with purchasing nothing. With the remaining probability $w_g^0 / [\sum_{\ell \in \mathcal{J}_g} w_\ell \, u_{\ell t} + w_g^0]$, the customer leaves without purchasing anything.

The logit model is crucial for the computational tractability of several problems that we work with. As mentioned before, the number of decision variables in problem (7)-(10) increases exponentially with the number of itineraries. However, Liu and van Ryzin (2008) show that the column generation subproblem for problem (7)-(10) is tractable whenever the customer choices are governed by the logit model. Similarly, noting the definition of $\mathcal{U}_i(x_{it})$, the number of possible values for the decision variable $z_{it}$ in problem (12) can be as large as $2^{|\mathcal{J}|}$, but Liu and van Ryzin (2008) show that problem (12) is tractable under the logit model.

We present computational experiments on three groups of test problems. The first group involves parallel flight legs that operate between the same origin-destination pair, whereas the second group involves an airline network with one hub serving multiple spokes. These two groups of test problems closely follow the experimental setup in Zhang and Adelman (2009). Finally, the third group involves an airline network with two hubs serving multiple spokes.

### 5.3 Computational Results on Parallel Flight Legs

We consider $n$ flight legs that operate between the same origin-destination pair. There is a high-fare and a low-fare itinerary associated with each flight leg so that the number of itineraries is $2\,n$. There are two customer types. The first customer type is interested in the high-fare itineraries, whereas the second customer type is interested in the low-fare itineraries. The preference weights associated with the itineraries are generated from the Poisson distribution with mean 100. The preference weight associated with purchasing nothing is set to $\sum_{j \in \mathcal{J}} w_j / 2$. The revenues associated with the low-fare itineraries are generated from the uniform distribution over $[10, 100]$. The revenue associated with a high-fare itinerary is obtained by multiplying the revenue associated with the corresponding low-fare itinerary by $\kappa$. We measure the tightness of the leg capacities by following the same approach as in Zhang and Adelman (2009). In particular, we let $\hat{u}_t$ be the optimal solution to the problem $\max_{u_t \in \{0,1\}^{|\mathcal{J}|}} \sum_{j \in \mathcal{J}} P_j(u_t)\, r_j$ so that $\hat{u}_t$ captures the set of itineraries that should be offered to the customers so as to maximize the immediate expected revenue. In this case, we use

$$\gamma = \frac{\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} P_j(\hat{u}_t)\, a_{ij}}{\sum_{i \in \mathcal{L}} c_i} \tag{18}$$

to measure the tightness of the leg capacities. We note that the expression above computes the ratio between the total expected capacity consumption and the total capacity over all flight legs. We label our test problems by the triplet $(n, \kappa, \gamma) \in \{4, 5, 6\} \times \{2, 4\} \times \{1.2, 1.6\}$, where $n$ is the number of flight legs, $\kappa$ is the revenue difference between the high-fare and low-fare itineraries and $\gamma$ is the measure of the tightness of the leg capacities.

We present our computational results in Table 1. The first column in this table shows the problem characteristics by using the triplet $(n, \kappa, \gamma)$. The second, third and fourth columns respectively show the upper bounds on the optimal total expected revenue that are obtained by AP decomposition, DLP and LvR decomposition. The fifth and sixth columns show the percent gaps between the upper bounds obtained by AP decomposition and the remaining two solution methods. The seventh, eighth and ninth columns respectively show the total expected revenues obtained by AP decomposition, DLP and

LvR decomposition. We estimate these total expected revenues by simulating the decisions made by the three solution methods under multiple customer arrival trajectories. The tenth column shows the percent gap between the total expected revenues obtained by AP decomposition and DLP. This column also includes a "✓" whenever the total expected revenue obtained by AP decomposition is better than the one obtained by DLP, a "×" whenever the total expected revenue obtained by DLP is better than the one obtained by AP decomposition and a "⊙" whenever there is no statistically significant difference between the total expected revenues obtained by AP decomposition and DLP at 95% significance level. The interpretation of the eleventh column is the same as that of the tenth column, but the eleventh column compares the performances of AP and LvR decomposition.

Comparing the upper bounds obtained by the different solution methods, we observe that the upper bounds obtained by AP decomposition are tighter than those obtained by LvR decomposition, which are, in turn, tighter than those obtained by DLP. This ordering agrees with Proposition 2. The gaps between the upper bounds obtained by AP decomposition and DLP range between 1.84% and 3.73%, whereas the gaps between the upper bounds obtained by AP and LvR decomposition range between 1.11% and 1.74%. Comparing the total expected revenues obtained by the different solution methods, AP decomposition improves on DLP by a margin that ranges between 2.54% and 4.74%. For a majority of the test problems, the performances of AP and LvR decomposition are quite close. There are three test problems where AP decomposition performs significantly better than LvR decomposition and the performances of the two solution methods are indistinguishable for the remaining test problems. There is one test problem where the performance of AP decomposition lags behind that of LvR decomposition by a small margin, but the performance gap for this test problem is not statistically significant. The number of flight legs emerges as an important factor that affects the performance gaps between AP and LvR decomposition. For the test problems with three, four and five flight legs, the average performance gaps between AP and LvR decomposition are respectively 0.20%, 0.53% and 0.96%. Finally, it is worthwhile to note that the gaps between the upper bounds and the total expected revenues obtained by AP decomposition are extremely small. Therefore, AP decomposition essentially obtains the optimal solution for this group of test problems.

### 5.4 Computational Results on an Airline Network with One Hub

In this group of test problems, we consider an airline network with one hub serving $n$ spokes. Half of the spokes have two parallel flight legs to the hub and the other half have two parallel flight legs from the hub. Figure 1 shows the structure of the airline network with $n = 8$. There is a high-fare and a low-fare itinerary associated with every possible sequence of connecting flight legs in the airline network. We associate a customer type with every origin-destination pair. Depending on its type, a customer chooses among the itineraries that connect a particular origin-destination pair. The preference weights associated with the high-fare and low-fare itineraries are respectively generated from the Poisson distributions with means 50 and 200. The revenues associated with the low-fare spoke-to-hub and hub-to-spoke itineraries are respectively generated from the uniform distributions over $[1, 10]$ and $[10, 100]$. The revenue associated with a low-fare spoke-to-spoke itinerary is 95% of the sum of the revenues associated with the corresponding spoke-to-hub and hub-to-spoke itineraries. Using $\kappa$ and $\gamma$ with the same interpretation as in the test problems with parallel flight legs, we label our test problems by the triplet $(n, \kappa, \gamma) \in \{8, 10, 12\} \times \{2, 4\} \times \{1.2, 1.6\}$, where $n$ is the number of spokes. This experimental setup closely parallels the one in Zhang and Adelman (2009).

We present our computational results in Table 2. The organization of Table 2 is the same as that of Table 1. For this group of test problems, the upper bounds on the optimal total expected revenue that are obtained by AP decomposition can significantly improve those obtained by DLP and LvR decomposition. The average gap between the upper bounds obtained by AP decomposition and DLP is 3.06%. On the other hand, the average gap between the upper bounds obtained by AP and LvR decomposition is 2.26%. Comparing the total expected revenues obtained by the different solution methods, we observe that AP decomposition provides improvements over DLP that range between 1.30% and 6.36%. For a majority of the test problems, the gaps between the total expected revenues obtained by AP and LvR decomposition exceed 2.00%. Such gaps are considered quite significant in the network revenue management setting. There is one test problem where the performance of LvR decomposition is slightly better than that of AP decomposition, but the performance gap for this test problem is not statistically significant. Finally, we note that the performance gaps between AP decomposition and the other two solution methods tend to increase as the leg capacities get tighter. If the leg capacities are very large, then we do not need to pay attention to the effects of the decisions at the current time period on the future time periods and it becomes trivially optimal to offer the set of itineraries that maximize the immediate expected revenue. Therefore, we intuitively expect the test problems with tight leg capacities to be more difficult and it is encouraging that AP decomposition provides especially good performance for such test problems.

### 5.5 Computational Results on an Airline Network with Two Hubs

We consider an airline network with two hubs serving $n$ spokes. There are four parallel flight legs that connect the first hub to the second hub. Half of the spokes have two parallel flight legs to the first hub and the other half have two parallel flight legs from the second hub. Figure 2 shows the structure of the airline network with $n = 8$. Among the set of all possible sequences of connecting flight legs in the airline network, we randomly sample about 100 and associate a high-fare and a low-fare itinerary with these sequences of connecting flight legs. Similar to the test problems with one hub, there is a customer type associated with every origin-destination pair. The preference weights are generated in the same fashion as in the test problems with one hub. The revenues associated with the low-fare spoke-to-hub and hub-to-spoke itineraries are generated from the uniform distribution over $[10, 100]$, whereas the revenues associated with the low-fare hub-to-hub itineraries are generated from the uniform distribution over $[1, 10]$. The revenue associated with an itinerary that involves more than one flight leg is 95% of the sum of the revenues associated with the corresponding single-leg itineraries. Using $\kappa$ and $\gamma$ with the same interpretation as in the test problems with parallel flight legs, we label our test problems by the triplet $(n, \kappa, \gamma) \in \{4, 6, 8\} \times \{2, 4\} \times \{1.2, 1.6\}$, where $n$ is the number of spokes.

We present our computational results in Table 3. This table is organized in the same fashion as Table 1. AP decomposition continues to provide significantly tighter upper bounds on the optimal total expected revenue when compared with DLP and LvR decomposition. Furthermore, the gaps between the upper bounds for this group of test problems are noticeably larger than those for the previous two groups. On the average, the upper bounds obtained by AP decomposition improve the upper bounds obtained by DLP and LvR decomposition by respectively 5.58% and 4.13%. The average gap between the total expected revenues obtained by AP decomposition and DLP is 4.60%. The total expected revenues obtained by AP decomposition are significantly better than those obtained by LvR decomposition for seven test problems and the performance gaps for these test problems can exceed 3.00%. There is one test problem where LvR decomposition performs better than AP decomposition with a statistically significant performance gap. Although both AP and LvR decomposition provide upper bounds on the optimal total expected revenue, making the decisions by using $\{\sum_{i \in \mathcal{L}} v_{it}(x_{it} \,|\, \hat{\alpha}, \hat{\beta}) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ and $\{\sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it}) : x_t \in \mathcal{C}^{|\mathcal{L}|},\ t \in \mathcal{T}\}$ as approximations to the value functions is essentially a heuristic idea. Therefore, LvR decomposition may perform better than AP decomposition, even though the upper bounds provided by AP decomposition are tighter than those provided by LvR decomposition. Similar to the test problems with one hub, the performance gaps between AP decomposition and the other two solution methods tend to increase as the leg capacities get tighter.

Table 4 shows the CPU seconds required for the different solution methods on a Pentium Core 2 Duo PC with 3 GHz CPU and 4 GB RAM. The three portions of the table correspond to the three groups of test problems in our experimental setup. The CPU seconds for AP decomposition correspond to the time required to solve problem (6) through 1,500 iterations of subgradient search. On the other hand, the CPU seconds for DLP correspond to the time required to solve problem (7)-(10), and the CPU seconds for LvR decomposition correspond to the time required to solve problem (7)-(10) and the optimality equation in (12). Since $n$ is the primary factor that affects the CPU seconds, we only provide the average CPU seconds over the test problems. For larger test problems, the CPU seconds for AP decomposition are larger than those for DLP and LvR decomposition by a factor of five to nine, but the extra computational effort can be worthwhile noting that AP decomposition provides improvements in both the upper bounds and the total expected revenues. We also note that as the number of parallel flight legs or the number of spokes increases, the CPU seconds for AP decomposition scale essentially in the same manner as the CPU seconds for DLP and LvR decomposition.

## 6 Conclusions

In this paper, we presented an alternative approach for decomposing the network revenue management problem with customer choice behavior. Similar to LvR decomposition that appears in the earlier literature, our approach allocates the revenue associated with an itinerary among the different flight legs and solves a single-leg revenue management problem for each flight leg in the airline network. The novel aspect of our decomposition approach is that it uses an auxiliary optimization problem to choose the revenue allocations and this problem takes the probabilistic nature of the customer choices into consideration. We showed that our decomposition approach provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the ones obtained by the deterministic linear program and LvR decomposition. Computational experiments showed that our approach can perform noticeably better than the deterministic linear program and LvR decomposition.

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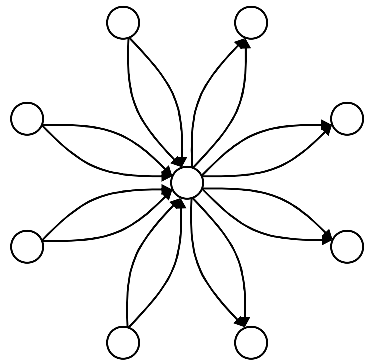

Figure 1: Structure of the airline network with one hub and eight spokes.

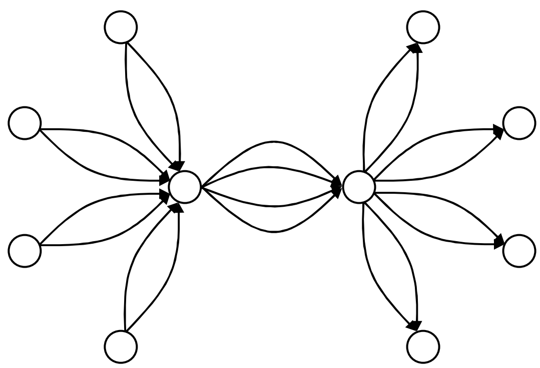

Figure 2: Structure of the airline network with two hubs and eight spokes.

| Problem | Upper Bnd. on Opt. Total Exp. Rev. | | | % Gap with AP | | Total Exp. Rev. Obtained by | | | % Gap with AP | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(n, \kappa, \gamma)$ | AP | DLP | LvR | DLP | LvR | AP | DLP | LvR | DLP | LvR |
| $(4, 2, 1.2)$ | 3,302 | 3,397 | 3,347 | 2.88 | 1.36 | 3,267 | 3,140 | 3,253 | 3.88 ✓ | 0.44 ⊙ |
| $(4, 2, 1.6)$ | 2,649 | 2,726 | 2,688 | 2.91 | 1.47 | 2,623 | 2,499 | 2,625 | 4.74 ✓ | -0.08 ⊙ |
| $(4, 4, 1.2)$ | 5,016 | 5,151 | 5,093 | 2.69 | 1.54 | 4,961 | 4,835 | 4,945 | 2.54 ✓ | 0.32 ⊙ |
| $(4, 4, 1.6)$ | 4,322 | 4,483 | 4,395 | 3.73 | 1.69 | 4,282 | 4,171 | 4,278 | 2.60 ✓ | 0.10 ⊙ |
| $(5, 2, 1.2)$ | 3,978 | 4,051 | 4,024 | 1.84 | 1.16 | 3,933 | 3,781 | 3,897 | 3.86 ✓ | 0.91 ✓ |
| $(5, 2, 1.6)$ | 3,229 | 3,301 | 3,265 | 2.23 | 1.11 | 3,192 | 3,077 | 3,190 | 3.60 ✓ | 0.06 ⊙ |
| $(5, 4, 1.2)$ | 5,986 | 6,116 | 6,064 | 2.17 | 1.30 | 5,923 | 5,751 | 5,873 | 2.91 ✓ | 0.84 ⊙ |
| $(5, 4, 1.6)$ | 5,211 | 5,366 | 5,283 | 2.97 | 1.38 | 5,155 | 5,009 | 5,138 | 2.83 ✓ | 0.32 ⊙ |
| $(6, 2, 1.2)$ | 4,960 | 5,053 | 5,027 | 1.88 | 1.35 | 4,879 | 4,712 | 4,802 | 3.42 ✓ | 1.58 ✓ |
| $(6, 2, 1.6)$ | 3,960 | 4,042 | 4,012 | 2.07 | 1.31 | 3,913 | 3,773 | 3,901 | 3.57 ✓ | 0.31 ⊙ |
| $(6, 4, 1.2)$ | 7,394 | 7,557 | 7,508 | 2.20 | 1.54 | 7,301 | 7,102 | 7,198 | 2.72 ✓ | 1.42 ✓ |
| $(6, 4, 1.6)$ | 6,368 | 6,546 | 6,479 | 2.80 | 1.74 | 6,299 | 6,125 | 6,266 | 2.76 ✓ | 0.52 ⊙ |
| Average | | | | 2.53 | 1.41 | | | | 3.28 | 0.56 |

Table 1: Comparison between the three solution methods on parallel flight legs.

| Problem | Upper Bnd. on Opt. Total Exp. Rev. | | | % Gap with AP | | Total Exp. Rev. Obtained by | | | % Gap with AP | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(n, \kappa, \gamma)$ | AP | DLP | LvR | DLP | LvR | AP | DLP | LvR | DLP | LvR |
| $(8, 2, 1.2)$ | 3,579 | 3,821 | 3,742 | 6.77 | 4.56 | 3,423 | 3,309 | 3,437 | 3.32 ✓ | -0.41 ⊙ |
| $(8, 2, 1.6)$ | 3,207 | 3,490 | 3,426 | 8.81 | 6.82 | 2,993 | 2,871 | 2,986 | 4.10 ✓ | 0.24 ⊙ |
| $(8, 4, 1.2)$ | 9,016 | 9,534 | 9,396 | 5.75 | 4.22 | 8,436 | 8,239 | 8,424 | 2.35 ✓ | 0.15 ⊙ |
| $(8, 4, 1.6)$ | 7,204 | 7,337 | 7,315 | 1.85 | 1.55 | 6,788 | 6,356 | 6,500 | 6.36 ✓ | 4.25 ✓ |
| $(10, 2, 1.2)$ | 7,944 | 8,257 | 8,170 | 3.94 | 2.85 | 7,361 | 7,172 | 7,212 | 2.57 ✓ | 2.02 ✓ |
| $(10, 2, 1.6)$ | 6,131 | 6,160 | 6,151 | 0.47 | 0.32 | 5,765 | 5,472 | 5,514 | 5.08 ✓ | 4.37 ✓ |
| $(10, 4, 1.2)$ | 16,027 | 16,574 | 16,404 | 3.42 | 2.36 | 14,869 | 14,477 | 14,570 | 2.64 ✓ | 2.02 ✓ |
| $(10, 4, 1.6)$ | 12,307 | 12,325 | 12,307 | 0.15 | 0.00 | 11,672 | 10,960 | 11,186 | 6.11 ✓ | 4.16 ✓ |
| $(12, 2, 1.2)$ | 8,649 | 8,872 | 8,834 | 2.58 | 2.14 | 8,039 | 7,935 | 8,010 | 1.30 ✓ | 0.36 ⊙ |
| $(12, 2, 1.6)$ | 6,779 | 6,795 | 6,787 | 0.23 | 0.11 | 6,355 | 6,102 | 6,064 | 3.99 ✓ | 4.58 ✓ |
| $(12, 4, 1.2)$ | 17,293 | 17,744 | 17,667 | 2.61 | 2.16 | 16,160 | 15,854 | 16,018 | 1.89 ✓ | 0.88 ✓ |
| $(12, 4, 1.6)$ | 13,571 | 13,590 | 13,573 | 0.14 | 0.02 | 12,820 | 12,316 | 12,141 | 3.93 ✓ | 5.30 ✓ |
| Average | | | | 3.06 | 2.26 | | | | 3.64 | 2.33 |

Table 2: Comparison between the three solution methods on an airline network with one hub.

| Problem | Upper Bnd. on Opt. Total Exp. Rev. | | | % Gap with AP | | Total Exp. Rev. Obtained by | | | % Gap with AP | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(n, \kappa, \gamma)$ | AP | DLP | LvR | DLP | LvR | AP | DLP | LvR | DLP | LvR |
| $(4, 2, 1.2)$ | 3,654 | 4,023 | 3,934 | 10.10 | 7.66 | 3,477 | 3,339 | 3,499 | 3.97 ✓ | -0.63 ⊙ |
| $(4, 2, 1.6)$ | 3,093 | 3,431 | 3,332 | 10.93 | 7.73 | 2,932 | 2,839 | 2,944 | 3.17 ✓ | -0.39 ⊙ |
| $(4, 4, 1.2)$ | 8,051 | 8,777 | 8,542 | 9.02 | 6.10 | 7,657 | 7,025 | 7,675 | 8.26 ✓ | -0.24 ⊙ |
| $(4, 4, 1.6)$ | 6,710 | 7,162 | 7,004 | 6.74 | 4.38 | 6,354 | 5,935 | 6,155 | 6.59 ✓ | 3.13 ✓ |
| $(6, 2, 1.2)$ | 5,982 | 6,444 | 6,337 | 7.72 | 5.93 | 5,591 | 5,381 | 5,633 | 3.75 ✓ | -0.76 × |
| $(6, 2, 1.6)$ | 4,964 | 5,155 | 5,122 | 3.84 | 3.18 | 4,662 | 4,435 | 4,538 | 4.88 ✓ | 2.65 ✓ |
| $(6, 4, 1.2)$ | 12,998 | 13,561 | 13,466 | 4.33 | 3.60 | 12,210 | 11,834 | 11,889 | 3.08 ✓ | 2.63 ✓ |
| $(6, 4, 1.6)$ | 10,326 | 10,407 | 10,371 | 0.78 | 0.44 | 9,819 | 9,184 | 9,577 | 6.47 ✓ | 2.46 ✓ |
| $(8, 2, 1.2)$ | 8,437 | 8,977 | 8,858 | 6.40 | 4.99 | 7,922 | 7,702 | 7,932 | 2.78 ✓ | -0.13 ⊙ |
| $(8, 2, 1.6)$ | 6,981 | 7,155 | 7,114 | 2.49 | 1.91 | 6,551 | 6,218 | 6,408 | 5.08 ✓ | 2.18 ✓ |
| $(8, 4, 1.2)$ | 17,897 | 18,644 | 18,554 | 4.17 | 3.67 | 16,780 | 16,395 | 16,621 | 2.29 ✓ | 0.95 ✓ |
| $(8, 4, 1.6)$ | 14,338 | 14,405 | 14,341 | 0.47 | 0.02 | 13,526 | 12,860 | 13,295 | 4.93 ✓ | 1.71 ✓ |
| Average | | | | 5.58 | 4.13 | | | | 4.60 | 1.13 |

Table 3: Comparison between the three solution methods on an airline network with two hubs.

| Parallel Flights | | | |
|---|---|---|---|
| | CPU seconds | | |
| $n$ | AP | DLP | LvR |
| 4 | 35 | 0.5 | 0.6 |
| 5 | 53 | 0.6 | 0.8 |
| 6 | 73 | 0.7 | 1.2 |

| Air. Netw. with One Hub | | | |
|---|---|---|---|
| | CPU seconds | | |
| $n$ | AP | DLP | LvR |
| 8 | 436 | 20 | 59 |
| 10 | 695 | 41 | 119 |
| 12 | 949 | 64 | 195 |

| Air. Netw. with Two Hubs | | | |
|---|---|---|---|
| | CPU seconds | | |
| $n$ | AP | DLP | LvR |
| 4 | 375 | 19 | 34 |
| 6 | 623 | 31 | 68 |
| 8 | 1,026 | 51 | 123 |

Table 4: CPU seconds required for the different solution methods.