Revenue Management with Calendar-Aware and Dependent Demands: Asymptotically Tight Fluid Approximations

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When modeling the demand in revenue management systems, a natural approach is to focus on a canonical interval of time, such as a week, so that we forecast the demand over each week in the selling horizon. Ideally, we would like to use random variables with general distributions to model the demand over each week. The current demand can give a signal for the future demand, so we also would like to capture the dependence between the demands over different weeks. Prevalent demand models in the literature, which are based on a discrete-time approximation to a Poisson process, are not compatible with these needs. In this paper, we focus on revenue management models that are compatible with a natural approach for forecasting the demand. Building such models through dynamic programming is not difficult. We divide the selling horizon into multiple stages, each stage being a canonical interval of time on the calendar. We have random number of customer arrivals in each stage, whose distribution is arbitrary and depends on the number of arrivals in the previous stage. The question we seek to answer is the form of the corresponding fluid approximation. We give the correct fluid approximation in the sense that it yields asymptotically optimal policies. The form of our fluid approximation is surprising as its constraints use expected capacity consumption of a resource up to a certain time period, conditional on the demand in the stage just before the time period in question. Letting K be the number of stages in the selling horizon, $c_{\min}$ be the smallest resource capacity and ϵ be a lower bound on the mass function of the demand in a stage, we use the fluid approximation to give a policy with a performance guarantee of $\Omega\Big(1 - \frac{\sqrt{(c_{\min}+K/\epsilon^6)\log c_{\min}}}{c_{\min}}\Big)$. Thus, as the resource capacities and number of stages increase with the same rate, the performance guarantee converges to one. To our knowledge, this result gives the first asymptotically optimal policy under dependent demands with arbitrary distributions. When the demands in different stages are independent, letting σ^2 be the variance proxy for the demand in each stage, a similar performance guarantee holds by replacing $\frac{1}{\epsilon^6}$ with σ^2. Our computational experiments indicate that using the correct fluid approximation can make a dramatic impact in practice.

1. Introduction

A natural approach for modeling the demand in revenue management systems is to focus on a canonical interval of time, such as a week, so that we forecast the demand over each week in the selling horizon. Ideally, we would like to model the demand over each week through a random variable with an arbitrary distribution. Indeed, it is common for revenue management systems to produce forecasts stating that the demand, for example, during the week of July 17-23, 2023 has the log-normal distribution with mean 500 and standard deviation 250. These forecasts involve arbitrary demand distributions. Also, these forecasts are aware of the calendar in the sense that they have a concept of when each week ends and the next one starts. Furthermore, the current demand usually gives a signal for the future demand. Thus, we also would like to capture the dependence between the demands over different weeks. Prevalent demand models in the literature are not compatible with such a natural approach for forecasting the demand. In particular, a common demand model is based on dividing the selling horizon into a number of time periods such that there is at most one customer arrival at each time period and the arrivals at successive time periods are independent. This model corresponds to a discrete-time approximation to a Poisson process, but it has shortcomings. Under this model, the demand over an interval of time will always be approximately Poisson. To make matters worse, using λ_t to denote the probability that we have a customer arrival at time period t, over an interval of T time periods, the mean and standard deviation of the number of customer arrivals are, respectively, $\sum_{t=1}^T \lambda_t$ and $\sqrt{\sum_{t=1}^T \lambda_t\,(1-\lambda_t)}$. Noting that $\sqrt{\sum_{t=1}^T \lambda_t\,(1-\lambda_t)} \leq \sqrt{\sum_{t=1}^T \lambda_t}$, the ratio between the standard deviation and mean of the demand is at most $1/\sqrt{\sum_{t=1}^T \lambda_t}$. Thus, as the mean demand gets large, the coefficient of variation of the demand gets smaller. In other words, large demand variability and large demand volume cannot co-exist in this demand model. Finally, because this demand model is based on a Poisson process, the demands over different time intervals must be independent.

Motivated by the shortcomings discussed in the previous paragraph, we focus on revenue management models that are intrinsically compatible with a natural approach for forecasting the demand. Such a natural approach for forecasting the demand may specify the distribution of the demand over, for example, different weeks in the selling horizon, possibly along with the correlation structure between the demands in successive weeks. It is not too difficult to build these revenue management models through dynamic programming. We can divide the selling horizon into a number of stages, each stage representing a canonical interval of time on the calendar, such as a week. The number of customer arrivals in each stage is a random variable, whose distribution is arbitrary and depends on the number of customer arrivals in the previous stage. Therefore, along with the remaining capacities of the resources, the state variable in the dynamic program needs to keep track of the number of customer arrivals so far in the current stage and the number of customer arrivals in the previous stage, so that as a function of these two quantities, we can compute the probability of having one more customer arrival in the current stage. This dynamic program would give a precise specification of our model, but it is not useful for computing the optimal policy in practice because it involves a high-dimensional state variable.

We focus on fluid approximations for our model so that we can construct practical policies with performance guarantees and compute upper bounds on the optimal total expected revenues.

Our Results and Contributions: We start with a revenue management model based on a dynamic programming formulation that can handle arbitrary distributions for the number of customer arrivals in each stage and allow dependence between the number of customer arrivals in successive stages. A stage may correspond to a canonical interval of time over which forecasts are produced. We capture the dependence between the demands through a Markov chain that specifies the distribution of the number of customer arrivals in one stage as a function of the number of customer arrivals in the previous stage. We give the correct fluid approximation for our revenue management model in the sense that the fluid approximation satisfies two properties. First, given that it is computationally difficult to find the optimal policy through a dynamic programming formulation, we can use our fluid approximation to construct approximate policies with performance guarantees. Second, we can use our fluid approximation to obtain an upper bound on the optimal total expected revenue, so that we can compare the total expected revenue of a heuristic policy with the upper bound to assess the optimality gap of the heuristic policy.

Structure of the Fluid Approximation. The structure of our fluid approximation turns out to be novel. In our fluid approximation, we use decision variables that allow the probability of accepting a customer request at a time period to depend on the number of customer arrivals in the previous stage. Due to the dependence between the demands in successive stages, this form of the decision variables is perhaps not surprising, but we are not aware of other fluid approximations with similar decision variables. More importantly though, the constraints in our fluid approximation keep track of the expected capacity consumption of a resource up to a certain time period in a particular stage, conditional on the demand in the stage just before the time period in question. The conditional form of these constraints is surprising and does not appear in the literature. We show that the optimal objective value of our fluid approximation is an upper bound on the optimal total expected revenue. Thus, we can use this upper bound to assess the optimality gaps of heuristic policies.

Policies under Dependent Demands. Using our fluid approximation, we give an approximate policy. Letting K be the number of stages in the selling horizon, $c_{\min}$ be the smallest resource capacity, L be the maximum number of resources used by a product and ϵ be a lower bound on the mass function of the demand in a stage given the demand in the previous stage, we show that our approximate policy has a performance guarantee of $\max\left\{\frac{1}{4L}, \left(1 - 4\,\frac{\sqrt{(c_{\min}+(K-1)/\epsilon^6)\log c_{\min}}}{c_{\min}} - \frac{L}{c_{\min}}\right)\right\}$. In many applications, the number of resources used by a particular product is usually small. In the airline setting, for example, the number of flight legs in an itinerary rarely exceeds two, corresponding to $L=2$. By the first term in the max operator, our approximate policy has a constant-factor performance guarantee when L is uniformly bounded. By the second term in the max operator, as the number of stages and the capacities of the resources both increase linearly with rate θ, our performance guarantee converges to one with rate $1-\frac{1}{\sqrt{\theta}}$.

Thus, our approximate policy is asymptotically optimal for systems that command large demands for the products and involve large capacities for the resources. During the course of the proof of this result, we also show that the upper bound provided by our fluid approximation is asymptotically tight in the same regime. To our knowledge, our approximate policy is the first to yield asymptotic optimality guarantee under dependent demands with arbitrary distributions. The proof for our performance guarantee uses techniques that have not been used in the related literature. In particular, under the standard demand model, to analyze policies from fluid approximations, we upper bound the total consumption of a resource by using a random variable expressed as a sum of independent random variables. In this case, we can use a concentration inequality for sums of independent random variables to upper bound the tail probability of the total consumption of a resource, which in turn, yields a lower bound on the probability that the policy has enough resource availabilities to accept the product request at each time period. The lower bound on the resource availability probabilities are used to lower bound the performance of the policy.

Because the demands in successive stages are dependent in our setting, concentration inequalities for sums of independent random variables are not helpful to us, so we explicitly construct the concentration inequalities that we need. It is standard to use the moment generating function of a random variable to bound its tail probabilities. In particular, if the moment generation function of the random variable Z satisfies $\mathbb{E}\{e^{\lambda Z}\} \leq f(\lambda)$ for all $\lambda \geq 0$, then we can bound its tail probabilities as $\mathbb{P}\{Z \geq c\} = \mathbb{P}\{e^{\lambda Z} \geq e^{\lambda c}\} \leq \frac{1}{e^{\lambda c}}\mathbb{E}\{e^{\lambda Z}\} \leq \frac{f(\lambda)}{e^{\lambda c}}$, where the first inequality is the Markov inequality. We use martingales and the method of bounded differences to bound the moment generating functions of the capacity consumptions of the resources; see Dubhashi and Panconesi (2009). This technique has been used in analyzing randomized algorithms, but their use in revenue management appears to be new. We hope that our use of this technique will further motivate other fluid approximations under even more sophisticated demand models.

Policies under Independent Demands. When the demands in different stages are independent, our revenue management model captures the case where the number of customer arrivals in, say, each week has an arbitrary distribution and the decision maker has a concept of when each week ends and the next one starts. Thus, this demand model is different from a demand model, where there is a single stage consisting of possibly multiple weeks and the number of customer arrivals in the single stage has an arbitrary distribution. In particular, the uncertainty in the demand in our model resolves sequentially over multiple stages. Recalling that the variance proxy of a sub-Gaussian random variable is an upper bound on its variance, under the assumption that the demand in each stage is sub-Gaussian with variance proxy $\sigma^2/200$, we give an approximate policy with a performance guarantee of $\max\left\{\frac{1}{4L}, \left(1 - 4\,\frac{\sqrt{(c_{\min}+\sigma^2(K-1))\log c_{\min}}}{c_{\min}} - \frac{L}{c_{\min}}\right)\right\}$. Variance may be a more intuitive statistic to work with than a lower bound on the mass function.

Even when the numbers of customer arrivals in different stages are independent, establishing the performance guarantee in the previous paragraph requires going one step beyond concentration inequalities for sums of independent random variables. In particular, the random variable that captures the demand in a particular stage creates dependence between the capacity consumptions of a resource at different time periods in the stage. Thus, as far as we can see, our performance guarantee with independent demands in each stage does not follow from the proof techniques used for the existing asymptotic optimality results under a discrete-time approximation to a Poisson process. We end up constructing the concentration inequalities that we need by exploiting the assumption of sub-Gaussian demand random variables. The assumption of sub-Gaussian demand random variables is relatively mild, as this class is rather general; see Section 2.1.2 in Wainwright (2019). Any bounded random variable, for example, is sub-Gaussian.

Computational Performance. To our knowledge, there is no work on asymptotically tight fluid approximations and asymptotically optimal policies for revenue management problems in which the demands over different time intervals are dependent and have arbitrary distributions. Building such fluid approximations and approximate policies is theoretically interesting, but fluid approximations with a sound theoretical footing can also make a significant impact in practice. In our computational experiments, we make comparisons with existing fluid approximations. While we can show that these fluid approximations do provide upper bounds, they do not provide asymptotically optimal policies. Our fluid approximation, owing to its sound theoretical footing, provides significantly tighter upper bounds and better approximate policies for a range of test problems.

Related Literature: There is significant work on fluid approximations in revenue management, but this work is under demand models that use a discrete-time approximation to a Poisson process, ruling out the possibility of arbitrary demand distributions and dependence between demands over different time intervals. Considering a revenue management problem with a single resource, Gallego and van Ryzin (1994) show that if we scale the expected demand and the capacity of the resource with the same rate θ, then a policy from a fluid approximation has a performance guarantee of $\Omega\Big(1-\frac{1}{\sqrt{\theta}}\Big)$. Gallego and van Ryzin (1997) generalize this result to a network of resources, where the sale of different products consumes capacities of different combinations of resources. The policies in these papers use the primal solution to a fluid approximation. Talluri and van Ryzin (1998) use the dual solution to construct an asymptotically optimal policy in the same regime. Liu and van Ryzin (2008) and Gallego et al. (2004) construct similar asymptotically optimal policies under customer choice behavior, where the customers choose among the sets of products offered to them. Considering the case where the customers with bookings do not necessarily show up at the time of service, Kunnumkal et al. (2012) give an asymptotically optimal policy that allows overbooking. The papers discussed so far solve the fluid approximation once at the beginning of the selling horizon. Jasin and Kumar (2012) show that solving the fluid approximation periodically over the selling horizon provides policies with substantially better performance guarantees. Letting $c_{\min}$ be the smallest resource capacity, Rusmevichientong et al. (2020) give a policy with a performance guarantee of $\Omega\left(1-\frac{1}{\sqrt[3]{c_{\min}}}\right)$. The asymptotic regime in this paper is different in the sense that the authors allow the expected demand to be scaled in an arbitrary fashion. In the same asymptotic regime, Bai et al. (2022) and Feng et al. (2022) give policies both with a performance guarantee of $\Omega\left(1-\frac{1}{\sqrt{c_{\min}}}\right)$. Balseiro et al. (2023) give a unified analysis for fluid approximations for a wide range of revenue management problems, while allowing the possibility of solving the fluid approximation periodically over the selling horizon.

The work on models that allow random number of customer arrivals with arbitrary distributions is relatively recent. Considering the setting where the capacities of the resources and the points in the support of the number of customer arrivals are scaled with the same rate θ, Besbes and Saure (2014) give a policy with a performance guarantee of $\Omega\left(1-\frac{1}{\sqrt{\theta}}\right)$. Under random number of customer arrivals, Aouad and Ma (2022) and Bai et al. (2023) give policies with a performance guarantee of $\Omega\left(1-\frac{1}{\sqrt{c_{\min}}}\right)$. In these papers, the number of customer arrivals is random, but conditioned on the number of customer arrivals, the products requested by the different customers are independent of each other. The customer arrivals occur in one stage, so there is no possibility of introducing dependence between the numbers of customer arrivals in successive stages. Letting L be the maximum number of resources used by a product, Jiang (2023) gives a policy with a performance guarantee of $1/(1+L)$ when the distribution governing the products requested by the customers evolves from one time period to the next according to a Markov chain that transitions independently from the decisions of the policy. The performance guarantee for the policy does not improve when we deal with systems commanding large demands for the products and involving large capacities for the resources. Ma et al. (2021) use fluid approximations to show that following static rules that do not pay attention to the state of the system can yield performance guarantees. The terminology of static calendar appears in their paper, but their goal is to emphasize that they follow static rules, rather than to work with demand models that are aware of when each stage ends and the next one starts on the calendar.

Organization: In Section 2, we formulate our revenue management model with arbitrary demand distributions in each stage and dependence between the demands in successive stages. In Section 3, we give the fluid approximation corresponding to our model and show that its optimal objective value is an upper bound on the optimal total expected revenue. In Section 4, we describe the approximate policy from the fluid approximation and give a performance guarantee for the approximate policy. In Section 5, we prove our performance guarantee using martingales and the method of bounded differences. In Section 6, we focus our results to the case with independent demands. In Section 7, we give computational experiments. In Section 8, we conclude.

2. Problem Formulation

The set of resources is $\mathcal{L}$. The capacity of resource i is c_i. The set of products is $\mathcal{J}$. The revenue of product j is f_j. The resources used by product j are given by the vector $\boldsymbol{a}_j = (a_{ij} : i \in \mathcal{L}) \in \{0,1\}^{|\mathcal{L}|}$, where $a_{ij} = 1$ if and only if product j uses resource i. We divide the selling horizon into K stages indexed by $\mathcal{K} = \{1,\ldots,K\}$. We use the random variable D^k to capture the number of customer arrivals in stage k. There are at most T customer arrivals in each stage. We divide each stage into T time periods indexed by $\mathcal{T} = \{1,\ldots,T\}$. We use λ^k_{jt} to denote the probability that the customer arriving at time period t in stage k requests product j, so we have $\sum_{j\in\mathcal{J}} \lambda^k_{jt} = 1$. We refer D^k as the demand in stage k. The demands in successive stages follow a Markov process. Thus, conditional on D^k, D^{k+1} is independent of $D^1,\ldots,D^{k-1}$. We characterize the evolution of the demands by the survival rate function $\theta^k_t(q) = \mathbb{P}\{D^k \geq t+1 \mid D^k \geq t,\ D^{k-1} = q\}$, capturing the probability that the demand in stage k is at least $t+1$, given that the demand in stage k is at least t and the demand in the previous stage was q. We assume that $\mathbb{P}\{D^{k+1} = p \mid D^k = q\} \geq \epsilon$ for all $p,q \in \mathcal{T}$ and $k \in \mathcal{K}$ for some $\epsilon > 0$, so the demand in any stage takes values over its full support.

Each stage is a canonical interval of time on the calendar, such as, a day, a week or a month. We are aware of the calendar in the sense that we know when the current stage starts, but we only have probabilistic information about the number of customer arrivals in each stage. Customers in the current stage arrive one by one. Each arriving customer makes a request for a product. We decide whether to accept the product request. Our goal is to find a policy to decide which customer requests to accept so that we maximize the total expected revenue over the selling horizon. We give a dynamic program to compute the optimal policy. We use $\boldsymbol{y} = (y_i : i \in \mathcal{L}) \in \mathbb{Z}^{|\mathcal{L}|}_+$ to capture the state of the system, where y_i is the remaining capacity of resource i. We use $\boldsymbol{u} = (u_j : j \in \mathcal{J}) \in \{0,1\}^{|\mathcal{J}|}$ to capture the decisions, where $u_j = 1$ if and only if we accept a request for product j. The set of feasible decisions is given by $\mathcal{F}(\boldsymbol{y}) = \{\boldsymbol{u} \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, u_j \leq y_i \quad \forall\, i \in \mathcal{L},\ j \in \mathcal{J}\}$, ensuring that if we want to accept a request for product j and the product uses resource i, then we need to have at least one unit of remaining capacity for resource i. We can find the optimal policy by computing the value functions $\{J^k_t : t \in \mathcal{T},\ k \in \mathcal{K}\}$ through the dynamic program

$$J^k_t(\boldsymbol{y},q) = \max_{\boldsymbol{u}\in\mathcal{F}(\boldsymbol{y})} \left\{ \sum_{j\in\mathcal{T}} \lambda^k_{jt} \left\{ f_j\, u_j + \theta^k_t(q)\, J^k_{t+1}(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, q) + (1-\theta^k_t(q))\, J^{k+1}_1(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, t) \right\} \right\}, \quad (1)$$

with the boundary condition that $J^{K+1}_1 = 0$. Note that the state variable above keeps both the remaining capacities of the resources and the demand in the previous stage.

In the dynamic program above, we have a request for product j at time period t in stage k with probability λ^k_{jt}. If we accept this request, then we generate a revenue of f_j and consume the capacities of the resources used by the product. Given that the demand in the current stage is already t and the demand in the previous stage was q, we have one more demand in the current stage with probability $\theta_t^k(q)$. The demand in the stage right before the beginning of the selling horizon is D^0 and it is a part of the problem data. Using $\boldsymbol{c} = (c_i : i \in \mathcal{L})$ to denote the initial resource capacities, the optimal total expected revenue is $\mathsf{OPT} = J_1^1(\boldsymbol{c}, D^0)$. Our model is unique in the sense that it allows dependent demands in different stages. The form of the survival rate function is general, so the demand in each stage can have an arbitrary distribution. Therefore, our model can capture general demand distributions, while allowing dependent demands in successive stages. In our model, we decide whether to accept the product request from each customer, but our results hold when we make pricing or assortment offer decisions for each customer.

We focus on developing fluid approximations corresponding to the dynamic program in (1) with the goal of obtaining tractable upper bounds and policies with performance guarantees.

3. Fluid Approximation

The dynamic program in (1) involves a high-dimensional state variable, so it is computationally difficult to solve this dynamic program. We give a fluid approximation that will serve two purposes. First, we will use the fluid approximation to obtain an upper bound on the optimal total expected revenue. In this case, we can compare the total expected revenue obtained by any policy with the upper bound to assess the optimality gap of the policy. Second, we will use the fluid approximation to construct a policy that is asymptotically optimal as the capacities of the resources and the demands get large. In our fluid approximation, we use the decision variable $x_{jt}^k(q)$ to capture the probability of accepting a request for product j at time period t in stage k given that the demand in the previous stage was q. Using the vector $\boldsymbol{x} = (x_{jt}^k(q) : j \in \mathcal{J},\ t, q \in \mathcal{T},\ k \in \mathcal{K})$, to approximate the optimal total expected revenue over the selling horizon, consider linear program

$$
\begin{aligned}
\overline{Z}_{\mathsf{LP}} \;=\; \max_{\boldsymbol{x} \in \mathbb{R}_+^{|\mathcal{J}|T^2K}} \quad & \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \sum_{q \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \, \mathbb{P}\{D^k \geq t,\, D^{k-1} = q\} \, x_{jt}^k(q) \\
\text{st} \quad & \sum_{\ell=1}^{k-1} \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} \sum_{j \in \mathcal{J}} a_{ij} \, \mathbb{P}\{D^\ell \geq s,\, D^{\ell-1} = p \,|\, D^k \geq t,\, D^{k-1} = q\} \, x_{js}^\ell(p) \\
& \qquad + \sum_{s=1}^{t} \sum_{j \in \mathcal{J}} a_{ij} \, x_{js}^k(q) \leq c_i \qquad \forall\, i \in \mathcal{L},\ t, q \in \mathcal{T},\ k \in \mathcal{K} \\
& x_{jt}^k(q) \leq \lambda_{jt}^k \qquad \forall\, j \in \mathcal{J},\ t, q \in \mathcal{T},\ k \in \mathcal{K}.
\end{aligned} \tag{2}
$$

In the linear program above, the objective function accounts for the total expected revenue over the selling horizon. In particular, we can make a sale for product j at time period t in stage k

only if the demand in stage k is at least t. Furthermore, if the demand in stage $k-1$ is q, then we make a sale for product j at time period t in stage k with probability $x^k_{jt}(q)$. Therefore, the expression $\sum_{q\in\mathcal{T}} \mathbb{P}\{D^k \geq t,\, D^{k-1}=q\}\, x^k_{jt}(q)$ corresponds to the expected sales for product j at time period t in stage k. The left side of the first constraint corresponds to the total expected capacity consumption of resource i up to and including time period t in stage k conditional on the fact that the demand in stage k is at least t and the demand in stage $k-1$ is q. In the first sum, similar to the objective function, the expression $\sum_{p\in\mathcal{T}} \mathbb{P}\{D^\ell \geq s,\, D^{\ell-1}=p \,|\, D^k \geq t,\, D^{k-1}=q\}\, x^\ell_{js}(p)$ corresponds to the expected sales for product j at time period s in stage ℓ conditional on the fact that the demand in stage k is at least t and the demand in stage $k-1$ is q. If product j uses resource i, then these sales consume the capacity of resource i. In the second sum, conditional on the fact that the demand in stage k is at least t and the demand in stage $k-1$ is q, we can make a sale for product j at all time periods in stage k up to and including time period t. Furthermore, we accept a request for product j at time period s in stage k with probability $x^k_{js}(q)$. The second constraint is the demand constraint, ensuring that the probability of accepting a request for a product at any time period in any stage does not exceed the probability of getting the request.

We emphasize two novel aspects of the linear program above. First, because the demand in stage k depends on the demand in stage $k-1$, the probability of accepting a request for a product at any time period in stage k depends on the demand in stage $k-1$ as well. Second, perhaps more surprisingly, the first constraint keeps track of the conditional total expected capacity consumption of a resource up to and including time period t in stage k, where we condition on the fact that demand in stage k is at least t and the demand in stage $k-1$ is q. The form of this conditioning is unexpected. We can use the Markovian structure of the demands to slightly simplify the first constraint. If $\ell \leq k-1$, then conditional on D^{k-1}, D^ℓ and $D^{\ell-1}$ are independent of D^k. Thus, we can replace the probability $\mathbb{P}\{D^\ell \geq s,\, D^{\ell-1}=p \,|\, D^k \geq t,\, D^{k-1}=q\}$ in the first sum with $\mathbb{P}\{D^\ell \geq s,\, D^{\ell-1}=p \,|\, D^{k-1}=q\}$. Furthermore, the second sum is increasing in t, so we can replace the sum $\sum_{s=1}^t \sum_{j\in\mathcal{J}} a_{ij}\, x^k_{js}(q)$ with $\sum_{s\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, x^k_{js}(q)$. Therefore, we can express the first constraint equivalently as $\sum_{\ell=1}^{k-1} \sum_{s\in\mathcal{T}} \sum_{p\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^\ell \geq s,\, D^{\ell-1}=p \,|\, D^{k-1}=q\}\, x^\ell_{js}(p) + \sum_{s\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, x^k_{js}(q) \leq c_i$ for all $i \in \mathcal{L}$, $q \in \mathcal{T}$ and $k \in \mathcal{K}$. In this way, we reduce the number of constraints in the first constraint by a factor of T. Nevertheless, we believe that our fluid approximation, as stated in (2), is more instructive, so we keep it in its full form.

We discuss using our fluid approximation to upper bound the optimal total expected revenue.

Upper Bound on the Optimal Total Expected Revenue:

We can show that the optimal objective value of the linear program in (2) is an upper bound on the optimal total expected revenue. From the practical side, it is difficult to compute the optimal policy, but we can compare the performance of any heuristic policy with the upper bound on the optimal total expected revenue to assess the optimality gap of the heuristic policy. From the theoretical side, we will give performance guarantees for a policy that is obtained by using the linear program in (2). Because it is difficult to compute the optimal policy, we establish these performance guarantees by comparing the total expected revenue of the policy with the upper bound on the optimal total expected revenue. In the next theorem, we show that the optimal objective value of problem (2) is indeed an upper bound on the optimal total expected revenue.

Theorem 3.1 (Upper Bound) *Using* OPT *to denote the optimal total expected revenue and* $\overline{Z}_{\mathsf{LP}}$ *to denote the optimal objective value of problem* (2)*, we have* $\overline{Z}_{\mathsf{LP}} \geq \mathsf{OPT}$*.*

We give the proof of the theorem in Appendix A. The proof has two parts. The first part is somewhat standard. In the dynamic program in (1), we have the capacity constraints $a_{ij}\, u_j \leq y_i$ for all $i \in \mathcal{L}$ and $j \in \mathcal{J}$. We relax these constraints by associating non-negative Lagrange multipliers with them, in which case, we obtain a relaxed dynamic program. For fixed choice of Lagrange multipliers, we can compute the value functions of the relaxed dynamic program in closed form. Also, for any choice of non-negative Lagrange multipliers, the value functions of the relaxed dynamic program provide an upper bound on the value functions in (1). Thus, we can use the relaxed dynamic program to obtain an upper bound on the optimal total expected revenue. We formulate the problem of choosing the Lagrange multipliers to obtain the tightest possible upper bound as a linear program. We refer to this linear program as the auxiliary linear program. The second part requires more care. It is difficult to interpret the auxiliary linear program, but we show that any feasible solution to the auxiliary linear program can be transformed into a feasible solution to problem (2) such that the objective values provided by these two solutions match. Thus, the optimal objective value of problem (2) is an upper bound on that of the auxiliary linear program, which is, in turn, an upper bound on the optimal total expected revenue.

4. Asymptotic Optimality

We use an optimal solution to problem (2) to construct an approximate policy. We show that this policy has a constant-factor performance guarantee, but if both the number of stages in the selling horizon and capacities of the resources increase with the same rate, then the policy is asymptotically optimal. Thus, we expect the approximate policy to perform particularly well for systems that command large demands for the products and involve large capacities for the resources. Because of its constant-factor guarantee, however, the approximate policy can never perform arbitrarily badly, even when the product demands and resource capacities are small. In our approximate policy, we solve the linear program in (2) once at the beginning of the selling horizon. Letting $\overline{\boldsymbol{x}} = (\overline{x}^k_{jt}(q) : j \in \mathcal{J},\ t, q \in \mathcal{T},\ k \in \mathcal{K})$ be an optimal solution, we make the decisions as follows.

Approximate Policy from the Fluid Approximation:

Using $\gamma \in [0,1]$ to denote a tuning parameter, if we have a request for product j at time period t in stage k and the demand in stage $k-1$ was q, then we are willing to accept the request with probability $\gamma\, \frac{\overline{x}^k_{jt}(q)}{\lambda^k_{jt}}$. If we are willing to accept the request and there are enough resource capacities to accept the request, then we accept the request. Otherwise, we reject.

Letting $c_{\min} = \min_{i \in \mathcal{L}} c_i$ be the smallest resource capacity and $L = \max_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} a_{ij}$ be the maximum number of resources used by a product, we have the next performance guarantee.

Theorem 4.1 (Performance Guarantee) *Using* APX *to denote the total expected revenue of the approximate policy, there exists a choice of the tuning parameter* γ *such that we have*

$$\frac{\mathsf{APX}}{\mathsf{OPT}} \ \geq\ \frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \ \geq\ \max\left\{ \frac{1}{4L}, \left(1 - 4\, \frac{\sqrt{(c_{\min} + \frac{1}{\epsilon^6}\,(K-1))\log c_{\min}}}{c_{\min}} - \frac{L}{c_{\min}} \right) \right\}.$$

We devote the next section to the proof of the theorem. To our knowledge, this theorem gives the first policy with an asymptotic performance guarantee under dependent demands. Furthermore, the proof involves ideas that have not been used in the related literature. There are two parts in the performance guarantee in the theorem, corresponding to the two terms in the max operator. The proof of the first part of the performance guarantee uses the Markov inequality to lower bound the probability that we have enough resource capacities to accept a product request. The Markov inequality uses only the first moment of a random variable to upper bound its tail probabilities, so the dependence between the demands in different stages does not introduce much complication, but it is still important to use the fluid approximation with the right structure, as given in (2). The proof of the second part of the performance guarantee is more involved. This part uses moment generating function of the resource capacity consumptions to lower bound the probability that we have enough resource capacities to accept a product request. Because of the dependence between the demands in different stages, it is difficult to characterize the moment generation function of the resource capacity consumptions. We bound the moment generating functions by using martingales and the method of bounded differences; see Chapter 5 in Dubhashi and Panconesi (2009). Dependence between the demands requires us to derive our own moment generating function bounds, which ultimately yield the tail probability bounds needed for our performance guarantee.

We proceed to interpreting the two parts in the performance guarantee in Theorem 4.1. In many network revenue management settings, the number of resources and number of products can be large, but the number of resources used by a particular product remains bounded. In the airline setting, for example, we may have hundreds of flight legs and thousands of itineraries, but the number of flight legs in an itinerary rarely exceeds two, corresponding to $L=2$. Thus, the first part in the performance guarantee provides a constant-factor performance guarantee for the approximate policy when L is uniformly bounded. On the other hand, consider a regime where we scale both the number of stages and resource capacities with the same rate θ, so that $K=\theta\,\overline{K}$ and $c_{\min}=\theta\,\bar{c}$ for some fixed $\overline{K},\bar{c}\in\mathbb{Z}_+$. If θ gets large, then the expected demands for the products and capacities for the resources both get large. Letting APX^θ be the total expected revenue from the approximate policy, OPT^θ be the optimal total expected revenue and $\overline{Z}^\theta_{\mathsf{LP}}$ be the optimal objective value of problem (2) when we scale the number of stages and resource capacities with θ, by Theorem 4.1, we have $1\geq\frac{\mathsf{APX}^\theta}{\mathsf{OPT}^\theta}\geq\frac{\mathsf{APX}^\theta}{\overline{Z}^\theta_{\mathsf{LP}}}\geq 1-\frac{4}{\epsilon^3}\frac{\sqrt{(\bar{c}+\overline{K})\,\log(\theta\,\bar{c})}}{\bar{c}\sqrt{\theta}}-\frac{L}{\theta\,\bar{c}}$. In this case, ignoring the logarithmic terms, as θ gets large, the relative gap between the total expected revenue of the approximate policy and the optimal total expected revenue converges to one with rate $1-\frac{1}{\sqrt{\theta}}$. Therefore, as the number of stages in the selling horizon and capacities of the resources increase with the same rate, the approximate policy is asymptotically optimal. Similarly, as θ gets large, the relative gap between the total expected revenue of the approximate policy and the optimal objective value of the fluid approximation converges to one with rate $1-\frac{1}{\sqrt{\theta}}$ as well.

The scaling regime in the previous paragraph increases the number of stages and resource capacities. Because the demands in different stages are dependent, increasing the number of stages in the selling horizon is perhaps the most natural approach to increase the expected demands for the products. In this way, we can increase the expected demands for the products without distorting the correlation structure for the demands in different stages. Another approach to increase the expected demands for the products could be to increase the support of the demand in each stage, while keeping the number of stages in the selling horizon fixed. Scaling the expected demand in this fashion can potentially distort the correlation structure for the demands in different stages. In Appendix B, we also give a counterexample to demonstrate that the relative gap between the total expected revenue of the approximate policy and the optimal objective value of the fluid approximation in (2) does not necessarily converge to one as we increase the support of the demand in each stage and capacities of the resources with the same rate, while keeping the number of stages constant. In our counterexample, we give a problem instance with three stages. There is a single resource with a capacity of $C+1$. The largest value of the demand in a week is C. There are two products. We show that the optimal total expected revenue is C, whereas the optimal objective value of problem (2) is $\frac{5}{4}C$. Thus, we have $\frac{\mathsf{APX}^\theta}{\overline{Z}^\theta_{\mathsf{LP}}}\leq\frac{\mathsf{OPT}^\theta}{\overline{Z}^\theta_{\mathsf{LP}}}=\frac{4}{5}$. For this problem instance, no matter how large C is, the ratio $\frac{\mathsf{APX}^\theta}{\overline{Z}^\theta_{\mathsf{LP}}}$ always stays away from one.

5. Performance Guarantee

In this section, we give a proof for Theorem 4.1. We focus on showing the performance guarantee $\frac{\mathsf{APX}}{\mathsf{OPT}} \geq \frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \geq 1 - 4\,\frac{\sqrt{(c_{\min} + \frac{1}{\epsilon^6}\,(K-1))\log c_{\min}}}{c_{\min}} - \frac{L}{c_{\min}}$. To establish this performance guarantee, we use ideas that have not been used in the revenue management literature to analyze fluid approximations. In Appendix C, we turn our attention to showing the performance guarantee $\frac{\mathsf{APX}}{\mathsf{OPT}} \geq \frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \geq \frac{1}{4L}$, which is more straightforward. A common approach for analyzing approximate policies from fluid approximations involves using an auxiliary random variable to upper bound the capacity consumption of a resource under the approximate policy. Thus, we can use a concentration inequality to upper bound the tail probabilities of the auxiliary random variable, in which case, we can lower bound the probability that there is enough capacity to accept different product requests at different time periods in the selling horizon; see, for example, Feng et al. (2022). This approach usually exploits the fact that the auxiliary random variable can be expressed as a sum of independent random variables, which facilitates using concentration inequalities for sums of independent random variables. Because the demands in different stages are dependent in our setting, we cannot construct similar auxiliary random variables that can be expressed as sums of independent random variables. Thus, we resort to new ideas.

Preliminary Random Variables and Availability Probabilities:

We define four classes of Bernoulli random variables for each $k \in \mathcal{K}$ and $t \in \mathcal{T}$. Analogues of these random variables have been used in the analysis of other fluid approximations.

- *Demand in Each Stage.* For each $q \in \mathcal{T}$, the random variable $\Psi_t^k(q)$ takes value one if we reach time period t in stage k before this stage is over and the demand in stage $k-1$ is q. In other words, letting $\mathbf{1}(\cdot)$ be the indicator function, $\Psi_t^k(q) = \mathbf{1}(D^k \geq t,\ D^{k-1} = q)$.

- *Product Request.* For each $j \in \mathcal{J}$, the random variable A_{jt}^k takes value one if the customer arriving at time period t in stage k requests product j. We have $\mathbb{P}\{A_{jt}^k = 1\} = \lambda_{jt}^k$. The random variables $\{A_{jt}^k : t \in \mathcal{T},\ k \in \mathcal{K}\}$ are independent of each other.

- *Policy Decision.* For each $j \in \mathcal{J}$ and $q \in \mathcal{T}$, the random variable $X_{jt}^k(q)$ takes value one if the approximate policy is willing to accept a request for product j at time period t in stage k when the demand in stage $k-1$ was q. By our approximate policy, $\mathbb{P}\{X_{jt}^k(q) = 1\} = \gamma\,\frac{\overline{x}_{jt}^k(q)}{\lambda_{jt}^k}$.

- *Availability.* For each $j \in \mathcal{J}$, the random variable G_{jt}^k takes value one if we have enough capacity to accept a request for product j at time period t in stage k under the approximate policy. Instead of calculating the probability $\mathbb{P}\{G_{jt}^k = 1\}$, we will lower bound $\mathbb{P}\{G_{jt}^k = 1 \mid \Psi_t^k(q) = 1\}$.

The random variables A_{jt}^k and $X_{jt}^k(q)$ are both simple Bernoulli draws independent of the decisions of the approximate policy, remaining capacities of the resources or realizations of the demands in different stages. Under the approximate policy, the sales for product j at time period t in stage k is given by $\sum_{q\in\mathcal{T}} \Psi_t^k(q)\, G_{jt}^k\, A_{jt}^k\, X_{jt}^k(q)$, where we use the fact that we sell product j at time period t in stage k if we reach time period t in stage k, there is enough capacity to accept a request for product j, we have a request for the product and the approximate policy is willing to accept the product request. The remaining capacities of the resources at time period t in stage k depend on the requests for the products and decisions of the approximate policy at the earlier time periods, but not at time period t in stage k. Thus, taking expectations, the expected sales for product j at time period t in stage k is $\sum_{q\in\mathcal{T}} \mathbb{P}\{\Psi_t^k(q)=1\}\,\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}\,\mathbb{P}\{A_{jt}^k=1\}\,\mathbb{P}\{X_{jt}^k(q)=1\}$. In this case, noting that $\mathbb{P}\{A_{jt}^k=1\}=\lambda_{jt}^k$ and $\mathbb{P}\{X_{jt}^k(q)=1\}=\gamma\,\frac{\overline{x}_{jt}^k(q)}{\lambda_{jt}^k}$, we can write the last expectation equivalently as $\sum_{q\in\mathcal{T}} \mathbb{P}\{D^k\ge t,\ D^{k-1}=q\}\,\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}\,\gamma\,\overline{x}_{jt}^k(q)$, capturing the expected sales for product j at time period t in stage k under the approximate policy. Thus, the total expected revenue of the approximate policy is given by

$$\mathsf{APX} \;=\; \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}\sum_{q\in\mathcal{T}} f_j\,\mathbb{P}\{D^k\ge t,\ D^{k-1}=q\}\ \mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}\ \gamma\,\overline{x}_{jt}^k(q). \tag{3}$$

By the definition of $\overline{\boldsymbol{x}}$, we have $\overline{Z}_{\mathsf{LP}}=\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}\sum_{q\in\mathcal{T}} f_j\,\mathbb{P}\{D^k\ge t,\,D^{k-1}=q\}\,\overline{x}_{jt}^k(q)$, so if we can show that $\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}\ge\alpha$, then we get $\mathsf{APX}\ge\gamma\,\alpha\,\overline{Z}_{\mathsf{LP}}$.

Motivated by the discussion in the previous paragraph, we focus on lower bounding the availability probability $\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}$. Under the approximate policy, the sales for product j at time period t in stage k is $\sum_{q\in\mathcal{T}}\Psi_t^k(q)\,G_{jt}^k\,A_{jt}^k\,X_{jt}^k(q)$, so $\sum_{q\in\mathcal{T}}\Psi_t^k(q)\,A_{jt}^k\,X_{jt}^k(q)$ is an upper bound on these sales for product j at time period t in stage k. In this case, $\sum_{j\in\mathcal{J}}\sum_{q\in\mathcal{T}} a_{ij}\,\Psi_t^k(q)\,A_{jt}^k\,X_{jt}^k(q)$ is an upper bound on the capacity consumption of resource i at time period t in stage k. Letting $N_{it}^k(q)=\sum_{j\in\mathcal{J}} a_{ij}\,A_{jt}^k\,X_{jt}^k(q)$, we express our upper bound on the capacity consumption of resource i at time period t in stage k as $\sum_{q\in\mathcal{T}}\Psi_t^k(q)\,N_{it}^k(q)$. Note that $\{N_{it}^k(q): t\in\mathcal{T},\ k\in\mathcal{K}\}$ are Bernoulli random variables and they are independent of each other. Having $\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{q\in\mathcal{T}}\Psi_s^\ell(q)\,N_{is}^\ell(q)+\sum_{s=1}^{t}\sum_{q\in\mathcal{T}}\Psi_s^k(q)\,N_{is}^k(q)<c_i$ implies that the total capacity consumption of resource i up to and including time period t in stage k does not exceed the capacity of the resource, in which case, we have capacity available for resource i at time period t in stage k. Therefore, letting $\mathcal{L}_j=\{i\in\mathcal{L}: a_{ij}=1\}$ to capture the set of resources used by product j, we obtain

$$\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\}\ge\mathbb{P}\left\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}\sum_{p\in\mathcal{T}}\Psi_s^k(p)\,N_{is}^k(p)<c_i\ \forall\, i\in\mathcal{L}_j\ \middle|\ \Psi_t^k(q)=1\right\},$$

where we use the fact if the upper bounds on the consumption of the resources used by product j do not exceed their capacities, then we have capacities to accept a request for product j.

By the definition of $\Psi_t^k(q)$, having $\Psi_t^k(q)=1$ is equivalent to having $D^k\ge t$ and $D^{k-1}=q$. Thus, if $\Psi_t^k(q)=1$, then we have $D^k\ge s$ for all $s=1,\ldots,t$, $D^{k-1}=q$ and $D^{k-1}\ne p$ for all $p\in\mathcal{T}\setminus\{q\}$.

Therefore, if $\Psi_t^k(q)=1$, then $\Psi_s^k(q)=1$ for all $s=1,\dots,t$ and $\Psi_s^k(p)=0$ for all $p\in\mathcal{T}\setminus\{q\}$. In this case, the inequality above is equivalent to

$$\begin{aligned}
\mathbb{P}\{G_{jt}^k=1\,|\,\Psi_t^k(q)=1\} \;&\geq\; \mathbb{P}\bigg\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)<c_i\;\forall\, i\in\mathcal{L}_j\;\bigg|\;\Psi_t^k(q)=1\bigg\}\\
&=\;1-\mathbb{P}\bigg\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)\geq c_i\text{ for some } i\in\mathcal{L}_j\;\bigg|\;\Psi_t^k(q)=1\bigg\}\\
&\geq\;1-\sum_{i\in\mathcal{L}_j}\mathbb{P}\bigg\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)\geq c_i\;\bigg|\;\Psi_t^k(q)=1\bigg\}\\
&=\;1-\sum_{i\in\mathcal{L}_j}\mathbb{P}\bigg\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)\geq c_i\;\bigg|\;D^{k-1}=q\bigg\},
\end{aligned}\tag{4}$$

where the second inequality is the union bound and the second equality holds because given D^{k-1}, $D^1,\dots,D^{k-1}$ are independent of D^k. Thus, it is enough to upper bound the last probability in (4).

<u>**Moment Generating Function Bounds:**</u>

The discussion so far in this section has been following standard arguments, but we proceed to introducing new ideas. To upper bound the last probability on the right side of (4), letting $n_{it}^k(q)=\mathbb{E}\{N_{it}^k(q)\}$, for all $i\in\mathcal{L}$ and $k\in\mathcal{K}$, we define $U_i^k=\sum_{\ell=1}^k\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)$ and $V_i^k=\sum_{\ell=1}^k\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,n_{is}^\ell(p)$. Using the vector of random variables $\boldsymbol{D}^{[\ell,k]}=(D^\ell,\dots,D^k)$ for notational brevity, note that the random variable V_i^k is a deterministic function of $\boldsymbol{D}^{[1,k]}$. Because e^x is convex in x, using the Jensen inequality, it is simple to show that $\mathbb{E}\{e^{\lambda V_i^k}\,|\,D^k\}\leq\mathbb{E}\{e^{\lambda U_i^k}\,|\,D^k\}$ for all $\lambda\geq 0$, so the moment generating function of the random variable U_i^k conditional on D^k upper bounds its counterpart for the random variable V_i^k. In the next lemma, we characterize the gap between the two moment generating functions.

Lemma 5.1 (Moment Generating Function Gap) *For all $k\in\mathcal{K}$, $i\in\mathcal{L}$ and $\lambda\geq 0$, we have $\mathbb{E}\{e^{\lambda U_i^k}\,|\,D^k\}\leq e^{\frac{1}{2\epsilon}k\lambda^2}\,\mathbb{E}\{e^{\lambda V_i^k}\,|\,D^k\}$.*

Proof: The random variables $\{N_{is}^\ell(p): s\in\mathcal{T},\ \ell=1,\dots,k\}$ are independent of each other. Also, for $\ell=1,\dots,k$ and $s,p\in\mathcal{T}$, $\Psi_s^\ell(p)$ is a deterministic function of $\boldsymbol{D}^{[1,k]}$. Thus, we have

$$\begin{aligned}
\mathbb{E}\{e^{\lambda(U_i^k-V_i^k)}\,|\,\boldsymbol{D}^{[1,k]}\}\;&=\;\mathbb{E}\{e^{\lambda\sum_{\ell=1}^k\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)(N_{is}^\ell(p)-n_{is}^\ell(p))}\,|\,\boldsymbol{D}^{[1,k]}\}\\
&=\;\prod_{\ell=1}^k\prod_{s\in\mathcal{T}}\mathbb{E}\{e^{\lambda\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)(N_{is}^\ell(p)-n_{is}^\ell(p))}\,|\,\boldsymbol{D}^{[1,k]}\}.
\end{aligned}\tag{5}$$

Because $\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\leq 1$ and the random variable $N_{is}^\ell(p)$ is Bernoulli with expectation $n_{is}^\ell(p)$, we have $\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)(N_{is}^\ell(p)-n_{is}^\ell(p))\in[-1,1]$. If the mean-zero random variable Z is bounded by

$[a,b]$, then we have $\mathbb{E}\{e^{\lambda Z}\} \leq e^{\frac{1}{8}(b-a)^2\lambda^2}$ for any $\lambda \geq 0$; see Lemma 5.1 in Dubhashi and Panconesi (2009). Thus, the last conditional expectation on the right side of (5) is upper bounded by $e^{\frac{1}{2}\lambda^2}$, so by (5), we obtain $\mathbb{E}\{e^{\lambda(U_i^k - V_i^k)} \,|\, \boldsymbol{D}^{[1,k]}\} \leq e^{\frac{1}{2}kT\lambda^2}$. The random variable V_i^k is a deterministic function of $\boldsymbol{D}^{[1,k]}$. In this case, using the tower property of conditional expectations, we get $\mathbb{E}\{e^{\lambda U_i^k} \,|\, D^k\} = \mathbb{E}\{\,\mathbb{E}\{e^{\lambda V_i^k}\, e^{\lambda(U_i^k - V_i^k)} \,|\, \boldsymbol{D}^{[1,k]}\} \,|\, D^k\} = \mathbb{E}\{e^{\lambda V_i^k}\, \mathbb{E}\{e^{\lambda(U_i^k - V_i^k)} \,|\, \boldsymbol{D}^{[1,k]}\} \,|\, D^k\}$. Using the fact that $\mathbb{E}\{e^{\lambda(U_i^k - V_i^k)} \,|\, \boldsymbol{D}^{[1,k]}\} \leq e^{\frac{1}{2}kT\lambda^2}$, we obtain $\mathbb{E}\{e^{\lambda U_i^k} \,|\, D^k\} \leq e^{\frac{1}{2}kT\lambda^2}\mathbb{E}\{e^{\lambda V_i^k} \,|\, D^k\}$ by the last chain of equalities. The result follows by noting the assumption that $\mathbb{P}\{D^{k+1} = p \,|\, D^k = q\} \geq \epsilon$ for all $p, q \in \mathcal{T}$, so $1 = \sum_{p\in\mathcal{T}} \mathbb{P}\{D^{k+1} = p \,|\, D^k = q\} \geq T\epsilon$, which implies that $T \leq 1/\epsilon$. ■

If the random variable Z satisfies $\mathbb{E}\{e^{\lambda Z}\} \leq f(\lambda)$ for all $\lambda \geq 0$, then we can upper bound its tail probabilities as $\mathbb{P}\{Z \geq c\} = \mathbb{P}\{e^{\lambda Z} \geq e^{\lambda c}\} \leq \frac{1}{e^{\lambda c}}\mathbb{E}\{e^{\lambda Z}\} \leq \frac{f(\lambda)}{e^{\lambda c}}$, where the first inequality uses the Markov inequality. In the last probability in (4), we have $U_i^{k-1} = \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}} \Psi_s^\ell(p)\, N_{is}^\ell(p)$, so we may use the moment generation function of U_i^k to lower bound the availability probabilities. By Lemma 5.1, the moment generating function of V_i^k can be a proxy for the moment generating function of U_i^k. For all $i \in \mathcal{L}$, $k \in \mathcal{K}$ and $\ell = 1, \ldots, k$, we define $M_i^k(\ell) = \mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[\ell,k]}\}$. Therefore, the random variable $M_i^k(\ell)$ is a deterministic function of $\boldsymbol{D}^{[\ell,k]}$. Noting that V_i^k is a deterministic function of $\boldsymbol{D}^{[1,k]}$, we have $M_i^k(1) = \mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[1,k]}\} = V_i^k$ with probability one.

In the next lemma, we upper bound the moment generating function of $M_i^k(\ell)$ for all $\ell = 1, \ldots, k$, which, noting that $M_i^k(1) = V_i^k$, will yield an upper bound on the same for V_i^k.

Lemma 5.2 (Moment Generating Function Bound) *Letting $M_i^0(0) = 0$, for all $k \in \mathcal{K}$, $i \in \mathcal{L}$, $\ell = 0, \ldots, k-1$ and $\lambda \geq 0$, we have*

$$\mathbb{E}\{e^{\lambda(M_i^{k-1}(\ell) + \sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1} = q\} \;\leq\; e^{\frac{2}{\epsilon^6}(k-1-\ell)\lambda^2 + \gamma\, c_i \lambda}.$$

Proof: We show the result by using induction over $\ell = 1, \ldots, k-1$. Consider the case $\ell = k-1$. We have $\mathbb{E}\{M_i^{k-1}(k-1) \,|\, D^{k-1}\} = \mathbb{E}\{\,\mathbb{E}\{V_i^{k-1} \,|\, D^{k-1}\} \,|\, D^{k-1}\} = \mathbb{E}\{V_i^{k-1} \,|\, D^{k-1}\}$, so we get

$$\begin{aligned}
\mathbb{E}\{M_i^{k-1}(k-1) \,|\, D^{k-1} = q\} + \sum_{s=1}^t n_{is}^k(q) \;&=\; \mathbb{E}\{V_i^{k-1} \,|\, D^{k-1} = q\} + \sum_{s=1}^t n_{is}^k(q) \\
&\overset{(a)}{=} \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}} \mathbb{P}\{D^\ell \geq s,\; D^{\ell-1} = p \,|\, D^{k-1} = q\}\, n_{is}^\ell(p) + \sum_{s=1}^t n_{is}^k(q) \\
&\overset{(b)}{=} \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^\ell \geq s,\; D^{\ell-1} = p \,|\, D^{k-1} = q\}\, \gamma\, \overline{x}_{js}^\ell(p) + \sum_{s=1}^t\sum_{j\in\mathcal{J}} a_{ij}\, \gamma\, x_{js}^k(q) \;\overset{(c)}{\leq}\; \gamma\, c_i,
\end{aligned}$$

where (a) uses the definition of V_i^{k-1}, (b) follows because we have $n_{is}^\ell(p) = \mathbb{E}\{N_{is}^\ell(p)\}$, in which case, by the definition of $N_{is}^\ell(p)$, we get $n_{is}^\ell(p) = \mathbb{E}\{N_{is}^\ell(p)\} = \sum_{j\in\mathcal{J}} a_{ij}\, \gamma\, \overline{x}_{js}^\ell(p)$ and (c) holds by the first constraint in problem (2), as well as noting that conditional on D^{k-1}, D^ℓ is independent of D^k for $\ell \le k-1$, so $\mathbb{P}\{D^\ell \ge s,\, D^{\ell-1}=p \,|\, D^k \ge t,\, D^{k-1}=q\} = \mathbb{P}\{D^\ell \ge s,\, D^{\ell-1}=p \,|\, D^{k-1}=q\}$. By its definition, $M_i^{k-1}(k-1)$ is a deterministic function of D^{k-1}, which implies that given $D^{k-1}=q$, $M_i^{k-1}(k-1)$ is a deterministic quantity. Thus, using the chain of inequalities above, we obtain $\mathbb{E}\{e^{\lambda(M_i^{k-1}(k-1)+\sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1}=q\} = e^{\mathbb{E}\{\lambda(M_i^{k-1}(k-1)+\sum_{s=1}^t n_{is}^k(q)) \,|\, D^{k-1}=q\}} \le e^{\gamma\, c_i \lambda}$, so the result holds for $\ell = k-1$. Assuming that the result holds for $\ell+1 \le k-1$, we show that the result holds for $\ell \le k-1$. In Lemma D.3 in Appendix D, we show that $|M_i^{k-1}(\ell) - M_i^{k-1}(\ell+1)| \le \frac{2}{\epsilon^3}$ with probability one. Also, by the tower property of conditional expectations, using the definition of $M_i^{k-1}(\ell)$, we have $\mathbb{E}\{M_i^{k-1}(\ell) \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} = \mathbb{E}\{\,\mathbb{E}\{V_i^{k-1} \,|\, \boldsymbol{D}^{[\ell,k-1]}\} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} = \mathbb{E}\{V_i^{k-1} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\}$. Using precisely the same argument, we can verify that $\mathbb{E}\{M_i^{k-1}(\ell+1) \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} = \mathbb{E}\{V_i^{k-1} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\}$ as well. Thus, conditional on $\boldsymbol{D}^{[\ell+1,k-1]}$, the random variable $M_i^{k-1}(\ell) - M_i^{k-1}(\ell+1)$ is mean-zero and bounded by $[-\frac{2}{\epsilon^3}, \frac{2}{\epsilon^3}]$. Recall that if the mean-zero random variable Z is bounded by $[a,b]$, then we have $\mathbb{E}\{e^{\lambda Z}\} \le e^{\frac{1}{8}(b-a)^2\lambda^2}$. In this case, we get $\mathbb{E}\{e^{\lambda(M_i^{k-1}(\ell)-M_i^{k-1}(\ell+1))} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} \le e^{\frac{2}{\epsilon^6}\lambda^2}$. Thus, using the fact that $M_i^{k-1}(\ell+1)$ is a deterministic function of $\boldsymbol{D}^{[\ell+1,k-1]}$, we have

$$
\begin{aligned}
\mathbb{E}\{e^{\lambda M_i^{k-1}(\ell)} \,|\, D^{k-1}\} &= \mathbb{E}\{\,\mathbb{E}\{e^{\lambda M_i^{k-1}(\ell+1)}\, e^{\lambda(M_i^{k-1}(\ell)-M_i^{k-1}(\ell+1))} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} \,|\, D^{k-1}\} \\
&= \mathbb{E}\{e^{\lambda M_i^{k-1}(\ell+1)}\, \mathbb{E}\{e^{\lambda(M_i^{k-1}(\ell)-M_i^{k-1}(\ell+1))} \,|\, \boldsymbol{D}^{[\ell+1,k-1]}\} \,|\, D^{k-1}\} \;\le\; e^{\frac{2}{\epsilon^6}\lambda^2}\, \mathbb{E}\{e^{\lambda M_i^{k-1}(\ell+1)} \,|\, D^{k-1}\}.
\end{aligned}
$$

Thus, we get $\mathbb{E}\{e^{\lambda(M_i^{k-1}(\ell)+\sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1}=q\} \le e^{\frac{2}{\epsilon^6}\lambda^2}\, \mathbb{E}\{e^{\lambda(M_i^{k-1}(\ell+1)+\sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1}=q\} \le e^{\frac{2}{\epsilon^6}\lambda^2}\, e^{\frac{2}{\epsilon^6}(k-2-\ell)\lambda^2+\gamma c_i \lambda}$, where the last inequality is by the induction assumption. ■

Note that $\{M_i^k(\ell) : \ell = 1,\dots,k\}$ is a martingale adapted to $\{\boldsymbol{D}^{[\ell,k]} : \ell=1,\dots,k\}$ in the sense that $\mathbb{E}\{M_i^k(\ell) \,|\, \boldsymbol{D}^{[\ell+1,k]}\} = \mathbb{E}\{\,\mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[\ell,k]}\} \,|\, \boldsymbol{D}^{[\ell+1,k]}\} = \mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[\ell+1,k]}\} = M_i^k(\ell+1)$.

Performance Guarantee for the Approximate Policy:

In the next lemma, we use the moment generating function bounds given in Lemmas 5.1 and 5.2 to lower bound the availability probability on the right side of (4).

Lemma 5.3 (Availability Probability Bound) *Letting $U_i^0 = 0$, for all $k \in \mathcal{K}$, $i \in \mathcal{L}$, $t,q \in \mathcal{T}$ and $\lambda \in [0,1]$, we have*

$$
\mathbb{P}\left\{U_i^{k-1} + \sum_{s=1}^t N_{is}^k(q) \ge c_i \;\middle|\; D^{k-1}=q\right\} \;\le\; e^{(c_i + \frac{3}{\epsilon^6}(k-1))\lambda^2 - (1-\gamma)c_i\lambda}.
$$

Proof: By the discussion just after the definition of $M_i^k(\ell)$, we have $V_i^{k-1} = M_i^{k-1}(1)$ with probability one, so $\mathbb{E}\{e^{\lambda(V_i^{k-1}+\sum_{s\in\mathcal{T}} n_{is}^k(q))} \,|\, D^{k-1}=q\} = \mathbb{E}\{e^{\lambda(M_i^{k-1}(1)+\sum_{s\in\mathcal{T}} n_{is}^k(q))} \,|\, D^{k-1}=q\} \le e^{\frac{2}{\epsilon^6}(k-2)\lambda^2+\gamma c_i\lambda}$, where the last inequality uses Lemma 5.2. On the other hand, by the first constraint in (2), we have $\sum_{s=1}^t n_{is}^k(q) = \sum_{s=1}^t \sum_{j\in\mathcal{J}} a_{ij}\,\gamma\,\overline{x}_{js}^k(q) \le \gamma\, c_i$. By a simple lemma, given as Lemma D.4 in Appendix

D, if the Bernoulli random variable Z has mean μ, then $\mathbb{E}\{e^{\lambda(Z-\mu)}\} \le e^{\mu\lambda^2}$ for all $\lambda \in [0,1]$, so because $n_{is}^k(q) = \mathbb{E}\{N_{is}^k(q)\}$, we get $\mathbb{E}\{e^{\lambda \sum_{s=1}^t (N_{is}^k(q) - n_{is}^k(q))}\} \le e^{\lambda^2 \sum_{s=1}^t n_{is}^k(q)} \le e^{\gamma\, c_i\, \lambda^2}$. Thus, we have

$$\begin{aligned}
\mathbb{P}\Big\{U_i^{k-1} + \sum_{s=1}^t N_{is}^k(q) \ge c_i \,\Big|\, D^{k-1} = q\Big\} &= \mathbb{P}\{e^{\lambda(U_i^{k-1} + \sum_{s=1}^t N_{is}^k(q))} \ge e^{\lambda\, c_i} \,|\, D^{k-1} = q\} \\
&\overset{(a)}{\le} \frac{1}{e^{\lambda c_i}}\, \mathbb{E}\{e^{\lambda(U_i^{k-1} + \sum_{s=1}^t N_{is}^k(q))} \,|\, D^{k-1} = q\} \\
&\overset{(b)}{=} \frac{1}{e^{\lambda c_i}}\, \mathbb{E}\{e^{\lambda \sum_{s=1}^t (N_{is}^k(q) - n_{is}^k(q))}\}\, \mathbb{E}\{e^{\lambda(U_i^{k-1} + \sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1} = q\} \\
&\overset{(c)}{\le} \frac{1}{e^{\lambda c_i}}\, \mathbb{E}\{e^{\lambda \sum_{s=1}^t (N_{is}^k(q) - n_{is}^k(q))}\}\, e^{\frac{1}{2\epsilon}(k-1)\lambda^2}\, \mathbb{E}\{e^{\lambda(V_i^{k-1} + \sum_{s=1}^t n_{is}^k(q))} \,|\, D^{k-1} = q\} \\
&\overset{(d)}{\le} \frac{1}{e^{\lambda c_i}}\, e^{\gamma\, c_i\, \lambda^2}\, e^{\frac{1}{2\epsilon}(k-1)\lambda^2}\, e^{\frac{2}{\epsilon^6}(k-2)\lambda^2 + \gamma\, c_i \lambda} \\
&= e^{(\gamma c_i + \frac{1}{2\epsilon}(k-1) + \frac{2}{\epsilon^6}(k-2))\lambda^2 - (1-\gamma) c_i \lambda} \overset{(e)}{\le} e^{(c_i + \frac{3}{\epsilon^6}(k-1))\lambda^2 - (1-\gamma) c_i \lambda},
\end{aligned}$$

where (a) uses the Markov inequality, (b) holds because $N_{is}^k(q)$ is independent of D^{k-1}, (c) is by Lemma 5.1, (d) uses the two inequalities at the beginning of the proof and (e) uses $\epsilon \le 1$. ■

Using specific values for γ and λ in Lemma 5.3, we will bound the availability probabilities. Using this bound in (3) will yield the performance guarantee in Theorem 4.1.

Proof of Theorem 4.1:

We use Lemma 5.3 with specific values of γ and λ. Letting $\delta = \frac{1}{\epsilon^6}$ for notational brevity, fix $\overline{\gamma} = 1 - \frac{\sqrt{4\,(c_{\min} + 3\,\delta\,(K-1))\log c_{\min}}}{c_{\min}}$ and $\overline{\lambda} = \frac{(1-\overline{\gamma})\,c_i}{2(c_i + 3\,\delta\,(K-1))}$. For these values of $\overline{\gamma}$ and $\overline{\lambda}$, we have

$$(c_i + 3\,\delta(K-1))\overline{\lambda}^2 - (1-\overline{\gamma})\,c_i \overline{\lambda} \overset{(a)}{=} -\frac{[(1-\overline{\gamma})\,c_i]^2}{4(c_i + 3\,\delta\,(K-1))} \overset{(b)}{\le} -\frac{[(1-\overline{\gamma})\,c_{\min}]^2}{4(c_{\min} + 3\,\delta\,(K-1))} \overset{(c)}{=} -\log c_{\min}, \quad (6)$$

where (a) follows by direct computation with the specific value of $\overline{\lambda}$, (b) holds because we can check the first derivative to verify that $\frac{[(1-\gamma)\,x]^2}{4(x+3\,\delta\,(K-1))}$ is increasing in x for $x \ge 0$ and (c) follows by noting that the value of $\overline{\gamma}$ satisfies $(1-\overline{\gamma})^2\, c_{\min}^2 = 4\,(c_{\min} + 3\,\delta\,(K-1))\log c_{\min}$. Without loss of generality, we can assume that $c_{\min} > 4\sqrt{(c_{\min} + \delta\,(K-1))\log c_{\min}}$. Otherwise, the second term in the max operator in the theorem becomes a negative number and $\frac{\mathsf{APX}}{Z_{\mathsf{LP}}}$ is trivially lower bounded by a negative number, so the result immediately holds. Therefore, we have $c_{\min} > 4\sqrt{(c_{\min} + \delta\,(K-1))\log c_{\min}} \ge \sqrt{4\,(c_{\min} + 3\,\delta\,(K-1))\log c_{\min}}$. In this case, our choice of $\overline{\gamma}$ satisfies $\overline{\gamma} \in [0,1]$. If $\overline{\gamma} \in [0,1]$, then our choice of $\overline{\lambda}$ satisfies $\overline{\lambda} \in [0,1]$ as well. Therefore, we can use Lemma 5.3 with $\gamma = \overline{\gamma}$ and $\lambda = \overline{\lambda}$, so noting (6), we obtain

$$\mathbb{P}\Big\{U_i^{k-1} + \sum_{s=1}^t N_{is}^k(q) \ge c_i \,\Big|\, D^{k-1} = q\Big\} \le e^{(c_i + \frac{3}{\epsilon^6}(K-1))\overline{\lambda}^2 - (1-\overline{\gamma}) c_i \overline{\lambda}} \le e^{-\log c_{\min}} = \frac{1}{c_{\min}}.$$

By the definition of U_i^{k-1}, the probability on the left side above is the same as the probability on the right side of (4). Using the inequality above on the right side of (4), because $|\mathcal{L}_j| \le L$, we get $\mathbb{P}\{G^k_{jt}=1 \,|\, \Psi^k_t(q)=1\} \geq 1-\frac{L}{c_{\min}}$. By the discussion just after (3), if $\mathbb{P}\{G^k_{jt}=1 \,|\, \Psi^k_t(q)=1\} \geq \alpha$, then $\frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \geq \gamma\,\alpha$. Thus, using the specific value of $\overline{\gamma}$, we get

$$\frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \geq \left(1-\frac{\sqrt{4\,(c_{\min}+3\,\delta\,(K-1))\log c_{\min}}}{c_{\min}}\right)\left(1-\frac{L}{c_{\min}}\right)$$
$$\geq \left(1-4\frac{\sqrt{(c_{\min}+\delta\,(K-1))\log c_{\min}}}{c_{\min}}\right)\left(1-\frac{L}{c_{\min}}\right).$$

The result follows by noting that the right side of the chain of inequalities above is lower bounded by $1-4\frac{\sqrt{(c_{\min}+\delta\,(K-1))\log c_{\min}}}{c_{\min}}-\frac{L}{c_{\min}}$, as well as using the fact that $\frac{\mathsf{APX}}{\mathsf{OPT}} \geq \frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}}$ by Theorem 3.1. ■

6. Independent Demands in Different Stages

We focus on the case where the demand random variables $D^1,\ldots,D^K$ in different stages are independent of each other. Note that having the demand random variables $D^1,\ldots,D^K$ independent of each other is not equivalent to having one stage with the number of customer arrivals given by the random variable $D^1+\ldots+D^K$. In particular, the resolution of the demand uncertainty in the two cases are different. Considering the case with K stages and the demand random variables $D^1,\ldots,D^K$ being independent of each other, if we are in stage k and there have been t customer arrivals in the current stage, then the probability of having one more customer arrival is given by $\mathbb{P}\{D^k \geq t+1 \,|\, D^k \geq t\} + \mathbb{P}\{D^k = t \,|\, D^k \geq t\}\; \mathbb{P}\{D^{k+1}+\ldots+D^K \geq 1\}$, which corresponds to the probability of having one more customer arrival in the current stage plus the probability of having no more customer arrivals in the current stage and one more customer arrival in the remaining stages. If the stages correspond to, for example, weeks, then this first demand model is aware of the calendar in the sense that it distinguishes between having one more customer arrival in the current week and finishing the current week with no more customer arrivals. On the other hand, considering the case with one stage and the number of customer arrivals being given by the random variable $D^1+\ldots+D^K$, if there have been v customer arrivals so far, then the probability of having one more customer arrival is given by $\mathbb{P}\{D^1+\ldots+D^K \geq v+1 \,|\, D^1+\ldots+D^K \geq v\}$, which does not explicitly consider the distribution of the random demand over different stages. If the stages correspond to weeks, then this second demand model is not aware of the calendar in the sense that it does not pay attention to the beginning and end of each week. To our knowledge, demand models with uncertainty in the demand resolving sequentially over multiple stages have not been considered in the revenue management literature. Even when the demand random variables in different stages are independent, our demand model and fluid approximation yield new results.

When the demand random variables in different stages are independent, we drop the assumption that $\mathbb{P}\{D^{k+1}=p \,|\, D^k=q\} \geq \epsilon$ for all $p,q \in \mathcal{T}$ and $k \in \mathcal{K}$ for some $\epsilon > 0$. Instead, we assume that the demand random variable in each stage is sub-Gaussian, so we have $\mathbb{E}\{e^{\lambda(D^k-\mathbb{E}\{D^k\})}\} \le e^{\frac{\sigma^2}{200}\lambda^2}$ for all $k \in \mathcal{K}$ and $\lambda \ge 0$ for some $\sigma^2 > 0$. The assumption of sub-Gaussian demand is mild, as the class of sub-Gaussian random variables is large; see, for example, Section 2.1.2 in Wainwright (2019). The parameter $\frac{\sigma^2}{200}$ is known as the variance proxy because if D^k satisfies the sub-Gaussian assumption, then the variance of D^k is at most $\frac{\sigma^2}{200}$. We scale the variance proxy by 1/200 for notational uniformity in our proofs. Under independent demands, we continue using the fluid approximation in (2), but this linear program admits obvious simplifications under independent demands. For example, we can write the probability $\mathbb{P}\{D^\ell \ge s,\, D^{\ell-1} = p \,|\, D^k \ge t,\, D^{k-1} = q\}$ in the first constraint as $\mathbb{P}\{D^\ell \ge s\}\, \mathbb{P}\{D^{\ell-1} = p\}$. To construct an approximate policy from the fluid approximation, we solve problem (2) once at the beginning of the selling horizon. Letting $\overline{\boldsymbol{x}} = (\overline{x}^k_{jt}(q) : j \in \mathcal{J},\ t, q \in \mathcal{T},\ k \in \mathcal{K})$ be an optimal solution, we make the decisions as follows.

Approximate Policy under Independent Demands:

Using $\gamma \in [0,1]$ to denote a tuning parameter, if we have a request for product j at time period t in stage k, then we are willing to accept the request with probability $\gamma \sum_{q \in \mathcal{T}} \mathbb{P}\{D^{k-1} = q\} \frac{\overline{x}^k_{jt}(q)}{\lambda^k_{jt}}$. If we are willing to accept the request and there are enough resource capacities to accept the request, then we accept the request. Otherwise, we reject.

In the next theorem, we give a performance guarantee for the approximate policy. We continue using APX to denote the total expected revenue of the approximate policy.

Theorem 6.1 (Independent Demands) *Under independent demands, there exists a choice of the tuning parameter γ such that we have*

$$\frac{\mathsf{APX}}{\mathsf{OPT}} \;\ge\; \frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}} \;\ge\; \max\left\{ \frac{1}{4L}, \left(1 - 4\, \frac{\sqrt{(c_{\min} + \sigma^2\,(K-1)) \log c_{\min}}}{c_{\min}} - \frac{L}{c_{\min}} \right) \right\}.$$

We give the proof of the theorem in Appendix E. Although the demand random variables in the theorem above are independent, we are unable to use concentration inequalities for sums of independent random variables in the proof. In particular, using N^k_{it} to capture the Bernoulli random variable that takes value one if there is a request for a product at time period t in stage k that uses resource i and the approximate policy is willing to accept the request, we can use $\sum_{\ell=1}^{k-1} \sum_{s \in \mathcal{T}} \mathbf{1}(D^\ell \ge s)\, N^\ell_{is} + \sum_{s=1}^{t} \mathbf{1}(D^k \ge s) N^k_{is}$ to upper bound the capacity consumption of resource i up to and including time period t in stage k. This random variable is the analogue of $\sum_{\ell=1}^{k-1} \sum_{s \in \mathcal{T}} \sum_{q \in \mathcal{T}} \Psi^\ell_s(q)\, N^\ell_{is}(q) + \sum_{s=1}^{t} \sum_{q \in \mathcal{T}} \Psi^k_s(q)\, N^k_{is}(q)$ used right after (3) in the previous section. Considering the sum $\sum_{\ell=1}^{k-1} \sum_{s \in \mathcal{T}} \mathbf{1}(D^\ell \ge s)\, N^\ell_{is} + \sum_{s=1}^{t} \mathbf{1}(D^k \ge s) N^k_{is}$, even if the demands are independent, for fixed stage ℓ, the terms $\{\mathbf{1}(D^\ell \ge s)\, N^\ell_{is} : s \in \mathcal{T}\}$ in this sum are not independent due to the random variable $\mathbf{1}(D^\ell \ge s)$. Thus, we cannot bound the tail probabilities of the capacity consumptions by using concentration inequalities for sums of independent random variables. We derive our concentration inequalities through moment generating function bounds.

If we assume that $\mathbb{P}\{D^k = t\} \geq \epsilon$ for all $t \in \mathcal{T}$ and $k \in \mathcal{K}$ for some $\epsilon > 0$, then the performance guarantee in Theorem 4.1 continues to apply under independent demands, but this assumption requires that the demand in each stage takes values over is full support with non-negligible probability. In contrast, the performance guarantee in Theorem 6.1, which specifically focuses on independent demands, only requires the demands to be sub-Gaussian. If D^k is bounded by $[a, b]$, then we have $\mathbb{E}\{e^{\lambda(D^k - \mathbb{E}\{D^k\})}\} \leq e^{\frac{1}{8}\lambda^2(b-a)^2}$ for all $\lambda \geq 0$; see Lemma 5.1 in Dubhashi and Panconesi (2009). Thus, the sub-Gaussian demand assumption is immediately satisfied when the demand random variables are bounded. Lastly, the interpretation of the performance guarantee in Theorem 6.1 is similar to that in Theorem 4.1. The approximate policy has a constant-factor performance guarantee, as long as the number of resources used by each product is uniformly bounded. Furthermore, if we scale both the number of stages and resource capacities with the same rate θ, then the relative gap between the total expected revenue of the approximate policy and the optimal total expected revenue converges to one with rate $1 - \frac{1}{\sqrt{\theta}}$.

7. Computational Experiments

We conduct a numerical study to investigate the benefits from our fluid approximation when the demands in different stages have arbitrary distributions and are dependent on each other.

Experimental Setup: We consider an airline network with one hub and three spokes. We have a flight leg that connects the hub to each spoke and each spoke to the hub. Thus, there are six flight legs. We have a high-fare and a low-fare itinerary that connects each origin-destination pair. Thus, there are $2 \times 4 \times 3 = 24$ itineraries. The itineraries that connect the hub to a spoke or a spoke to the hub are direct, including one flight leg, whereas the itineraries that connect a spoke to another spoke connect at the hub, including two flight legs. In this setting, the resources correspond to the flight legs and the products correspond to the itineraries. To generate the revenues associated with the itineraries, we place the hub at the center of a 100×100 square. We place the spokes over the square uniformly at random. The revenue of a low-fare itinerary is the Euclidean distance between the origin and destination locations of the itinerary. The revenue of a high-fare itinerary is κ times the revenue of the corresponding low-fare itinerary. The experimental setup so far closely follows Topaloglu (2009), where the author does not consider calendar-aware or dependent demands. We proceed to discussing how we come up with the demand random variables.

We have K stages in the selling horizon. We will vary the parameter K. The demand in each stage has a truncated and discretized log-normal distribution. In particular, given that the demand in stage $k-1$ takes the value $\overline{D}^{k-1}$, the mean and standard deviation of the demand in stage k are, respectively, $\mu^k = \rho\,\overline{D}^{k-1} + (1-\rho)\,100$ and $\sigma^k = 0.3\,(\rho\,\overline{D}^{k-1} + (1-\rho)\,100)$. In this case, the demand in stage k is obtained by rounding a log-normal random variable with mean μ^k and standard deviation σ^k up to the closest integer and truncating at $\lceil \mu^k + 3\,\sigma^k \rceil$. We fix $D^0 = 100$. Thus, the demand in each stage always has the mean of 100 and coefficient of variation of 0.3. The parameter ρ controls the correlation between the demands in successive stages. Because a linear transformation of a log-normal random variable is not log-normal, it is not guaranteed that the correlation coefficient between the demands in successive stages is ρ. After generating each test problem, however, we computed the correlation coefficient between the demands in successive stages and the correlation coefficient came out to be quite close to ρ. We will vary the parameter ρ.

To come up with the request arrival probabilities $\{\lambda^k_{jt} : j \in \mathcal{J},\ t \in \mathcal{T},\ k \in \mathcal{K}\}$, we assume that the requests for the low-fare itineraries tend to arrive towards the beginning of the selling horizon, whereas the requests for the high-fare itineraries tend to arrive towards the end of the selling horizon. In this way, we generate test problems in which it is important to protect capacities for the high-fare itinerary requests that tend to arrive later. In particular, the probability of getting a request for a low-fare itinerary linearly decreases over time. There are K stages, each with at most T time periods. We sample a threshold from the uniform distribution over $\{\lceil \frac{1}{2}KT \rceil, \ldots, \lceil \frac{2}{3}KT \rceil\}$ so that the requests for the high-fare itineraries arrive only after the threshold and the probability of getting a request for a high-fare itinerary linearly increases over time. In Appendix F, we give the details of our approach for generating the request arrival probabilities.

Once we generate the request arrival probabilities $\{\lambda^k_{jt} : j \in \mathcal{J},\ t \in \mathcal{T},\ k \in \mathcal{K}\}$, the total expected demand for the capacity on fight leg i is $\xi_i = \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^k \geq t\}\, \lambda^k_{jt}$. We set the capacity of flight leg i as $c_i = \lceil \xi_i / \beta \rceil$, where the parameter β controls the tightness of the capacities. Thus, the total expected demand for the capacity on a flight leg exceeds the capacity of the flight leg by a factor of β. In our test problems, we fix $\kappa = 8$ and $\beta = 1.6$. We experimented with different values for these two parameters and our computational results remained qualitatively the same. We build on the log-normal distribution to capture the demand random variables. This distribution takes values over the positive real line and its two parameters allow us to increase the coefficient of variation of the demand as much as we like and not be concerned about negative values. Poisson or normal distributions, for example, do not provide such flexibility.

We vary $K \in \{5, 10, 15, 20, 25, 30\}$ and $\rho \in \{0.2, 0.4, 0.6, 0.8\}$. For each parameter combination (K, ρ), we generate a test problem using the approach described in this section.

Benchmark: We use a benchmark fluid approximation that uses only the expected values of the demands. The total expected demand for product j is $\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \mathbb{P}\{D^k \geq t\}\, \lambda^k_{jt}$. Using the decision variable w_j to capture the total number of requests for product j that we accept, using the vector $\boldsymbol{w} = (w_j : j \in \mathcal{J})$, we consider the linear program

$$\max_{\boldsymbol{w} \in \mathbb{R}_+^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} f_j \, w_j \; : \; \sum_{j \in \mathcal{J}} a_{ij} \, w_j \le c_i \quad \forall \, i \in \mathcal{L}, \quad w_j \le \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \mathbb{P}\{D^k \ge t\} \, \lambda_{jt}^k \quad \forall \, j \in \mathcal{J} \right\}. \tag{7}$$

The first constraint ensures that the expected capacity consumption of each resource does not exceed its capacity, whereas the second constraint ensures that the expected number of accepted requests for each product does not exceed its expected demand. We can argue that optimal objective value of problem (7) is an upper bound on the optimal total expected revenue. In particular, letting $\overline{\boldsymbol{x}}$ be an optimal solution to (2), we set $\overline{w}_j = \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \sum_{q \in \mathcal{T}} \mathbb{P}\{D^k \ge t, \; D^{k-1} = q\} \, \overline{x}_{jt}^k(q)$. In this case, considering the first constraint in (2) for $k = K$ and $t = T$, if we multiply this constraint with $\mathbb{P}\{D^k \ge t, \; D^{k-1} = q\}$ and add over all $t, q \in \mathcal{T}$, then we can verify that $\overline{\boldsymbol{w}}$ satisfies the first constraint in (7). Similarly, considering the second constraint in (2), if we multiply this constraint with $\mathbb{P}\{D^k \ge t, \; D^{k-1} = q\}$ and add over all $t, q \in \mathcal{T}$, then we can verify that $\overline{\boldsymbol{w}}$ satisfies the second constraint in (7). In this way, the solution $\overline{\boldsymbol{w}}$ is feasible to problem (7) and provides an objective value that is equal to the optimal objective value of problem (2). Thus, the optimal objective value of (7) is an upper bound on that of (2), which is, in turn, an upper bound on the optimal total expected revenue. Using problem (7), we can also obtain an approximate policy. Letting $\overline{\boldsymbol{w}}$ be an optimal solution to (7), the approximate policy is willing to accept a request for product j at any time period in any stage with probability $\overline{w}_j / \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \mathbb{P}\{D^k \ge t\} \, \lambda_{jt}^k$.

Computational Results: We refer to the fluid approximation in (2) as PRF to indicate that this fluid approximation uses information on the full probability distribution of the demands, whereas we refer to the fluid approximation in (7) as EXF to indicate that this fluid approximation uses information on only the expected values of the demands. In our computational experiments, we generate 24 test problems using the approach discussed earlier in this section. For each test problem, we solve PRF and EXF to compute the upper bounds on the optimal total expected revenue provided by the two fluid approximations. Furthermore, we simulate the decisions of the approximate policies provided by PRF and EXF for 1000 sample paths to estimate the total expected revenues of these two approximate policies. We use common random numbers when simulating the decisions of the two approximate policies. We give our computational results in Table 1. In this table, the first column gives the parameter combinations for our test problems using the pair (K, ρ). The second column gives the upper bound on the optimal total expected revenue provided by PRF, whereas the third column gives the total expected revenue of the approximate policy from PRF. The fourth column gives the ratio between the total expected revenue of the approximate policy and the upper bound provided by PRF multiplied by 100. The fifth, sixth and seventh columns give

Params.	PRF			EXF			Bnd.	Plcy.
(K, ρ)	Bound	Policy	Ratio	Bound	Policy	Ratio	Gap	Gap
$(5, 0.2)$	32,752	30,185	92.16%	34,533	28,206	81.68%	5.16%	7.01%
$(5, 0.4)$	31,755	28,700	90.38%	33,529	26,603	79.34%	5.29%	7.88%
$(5, 0.6)$	30,244	26,866	88.83%	31,983	24,404	76.30%	5.44%	10.09%
$(5, 0.8)$	27,690	24,285	87.70%	29,449	20,492	69.58%	5.97%	18.51%
$(10, 0.2)$	71,923	68,361	95.05%	74,333	65,324	87.88%	3.24%	4.65%
$(10, 0.4)$	69,342	64,991	93.73%	72,099	61,155	84.82%	3.82%	6.27%
$(10, 0.6)$	64,802	59,938	92.49%	67,861	55,599	81.93%	4.51%	7.80%
$(10, 0.8)$	55,250	49,893	90.30%	58,482	42,911	73.37%	5.53%	16.27%
$(15, 0.2)$	111,579	106,140	95.13%	114,223	102,259	89.53%	2.31%	3.80%
$(15, 0.4)$	107,571	102,204	95.01%	110,808	97,262	87.78%	2.92%	5.08%
$(15, 0.6)$	100,060	93,949	93.89%	104,080	87,636	84.20%	3.86%	7.20%
$(15, 0.8)$	82,458	76,242	92.46%	87,337	66,527	76.17%	5.59%	14.60%
$(20, 0.2)$	151,270	144,609	95.60%	154,021	140,783	91.40%	1.79%	2.72%
$(20, 0.4)$	145,995	139,229	95.37%	149,455	133,671	89.44%	2.31%	4.16%
$(20, 0.6)$	135,759	129,531	95.41%	140,308	122,021	86.97%	3.24%	6.15%
$(20, 0.8)$	110,234	105,274	95.50%	116,391	92,082	79.11%	5.29%	14.33%
$(25, 0.2)$	191,025	181,896	95.22%	193,835	177,921	91.79%	1.45%	2.23%
$(25, 0.4)$	184,421	176,305	95.60%	188,005	170,534	90.71%	1.91%	3.38%
$(25, 0.6)$	171,628	164,427	95.80%	176,483	156,205	88.51%	2.75%	5.26%
$(25, 0.8)$	138,672	131,287	94.67%	145,754	117,589	80.68%	4.86%	11.65%
$(30, 0.2)$	230,756	220,695	95.64%	233,608	216,844	92.82%	1.22%	1.78%
$(30, 0.4)$	222,995	214,150	96.03%	226,656	207,869	91.71%	1.62%	3.02%
$(30, 0.6)$	207,655	196,766	94.76%	212,704	188,936	88.83%	2.37%	4.14%
$(30, 0.8)$	167,445	158,680	94.77%	175,196	143,432	81.87%	4.42%	10.63%
Avg.			93.81%			84.43%	3.62%	7.44%

Table 1 Performance of the fluid approximations.

the same statistics for EXF. The eighth column gives the percent gap between the upper bounds provided by PRF and EXF, whereas the ninth column gives the percent gap between the total expected revenues of the approximate policies provided by the two fluid approximations.

Our results indicate that the approximate policy from PRF performs quite well. Over all of our test problems, the average gap between the total expected revenues of the approximate policy and the upper bounds provided by PRF is 6.19%. In other words, noting that we compare the performance of the approximate policy with an upper bound on the optimal total expected revenue, rather than the optimal total expected revenue itself, a conservative estimate of the average optimality gap of the approximate policy provided by PRF is 6.19%. In alignment with Theorem 4.1, the relative gaps between the upper bounds and the total expected revenues of the approximate policy from PRF tend to diminish as the number of stages increases. Considering groups of test problems with 5, 10, 15, 20, 25 and 30 stages, on average, the ratios between the total expected revenues of the approximate policy and the upper bounds provided by PRF are, respectively, 89.77, 92.89, 94.12, 95.47, 95.32 and 95.30. The performance of EXF, both in terms of the tightness of the upper bounds and the total expected revenues of the approximate policy that it provides, is noticeably inferior to PRF. Over all of our test problems, the average gap between the total expected revenues of the approximate policy and the upper bounds provided by EXF is 15.57%. On average, the upper bounds on the optimal total expected revenue provided by PRF improve those provided by EXF by 3.62%. Similarly, the total expected revenues of the approximate policy provided by PRF improve those provided by EXF by 7.44%. There are test problems where the gap between the two upper bounds can reach 5.97% and the gap between the total expected revenues of the two approximate policies can reach 18.51%. The fluid approximation used by EXF builds on the standard fluid approximation framework by using only the expected values of the demands, but there are significant benefits from using a fluid approximation that is specifically designed to handle dependent demands with arbitrary distributions.

8. Conclusions

Our starting point in this paper is the fact that modeling the demand in revenue management systems often requires focusing on a canonical interval of time, such as a week, so that we forecast the demand over each week in the selling horizon. As we use such a forecasting approach, we would like to use arbitrary distributions to capture the demand over each week and handle the demand dependence between successive weeks. We gave a fluid approximation that provides asymptotically tight upper bounds and asymptotically optimal approximate policies in such a demand environment. Our fluid approximation, as far as we are aware, provides the first asymptotically optimal policy under dependent demands with arbitrary distributions. The proof of our performance guarantee uses techniques that have not been used in the related literature. Our work opens up several research directions. In our model of dependence, the demand in a stage depends on the demand only on the previous stage. One can focus on more complicated dependence structures. In our asymptotic regime, we scale the number of stages and the capacities of the resources. One can consider other asymptotic regimes. Lastly, it would be useful to test our approach in practical revenue management systems.

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Online Supplement: Revenue Management with Calendar-Aware and Dependent Demands: Asymptotically Tight Fluid Approximations

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Appendix A: Proof of Theorem 3.1

We relax the constraint $a_{ij}\, u_j \le y_i$ at time period t in stage k in (1) using the Lagrange multiplier $\mu^k_{it}(q)$. Letting $\boldsymbol{\mu} = (\mu^k_{it}(q) : i \in \mathcal{L},\ t, q \in \mathcal{T},\ k \in \mathcal{K})$, we obtain the relaxed dynamic program

$$
\begin{aligned}
\widetilde{J}^k_t(\boldsymbol{y}, q; \boldsymbol{\mu}) &= \max_{\boldsymbol{u} \in \{0,1\}^{|\mathcal{J}|}} \Bigg\{ \sum_{j \in \mathcal{T}} \lambda^k_{jt} \Big\{ f_j\, u_j + \theta^k_t(q)\, \widetilde{J}^k_{t+1}(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, q; \boldsymbol{\mu}) \\
&\qquad\qquad + (1 - \theta^k_t(q))\, \widetilde{J}^{k+1}_1(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, t; \boldsymbol{\mu}) \Big\} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \lambda^k_{jt}\, \mu^k_{it}(q)\, [y_i - a_{ij}\, u_j] \Bigg\} \\
&= \max_{\boldsymbol{u} \in \{0,1\}^{|\mathcal{J}|}} \Bigg\{ \sum_{j \in \mathcal{T}} \lambda^k_{jt} \Big\{ \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \mu^k_{it}(q) \Big] u_j + \theta^k_t(q)\, \widetilde{J}^k_{t+1}(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, q; \boldsymbol{\mu}) \\
&\qquad\qquad + (1 - \theta^k_t(q))\, \widetilde{J}^{k+1}_1(\boldsymbol{y} - \boldsymbol{a}_j\, u_j, t; \boldsymbol{\mu}) \Big\} \Bigg\} + \sum_{i \in \mathcal{L}} \mu^k_{it}(q)\, y_i, \qquad (8)
\end{aligned}
$$

with the boundary condition that $\widetilde{J}^{K+1}_1 = 0$. Note that the value functions of the relaxed dynamic program depend on the choice of the Lagrange multipliers. In the first equality above, we scale the Lagrange multiplier $\mu^k_{it}(q)$ with λ^k_{jt} for notational uniformity. The second equality follows by arranging the terms and using the fact that $\sum_{j \in \mathcal{J}} \lambda^k_{jt} = 1$. If the Lagrange multipliers are non-negative, then the value functions from the relaxed dynamic program in (8) are upper bounds on the value functions from the dynamic program in (1). We do not show this result. This result is considered standard and analogues of this result have been shown in other settings; see Proposition 2 in Adelman and Mersereau (2008). Therefore, we have $\widetilde{J}^k_t(\boldsymbol{y}, q; \boldsymbol{\mu}) \ge J^k_t(\boldsymbol{y}, q)$ for all $\boldsymbol{y} \in \mathbb{Z}^{|\mathcal{L}|}_+$, $q \in \mathcal{T}$ as long as $\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{L}|T^2K}_+$. We can solve the problem $\min_{\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{L}|T^2K}_+} \widetilde{J}^1_1(\boldsymbol{c}, D^0; \boldsymbol{\mu})$ to obtain an upper bound on the optimal total expected revenue. One of the useful features of the relaxed dynamic program is that the value functions computed through this dynamic program are linear in the remaining capacities. In the next lemma, we show that $\widetilde{J}^k_t(\boldsymbol{y}, q; \boldsymbol{\mu}) = \sum_{i \in \mathcal{L}} \alpha^k_{it}(q; \boldsymbol{\mu})\, y_i + \beta^k_t(q; \boldsymbol{\mu})$, where the slope $\alpha^k_{it}(q; \boldsymbol{\mu})$ and the intercept $\beta^k_t(q; \boldsymbol{\mu})$ are recursively computed as

$$
\begin{aligned}
\alpha^k_{it}(q; \boldsymbol{\mu}) &= \mu^k_{it}(q) + \theta^k_t(q)\, \alpha^k_{i,t+1}(q; \boldsymbol{\mu}) + (1 - \theta^k_t(q))\, \alpha^{k+1}_{i1}(t; \boldsymbol{\mu}) \qquad (9) \\
\beta^k_t(q; \boldsymbol{\mu}) &= \sum_{j \in \mathcal{J}} \lambda^k_{jt} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij}\, \alpha^k_{it}(q; \boldsymbol{\mu}) \Big]^+ + \theta^k_t(q)\, \beta^k_{t+1}(q; \boldsymbol{\mu}) + (1 - \theta^k_t(q))\, \beta^{k+1}_1(t; \boldsymbol{\mu}),
\end{aligned}
$$

with the boundary condition that $\alpha^{K+1}_{i1} = 0$ and $\beta^{K+1}_1 = 0$. The linear form of the value functions from the relaxed dynamic program will be useful to show Theorem 3.1.

Lemma A.1 *Letting $\alpha^k_{it}(q;\boldsymbol{\mu})$ and $\beta^k_t(q;\boldsymbol{\mu})$ be as in* (9)*, the value functions computed through the dynamic program in* (8) *satisfy $\widetilde{J}^k_t(\boldsymbol{y},q;\boldsymbol{\mu}) = \sum_{i\in\mathcal{L}} \alpha^k_{it}(q;\boldsymbol{\mu})\, y_i + \beta^k_t(q;\boldsymbol{\mu})$ for all $t\in\mathcal{T}$ and $k\in\mathcal{K}$.*

Proof: We show the result by using induction over the time periods. At the last time period in the last stage, by (8), we have $J^K_T(\boldsymbol{y},q;\boldsymbol{\mu}) = \sum_{j\in\mathcal{T}} \lambda^K_{jT}\,[f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\mu^K_{iT}(q)]^+ + \sum_{i\in\mathcal{L}} \mu^K_{iT}(q)\, y_i = \beta^K_T(q;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^K_{iT}(q;\boldsymbol{\mu})\, y_i$, where the last equality uses (9). Therefore, the result holds at the last time period in the last stage. Assuming that the result holds at all time periods after time period t in stage k, we show that the result holds at time period t in stage k as well. Using the induction assumption on the right side of (8), we have

$$
\begin{aligned}
\widetilde{J}^k_t(\boldsymbol{y},q;\boldsymbol{\mu}) &= \max_{\boldsymbol{u}\in\{0,1\}^{|\mathcal{J}|}} \Bigg\{ \sum_{j\in\mathcal{T}} \lambda^k_{jt} \bigg\{ \Big[f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\mu^k_{it}(q) \Big] u_j + \theta^k_t(q) \Big[\beta^k_{t+1}(q;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^k_{i,t+1}(q;\boldsymbol{\mu})\,(y_i - a_{ij}\,u_j) \Big] \\
&\qquad + (1-\theta^k_t(q)) \Big[\beta^{k+1}_1(t;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^{k+1}_{i1}(t;\boldsymbol{\mu})\,(y_i - a_{ij}\,u_j) \Big] \bigg\} \Bigg\} + \sum_{i\in\mathcal{L}} \mu^k_{it}(q)\, y_i \\
&\overset{(a)}{=} \max_{\boldsymbol{u}\in\{0,1\}^{|\mathcal{J}|}} \Bigg\{ \sum_{j\in\mathcal{T}} \lambda^k_{jt} \Big[f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha^k_{it}(q;\boldsymbol{\mu}) \Big] u_j \Bigg\} \\
&\qquad + \theta^k_t(q)\,\beta^k_{t+1}(q;\boldsymbol{\mu}) + (1-\theta^k_t(q))\,\beta^{k+1}_1(t;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^k_{it}(q;\boldsymbol{\mu})\, y_i \\
&= \sum_{j\in\mathcal{T}} \lambda^k_{jt} \Big[f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha^k_{it}(q;\boldsymbol{\mu}) \Big]^+ + \theta^k_t(q)\,\beta^k_{t+1}(q;\boldsymbol{\mu}) + (1-\theta^k_t(q))\,\beta^{k+1}_1(t;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^k_{it}(q;\boldsymbol{\mu})\, y_i \\
&\overset{(b)}{=} \beta^k_t(q;\boldsymbol{\mu}) + \sum_{i\in\mathcal{L}} \alpha^k_{it}(q;\boldsymbol{\mu})\, y_i,
\end{aligned}
$$

where (a) follows by arranging the terms and using the definition of $\alpha^k_{it}(q;\boldsymbol{\mu})$, as well as noting the fact that $\sum_{j\in\mathcal{J}} \lambda^k_{jt} = 1$, whereas (b) uses the definition of $\beta^k_t(q;\boldsymbol{\mu})$. ■

By the lemma above, we have $\widetilde{J}^1_1(\boldsymbol{c},D^0;\boldsymbol{\mu}) = \sum_{i\in\mathcal{L}} \alpha^1_{i1}(D^0;\boldsymbol{\mu})\, c_i + \beta^1_1(D^0,\boldsymbol{\mu})$. In this case, the problem $\min_{\boldsymbol{\mu}\in\mathbb{R}^{|\mathcal{L}|T^2K}_+} \widetilde{J}^1_1(\boldsymbol{c},D^0;\boldsymbol{\mu})$ is equivalent the linear program

$$
\begin{aligned}
\min_{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\eta})\in\mathbb{R}^{T^2K(|\mathcal{L}|+1)}\times\mathbb{R}^{T^2K(|\mathcal{L}|+|\mathcal{J}|)}_+} \quad & \sum_{i\in\mathcal{L}} \alpha^1_{i1}(D^0)\, c_i + \beta^1_1(D^0) && (10)\\
\text{st} \quad \alpha^k_{it}(q) \;&=\; \mu^k_{it}(q) + \theta^k_t(q)\,\alpha^k_{i,t+1}(q) + (1-\theta^k_t(q))\,\alpha^{k+1}_{i1}(t) && \forall\, i\in\mathcal{L},\ t,q\in\mathcal{T},\ k\in\mathcal{K}\\
\beta^k_t(q) \;&=\; \sum_{j\in\mathcal{J}} \lambda^k_{jt}\,\eta^k_{jt}(q) + \theta^k_t(q)\,\beta^k_{t+1}(q) + (1-\theta^k_t(q))\,\beta^{k+1}_1(t) && \forall\, t,q\in\mathcal{T},\ k\in\mathcal{K}\\
\eta^k_{jt}(q) \;&\geq\; f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha^k_{it}(q) \qquad \forall\, j\in\mathcal{J},\ t,q\in\mathcal{T},\ k\in\mathcal{K}, &&
\end{aligned}
$$

where we use the decision variables $\boldsymbol{\alpha} = (\alpha^k_{it}(q) : i\in\mathcal{L},\ t,q\in\mathcal{T},\ k\in\mathcal{K})$, $\boldsymbol{\beta} = (\beta^k_t(q) : q,t\in\mathcal{T},\ k\in\mathcal{K})$, $\boldsymbol{\mu} = (\mu^k_{it}(q) : i\in\mathcal{L},\ t,q\in\mathcal{T},\ k\in\mathcal{K})$ and $\boldsymbol{\eta} = (\eta^k_{jt}(q) : j\in\mathcal{J},\ t,q\in\mathcal{T},\ k\in\mathcal{K})$. We follow the convention that $\alpha_{i1}^{K+1}(q) = 0$ and $\beta_1^{K+1}(q) = 0$ for all $i \in \mathcal{L}$ and $q \in \mathcal{T}$. In the linear program above, the first constraint computes the slopes of the value functions of the relaxed dynamic program. By the third constraint, noting the non-negativity constraints, we have $\eta_{jt}^k(q) = [f_j - \sum_{i\in\mathcal{L}} a_{ij}\,\alpha_{it}^k(q)]^+$ at an optimal solution to the linear program, in which case, the second constraint computes the intercepts of the value functions of the relaxed dynamic program. We write the objective function of problem (10) as $\sum_{q\in\mathcal{T}}\sum_{i\in\mathcal{L}} \mathbf{1}(D^0 = q)\,c_i\,\alpha_{i1}^1(q) + \sum_{q\in\mathcal{T}} \mathbf{1}(D^0 = q)\,\beta_1^1(q)$. We work with the dual of problem (10). We associate the dual variables $\boldsymbol{w} = (w_t^k(q) : t, q \in \mathcal{T},\ k \in \mathcal{K})$ with the second constraint in (10). The decision variables $(\beta_t^k(q) : t, q \in \mathcal{T},\ k \in \mathcal{K})$ appear only in the second constraint in problem (10), so in the dual of problem (10), the constraints associated with the decision variables $(\beta_t^k(q) : t, q \in \mathcal{T},\ k \in \mathcal{K})$ are given by $w_1^1(q) = \mathbf{1}(D^0 = q)$ for all $q \in \mathcal{T}$, $w_1^k(q) = \sum_{p\in\mathcal{T}}(1 - \theta_q^{k-1}(p))\,w_q^{k-1}(p)$ for all $q \in \mathcal{T},\ k \in \mathcal{K} \setminus \{1\}$ and $w_t^k(q) = \theta_{t-1}^k(q)\,w_{t-1}^k(q)$ for all $t \in \mathcal{T} \setminus \{1\}$, $q \in \mathcal{T}$, $k \in \mathcal{K}$. We capture these constraints by defining the set

$$\mathcal{W} = \Bigg\{ \boldsymbol{w} \in \mathbb{R}^{T^2 K} \ :\ w_t^k(q) = \theta_{t-1}^k(q)\,w_{t-1}^k(q) \quad \forall\, t \in \mathcal{T} \setminus \{1\},\ q \in \mathcal{T},\ k \in \mathcal{K},$$
$$w_1^k(q) = \sum_{p\in\mathcal{T}} (1 - \theta_q^{k-1}(p))\,w_q^{k-1}(p) \quad \forall\, q \in \mathcal{T},\ k \in \mathcal{K} \setminus \{1\}, \quad w_1^1(q) = \mathbf{1}(D^0 = q) \quad \forall\, q \in \mathcal{T} \Bigg\}.$$

In the next lemma, we show that if we have $\boldsymbol{w} \in \mathcal{W}$, then the vector $\boldsymbol{w}$ is closely related to the joint distribution of demands in a pair of successive stages.

Lemma A.2 *If $\boldsymbol{w} \in \mathcal{W}$, then we have $w_t^k(q) = \mathbb{P}\{D^k \geq t,\ D^{k-1} = q\}$ for all $t, q \in \mathcal{T}$ and $k \in \mathcal{K}$.*

Proof: We show the result by using induction over the time periods. At the first time period in the first stage, we have $w_1^1(q) = \mathbf{1}(D^0 = q) = \mathbb{P}\{D^0 = q\} = \mathbb{P}\{D^1 \geq 1,\ D^0 = q\}$, where the first equality holds by noting the third constraint in the definition of $\mathcal{W}$, the second equality follows by noting that D^0 is a deterministic quantity and the third equality holds because the support of D^1 is $\{1, \ldots, T\}$. Assuming that the result holds at all time periods before time period t in stage k, we show that the result holds at time period t in stage k as well. If $t \neq 1$, then using the first constraint in the definition of $\mathcal{W}$, we have the chain of equalities $w_t^k(q) = \theta_{t-1}^k(q)\,w_{t-1}^k(q) = \mathbb{P}\{D^k \geq t \,|\, D^k \geq t-1,\ D^{k-1} = q\}\,\mathbb{P}\{D^k \geq t-1,\ D^{k-1} = q\} = \mathbb{P}\{D^k \geq t,\ D^{k-1} = q\}$, where the second equality is by the definition of $\theta_{t-1}^k(q)$ and the induction assumption. Similarly, if $t = 1$, then using the second constraint in the definition of $\mathcal{W}$, we have $w_1^k(q) = \sum_{p\in\mathcal{T}}(1 - \theta_q^{k-1}(p))\,w_q^{k-1}(p) = \sum_{p\in\mathcal{T}} \mathbb{P}\{D^{k-1} = q \,|\, D^{k-1} \geq q,\ D^{k-2} = p\}\,\mathbb{P}\{D^{k-1} \geq q,\ D^{k-2} = p\}$, but the last sum expression is equal to $\mathbb{P}\{D^{k-1} = q\}$, so $w_1^k(q) = \mathbb{P}\{D^{k-1} = q\} = \mathbb{P}\{D^k \geq 1,\ D^{k-1} = q\}$. ■

By the lemma above, there exists a single element in $\mathcal{W}$. To write the dual of problem (10), we associate the dual variables $\boldsymbol{y} = (y_{it}^k(q) : i \in \mathcal{L},\ t, q \in \mathcal{T},\ k \in \mathcal{K})$, $\boldsymbol{w} = (w_t^k(q) : t, q \in \mathcal{T},\ k \in \mathcal{K})$ and

$\boldsymbol{u} = (u^k_{jt}(q) : j \in \mathcal{J},\ t, q \in \mathcal{T},\ k \in \mathcal{K}\}$ with the first, second and third constraints, respectively, in problem (10). In this case, the dual of problem (10) is given by

$$\begin{aligned}
\max_{(\boldsymbol{y},\boldsymbol{u},\boldsymbol{w}) \in \mathbb{R}_+^{T^2 K(|\mathcal{L}|+|\mathcal{J}|)} \times \mathcal{W}} \quad & \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \sum_{q\in\mathcal{T}} \sum_{j\in\mathcal{J}} f_j\, u^k_{jt}(q) && (11)\\
\text{st} \quad & y^k_{it}(q) + \sum_{j\in\mathcal{J}} a_{ij}\, u^k_{jt}(q) \;=\; \theta^k_{t-1}(q)\, y^k_{i,t-1}(q) \qquad \forall\, i\in\mathcal{L},\ t\in\mathcal{T}\setminus\{1\},\ q\in\mathcal{T},\ k\in\mathcal{K}\\
& y^k_{i1}(q) + \sum_{j\in\mathcal{J}} a_{ij}\, u^k_{j1}(q) \;=\; \sum_{p\in\mathcal{T}} (1-\theta^{k-1}_q(p))\, y^{k-1}_{iq}(p) \qquad \forall\, i\in\mathcal{L},\ q\in\mathcal{T},\ k\in\mathcal{K}\setminus\{1\}\\
& y^1_{i1}(q) + \sum_{j\in\mathcal{J}} a_{ij}\, u^1_{j1}(q) \;=\; \mathbf{1}(D^0 = q)\, c_i \qquad \forall\, i\in\mathcal{L},\ q\in\mathcal{T}\\
& u^k_{jt}(q) \le \lambda^k_{jt}\, w^k_t(q) \qquad \forall\, j\in\mathcal{J},\ t,q\in\mathcal{T},\ k\in\mathcal{K},
\end{aligned}$$

where the constraints above are associated with the decision variables $\boldsymbol{\alpha}$ and $\boldsymbol{\eta}$ in (10). The constraint for the decision variables $\boldsymbol{\mu}$ translates into the non-negativity constraint for $\boldsymbol{y}$.

In (11), we capture the constraint associated with the decision variables $\boldsymbol{\beta}$ as $\boldsymbol{w} \in \mathcal{W}$. In the next lemma, we give an equality that is satisfied by all feasible solutions to problem (11).

Lemma A.3 *Letting $(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{w})$ be a feasible solution to the linear program in* (11)*, for all $i \in \mathcal{L}$, $t, q \in \mathcal{T}$ and $k \in \mathcal{K}$, we have*

$$\begin{aligned}
\mathbb{P}\{D^k \ge t,\ D^{k-1} = q\}\, c_i - \sum_{\ell=1}^{k-1} \sum_{s\in\mathcal{T}} \sum_{p\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^k \ge t,\ D^{k-1} = q \,|\, D^\ell \ge s,\ D^{\ell-1} = p\}\, u^\ell_{js}(p)&\\
- \sum_{s=1}^{t} \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^k \ge t \,|\, D^k \ge s,\ D^{k-1} = q\}\, u^k_{js}(q) \;=\; y^k_{it}(q).&
\end{aligned}$$

Proof: We show the result by using induction over the time periods. At the first time period in the first stage, by the third constraint in (11), we have $y^1_{i1}(q) = \mathbf{1}(D^0 = q)\, c_i - \sum_{j\in\mathcal{J}} a_{ij}\, u^1_{j1}(q) = \mathbb{P}\{D^1 \ge 1,\ D^0 = q\}\, c_i - \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{P}\{D^1 \ge 1 \,|\, D^1 \ge 1,\ D^0 = q\}\, u^1_{j1}(q)$, where the last equality holds because D^0 is a deterministic quantity and the support of D^1 is $\{1, \ldots, T\}$. Assuming that the result holds at all time periods up to and including time period t in stage k, we show that the result holds at the subsequent time period as well. Consider the case $t \neq T$. We will use three identities. First, for $\ell \le k-1$, given D^{k-1}, D^k is independent of $D^1, \ldots, D^\ell$, in which case, we obtain

$$\begin{aligned}
&\theta^k_t(q)\, \mathbb{P}\{D^k \ge t,\ D^{k-1} = q \,|\, D^\ell \ge s,\ D^{\ell-1} = p\}\\
&\quad = \mathbb{P}\{D^k \ge t+1 \,|\, D^k \ge t,\ D^{k-1} = q\}\ \mathbb{P}\{D^k \ge t,\ D^{k-1} = q \,|\, D^\ell \ge s,\ D^{\ell-1} = p\}\\
&\quad = \mathbb{P}\{D^k \ge t+1,\ D^{k-1} = q \,|\, D^\ell \ge s,\ D^{\ell-1} = p\}.
\end{aligned}$$

Second, by the Bayes rule and definition of $\theta^k_t(q)$, we can show that $\theta^k_t(q)\, \mathbb{P}\{D^k \ge t, D^{k-1} = q\} = \mathbb{P}\{D^k \ge t+1, D^{k-1} = q\}$. Third, for $s \le t$, we can, once more, use the Bayes rule and definition of

$\theta_t^k(q)$ to show that $\theta_t^k(q)\,\mathbb{P}\{D^k \geq t \,|\, D^k \geq s,\ D^{k-1}=q\} = \mathbb{P}\{D^k \geq t+1 \,|\, D^k \geq s,\ D^{k-1}=q\}$. Noting that the solution $(\boldsymbol{y},\boldsymbol{u},\boldsymbol{w})$ is feasible to (11), it satisfies the first constraint. Thus, we obtain

$$\begin{aligned}
y_{i,t+1}^k(q) \;&=\; \theta_t^k(q)\, y_{it}^k(q) - \sum_{j\in\mathcal{J}} a_{ij}\, u_{j,t+1}^k(q) \\
&\overset{(a)}{=}\; \theta_t^k(q)\,\mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}\, c_i \\
&\quad - \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\,\theta_t^k(q)\,\mathbb{P}\{D^k \geq t,\ D^{k-1}=q \,|\, D^\ell \geq s,\ D^{\ell-1}=p\}\, u_{js}^\ell(p) \\
&\quad - \sum_{s=1}^{t}\sum_{j\in\mathcal{J}} a_{ij}\,\theta_t^k(q)\,\mathbb{P}\{D^k \geq t \,|\, D^k \geq s,\ D^{k-1}=q\}\, u_{js}^k(q) - \sum_{j\in\mathcal{J}} a_{ij}\, u_{j,t+1}^k(q) \\
&\overset{(b)}{=}\; \mathbb{P}\{D^k \geq t+1,\ D^{k-1}=q\}\, c_i \\
&\quad - \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbb{P}\{D^k \geq t+1,\ D^{k-1}=q \,|\, D^\ell \geq s,\ D^{\ell-1}=p\}\, u_{js}^\ell(p) \\
&\quad - \sum_{s=1}^{t}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbb{P}\{D^k \geq t+1 \,|\, D^k \geq s,\ D^{k-1}=q\}\; u_{js}^k(q) - \sum_{j\in\mathcal{J}} a_{ij}\, u_{j,t+1}^k(q) \\
&\overset{(c)}{=}\; \mathbb{P}\{D^k \geq t+1,\ D^{k-1}=q\}\, c_i \\
&\quad - \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbb{P}\{D^k \geq t+1,\ D^{k-1}=q \,|\, D^\ell \geq s,\ D^{\ell-1}=p\}\, u_{js}^\ell(p) \\
&\quad - \sum_{s=1}^{t+1}\sum_{j\in\mathcal{J}} a_{ij}\,\mathbb{P}\{D^k \geq t+1 \,|\, D^k \geq s,\ D^{k-1}=q\}\; u_{js}^k(q),
\end{aligned}$$

where (a) is by the induction assumption, (b) uses the three identities given earlier in the proof and (c) holds by noting that $\mathbb{P}\{D^k \geq t+1 \,|\, D^k \geq t+1,\ D^{k-1}=q\}=1$ and collecting the terms.

The chain of equalities above shows that if $t \neq T$, then the result holds at the subsequent time period. We can use a similar argument to show that the result holds when $t=T$ as well. ■

Using Lemmas A.2 and A.3, we give a proof for Theorem 3.1.

Proof of Theorem 3.1:

Any feasible solution $(\boldsymbol{y},\boldsymbol{u},\boldsymbol{w})$ to problem (11) satisfies $y_{it}^k(q) \geq 0$, in which case, dividing both sides of the equality in Lemma A.3 by $\mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}$, we obtain

$$\begin{aligned}
&\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\,\frac{\mathbb{P}\{D^k \geq t,\ D^{k-1}=q \,|\, D^\ell \geq s,\ D^{\ell-1}=p\}}{\mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}}\, u_{js}^\ell(p) \\
&\qquad\qquad - \sum_{s=1}^{t}\sum_{j\in\mathcal{J}} a_{ij}\,\frac{\mathbb{P}\{D^k \geq t \,|\, D^k \geq s,\ D^{k-1}=q\}}{\mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}}\, u_{js}^k(q) \;\leq\; c_i.
\end{aligned}$$

By the Bayes rule, the two fractions on the left side of the inequality above are, respectively, given by $\frac{\mathbb{P}\{D^\ell \geq s,\ D^{\ell-1}=p \,|\, D^k \geq t,\ D^{k-1}=q\}}{\mathbb{P}\{D^\ell \geq s,\ D^{\ell-1}=p\}}$ and $\frac{\mathbb{P}\{D^k \geq s,\ D^{k-1}=q \,|\, D^k \geq t,\ D^{k-1}=q\}}{\mathbb{P}\{D^k \geq s,\ D^{k-1}=q\}}$, but for $s \leq t$, the last probability is equal to $\frac{1}{\mathbb{P}\{D^k \geq s,\ D^{k-1}=q\}}$. In this case, any feasible solution to the linear program in (11) is also a feasible solution to the linear program

$$\max_{(\boldsymbol{u},\boldsymbol{w})\in\mathbb{R}_+^{|\mathcal{J}|T^2K}\times\mathcal{W}} \quad \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\sum_{q\in\mathcal{T}}\sum_{j\in\mathcal{J}} f_j\, u_{jt}^k(q) \tag{12}$$

$$\text{st} \quad \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\, \frac{\mathbb{P}\{D^\ell \geq s,\, D^{\ell-1}=p \,|\, D^k \geq t,\, D^{k-1}=q\}}{\mathbb{P}\{D^\ell \geq s,\ D^{\ell-1}=p\}}\, u_{js}^\ell(p)$$

$$+\sum_{s=1}^{t}\sum_{j\in\mathcal{J}} a_{ij}\, \frac{1}{\mathbb{P}\{D^k \geq s,\ D^{k-1}=q\}}\, u_{js}^k(q) \;\leq\; c_i \qquad \forall\, i\in\mathcal{L},\ t,q\in\mathcal{T},\ k\in\mathcal{K}$$

$$u_{jt}^k(q) \leq \lambda_{jt}^k\, w_t^k(q) \qquad \forall\, j\in\mathcal{J},\ t,q\in\mathcal{T},\ k\in\mathcal{K}.$$

Thus, the optimal objective value of problem (12) is an upper bound on that of problem (11), which is, in turn, an upper bound on the optimal total expected revenue.

By Lemma A.2, for any $\boldsymbol{w}\in\mathcal{W}$, we have $w_t^k(q) = \mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}$. In this case, making the change of variables $x_{jt}^k(q) = \frac{1}{\mathbb{P}\{D^k \geq t,\ D^{k-1}=q\}}\, u_{jt}^k(q)$, problem (12) is equivalent to problem (2). ■

Appendix B: Problem Instance with Increasing Support for Demands

When there is a single stage, so the demand model is not aware of the calendar and the dependence between the demands in different stages is not an issue, we can construct fluid approximations such that the relative gap between the optimal objective value of the fluid approximation and the optimal total expected revenue diminishes as the capacities of the resources get large, irrespective of how the demand is scaled; see, for example, Bai et al. (2023). We give a counterexample to demonstrate that if we scale the support of the demand as the capacities of the resources get large, then the relative gap between the optimal objective value of problem (2) and the optimal total expected revenue does not necessarily diminish, even when the demands at different stages are independent of each other. We consider a problem instance with $K=3$ stages. There is one resource with a capacity of $C+1$ for an even integer C. There are two products indexed by $\{1,2\}$. The revenue associated with the two products are $f_1 = 1$ and $f_2 = C/4$. The demand in the first stage can take two values with $\mathbb{P}\{D^1 = 0\} = 1/2$ and $\mathbb{P}\{D^1 = C\} = 1/2$. The demand in the second and third stages have the distributions $\mathbb{P}\{D^2 = 1\} = 1$ and $\mathbb{P}\{D^3 = C/2\} = 1$. The probabilities of getting requests for the different products are given by $\lambda_{1t}^1 = 1$ for all $t = 1,\ldots,C$, $\lambda_{11}^2 = 1$, $\lambda_{1t}^3 = 1$ for all $t = 1,\ldots,C/2-1$ and $\lambda_{2,C/2}^3 = 1$. All other request probabilities are zero. Thus, we have a request for the second product only at the last time period in the last stage.

The optimal policy for this problem instance is to accept all product requests until there is one unit of remaining capacity and save that unit of capacity for the request for the second product. In this case, if the demand in the first stage is C, then we obtain a total revenue of $C+\frac{1}{4}C=\frac{5}{4}C$. If the demand in the first stage is zero, then we obtain a total revenue of $\frac{1}{2}C+\frac{1}{4}C=\frac{3}{4}C$. Therefore, the optimal total expected revenue is C, so $\mathsf{OPT}=C$. In the linear program in (2), noting the last constraint, if $\lambda^k_{jt}=0$, then we can drop the decision variables $\{x^k_{jt}(q):q\in\mathcal{T}\}$. In this case, using D^0 to denote the demand right before the beginning of the selling horizon, for the problem instance in the previous paragraph, the linear program in (2) is given by

$$
\begin{aligned}
\overline{Z}_{\mathsf{LP}} \;=\; \max \quad & \frac{1}{2}\sum_{t=1}^{C} x^1_{1t}(D^0)+\frac{1}{2}x^2_{11}(0)+\frac{1}{2}x^2_{11}(C)+\sum_{t=1}^{C/2-1} x^3_{1t}(1)+\frac{1}{4}C\,x^3_{2,C/2}(1)\\
\text{st} \quad & \sum_{t=1}^{C} x^1_{1t}(D^0)\le C+1\\
& x^2_{11}(0)\le C+1\\
& \sum_{t=1}^{C} x^1_{1t}(D^0)+x^2_{11}(C)\le C+1\\
& \frac{1}{2}\sum_{t=1}^{C} x^1_{1t}(D^0)+\frac{1}{2}\,x^2_{11}(0)+\frac{1}{2}\,x^2_{11}(C)+\sum_{t=1}^{C/2-1} x^3_{1t}(1)+x^3_{2,C/2}(1)\le C+1,
\end{aligned}
$$

where we understand that all decision variables are restricted to be in $[0,1]$. Setting all decision variables to one yields a feasible solution, so $\overline{Z}_{\mathsf{LP}}=\frac{1}{2}C+1+\frac{1}{2}C-1+\frac{1}{4}C=\frac{5}{4}C$.

Thus, for this problem instance, we have $\frac{\mathsf{OPT}}{\overline{Z}_{\mathsf{LP}}}=\frac{4}{5}$. Therefore, the ratio $\frac{\mathsf{OPT}}{\overline{Z}_{\mathsf{LP}}}$ stays away from one for this problem instance as the resource capacity gets large.

Appendix C: Performance Guarantee for the Approximate Policy

We show that the total expected revenue of the approximate policy satisfies $\frac{\mathsf{APX}}{\mathsf{OPT}}\ge\frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}}\ge\frac{1}{4L}$. We use the random variables $\Psi^k_t(q)$, G^k_{jt} and $N^k_{it}(q)$ as defined at the beginning of Section 5. By the discussion right after (3), if we can show that $\mathbb{P}\{G^k_{jt}=1\,|\,\Psi^k_t(q)=1\}\ge\alpha$, then $\mathsf{APX}\ge\gamma\,\alpha\,\overline{Z}_{\mathsf{LP}}$. By (4), to lower bound the probability $\mathbb{P}\{G^k_{jt}=1\,|\,\Psi^k_t(q)=1\}$, it is enough to upper bound the probability $\mathbb{P}\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi^\ell_s(p)\,N^\ell_{is}(p)+\sum_{s=1}^{t}N^k_{is}(q)\ge c_i\,|\,D^{k-1}=q\}$. Recall that the random variables $\{N^k_{it}(q):i\in\mathcal{L},\ t\in\mathcal{T},\ k\in\mathcal{K}\}$ are independent of demands, so we get

$$
\begin{aligned}
&\mathbb{E}\Big\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi^\ell_s(p)\,N^\ell_{is}(p)+\sum_{s=1}^{t}N^k_{is}(q)\,\Big|\,D^{k-1}=q\Big\}\\
&\qquad\overset{(a)}{=}\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\mathbb{P}\{D^\ell\ge s,\ D^{\ell-1}=p\,|\,D^{k-1}=q\}\sum_{j\in\mathcal{J}}a_{ij}\,\gamma\,\overline{x}^\ell_{js}(p)+\sum_{s=1}^{t}\sum_{j\in\mathcal{J}}a_{ij}\,\gamma\,\overline{x}^k_{js}(q)\overset{(b)}{\le}\gamma\,c_i,
\end{aligned}
$$

where (a) holds by using the definition of $\Psi^k_t(p)$ and $\mathbb{E}\{N^k_{it}(q)\}=\sum_{j\in\mathcal{J}}a_{ij}\,\overline{x}^k_{jt}(q)$, whereas (b) holds by noting that $\overline{\boldsymbol{x}}$ satisfies the first constraint in problem (2), as well as using the fact that conditional on D^{k-1}, D^k is independent of $D^1,\ldots,D^{k-1}$, in which case, for $\ell \le k-1$, we have $\mathbb{P}\{D^\ell \ge s,\ D^{\ell-1}=p \,|\, D^{k-1}=q\} = \mathbb{P}\{D^\ell \ge s,\ D^{\ell-1}=p \,|\, D^k \ge t,\ D^{k-1}=q\}$. Therefore, we get

$$\mathbb{P}\left\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)\ \ge\ c_i \,|\, D^{k-1}=q\right\}$$
$$\stackrel{(c)}{\le}\frac{1}{c_i}\,\mathbb{E}\left\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\Psi_s^\ell(p)\,N_{is}^\ell(p)+\sum_{s=1}^{t}N_{is}^k(q)\ge c_i \,|\, D^{k-1}=q\right\}\le\gamma,$$

where (c) is the Markov inequality. Thus, by (4), we have $\mathbb{P}\{G_{jt}^k=1 \,|\, D^{k-1}=q\}\ge 1-L\,\gamma$, yielding $\mathsf{APX}\ge\gamma\,(1-L\,\gamma)\,\overline{Z}_{\mathsf{LP}}$. Setting $\gamma=\frac{1}{2L}$ and using Theorem 3.1, we get $\frac{1}{4L}\le\frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}}\le\frac{\mathsf{APX}}{\mathsf{OPT}}$. ■

Appendix D: Auxiliary Results for Concentration Inequalities

We give proofs for two results used in Section 5. First, we show that $|M_i^k(\ell)-M_i^k(\ell+1)|\le\frac{2}{\epsilon^3}$ with probability one. Letting $V^k(q)=\mathbb{P}\{D^k=q\}$ and $\theta=1-\epsilon$ for notational brevity, we define

$$Q^k(p,q)=\frac{1}{\theta}\big[\mathbb{P}\{D^k=q \,|\, D^{k-1}=p\}-(1-\theta)\,V^k(q)\big]. \tag{13}$$

By the assumption that $\mathbb{P}\{D^k=q \,|\, D^{k-1}=p\}\ge\epsilon$, we have $Q^k(p,q)\ge\frac{1}{\theta}\,(\epsilon-(1-\theta))=0$. Also, we have $\sum_{q\in\mathcal{T}}Q^k(p,q)=\frac{1}{\theta}\big[\sum_{q\in\mathcal{T}}\mathbb{P}\{D^k=q \,|\, D^{k-1}=p\}-(1-\theta)\sum_{q\in\mathcal{T}}V^k(q)\big]=1$. Thus, we can use $Q^k(p,q)$ to characterize the transition probabilities of a non-stationary Markov chain. Consider the non-stationary Markov chain $Y^0,Y^1,Y^2,\ldots$ over the state space $\mathcal{T}$ characterized by the transition probabilities $\mathbb{P}\{Y^k=q \,|\, Y^{k-1}=p\}=Q^k(p,q)$ for all $p,q\in\mathcal{T}$ with $\mathbb{P}\{Y^0=q\}=\mathbf{1}(D^0=q)$. To show that $|M_i^{k-1}(\ell)-M_i^{k-1}(\ell+1)|\le\frac{2}{\epsilon^3}$ with probability one, we will use two preliminary lemmas. In the next lemma, we slightly extend Theorem 4.9 in Levin and Peres (2017), which characterizes the mixing times of Markov chains, to non-stationary Markov chains.

Lemma D.1 *For all $p,q\in\mathcal{T}$ and $k,\ell\in\mathcal{K}$ with $\ell\ge k+1$, we have*

$$\mathbb{P}\{D^\ell=q \,|\, D^k=p\}-\mathbb{P}\{D^\ell=q\}\ =\ (1-\epsilon)^{(\ell-k)}\left[\mathbb{P}\{Y^\ell=q \,|\, Y^k=p\}-\mathbb{P}\{D^\ell=q\}\right].$$

Proof: Letting $\theta=1-\epsilon$ for notational brevity, for all $p,q\in\mathcal{T}$ and $k,\ell\in\mathcal{K}$ with $\ell\ge k+1$, we claim that $\mathbb{P}\{D^\ell=q \,|\, D^k=p\}=(1-\theta^{\ell-k})\,\mathbb{P}\{D^\ell=q\}+\theta^{\ell-k}\,\mathbb{P}\{Y^\ell=q \,|\, Y^k=p\}$. We show the claim by using induction over $\ell=k+1,\ldots,K$. Consider the case $\ell=k+1$. By the definition of $Q^{k+1}(p,q)$, we have $\mathbb{P}\{Y^{k+1}=q \,|\, Y^k=p\}=Q^{k+1}(p,q)=\frac{1}{\theta}\big[\mathbb{P}\{D^{k+1}=q \,|\, D^k=p\}-(1-\theta)\,\mathbb{P}\{D^{k+1}=q\}\big]$, so arranging the terms, we get $(1-\theta)\,\mathbb{P}\{D^{k+1}=q\}+\theta\,\mathbb{P}\{Y^{k+1}=q \,|\, Y^k=p\}=\mathbb{P}\{D^{k+1}=q \,|\, D^k=p\}$, establishing the claim for $\ell=k+1$. Assuming that the claim holds for $\ell\ge k+1$, we show that the claim holds for $\ell+1\ge k+1$ as well. Arranging the terms in the definition of $Q^{\ell+1}(s,q)$ in (13), we have $\mathbb{P}\{D^{\ell+1}=q \,|\, D^\ell=s\}=\theta\,Q^{\ell+1}(s,q)+(1-\theta)\,V^{\ell+1}(q)$. In this case, noting the identity $\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^k=p\} = \sum_{s\in\mathcal{T}}\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^\ell=s\}\,\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}$, it follows that we have the chain of equalities

$$\begin{aligned}
&\sum_{s\in\mathcal{T}}\mathbb{P}\{D^{\ell+1}=q\,|\,D^\ell=s\}\;\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}\\
&\quad= \sum_{s\in\mathcal{T}}\Big[\theta\,Q^{\ell+1}(s,q)+(1-\theta)\,V^{\ell+1}(q)\Big]\,\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}\\
&\quad\stackrel{(a)}{=}\theta\sum_{s\in\mathcal{T}}\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^\ell=s\}\,\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}+(1-\theta)\,V^{\ell+1}(q)\\
&\quad\stackrel{(b)}{=}\theta\,\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^k=p\}+(1-\theta)\,\mathbb{P}\{D^{\ell+1}=q\},
\end{aligned}\tag{14}$$

where (a) holds because $Q^{\ell+1}(s,q)=\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^\ell=s\}$ and $\sum_{s\in\mathcal{T}}\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}=1$, whereas (b) uses the fact that we have $V^{\ell+1}(q)=\mathbb{P}\{D^{\ell+1}=q\}$ by its definition.

We have $\mathbb{P}\{D^\ell=s\,|\,D^k=p\}=(1-\theta^{\ell-k})\,\mathbb{P}\{D^\ell=s\}+\theta^{\ell-k}\,\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}$ by the induction assumption, so noting that $\mathbb{P}\{D^{\ell+1}=q\}=\sum_{s\in\mathcal{T}}\mathbb{P}\{D^{\ell+1}=q\,|\,D^\ell=s\}\,\mathbb{P}\{D^\ell=s\}$, we have

$$\begin{aligned}
\mathbb{P}\{D^{\ell+1}=q\,|\,D^k=p\} &= \sum_{s\in\mathcal{T}}\mathbb{P}\{D^{\ell+1}=q\,|\,D^\ell=s\}\;\mathbb{P}\{D^\ell=s\,|\,D^k=p\}\\
&= \sum_{s\in\mathcal{T}}\mathbb{P}\{D^{\ell+1}=q\,|\,D^\ell=s\}\,\Big[(1-\theta^{\ell-k})\,\mathbb{P}\{D^\ell=s\}+\theta^{\ell-k}\,\mathbb{P}\{Y^\ell=s\,|\,Y^k=p\}\Big]\\
&\stackrel{(c)}{=}(1-\theta^{\ell-k})\,\mathbb{P}\{D^{\ell+1}=q\}+\theta^{\ell-k}\,\Big[\theta\,\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^k=p\}+(1-\theta)\,\mathbb{P}\{D^{\ell+1}=q\}\Big]\\
&= (1-\theta^{\ell+1-k})\,\mathbb{P}\{D^{\ell+1}=q\}+\theta^{\ell+1-k}\,\mathbb{P}\{Y^{\ell+1}=q\,|\,Y^k=p\},
\end{aligned}$$

where (c) is by (14). Thus, the claim holds for $\ell+1\ge k$ as well. By the claim, $\mathbb{P}\{D^\ell=q\,|\,D^k=p\}=(1-\theta^{\ell-k})\,\mathbb{P}\{D^\ell=q\}+\theta^{\ell-k}\,\mathbb{P}\{Y^\ell=q\,|\,Y^k=p\}$, so the lemma follows by arranging the terms. ■

In the next lemma, we build on Lemma D.1 to bound the gap between the probabilities $\mathbb{P}\{D^k\ge s,\ D^{k-1}=p\,|\,D^\ell=q\}$ and $\mathbb{P}\{D^k\ge s,\ D^{k-1}=p\}$ for $\ell\ge k+1$.

Lemma D.2 *For all $p,q,s\in\mathcal{T}$ and $k,\ell\in\mathcal{K}$ with $\ell\ge k+1$, we have*

$$\sum_{p\in\mathcal{T}}\Big|\mathbb{P}\{D^k\ge s,\ D^{k-1}=p\,|\,D^\ell=q\}-\mathbb{P}\{D^k\ge s,\ D^{k-1}=p\}\Big|\;\le\;\frac{1}{\epsilon}\,(1-\epsilon)^{\ell-k}.$$

Proof: Using the fact that $\mathbb{P}\{D^{k+1}=q\,|\,D^k=p\}\ge\epsilon$ for all $p,q\in\mathcal{T}$ and $k\in\mathcal{K}$, it is simple to check that $\mathbb{P}\{D^k=q\}\ge\epsilon$ for all $q\in\mathcal{T}$ and $k\in\mathcal{K}$. Also, for $\ell\ge k+1$, by the Bayes rule, we have

$$\begin{aligned}
\frac{\mathbb{P}\{D^k=p,\ D^{k-1}=r\}}{\mathbb{P}\{D^\ell=q\}}\,\mathbb{P}\{D^\ell=q\,|\,D^k=p\} &= \frac{\mathbb{P}\{D^k=p,\ D^{k-1}=r\}}{\mathbb{P}\{D^\ell=q\}}\,\frac{\mathbb{P}\{D^\ell=q,\ D^k=p\}}{\mathbb{P}\{D^k=p\}}\\
&= \mathbb{P}\{D^{k-1}=r\,|\,D^k=p\}\,\mathbb{P}\{D^k=p\,|\,D^\ell=q\} = \mathbb{P}\{D^k=p,\ D^{k-1}=r\,|\,D^\ell=q\},
\end{aligned}\tag{15}$$

where the last equality uses the fact that given D^k, D^{k-1} is independent of D^ℓ. Noting that we have $\mathbb{P}\{Y^\ell=q\,|\,Y^k=p\}-\mathbb{P}\{D^\ell=q\}\in[-1,1]$, by Lemma D.1, we obtain the chain of inequalities

$-(1-\epsilon)^{\ell-k} \leq \mathbb{P}\{D^\ell = q \,|\, D^k = p\} - \mathbb{P}\{D^\ell = q\} \leq (1-\epsilon)^{\ell-k}$. In this case, multiplying this chain of inequalities with $\frac{\mathbb{P}\{D^k = p,\ D^{k-1}=r\}}{\mathbb{P}\{D^\ell = q\}}$, as well as using (15), we have

$$
\begin{aligned}
&-(1-\epsilon)^{\ell-k}\,\frac{\mathbb{P}\{D^k = p,\ D^{k-1}=r\}}{\mathbb{P}\{D^\ell = q\}} \\
&\qquad \leq\ \mathbb{P}\{D^k = p,\ D^{k-1} = r \,|\, D^\ell = q\} - \mathbb{P}\{D^k = p,\ D^{k-1}=r\} \\
&\qquad\qquad \leq\ (1-\epsilon)^{\ell-k}\,\frac{\mathbb{P}\{D^k = p,\ D^{k-1}=r\}}{\mathbb{P}\{D^\ell = q\}}.
\end{aligned}
$$

Because $\mathbb{P}\{D^\ell = q\} \geq \epsilon$, the chain of inequalities above yields $-\frac{1}{\epsilon}(1-\epsilon)^{\ell-k}\,\mathbb{P}\{D^k = p,\ D^{k-1}=r\} \leq \mathbb{P}\{D^k = p,\ D^{k-1} = r \,|\, D^\ell = q\} - \mathbb{P}\{D^k = p,\ D^{k-1}=r\} \leq \frac{1}{\epsilon}(1-\epsilon)^{\ell-k}\,\mathbb{P}\{D^k = p,\ D^{k-1}=r\}$.

To conclude the proof, if we add the last chain of inequalities in the previous paragraph over all $p \geq s$, then we obtain the chain of inequalities

$$
\begin{aligned}
&-\frac{1}{\epsilon}(1-\epsilon)^{\ell-k}\,\mathbb{P}\{D^k \geq s,\ D^{k-1}=r\} \\
&\qquad \leq\ \mathbb{P}\{D^k \geq s,\ D^{k-1} = r \,|\, D^\ell = q\} - \mathbb{P}\{D^k \geq s,\ D^{k-1}=r\} \\
&\qquad\qquad \leq\ \frac{1}{\epsilon}(1-\epsilon)^{\ell-k}\,\mathbb{P}\{D^k \geq s,\ D^{k-1}=r\}.
\end{aligned}
$$

Using the fact that $\sum_{r\in\mathcal{T}} \mathbb{P}\{D^k \geq s,\ D^{k-1}=r\} = \mathbb{P}\{D^k \geq s\} \leq 1$, the chain of inequalities above yields $\sum_{r\in\mathcal{T}} |\mathbb{P}\{D^k \geq s,\ D^{k-1} = r \,|\, D^\ell = q\} - \mathbb{P}\{D^k \geq s,\ D^{k-1}=r\}| \leq \frac{1}{\epsilon}(1-\epsilon)^{\ell-k}$. ■

Using the lemma above, we can show the first result that we are interested in. In the next lemma, we show that $|M_i^k(\ell) - M_i^k(\ell+1)| \leq \frac{2}{\epsilon^3}$ with probability one.

Lemma D.3 *For all $i \in \mathcal{L}$ and $k, \ell \in \mathcal{K}$ and $\ell + 1 \leq k$, with probability one, we have*

$$
|M_i^k(\ell) - M_i^k(\ell+1)| \ \leq\ \frac{2}{\epsilon^3}.
$$

Proof: By the definition of $\Psi_t^k(q)$, we have $\mathbb{E}\{\Psi_t^k(q) \,|\, \boldsymbol{D}^{[v,\ell]}\} = \mathbb{P}\{D^k \geq t,\ D^{k-1} = q \,|\, \boldsymbol{D}^{[v,\ell]}\}$ for any $v \in \mathcal{K}$. Noting that $V_i^k = \sum_{v=1}^k \sum_{s\in\mathcal{T}} \sum_{p\in\mathcal{T}} \Psi_s^v(p)\, n_{is}^v(p)$, we obtain

$$
\begin{aligned}
|\,M_i^k(\ell) - M_i^k(\ell+1)\,| \ &=\ |\,\mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[\ell,k]}\} - \mathbb{E}\{V_i^k \,|\, \boldsymbol{D}^{[\ell+1,k]}\}\,| \\
&\leq \sum_{v=1}^{k} \sum_{s\in\mathcal{T}} \sum_{p\in\mathcal{T}} n_{is}^v(p) \left| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell,k]}\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\} \right| \\
&\overset{(a)}{=} \sum_{v=1}^{\ell+1} \sum_{s\in\mathcal{T}} \sum_{p\in\mathcal{T}} n_{is}^v(p) \left| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell,k]}\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\} \right|, \qquad (16)
\end{aligned}
$$

where (a) follows from the fact that if $v \geq \ell + 2$, then both D^v and D^{v-1} are deterministic functions of $\boldsymbol{D}^{[\ell+1,k]}$. Therefore, if $v \geq \ell + 2$, then the probabilities $\mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell,k]}\}$

and $\mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\}$ take the same value of zero or one. For $v \leq \ell$, given D^ℓ, both D^v and D^{v-1} are independent of $D^{\ell+1}, \ldots, D^k$. Thus, if $v \leq \ell$, then $\mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell,k]}\} = \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^\ell\}$. Similarly, for $v \leq \ell$, given $D^{\ell+1}$, both D^v and D^{v-1} are independent of $D^{\ell+2}, \ldots, D^k$. Thus, if $v \leq \ell$, then $\mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\} = \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^{\ell+1}\}$. Lastly, by the definition of $n_{it}^k(q)$, we have $n_{it}^k(q) = \mathbb{E}\{N_{it}^k(q)\} = \sum_{j \in \mathcal{J}} a_{ij}\, \gamma\, \overline{x}_{jt}^k(q) \leq \gamma \sum_{j \in \mathcal{J}} \lambda_{jt}^k \leq 1$, where the first inequality uses the fact that $\overline{\boldsymbol{x}}$ satisfies the second constraint in problem (2). In this case, we can upper bound the expression on the right side of (16) as

$$
\begin{aligned}
&\sum_{v=1}^{\ell+1} \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} n_{is}^v(p) \Big| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell,k]}\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\} \Big| \\
&\quad = \sum_{v=1}^{\ell} \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} n_{is}^v(p) \Big| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^\ell\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^{\ell+1}\} \Big| \\
&\qquad + \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} n_{is}^{\ell+1}(p) \Big| \mathbb{P}\{D^{\ell+1} \geq s,\ D^\ell = p \,|\, \boldsymbol{D}^{[\ell,k]}\} - \mathbb{P}\{D^{\ell+1} \geq s,\ D^\ell = p \,|\, \boldsymbol{D}^{[\ell+1,k]}\} \Big| \\
&\quad \leq \sum_{v=1}^{\ell} \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} \Big| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^\ell\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p\} \Big| \\
&\qquad + \sum_{v=1}^{\ell} \sum_{s \in \mathcal{T}} \sum_{p \in \mathcal{T}} \Big| \mathbb{P}\{D^v \geq s,\ D^{v-1} = p \,|\, D^{\ell+1}\} - \mathbb{P}\{D^v \geq s,\ D^{v-1} = p\} \Big| + T^2 \\
&\quad \stackrel{(b)}{\leq} \frac{1}{\epsilon} T \sum_{v=1}^{\ell} (1-\epsilon)^{\ell-v} + \frac{1}{\epsilon} T \sum_{v=1}^{\ell} (1-\epsilon)^{\ell+1-v} + T^2 \\
&\quad \leq \frac{1}{\epsilon} T \left[\frac{1}{\epsilon} + \frac{1-\epsilon}{\epsilon} \right] + T^2, \qquad (17)
\end{aligned}
$$

where (b) uses Lemma D.2. Because $1 = \sum_{p \in \mathcal{T}} \mathbb{P}\{D^{k+1} = p \,|\, D^k = q\} \geq T\, \epsilon$, we get $T \leq 1/\epsilon$, so $\frac{1}{\epsilon} T \left[\frac{1}{\epsilon} + \frac{1-\epsilon}{\epsilon} \right] + T^2 \leq \frac{1}{\epsilon^2} \left[\frac{1}{\epsilon} + \frac{1-\epsilon}{\epsilon} \right] + \frac{1}{\epsilon^2} = \frac{2}{\epsilon^3}$. Thus, the result follows from (16) and (17). ■

In the next lemma, which is the second result that we are interested in, we give an upper bound on the moment generating function of a Bernoulli random variable.

Lemma D.4 *If Z is Bernoulli with mean μ, then we have $\mathbb{E}\{e^{\lambda(Z-\mu)}\} \leq e^{\mu \lambda^2}$ for all $\lambda \in [0,1]$.*

Proof: Because e^x is convex in x, over the interval [0,1], the function e^x lies below the line segment that connects the points $(0,1)$ and $(1,e)$. Thus, we have $e^x \leq 1 + x\,(e-1)$ for all $x \in [0,1]$. In this case, noting that $e^\lambda = \int_0^\lambda e^x\, dx + 1$, for all $\lambda \in [0,1]$, we get $e^\lambda \leq \int_0^\lambda (1 + x\,(e-1))\, dx + 1 = \lambda + \frac{1}{2}(e-1)\lambda^2 + 1 \leq 1 + \lambda + \lambda^2$, yielding $e^\lambda - 1 - \lambda \leq \lambda^2$ for all $\lambda \in [0,1]$. Because Z is Bernoulli with mean μ, we have $\mathbb{P}\{Z = 1\} = \mu$, so $\mathbb{E}\{e^{\lambda Z}\} = (1-\mu) + \mu\, e^\lambda = 1 + \mu\,(e^\lambda - 1) \leq e^{\mu\,(e^\lambda - 1)}$, where the last inequality holds because $1 + x \leq e^x$ for all $x \in \mathbb{R}$. By the last chain of inequalities, we obtain $\mathbb{E}\{e^{\lambda(Z-\mu)}\} \leq e^{\mu(e^\lambda - 1 - \lambda)} \leq e^{\mu\lambda^2}$, where we use the fact that $e^\lambda - 1 - \lambda \leq \lambda^2$. ■

Appendix E: Proof of Theorem 6.1

We give a proof for Theorem 6.1. Our starting point is similar to the outline of Section 5, but we will construct different concentration inequalities to exploit the sub-Gaussian property.

Preliminary Random Variables and Availability Probabilities:

We define four classes of Bernoulli random variables for each $k \in \mathcal{K}$ and $t \in \mathcal{T}$. We worked with the analogues of these random variables in Section 5.

- *Demand in Each Stage.* The random variable Ψ_t^k takes value one if we reach time period t in stage k before this stage is over. In other words, recalling that we use $\mathbf{1}(\cdot)$ to denote the indicator function, we simply have $\Psi_t^k = \mathbf{1}(D^k \geq t)$.

- *Product Request.* For each $j \in \mathcal{J}$, the random variable A_{jt}^k takes value one if the customer arriving at time period t in stage k requests product j. We have $\mathbb{P}\{A_{jt}^k = 1\} = \lambda_{jt}^k$. The random variables $\{A_{jt}^k : t \in \mathcal{T},\ k \in \mathcal{K}\}$ are independent of each other.

- *Policy Decision.* For each $j \in \mathcal{J}$, the random variable X_{jt}^k takes value one if the approximate policy is willing to accept a request for product j at time period in stage k. By the definition of the approximate policy under independent demands, $\mathbb{P}\{X_{jt}^k = 1\} = \gamma \sum_{q \in \mathcal{T}} \mathbb{P}\{D^{k-1} = q\} \frac{\overline{x}_{jt}^k(q)}{\lambda_{jt}^k}$.

- *Availability.* For each $j \in \mathcal{J}$, the random variable G_{jt}^k takes value one if we have enough capacity to accept a request for product j at time period t in stage k under the approximate policy. Instead of calculating the probability $\mathbb{P}\{G_{jt}^k = 1\}$, we will lower bound $\mathbb{P}\{G_{jt}^k = 1 \,|\, \Psi_t^k = 1\}$.

Under independent demands, we drop the assumption that $\mathbb{P}\{D^{k+1} = p \,|\, D^k = q\} \geq \epsilon$ for all $p, q \in \mathcal{T}$ and $k \in \mathcal{K}$ for some $\epsilon > 0$. Instead, we assume that $\mathbb{E}\{e^{\lambda(D^k - \mathbb{E}\{D^k\})}\} \leq e^{\frac{\sigma^2}{200}\lambda^2}$ for all $k \in \mathcal{K}$ and $\lambda \geq 0$ for some $\sigma^2 > 0$. Throughout this section, when we refer to the approximate policy, we mean the approximate policy under independent demands as described in Section 6. Under the approximate policy, the sales for product j at time period t in stage k is given by the random variable $\Psi_t^k \, G_{jt}^k \, A_{jt}^k \, X_{jt}^k$. Taking expectations, the expected sales for product j at time period t in stage k is $\mathbb{P}\{\Psi_t^k = 1\}\, \mathbb{P}\{G_{jt}^k = 1 \,|\, \Psi_t^k = 1\}\, \mathbb{P}\{A_{jt}^k = 1\}\, \mathbb{P}\{X_{jt}^k = 1\}$, where we use the fact that the random variables A_{jt}^k and X_{jt}^k are independent of the demands and remaining capacities of the resources. Noting that $\mathbb{P}\{A_{jt}^k = 1\} = \lambda_{jt}$ and $\mathbb{P}\{X_{jt}^k = 1\} = \gamma \sum_{q \in \mathcal{T}} \mathbb{P}\{D^{k-1} = q\} \frac{\overline{x}_{jt}^k(q)}{\lambda_{jt}^k}$, the last expectation is given by $\mathbb{P}\{D^k \geq t\}\, \mathbb{P}\{G_{jt}^k = 1 \,|\, \Psi_t^k = 1\} \sum_{q \in \mathcal{T}} \gamma\, \mathbb{P}\{D^{k-1} = q\}\, \overline{x}_{jt}^k(q)$. Therefore, the total expected revenue of the approximate policy is

$$\mathsf{APX} \;=\; \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{q \in \mathcal{T}} f_j\, \mathbb{P}\{D^k \geq t\}\; \mathbb{P}\{G_{jt}^k = 1 \,|\, \Psi_t^k = 1\}\; \gamma\, \mathbb{P}\{D^{k-1} = q\}\, \overline{x}_{jt}^k(q).$$

Because $\overline{\boldsymbol{x}}$ is an optimal solution to problem (2), the optimal objective value of problem (2) is $\overline{Z}_{\mathsf{LP}} = \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{q \in \mathcal{T}} f_j\, \mathbb{P}\{D^k \geq t\}\, \mathbb{P}\{D^{k-1} = q\}\, \overline{x}_{jt}^k(q)$, where we use the fact that the demands in different stages are independent, so $\mathbb{P}\{D^k \geq t,\ D^{k-1} = q\} = \mathbb{P}\{D^k \geq t\}\,\mathbb{P}\{D^{k-1} = q\}$. Thus, comparing the expressions for the total expected revenue APX of the approximate policy and the optimal objective value $\overline{Z}_{\mathsf{LP}}$ of the linear program in (2), it follows that if we can show that $\mathbb{P}\{G^k_{jt} = 1 \,|\, \Psi^k_t = 1\} \geq \alpha$, then we obtain $\mathsf{APX} \geq \gamma\,\alpha \overline{Z}_{\mathsf{LP}}$. We focus on lower bounding the availability probability $\mathbb{P}\{G^k_{jt} = 1 \,|\, \Psi^k_t = 1\}$. Under the approximate policy, the capacity consumption of resource i at time period t in stage k is given by the random variable $\sum_{j\in\mathcal{J}} a_{ij}\,\Psi^k_t\,G^k_{jt}\,A^k_{jt}\,X^k_{jt}$, which implies that the capacity consumption of resource i at time period t in stage k is upper bounded by $\sum_{j\in\mathcal{J}} a_{ij}\,\Psi^k_t\,A^k_{jt}\,X^k_{jt}$. Letting $N^k_{it} = \sum_{j\in\mathcal{J}} a_{ij}\,A^k_{jt}\,X^k_{jt}$, we write the upper bound on the capacity consumption of resource i at time period t in stage k as $\Psi^k_t\,N^k_{it}$. In this case, if we have $\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}} \Psi^\ell_s\,N^\ell_{is} + \sum_{s=1}^{t} \Psi^k_s\,N^k_{is} < c_i$, then the total capacity consumption of resource i up to and including time period t in stage k does not exceed the capacity of the resource, in which case, we must have capacity available for resource i at time period t in stage k. Therefore, using the same line of reasoning that we followed to obtain the chain of inequalities in (4), we can lower bound the availability probability as

$$\mathbb{P}\{G_{jt} = 1 \,|\, \Psi^k_t = 1\} \;\geq\; 1 - \sum_{i\in\mathcal{L}_j} \mathbb{P}\left\{ \sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}} \Psi^\ell_s\,N^\ell_{is} + \sum_{s=1}^{t} N^k_{is} \,\geq\, c_i \;\middle|\; D^k \geq t \right\}. \tag{18}$$

The discussion so far closely followed the one at the beginning of Section 5, but we need to deviate from that discussion to exploit the sub-Gaussian assumption.

Moment Generating Function Bounds:

Note that N^k_{it} is a Bernoulli random variable. We define $n^k_{it} = \mathbb{E}\{N^k_{it}\}$. In the next lemma, we bound the moment generating function of the squared deviation of the demand around its mean.

Lemma E.1 *Letting $\mu^k = \mathbb{E}\{D^k\}$, for all $k \in \mathcal{K}$, if $|\lambda| \leq \frac{2}{\sigma}$, then we have*

$$\mathbb{E}\{e^{\lambda^2 (D^k - \mu^k)^2}\} \;\leq\; e^{\frac{1}{4}\,\sigma^2\lambda^2}.$$

Proof: In an auxiliary lemma, labeled as Lemma E.4, given at the end of this section, we show that if Z is a mean-zero sub-Gaussian random variable with variance proxy M^2 so that $\mathbb{E}\{e^{\lambda Z}\} \leq e^{M^2\lambda^2}$ for all $\lambda \in \mathbb{R}$, then we have $\mathbb{E}\{e^{\theta^2 Z^2}\} \leq e^{(7M\theta)^2}$ for all $|\theta| \leq \frac{1}{7M}$. Because $D^k - \mu^k$ is a mean-zero sub-Gaussian random variable with variance proxy $\frac{\sigma^2}{200}$, using Lemma E.4 with $M = \frac{\sigma}{\sqrt{200}}$, we have $\mathbb{E}\{e^{\theta^2\,(D^k-\mu^k)^2}\} \leq e^{(7\frac{\sigma}{\sqrt{200}}\theta)^2}$ for all $|\theta| \leq \frac{1}{7\,\sigma/\sqrt{200}}$. We have $\frac{1}{7\,\sigma/\sqrt{200}} \geq \frac{2}{\sigma}$. Thus, if we have $|\theta| \leq \frac{2}{\sigma}$, then we also have $|\theta| \leq \frac{1}{7\,\sigma/\sqrt{200}}$, in which case, $\mathbb{E}\{e^{\theta^2\,(D^k-\mu^k)^2}\} \leq e^{(7\frac{\sigma}{\sqrt{200}}\theta)^2} = e^{\frac{49}{200}\sigma^2\theta^2} \leq e^{\frac{1}{4}\sigma^2\theta^2}$. ∎

In the next lemma, we use the lemma above to bound the moment generating function of $\sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\,n^k_{is}$, which will be key to bounding the availability probabilities.

Lemma E.2 *For all $k \in \mathcal{K}$, $i \in \mathcal{L}$ and $\lambda \in \mathbb{R}$, we have*

$$\mathbb{E}\{e^{\lambda \sum_{s\in\mathcal{T}} n_{is}^k \mathbf{1}(D^k \geq s)}\} \leq e^{\sigma^2 \lambda^2 + \lambda \sum_{s\in\mathcal{T}} n_{is}^k \mathbb{P}\{D^k \geq s\}}.$$

Proof: For the moment, we assume that $|\lambda| \leq \frac{1}{\sigma}$. We will need two inequalities. First, we use the random variable R^k to denote an independent and identically distributed copy of D^k. Noting that $\sum_{s\in\mathcal{T}} n_{is}^k \mathbf{1}(D^k \geq s) = \sum_{s=1}^{D^k} n_{is}^k$, using the fact that $n_{is}^k \in [0,1]$, we get the chain of inequalities $|\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbf{1}(R^k \geq s))| = |\sum_{s=1}^{D^k} n_{is}^k - \sum_{s=1}^{R^k} n_{is}^k| \leq |D^k - R^k|$. It is simple to verify the inequality $(a-b)^2 \leq 2(a-\delta)^2 + 2(b-\delta)^2$. Thus, letting $\mu^k = \mathbb{E}\{D^k\} = \mathbb{E}\{R^k\}$, we obtain $[\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbf{1}(R^k \geq s))]^2 \leq (D^k - R^k)^2 \leq 2\,(D^k - \mu^k)^2 + 2\,(R^k - \mu^k)^2$. Second, because $|\lambda| \leq \frac{1}{\sigma}$, we have $|\sqrt{2}\,\lambda| \leq \frac{\sqrt{2}}{\sigma} \leq \frac{2}{\sigma}$, so by Lemma E.1, $\mathbb{E}\{e^{2\lambda^2 (D^k - \mu^k)^2}\} \leq e^{\frac{1}{2}\sigma^2\lambda^2}$. Thus, we have

$$\begin{aligned}
\mathbb{E}\{e^{\lambda^2 [\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbb{P}\{D^k \geq s\})]^2}\} &= \mathbb{E}\{e^{\lambda^2 [\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbb{E}\{\mathbf{1}(R^k \geq s)\})]^2}\} \\
&\overset{(a)}{\leq} \mathbb{E}\{e^{\lambda^2 [\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbf{1}(R^k \geq s))]^2}\} \overset{(b)}{\leq} \mathbb{E}\{e^{2\lambda^2 (D^k - \mu^k)^2}\}\, \mathbb{E}\{e^{2\lambda^2 (R^k - \mu^k)^2}\} \overset{(c)}{\leq} e^{\sigma^2 \lambda^2}, \quad (19)
\end{aligned}$$

where (a) follows by the Jensen inequality and the fact that $e^{\lambda^2 x^2}$ is convex in x, (b) uses the first inequality in this paragraph, as well as the fact that the random variables D^k and R^k are independent and (c) uses the second inequality in this paragraph. In an auxiliary lemma, labeled as Lemma E.5, given at the end of this section, we show that if Z is a mean-zero random variable that satisfies $\mathbb{E}\{e^{\lambda^2 Z^2}\} \leq e^{M^2 \lambda^2}$ for all $|\lambda| \leq \frac{1}{M}$, then we have $\mathbb{E}\{e^{\theta Z}\} \leq e^{M^2 \theta^2}$ for all $\theta \in \mathbb{R}$. By (19), we have $\mathbb{E}\{e^{\lambda^2 [\sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbb{P}\{D^k \geq s\})]^2}\} \leq e^{\sigma^2 \lambda^2}$ for all $|\lambda| \leq \frac{1}{\sigma}$. Thus, using Lemma E.5 with $M = \sigma$, $\mathbb{E}\{e^{\theta \sum_{s\in\mathcal{T}} n_{is}^k (\mathbf{1}(D^k \geq s) - \mathbb{P}\{D^k \geq s\})}\} \leq e^{\sigma^2 \theta^2}$ for all $\theta \in \mathbb{R}$, which is the desired result. ■

Performance Guarantee for the Approximate Policy:

In the next lemma, we use the moment generating function bound in Lemma E.2 to give a lower bound on the availability probabilities on the right side of (18).

Lemma E.3 *For all $k \in \mathcal{K}$, $i \in \mathcal{L}$, $t \in \mathcal{T}$ and $\lambda \in [0,1]$, we have*

$$\mathbb{P}\left\{ \sum_{\ell=1}^{k-1} \sum_{s\in\mathcal{T}} \mathbf{1}(D^\ell \geq s)\, N_{is}^\ell + \sum_{s=1}^{t} N_{is}^k \geq c_i \,\middle|\, D^k \geq t \right\} \leq e^{(c_i + 4\sigma^2 (k-1))\lambda^2 - (1-\gamma)\, c_i \lambda}.$$

Proof: Given D^k, $\mathbf{1}(D^k \geq s)\, N_{is}^k$ is a Bernoulli random variable with mean $\mathbf{1}(D^k \geq s)\, n_{is}^k$, so by Lemma D.4, we have $\mathbb{E}\{e^{\lambda \mathbf{1}(D^k \geq s)\,(N_{is}^k - n_{is}^k)} \mid D^k\} \leq e^{\lambda^2 \mathbf{1}(D^k \geq s)\, n_{is}^k}$. In this case, we obtain

$$\mathbb{E}\{e^{\lambda \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\, N_{is}^k} \mid D^k\} = e^{\lambda \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\, n_{is}^k}\, \mathbb{E}\{e^{\lambda \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\,(N_{is}^k - n_{is}^k)} \mid D^k\} \leq e^{(\lambda + \lambda^2) \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\, n_{is}^k},$$

where we use the fact that $\{N_{is}^k : s \in \mathcal{T}\}$ are independent of each other. Taking expectations in the chain of inequalities above, we obtain $\mathbb{E}\{e^{\lambda \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\, N_{is}^k}\} \leq \mathbb{E}\{e^{(\lambda+\lambda^2) \sum_{s\in\mathcal{T}} \mathbf{1}(D^k \geq s)\, n_{is}^k}\}$. Using the same argument, we have $\mathbb{E}\{e^{\lambda\sum_{s=1}^t N_{is}^k}\} \le e^{(\lambda^2+\lambda)\sum_{s=1}^t n_{is}^k}$. On the other hand, by the discussion in Section 3, we can replace the probability $\mathbb{P}\{D^\ell \ge s,\, D^{\ell-1}=p \,|\, D^k \ge t,\, D^{k-1}=q\}$ in the first constraint in problem (2) with $\mathbb{P}\{D^\ell \ge s,\, D^{\ell-1}=p \,|\, D^{k-1}=q\}$. Furthermore, $\overline{\boldsymbol{x}}$ is a feasible solution to problem (2), so it satisfies the first constraint. Multiplying this constraint with $\mathbb{P}\{D^{k-1}=q\}$ and adding over all $q\in\mathcal{T}$, we get $\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\sum_{p\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\mathbb{P}\{D^\ell\ge s,\ D^{\ell-1}=p\}\,\overline{x}_{js}^\ell(p) + \sum_{s=1}^t\sum_{j\in\mathcal{J}}\sum_{q\in\mathcal{T}} a_{ij}\,\mathbb{P}\{D^{k-1}=q\}\,\overline{x}_{js}^k(q)\le c_i$. By the definition of N_{it}^k, we also have the identity $n_{it}^k=\mathbb{E}\{N_{it}^k\}=\gamma\sum_{j\in\mathcal{J}}a_{ij}\sum_{q\in\mathcal{T}}\mathbb{P}\{D^{k-1}=q\}\,\overline{x}_{jt}^k(q)$. Thus, noting that the demands in different stages are independent of each other, so that $\mathbb{P}\{D^\ell\ge s,\ D^{\ell-1}=p\}=\mathbb{P}\{D^\ell\ge s\}\,\mathbb{P}\{D^{\ell-1}=p\}$, the last inequality yields $\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbb{P}\{D^\ell\ge s\}\,n_{is}^\ell+\sum_{s=1}^t n_{is}^k\le\gamma\,c_i$. Noting that the random variables $\{N_{it}^k : t\in\mathcal{T},\ k\in\mathcal{K}\}$ and $\{D^k : k\in\mathcal{K}\}$ are all independent of each other, we get

$$
\begin{aligned}
\mathbb{P}\Big\{\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbf{1}(D^\ell\ge s)\,N_{is}^\ell+\sum_{s=1}^t N_{is}^k\ \ge\ c_i\,\Big|\,D^k\ge t\Big\}\ &=\ \mathbb{P}\{e^{\lambda\,(\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbf{1}(D^\ell\ge s)\,N_{is}^\ell+\sum_{s=1}^t N_{is}^k)}\ \ge e^{\lambda\,c_i}\}\\
&\overset{(a)}{\le}\ \frac{1}{e^{\lambda\,c_i}}\,\mathbb{E}\{e^{\lambda\,(\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbf{1}(D^\ell\ge s)\,N_{is}^\ell+\sum_{s=1}^t N_{is}^k)}\}\\
&=\ \frac{1}{e^{\lambda\,c_i}}\prod_{\ell=1}^{k-1}\mathbb{E}\{e^{\lambda\,(\sum_{s\in\mathcal{T}}\mathbf{1}(D^\ell\ge s)\,N_{is}^\ell}\}\ \mathbb{E}\{e^{\lambda\sum_{s=1}^t N_{is}^k}\}\\
&\overset{(b)}{\le}\ \frac{1}{e^{\lambda\,c_i}}\prod_{\ell=1}^{k-1}\mathbb{E}\{e^{(\lambda^2+\lambda)\sum_{s\in\mathcal{T}}\mathbf{1}(D^\ell\ge s)\,n_{is}^\ell}\}\ e^{(\lambda^2+\lambda)\sum_{s=1}^t n_{is}^k}\\
&\overset{(c)}{\le}\ \frac{1}{e^{\lambda\,c_i}}\prod_{\ell=1}^{k-1}e^{4\sigma^2\lambda^2+(\lambda^2+\lambda)\sum_{s\in\mathcal{T}}\mathbb{P}\{D^\ell\ge s\}\,n_{is}^\ell}\ e^{(\lambda^2+\lambda)\sum_{s=1}^t n_{is}^k}\\
&=\ \frac{1}{e^{\lambda\,c_i}}\,e^{4\sigma^2(k-1)\lambda^2}\ e^{(\lambda^2+\lambda)(\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbb{P}\{D^\ell\ge s\}\,n_{is}^\ell+\sum_{s=1}^t n_{is}^k)}\\
&\overset{(d)}{\le}\ e^{(c_i+4\,\sigma^2(k-1))\lambda^2-(1-\gamma)\,c_i\lambda},
\end{aligned}
$$

where (a) is the Markov inequality, (b) is by the discussion earlier in the proof, (c) uses Lemma E.2 and $4\lambda^2\ge(\lambda^2+\lambda)^2$ for $\lambda\in[0,1]$ and (d) holds by $\sum_{\ell=1}^{k-1}\sum_{s\in\mathcal{T}}\mathbb{P}\{D^\ell\ge s\}\,n_{is}^\ell+\sum_{s=1}^t n_{is}^k\le\gamma\,c_i$. ■

Using specific values for γ and λ in Lemma E.3 will provide a lower bound for the availability probabilities, which, in turn, will yield a performance guarantee for the approximate policy.

Proof of Theorem 6.1:

Identifying $4\,\sigma^2$ with $\frac{3}{\epsilon^6}$, the bounds in Lemmas 5.3 and E.3 have the same form. Thus, choosing $\gamma=1-\frac{\sqrt{4\,(c_{\min}+4\,\sigma^2\,(K-1))\log c_{\min}}}{c_{\min}}$ and $\lambda=\frac{(1-\gamma)\,c_i}{2(c_i+4\,\sigma^2\,(K-1))}$, following the proof of Theorem 4.1, we get

$$
\frac{\mathsf{APX}}{\overline{Z}_{\mathsf{LP}}}\ \ge\ \left(1-4\frac{\sqrt{(c_{\min}+\sigma^2\,(K-1))\log c_{\min}}}{c_{\min}}-\frac{L}{c_{\min}}\right).
$$

The proof of the inequality above precisely follows the argument at the end of Section 5 line by line. On the other hand, we can precisely follow the argument in Appendix C line by line to show that $\frac{\mathsf{APX}}{\bar{Z}_{\mathsf{LP}}} \geq \frac{1}{4L}$. In this case, the desired result follows by collecting these two inequalities together, as well as noting that we have $\frac{\mathsf{APX}}{\mathsf{OPT}} \geq \frac{\mathsf{APX}}{\bar{Z}_{\mathsf{LP}}}$ by Theorem 3.1. ∎

In the proofs of Lemmas E.1 and E.2, we used two results related to sub-Gaussian random variables. We used the next lemma in the proof of Lemma E.1.

Lemma E.4 *If Z is a mean-zero sub-Gaussian random variable such that $\mathbb{E}\{e^{\lambda Z}\} \leq e^{M^2\lambda^2}$ for all $\lambda \in \mathbb{R}$, then we have $\mathbb{E}\{e^{\theta^2 Z^2}\} \leq e^{(7M\theta)^2}$ for all $|\theta| \leq \frac{1}{7M}$.*

Proof: By the discussion that follows Definition 2.2 in Wainwright (2019), if $\mathbb{E}\{e^{\lambda Z}\} \leq e^{M^2\lambda^2}$ for all $\lambda \in \mathbb{R}$, then $\mathbb{P}\{|Z| \geq t\} \leq 2e^{-\frac{t^2}{4M^2}}$ for all $t \geq 0$. Letting $\|Z\|_{L^p} = (\mathbb{E}\{|Z|^p\})^{1/p}$ for $p \in \mathbb{Z}_{++}$, we get

$$\begin{aligned}\mathbb{E}\{|Z|^p\} &= \int_0^\infty \mathbb{P}\{|Z|^p \geq u\}\,du \overset{(a)}{=} \int_0^\infty \mathbb{P}\{|Z| \geq t\}\,p\,t^{p-1}\,dt \leq 2\int_0^\infty e^{-\frac{t^2}{4M^2}}\,p\,t^{p-1}\,dt \\ &\overset{(b)}{=} 2p\int_0^\infty e^{-z}(2M\sqrt{z})^{p-1}\,\frac{M}{\sqrt{z}}\,dz = p\,(2M)^p\int_0^\infty e^{-z}z^{\frac{p}{2}-1}\,dz \overset{(c)}{=} p\,(2M)^p\Gamma(p/2) \overset{(d)}{\leq} p\,(2M)^p\,(p/2)^{p/2},\end{aligned}$$

where (a) uses the change of variables $u = t^p$, (b) uses the change of variables $z = t^2/(4M^2)$, (c) is the definition of the gamma function and (d) holds because the gamma function satisfies $\Gamma(x) \leq x^x$. By the chain of inequalities above, we obtain $\|Z\|_{L^p} \leq p^{1/p}\,M\sqrt{2p}$. It is simple to verify that $\max_{p\in\mathbb{Z}_{++}} p^{1/p} = 3^{1/3}$, so the last inequality yields $\|Z\|_{L^p} \leq 3^{1/3}\,M\sqrt{2p} \leq 2.04\,M\sqrt{p}$ for all $p \in \mathbb{Z}_{++}$. In this case, we have $\mathbb{E}\{|Z|^{2p}\} \leq (2.04\,M\sqrt{2p})^{2p} = (2.04^2\,M^2\,2\,p)^p \leq (8.33M^2p)^p$. Also, by our assumption that $|\theta| \leq \frac{1}{7M}$, we get $8.33\,e\,M^2\theta^2 \leq \frac{8.33\,e}{49} \leq \frac{1}{2}$. Lastly, we can verify the inequality $e^{2x} \geq \frac{1}{1-x}$ for all $x \in [0, \frac{1}{2}]$. Thus, using the Taylor series expansion of e^x, we have

$$\begin{aligned}\mathbb{E}\{e^{\theta^2 Z^2}\} &= 1 + \sum_{p=1}^\infty \frac{\theta^{2p}\,\mathbb{E}\{|Z|^{2p}\}}{p!} \leq 1 + \sum_{p=1}^\infty \frac{(\theta^2\,8.33M^2p)^p}{p!} \overset{(e)}{\leq} 1 + \sum_{p=1}^\infty \frac{(\theta^2\,8.33M^2p)^p}{(p/e)^p} \\ &= 1 + \sum_{p=1}^\infty (\theta^2\,8.33\,eM^2)^p \overset{(f)}{=} \frac{1}{1-\theta^2\,8.33\,eM^2} \overset{(g)}{\leq} e^{(7M\theta)^2},\end{aligned}$$

where (e) holds because $p! \geq (p/e)^p$ by the Sterling approximation, (f) follows by $8.33\,e\,M^2\theta^2 \leq \frac{1}{2}$ and (g) holds because $8.33\,e\,M^2\theta^2 \leq \frac{1}{2}$, so $\frac{1}{1-\theta^2\,8.33\,eM^2} \leq e^{2\,\theta^2 8.33eM^2} \leq e^{49\,\theta^2M^2}$. ∎

We used the next lemma in the proof of Lemma E.2. Intuitively speaking, the result in this lemma is the converse of the one in Lemma E.4.

Lemma E.5 *If Z is a mean-zero random variable that satisfies $\mathbb{E}\{e^{\lambda^2 Z^2}\} \leq e^{M^2\lambda^2}$ for all $|\lambda| \leq \frac{1}{M}$, then we have $\mathbb{E}\{e^{\theta Z}\} \leq e^{M^2\theta^2}$ for all $\theta \in \mathbb{R}$.*

Proof: Consider any $\theta \in \mathbb{R}$. We claim that $\mathbb{E}\{e^{\frac{\theta}{M}Z}\} \leq e^{\theta^2}$. First, assume that $|\theta| \leq 1$. Because $|\frac{\theta}{M}| \leq \frac{1}{M}$, using the assumption of the lemma with $\lambda = \frac{\theta}{M}$, we get $\mathbb{E}\{e^{\frac{\theta^2}{M^2}Z^2}\} \leq e^{\theta^2}$. For any $x \in \mathbb{R}$, we have the inequality $e^x \leq x + e^{x^2}$, in which case, noting that $\mathbb{E}\{Z\} = 0$, we obtain $\mathbb{E}\{e^{\frac{\theta}{M}Z}\} \leq \mathbb{E}\{\frac{\theta}{M}Z\} + \mathbb{E}\{e^{\frac{\theta^2}{M^2}Z^2}\} \leq e^{\theta^2}$, establishing the claim when $|\theta| \leq 1$. Second, assume that $|\theta| \geq 1$. Because $\frac{1}{\sqrt{2}M} \leq \frac{1}{M}$, using the assumption of the lemma with $\lambda = \frac{1}{\sqrt{2}M}$, we get $\mathbb{E}\{e^{\frac{1}{2M^2}Z^2}\} \leq e^{\frac{1}{2}}$. We have the standard inequality $\frac{1}{2}a^2 + \frac{1}{2}b \geq ab$, which implies that $\frac{\theta^2}{2} + \frac{x^2}{2M^2} \geq \frac{\theta\, x}{M}$ for all $x \in \mathbb{R}$. In this case, we obtain the chain of inequalities $\mathbb{E}\{e^{\frac{\theta}{M}Z}\} \leq e^{\frac{\theta^2}{2}}\, \mathbb{E}\{e^{\frac{1}{2M^2}Z^2}\} \leq e^{\frac{\theta^2}{2}}\, e^{\frac{1}{2}} \leq e^{\theta^2}$, where the last inequality holds because we have $|\theta| \geq 1$. Therefore, the claim holds when $|\theta| \geq 1$ as well.

By the claim established in the previous paragraph, we have $\mathbb{E}\{e^{\frac{\theta}{M}Z}\} \leq e^{\theta^2}$ for all $\theta \in \mathbb{R}$, which is equivalent to having $\mathbb{E}\{e^{\theta Z}\} \leq e^{M^2\theta^2}$ for all $\theta \in \mathbb{R}$. ■

Appendix F: Request Arrival Probabilities for the Products

We give our approach for generating the request arrival probabilities $\{\lambda^k_{jt} : j \in \mathcal{J},\ t \in \mathcal{T},\ k \in \mathcal{K}\}$. For each origin-destination pair (f, g), we sample ζ_{fg} from the uniform distribution over $[0, 1]$. Using N to denote the set of all locations, we set $\gamma_{fg} = \frac{\zeta_{fg}}{\sum_{k \in N} \sum_{\ell \in N \setminus \{\ell\}} \zeta_{k\ell}}$ so that the parameters $\{\gamma_{fg} : f \in N,\ g \in N \setminus \{f\}\}$ are normalized to add up to one. At any time period in any stage, we have a request for an itinerary that connects origin-destination pair (f, g) with probability γ_{fg}. There is a high-fare and a low-fare itinerary that connects each origin-destination pair. The probability that we have a request for a low-fare itinerary linearly increases over time. The probability that we have a request for a high-fare itinerary is zero until a certain threshold and increases linearly over time after the threshold. Therefore, for each origin-destination pair (f, g), we sample the threshold τ_{fg} from the uniform distribution over $\{\lceil \frac{1}{2}KT \rceil, \ldots, \lceil \frac{2}{3}KT \rceil\}$. The function $F(t) = \frac{KT+1-t}{KT}$ is decreasing for $t \in [1, KT]$, whereas the function $H_{fg}(t) = \frac{[t-\tau_{fg}]^+}{KT-\tau_{fg}}$ takes the value zero for $t \in [1, \tau_{fg}]$ and is increasing for $t \in [\tau_{fg}, KT]$. In this case, noting that there are $(k-1)\,T$ time periods before stage k, if product j corresponds to a low-fare itinerary that connects origin-destination pair (f, g), then we set $\lambda^k_{jt} = \gamma_{fg}\, \frac{F((k-1)\,T+t)}{F((k-1)\,T+t) + H_{fg}((k-1)\,T+t)}$, whereas if product j corresponds to a high-fare itinerary that connects origin-destination pair (f, g), then we set $\lambda^k_{jt} = \gamma_{fg}\, \frac{H_{fg}((k-1)\,T+t)}{F((k-1)\,T+t) + H_{fg}((k-1)\,T+t)}$.