

# Revenue Management for Boutique Hotels: Resources with Unit Capacities and Itineraries over Intervals of Resources

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We consider the revenue management problem for a boutique hotel offering unique rooms. Customers arriving into the system make booking requests for different intervals of stay. The goal is to find a policy that determines an assortment of rooms to offer to each customer to maximize the total expected revenue. Because each room is unique and customers book intervals of days, the problem has two special features. First, each resource has a unit capacity. Second, the resources can be ordered such that each itinerary consumes an interval of resources. We consider static policies that offer each assortment of rooms with a fixed probability. We show that we can efficiently perform rollout on any static policy, allowing us to build on any static policy and construct an even better policy. Next, we develop two static policies, each of which is derived from linear and polynomial approximations of the value functions. We give performance guarantees for both policies, so the rollout policies based on these static policies inherit the same guarantee. Lastly, we develop an approach for computing an upper bound on the optimal total expected revenue. Our results for efficient rollout, static policies, and upper bounds all exploit the special features of the boutique hotel revenue management problem. We give computational experiments on both a real-world boutique hotel dataset and synthetic datasets, demonstrating that our rollout approach can provide remarkably good policies and the upper bounds provided by our approach can significantly improve those provided by existing techniques.

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## 1. Introduction

Network revenue management problems focus on managing resources with limited capacities to satisfy the requests for itineraries that arrive randomly over time. Satisfying a request for an itinerary generates a certain revenue and consumes the capacities of a combination of resources. These problems appear in airlines, hospitality, retail, railways, and broadcasting, where the meanings of a resource and an itinerary take different forms depending on the specific industry setting. The main tradeoff is between accepting a current itinerary request to generate immediate revenue and reserving the resource capacities for a more profitable itinerary request that may arrive in the future. Different itineraries consume different combinations of resources, so computing an optimal policy requires keeping track of the remaining capacities of all resources simultaneously, resulting in the curse of dimensionality as the number of resources increases.

We study a revenue management problem for boutique hotels. A boutique hotel has a collection of unique rooms. Customers arriving into the system make booking requests for different intervals

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**Figure 1** Room configurations for Villa Mahal.

of stay. Each customer is offered an assortment of rooms, and she chooses from those offered in the assortment. The goal is to find a policy for determining an assortment of rooms to offer each customer to maximize the total expected revenue. In the boutique hotel setting, each resource corresponds to a room on a particular night. Each itinerary corresponds to an interval of nights of stay in a particular room. There are two special features of the revenue management problem for a boutique hotel. First, each resource has a unit capacity because each room is unique. Second, we can order the resources such that each itinerary uses an interval of resources.

Boutique hotels are becoming more common in the hospitality industry. In Figure 1, we show the rooms offered by Villa Mahal (2020), a boutique hotel in Kalkan province of Turkey that motivated our study. The hotel offers six unique rooms: Moonlight Room, Moonlight Deluxe, Sunset Deluxe, Sunset Suite, Pool Room, and Cliff House. Each room is decorated differently and has different views, and the rooms are offered at drastically different prices. During the booking process, the customers are shown available rooms for their desired nights of stay, and they choose a specific room. Boutique Homes (2020) lists more than 150 boutique hotels with a similar setup.

**Main Contributions:** We make contributions in constructing policies with performance guarantees, and providing a novel upper bound on the optimal total expected revenue. Our results exploit the fact that resources have unit capacity and that itineraries use intervals of resources.

*Efficient Rollout of Static Policies:* A static policy offers a (possibly random) assortment of rooms, without paying attention to the current state of the system. Letting  $n$  be the number of rooms in the hotel and  $T$  be the number of nights in the booking horizon that a customer can

possibly choose to stay, to compute the value functions associated with a static policy, we need to solve a dynamic program with  $O(2^{nT})$  possible states that keep track of the availability of each room on each night. We show that we can compute the value functions of a static policy by using  $n$  dynamic programs, each with  $O(T^2)$  possible states (Theorem 3.1). Intuitively speaking, our dynamic program keeps track of the availabilities of intervals of nights and uses the fact that itineraries that straddle disjoint intervals cannot be accepted. To establish this result, we exploit both the unit capacities of the resources and the interval structure of the itineraries.

Once we compute the value function of a static policy, we can perform rollout on the static policy. Rolling out a static policy yields a policy that is guaranteed to perform at least as well as the static policy; see Section 6.4.1 in Bertsekas (2017). Because the static policy does not consider the current state of the system, it may offer an unavailable room. The rollout policy, however, never offers an unavailable room. In our experiments, rolling out a static policy improves the performance of the static policy by up to 15%. In most rollout applications, the value functions are approximated by simulation, but we can compute the value functions of our static policies exactly.

*Static Policies via Linear Approximations:* By the discussion in the two previous paragraphs, if we have a static policy with a performance guarantee, then we can efficiently perform rollout on this static policy to obtain another policy that is at least as good. Letting  $D_{\max}$  be the maximum number of nights of stay requested by the customers, we use linear approximations of the value function to give a static policy that is guaranteed to obtain  $1/(2D_{\max})$  fraction of the optimal total expected revenue (Theorem 4.1). Our result uses a characterization of feasible itineraries that holds under unit resource capacities (Lemma 5.1). There is a history of papers using linear value function approximations, but the available performance guarantees for them are in an asymptotic regime, where the capacities of the resources and the expected customer demand increase at the same rate; see, for example, Talluri and van Ryzin (1998). Our result does not require an asymptotic regime. Moreover, such an asymptotic regime is not relevant to our setting, because each room is unique, so resource capacities are always one. Before our study, it was not known if linear approximations could yield performance guarantees without asymptotic regimes.

*Static Policies via Polynomial Approximations:* Letting  $D_{\min}$  be the minimum number of nights of stay requested by the customers, we use polynomial approximations of the value functions to give a static policy that is guaranteed to obtain  $\frac{1}{2+\lceil(D_{\max}-1)/D_{\min}\rceil}$  fraction of the optimal total expected revenue. To establish this result, for each itinerary, we designate a so-called intersection preserving subset of resources with the following property. If the capacities are not available for the resources in the intersection preserving subset of resources, then we can immediately conclude that we cannot accommodate a request for the itinerary. As a function of the number of resources

in the intersection preserving subsets, we give a performance guarantee for the static policy from the polynomial value function approximations (Theorem 6.1). Next, using both the unit capacities of the resources and the interval structure of the itineraries once again, we bound the number of resources in the intersection preserving subsets by  $1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$  (Theorem 6.2). Putting these two results together yields the desired performance guarantee.

For network revenue management problems with non-unit resource capacities over general networks, letting  $L$  be the maximum number of resources used by an itinerary, Ma et al. (2020) give a policy with a performance guarantee of  $1/(1 + L)$ . Because  $D_{\min} \geq 1$ , we have  $\frac{1}{2 + \lceil (D_{\max} - 1)/D_{\min} \rceil} \geq \frac{1}{2 + \lceil (D_{\max} - 1) \rceil} = 1/(1 + D_{\max})$ , so our performance guarantee is at least as good as  $1/(1 + L)$ . The improvement provided by our performance guarantee is relevant in practice because some boutique hotels impose minimum nights of stay requirements, so  $D_{\min} > 1$ . For example, in addition to its six rooms, Villa Mahal offers three villas: Ying Yang, Gunbatimi, and Ruya. These villas require a minimum of seven nights of stay. Surprisingly, if all stays are for the same duration ( $D_{\max} = D_{\min}$ ), then our policy yields a constant  $1/3$  performance guarantee. Such a constant factor was not known before. In earlier work,  $D_{\min}$  plays no role in the guarantee, but we give a stronger guarantee using both  $D_{\max}$  and  $D_{\min}$ . Our linear approximations have a looser guarantee than our polynomial ones, but the practical performance of both policies, especially after rollout, is competitive.

*Upper Bound on the Optimal Policy Performance:* We give an efficiently computable upper bound on the optimal total expected revenue. When assessing the optimality gap of a policy, we can compare its total expected revenue with such an upper bound. Our upper bound is based on allocating the revenue from an itinerary over different rooms and solving a separate dynamic program to control the capacity of each room. We show that this approach yields an upper bound on the optimal total expected revenue (Proposition 7.1). A common approach to obtaining an upper bound is to formulate a linear programming approximation under the assumption that the arrivals and choices of the customers take on their expected values; see Gallego et al. (2004). We show how to choose the revenue allocations such that the upper bound from our approach is at least as tight as that from such a linear program (Theorem 7.2). The dynamic program that we solve for each room still has  $O(2^T)$  states, keeping track of the availability of the room on each night. We give an equivalent dynamic program with only  $O(T^2)$  state variables (Theorem 7.3).

One motivation for using a linear programming approximation to construct an upper bound is that such upper bounds are tight in the same asymptotic regime discussed above. This asymptotic regime does not apply to our problem because our resource capacities are always one, irrespective of the number of rooms in the hotel. In our computational experiments, we indeed observe that the upper bounds from the linear program can be substantially looser than our upper bounds,

with gaps reaching 29%. Another approach to obtaining an upper bound is based on decomposing the dynamic program for the problem by the resources; see Topaloglu (2009). The decomposition approach constructs piecewise linear approximations of the value functions that are separable by the resources. In our problem, the capacity of each resource is one, so a piecewise linear approximation separable by the resources is no different from linear approximations. In our computational experiments, the upper bound from the decomposition approach was indeed almost as loose as that from the linear programming approximation. Lastly, our upper bound requires solving a booking problem for each room separately, so our results indicate that if there is a single room, then we can use a dynamic program with  $O(T^2)$  states to compute the optimal policy.

*Validation on Real-World Hotel Data:* We test our policies on both a dataset from an actual boutique hotel and randomly generated datasets. Our policies can handle real problem sizes. Policies based on our linear and polynomial approximations outperform strong benchmarks. Moreover, our approach of rolling out a static policy can significantly improve the static policy on hand.

**Literature Review:** Rollout is a general approach to improving the performance of any policy; see Section 6.4.1 in Bertsekas (2017). The idea is to compute the value functions of the initial policy on hand and use the greedy policy with respect to the value functions. The policy from rollout is guaranteed to perform at least as well as the initial policy on hand. The difficulty of performing rollout is in computing the value functions of the initial policy. Computing these value functions can be as difficult as computing the optimal policy, so the value functions are often estimated using simulation. Our study exploits the unit capacities of the resources and the interval structure of the itineraries to compute exactly the value functions of a static policy and perform rollout on the static policy. The rollout approach has been applied to combinatorial optimization, scheduling, vehicle routing, and revenue management, but most existing work use approximations or simulations to estimate the value functions of the initial policy; see Bertsekas et al. (1997), Bertsekas and Castanon (1999), Secomandi (2001), and Bertsimas and Popescu (2003).

In bid price policies, one attaches a bid price to each resource characterizing the opportunity cost of a unit of resource. The decision for accepting an itinerary request is made by comparing the revenue from the itinerary with the total opportunity cost of the resources that the itinerary consumes. Simpson (1989) and Williamson (1992) compute bid prices using a linear programming approximation. Talluri and van Ryzin (1998), Cooper (2002), Maglaras and Meissner (2006), Jasin and Kumar (2012), and Jasin and Kumar (2013) characterize the optimality gaps of the policies derived from such a linear programming approximation, but these papers focus on the asymptotic regime discussed earlier in this section. Once again, such a regime is not relevant to our setting because the capacities of our resources are always one. The value of a unit of resource should

depend on the time left in the selling horizon to utilize the resource and the remaining capacity of the resource, so the bid prices should be dependent on time and capacity.

There is work on computing such time and capacity dependent bid prices. In particular, Adelman (2007), Zhang and Adelman (2009), Kunnumkal and Topaloglu (2010a), and Kirshner and Nediak (2015) develop methods that yield time dependent bid prices, whereas Cooper and de Mello (2007), Topaloglu (2009), and Zhang (2011) develop methods that yield capacity dependent bid prices. Tong and Topaloglu (2013), Vossen and Zhang (2015a,b), and Kunnumkal and Talluri (2016a) show that some of these approaches are equivalent, although their derivations appear to use unrelated paths. Intuitively speaking, for our boutique hotel setting, the advantage of using capacity dependent bid prices is unclear, because the resources already have unit capacities.

Rusmevichientong et al. (2020) give a policy with  $1/2$  performance guarantee for a dynamic assortment optimization problem. In their problem, there is no underlying network structure, so each itinerary uses exactly one resource. Thus, their results do not apply here. Ma et al. (2020) give a  $1/(1+L)$  performance guarantee for a general network setting, where  $L$  is the maximum number of resources used by an itinerary. As discussed earlier, our polynomial approximations use the special features of the boutique hotel setting to give a better performance guarantee. Moreover, our rollout approach ultimately provides nonseparable approximations of the value functions, yielding policies that perform noticeably better than those from linear and polynomial ones.

There is also work on incorporating customer choice, where customers choose among the itineraries; see Gallego et al. (2004), Liu and van Ryzin (2008), Kunnumkal and Topaloglu (2008), Bront et al. (2009), Mendez-Diaz et al. (2014), Meissner et al. (2012), Talluri (2014), and Strauss and Talluri (2017). Some papers have focused on approximating the value functions under customer choice; see Zhang and Cooper (2005, 2009), Zhang and Adelman (2009), Kunnumkal and Topaloglu (2010b), and Kunnumkal and Talluri (2016b). Other papers have used stochastic approximation to compute booking limits and bid prices; see van Ryzin and Vulcano (2008a,b), Topaloglu (2008), and Chaneton and Vulcano (2011). These papers do not give performance guarantees.

**Organization:** In Section 2, we give a dynamic programming formulation for our problem. In Section 3, we show how to perform rollout efficiently on a static policy. In Section 4, we give a static policy using linear approximations. In Section 5, we establish the performance guarantee for this policy by showing that we can use our approximations to bound the performance of both the optimal and static policy. In Section 6, we give a static policy using polynomial approximations. The performance guarantee for this policy also uses a bounding approach, but the specifics are different. In Section 7, we give a method to get an upper bound on the optimal total expected revenue. In Section 8, we give our computational experiments. In Section 9, we conclude.

## 2. Problem Formulation

We consider a boutique hotel that has  $n$  unique rooms indexed by  $\mathcal{N} = \{1, \dots, n\}$ . The rooms are available for stay during the nights indexed by  $\mathcal{T} = \{1, 2, \dots, T\}$ . Let  $\mathcal{F} = \{[s, f] : 1 \leq s \leq f \leq T\}$  denote the set of possible intervals of stay, where the interval  $[s, f]$  corresponds to a stay over nights  $\{s, \dots, f\}$ . The revenue from booking room  $i$  over interval  $[s, f]$  is  $r_{i,[s,f]}$ . The booking requests arrive over the time periods indexed by  $\mathcal{Q} = \{1, 2, \dots, Q\}$ . Each time period is a small enough interval of time that there is at most one booking request at each time period. At time period  $q$ , we have a booking request for stay over interval  $[s, f]$  with probability  $\lambda_{[s,f]}^q$ . With probability  $1 - \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q$ , there is no booking request at time period  $q$ . Given that we offer assortment  $S \subseteq \mathcal{N}$  of rooms at time period  $q$ , the customer making a booking request at time period  $q$  chooses room  $i$  with probability  $\phi_i^q(S)$ . The choice probability  $\phi_i^q(S)$  is governed by a general choice model, as long as the choice probabilities of the rooms in an assortment decrease as we add more rooms into the assortment; that is,  $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$  for all  $i \in S$  and  $j \notin S$ .

Our goal is to find a policy for deciding which assortment of rooms to make available for the customer arriving at each time period to maximize the total expected revenue from all booking requests. We formulate a dynamic program to compute an optimal policy. We use the vector  $\mathbf{x} = (x_{i,\ell} : i \in \mathcal{N}, \ell \in \mathcal{T}) \in \{0, 1\}^{n \times T}$  to capture the state of the system at a generic time period, where we have  $x_{i,\ell} = 1$  if and only if room  $i$  is available for stay on night  $\ell$ . To accommodate a booking request to stay in room  $i$  over the interval  $[s, f]$ , we need to have capacity available in this room on nights  $\{s, \dots, f\}$ . In other words, given that the state of the system is  $\mathbf{x}$ , we can accommodate a booking request to stay in room  $i$  over the interval  $[s, f]$  if and only if  $\prod_{\ell=s}^f x_{i,\ell} = 1$ . Let  $J^q(\mathbf{x})$  be the optimal total expected revenue over time periods  $\{q, \dots, Q\}$  given that the state of the system at time period  $q$  is  $\mathbf{x}$ . Using  $\mathbf{e}_{i,[s,f]} \in \{0, 1\}^{n \times T}$  to denote the vector with ones only in the components corresponding to room  $i$  and nights  $\{s, \dots, f\}$ , we can find the optimal policy by computing the value functions  $\{J^q : q \in \mathcal{Q}\}$  through the dynamic program

$$J^q(\mathbf{x}) = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + J^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) \right] \right. \\ \left. + \left[ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( 1 - \prod_{\ell=s}^f x_{i,\ell} \right) + 1 - \sum_{i \in \mathcal{N}} \phi_i^q(S) \right] J^{q+1}(\mathbf{x}) \right\} + \left[ 1 - \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \right] J^{q+1}(\mathbf{x}),$$

with the boundary condition that  $J^{Q+1} = 0$ . In this case, letting  $\mathbf{e} \in \{0, 1\}^{n \times T}$  be the vector of all ones, the optimal total expected revenue is given by  $J^1(\mathbf{e})$ .

In the dynamic program above, if we offer the assortment  $S$  of rooms to a customer requesting to stay over the interval  $[s, f]$ , then she chooses room  $i$  with probability  $\phi_i^q(S)$ . If  $\prod_{\ell=s}^f x_{i,\ell} = 1$ ,

so that room  $i$  is available to accommodate a booking over the interval  $[s, f]$ , then we generate a revenue of  $r_{i,[s,f]}$  and consume the capacity in room  $i$  over nights  $\{s, \dots, f\}$ . With probability  $\sum_{i \in \mathcal{N}} \phi_i^q(S) (1 - \prod_{\ell=s}^f x_{i,\ell})$ , the customer chooses a room for which we do not have capacity available to accommodate a booking over the interval  $[s, f]$ . With probability  $1 - \sum_{i \in \mathcal{N}} \phi_i^q(S)$ , the customer does not choose any of the rooms in the offered assortment. In either case, we do not consume capacity in any of the rooms. Lastly, with probability  $1 - \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q$ , we do not have a booking request, in which case, we do not consume capacity in any of the rooms either. In our dynamic program, we assume that if we have a customer making a booking request for the interval  $[s, f]$ , then we may offer a room that is not available on one of the nights  $\{s, \dots, f\}$ . If the customer chooses this room, then she leaves without making a booking. This assumption is innocuous because, as we argue shortly, there exists an optimal policy that never offers an unavailable room for a booking request. The dynamic program above can be unwieldy, but arranging the terms on the right side, we can write this dynamic program equivalently as

$$J^q(\mathbf{x}) = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + J^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J^{q+1}(\mathbf{x}) \right] \right\} + J^{q+1}(\mathbf{x}). \quad (1)$$

Given that the state of the system at time period  $q$  is  $\mathbf{x}$ , we interpret  $J^{q+1}(\mathbf{x}) - J^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]})$  as the opportunity cost of the capacities used by booking room  $i$  for nights  $\{s, \dots, f\}$ .

Using the dynamic program above, we can argue that there exists an optimal policy that never offers an unavailable room for a booking request. In particular, given that the state of the system at time period  $q$  is  $\mathbf{x}$  and we have a booking request for the interval  $[s, f]$ , we can compute the optimal assortment of rooms to offer by solving the maximization problem on the right side of (1). This maximization problem is of the form  $\max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) p_{i,[s,f]}^q(\mathbf{x})$ , where  $p_{i,[s,f]}^q(\mathbf{x}) = \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + J^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J^{q+1}(\mathbf{x}) \right]$ . In Appendix A, we consider the problem  $\max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) p_i$  and show that there exists an optimal solution  $S^*$  to this problem that satisfies  $S^* \subseteq \{i \in \mathcal{N} : p_i > 0\}$ . In other words, if  $p_i^* \leq 0$ , then  $S^*$  does not offer room  $i$ . This result follows from the assumption that the choice probabilities satisfy  $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$  for all  $i \in S$  and  $j \notin S$ . If room  $i$  is not available for some night over the interval  $[s, f]$ , then we have  $\prod_{\ell=s}^f x_{i,\ell} = 0$ , which implies that  $p_{i,[s,f]}^q(\mathbf{x}) = 0$ . Therefore, there exists an optimal solution to the maximization problem on the right side of (1) that does not offer room  $i$  when this room is unavailable for a booking request over the interval  $[s, f]$ .

The state variable  $\mathbf{x}$  in (1) has  $O(2^{nT})$  possible values, making an optimal policy difficult to compute. Thus, we focus on developing policies with performance guarantees.



### 3. Efficient Rollout of Static Policies

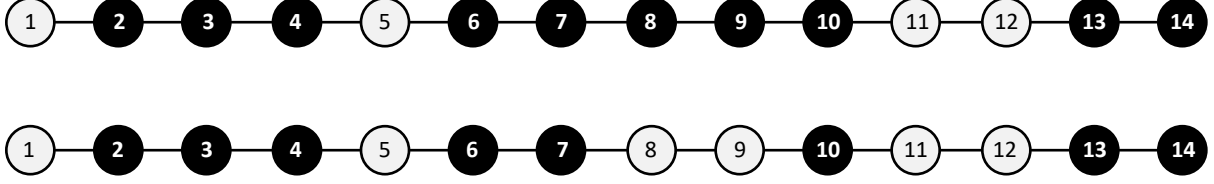
In this section, we show that we can compute the value functions of a static policy efficiently, which ultimately allows us to perform rollout on the static policy. A static policy  $\mu$  is a collection of offer probabilities  $\{\mu_{[s,f]}^q(S) : S \subseteq \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  such that if we have a booking request for interval  $[s,f]$  at time period  $q$ , then the policy offers the assortment  $S$  of rooms with probability  $\mu_{[s,f]}^q(S)$ . Naturally, we require  $\sum_{S \subseteq \mathcal{N}} \mu_{[s,f]}^q(S) = 1$ . Because the offer probabilities do not depend on the state of the system, a static policy may offer an unavailable room. If a customer chooses an unavailable room, then she leaves without making a booking. Nevertheless, we will ensure that, just like the optimal policy, the rollout policy based on a static policy never offers an unavailable room. To compute the value functions of the static policy  $\mu$ , we define  $\psi_{\mu,i,[s,f]}^q = \sum_{S \subseteq \mathcal{N}} \mu_{[s,f]}^q(S) \phi_i^q(S)$ , which is the probability that a customer arriving at time period  $q$  with a booking request for interval  $[s,f]$  chooses room  $i$  under the static policy  $\mu$ . Let  $J_{\mu}^q(\mathbf{x})$  be the total expected revenue obtained by the static policy  $\mu$  over the time periods  $\{q, \dots, Q\}$  given that the state of the system at time period  $q$  is  $\mathbf{x}$ . We compute the value functions  $\{J_{\mu}^q : q \in \mathcal{Q}\}$  through the dynamic program

$$J_{\mu}^q(\mathbf{x}) = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \psi_{\mu,i,[s,f]}^q \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + J_{\mu}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J_{\mu}^{q+1}(\mathbf{x}) \right] \right\} + J_{\mu}^{q+1}(\mathbf{x}), \quad (2)$$

with the boundary condition that  $J_{\mu}^{Q+1} = 0$ . In this case, the total expected revenue obtained by the static policy  $\mu$  is given by  $J_{\mu}^1(\mathbf{e})$ .

The dynamic program in (2) is similar to that in (1), but when computing the value functions of a static policy, the assortment that we offer is determined by the static policy, rather than being a decision variable. In particular, under the static policy  $\mu$ , if we have a booking request for interval  $[s,f]$  at time period  $q$ , then the customer chooses room  $i$  with probability  $\psi_{\mu,i,[s,f]}^q$ . The state variable  $\mathbf{x}$  in (2) has  $O(2^{nT})$  possible values, just like the state variable in (1), but two properties of static policies allow us to compute the value functions of a static policy efficiently. First, under the static policy  $\mu$ , given that we have a booking request for interval  $[s,f]$  at time period  $q$ , room  $i$  receives a booking with fixed probability  $\psi_{\mu,i,[s,f]}^q$ . Therefore, intuitively speaking, each room faces an exogenous stream of booking requests, allowing us to focus on each room separately. Second, if room  $i$  is available on nights  $\{a, \dots, b-1\}$  and nights  $\{b+1, \dots, c\}$ , but not on night  $b$ , then this room can never accommodate a booking request for an interval that starts before night  $b-1$  and ends after night  $b+1$ . Thus, we can separately compute the total expected revenue from each uninterrupted interval of available nights in a room.

Motivated by the discussion above, to compute the value functions of the static policy  $\mu$ , we let  $V_{\mu,i}^q(a,b)$  be the total expected revenue collected by the static policy  $\mu$  from room  $i$  over time



**Figure 2** Maximal available intervals for  $\mathbf{x}_i$  (top) and maximal intervals for  $\mathbf{x}_i - \mathbf{e}_{[8,9]}$  (bottom).

periods  $\{q, \dots, Q\}$  given that room  $i$  is available on all the nights in the interval  $[a, b]$ . We compute the value functions  $\{V_{\mu,i}^q : q \in \mathcal{Q}\}$  by solving the dynamic program

$$V_{\mu,i}^q(a, b) = \sum_{[s,f] \subseteq [a,b]} \lambda_{[s,f]}^q \left\{ \psi_{\mu,i,[s,f]}^q \left[ r_{i,[s,f]} + V_{\mu,i}^{q+1}(a, s-1) + V_{\mu,i}^{q+1}(f+1, b) - V_{\mu,i}^{q+1}(a, b) \right] \right\} + V_{\mu,i}^{q+1}(a, b), \quad (3)$$

with the boundary condition that  $V_{\mu,i}^{Q+1} = 0$ . Given that room  $i$  is available over all nights in the interval  $[a, b]$ , to compute the total expected revenue collected by the static policy  $\mu$  from this room over time periods  $\{q, \dots, Q\}$ , we consider the booking requests only for the intervals within the interval  $[a, b]$ . With probability  $\lambda_{[s,f]}^q$ , the customer arriving at time period  $q$  makes a booking request for interval  $[s, f]$ . Under the static policy  $\mu$ , this customer chooses room  $i$  with probability  $\psi_{\mu,i,[s,f]}^q$ , in which case we generate a revenue of  $r_{i,[s,f]}$ . Furthermore, room  $i$  becomes no longer available on all nights in the interval  $[a, b]$ , but it is still available for the intervals  $[a, s-1]$  and  $[f+1, b]$ . Given that room  $i$  is available on all nights in the interval  $[a, b]$  at time period  $q$ , we can thus interpret  $V_{\mu,i}^{q+1}(a, b) - V_{\mu,i}^{q+1}(a, s-1) - V_{\mu,i}^{q+1}(f+1, b)$  in (3) as the opportunity cost of the capacities used by booking room  $i$  for the nights  $\{s, \dots, f\}$ . If  $s = a$ , then we set  $V_{\mu,i}^{q+1}(a, s-1) = 0$ , whereas if  $f = b$ , then we set  $V_{\mu,i}^{q+1}(f+1, b) = 0$  in (3).

The state variable  $[a, b]$  in (3) has  $O(T^2)$  possible values. Therefore, we can solve the dynamic program in (3) efficiently. Our main result in this section shows that we can compute the value functions  $\{J_{\mu}^q : q \in \mathcal{Q}\}$  in (2) by using the value functions in  $\{V_{\mu,i}^q : q \in \mathcal{Q}\}$  in (3). In particular, we use  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,T}) \in \{0, 1\}^T$  to denote the state of room  $i$ , where  $x_{i,\ell} = 1$  if and only if room  $i$  is available for a stay on night  $\ell$ . We refer to the interval  $[a, b]$  as a *maximal available interval* with respect to  $\mathbf{x}_i$  if and only if room  $i$  is available on all nights in the interval  $[a, b]$ , but not available on nights  $a-1$  and  $b+1$ ; that is,  $\prod_{\ell=a}^b x_{i,\ell} = 1$ ,  $x_{i,a-1} = 0$  and  $x_{i,b+1} = 0$ . Let  $\mathcal{I}(\mathbf{x}_i)$  be the collection of maximal available intervals with respect to  $\mathbf{x}_i$ . In the top portion of Figure 2, we show the collection  $\mathcal{I}(\mathbf{x}_i)$  for a specific value of  $\mathbf{x}_i$  with  $T = 14$ . Each circle in the top portion corresponds to a night with  $x_{i,\ell} = 1$  for the black circles and  $x_{i,\ell} = 0$  for the white circles. The collection of maximal available intervals is  $\mathcal{I}(\mathbf{x}_i) = \{[2, 4], [6, 10], [13, 14]\}$ .

We use two properties of maximal available intervals. First, we have  $\prod_{\ell=s}^f x_{i,\ell} = 1$  if and only if there exists a maximal available interval  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$  such that  $[s, f] \subseteq [a, b]$ . In the top portion

of Figure 2, for example, we have  $\prod_{\ell=7}^9 x_{i,\ell} = 1$ , so there must exist a maximal available interval  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$  such that  $[7, 9] \subseteq [a, b]$ . This maximal available interval is  $[6, 10]$ . Second, using  $\mathbf{e}_{[s,f]} \in \{0, 1\}^T$  to denote the vector with ones only in the components corresponding to the nights  $\{s, \dots, f\}$ , if  $[a, b]$  is the maximal available interval that includes the interval  $[s, f]$ , then we have the identity  $\mathcal{I}(\mathbf{x}_i - \mathbf{e}_{[s,f]}) = (\mathcal{I}(\mathbf{x}_i) \setminus [a, b]) \cup \{[a, s-1], [f+1, b]\}$ . In this identity, if  $s = a$ , then we omit the interval  $[a, s-1]$ . Similarly, if  $f = b$ , then we omit the interval  $[f+1, b]$ . In the top portion of Figure 2, for example, we have  $\mathcal{I}(\mathbf{x}_i) = \{[2, 4], [6, 10], [13, 14]\}$  for the specific value of  $\mathbf{x}_i$ . Considering the interval  $[8, 9]$ , the maximal available interval that includes this interval is  $[6, 10]$ . In the bottom portion of Figure 2, we have  $\mathcal{I}(\mathbf{x}_i - \mathbf{e}_{[8,9]}) = \{[2, 4], [6, 7], [10, 10], [13, 14]\}$ , which is indeed equal to  $(\mathcal{I}(\mathbf{x}_i) \setminus [6, 10]) \cup \{[6, 7], [10, 10]\}$ .

In the next theorem, we use these properties to show that we can compute the value functions  $\{J_\mu^q : q \in \mathcal{Q}\}$  in (2) by using the value functions in  $\{V_{\mu,i}^q : q \in \mathcal{Q}\}$  in (3)

**Theorem 3.1 (Efficient Rollout)** *If the value functions  $\{J_\mu^q : q \in \mathcal{Q}\}$  are computed through (2) and the value functions  $\{V_{\mu,i}^q : q \in \mathcal{Q}\}$  are computed through (3), then we have*

$$J_\mu^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} V_{\mu,i}^q(a, b).$$

*Proof:* We show the result by using induction over the time periods. At time period  $Q+1$ , we have  $J_\mu^{Q+1} = 0 = V_{\mu,i}^{Q+1}$ , so the result holds at time period  $Q+1$ . Assuming that the result holds at time period  $q+1$ , we show that the result holds at time period  $q$  as well. By the first property just before the theorem, we have  $\prod_{\ell=s}^f x_{i,\ell} = 1$  if and only if there exists a maximal available interval  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$  such that  $[s, f] \subseteq [a, b]$ . Thus, using  $\mathbb{1}_{\{\cdot\}}$  to denote the indicator function, we have  $\prod_{\ell=s}^f x_{i,\ell} = 1$  if and only if  $\sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} \mathbb{1}_{\{[s,f] \subseteq [a,b]\}} = 1$ . Furthermore, by the second property just before the theorem, if  $[a, b]$  is the maximal available interval that includes the interval  $[s, f]$ , then we have  $\mathcal{I}(\mathbf{x}_i - \mathbf{e}_{[s,f]}) = (\mathcal{I}(\mathbf{x}_i) \setminus [a, b]) \cup \{[a, s-1], [f+1, b]\}$ . In this case, considering the maximal available interval  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$ , for each interval  $[s, f] \subseteq [a, b]$ , using the induction hypothesis, we obtain the chain of equalities

$$\begin{aligned} J_\mu^{q+1}(\mathbf{x}) - J_\mu^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) &= \sum_{[c,d] \in \mathcal{I}(\mathbf{x}_i)} V_{\mu,i}^{q+1}(c, d) - \sum_{[c,d] \in \mathcal{I}(\mathbf{x}_i - \mathbf{e}_{i,[s,f]})} V_{\mu,i}^{q+1}(c, d) \\ &= \sum_{[c,d] \in \mathcal{I}(\mathbf{x}_i)} V_{\mu,i}^{q+1}(c, d) - \left( \sum_{[c,d] \in \mathcal{I}(\mathbf{x}_i)} V_{\mu,i}^{q+1}(c, d) - V_{\mu,i}^{q+1}(a, b) + V_{\mu,i}^{q+1}(a, s-1) + V_{\mu,i}^{q+1}(f+1, b) \right) \\ &= V_{\mu,i}^{q+1}(a, b) - V_{\mu,i}^{q+1}(a, s-1) - V_{\mu,i}^{q+1}(f+1, b), \end{aligned} \quad (4)$$

where the first equality holds because we have  $J_\mu^{q+1}(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{[c,d] \in \mathcal{I}(\mathbf{x}_i)} V_{\mu,i}^{q+1}(c, d)$  by the induction hypothesis, whereas the second equality holds because, for all  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$  and

$[s, f] \subseteq [a, b]$ , we have the identity  $\mathcal{I}(\mathbf{x}_i - \mathbf{e}_{[s,f]}) = (\mathcal{I}(\mathbf{x}_i) \setminus [a, b]) \cup \{[a, s-1], [f+1, b]\}$ . Thus, using the fact that  $\prod_{\ell=s}^f x_{i,\ell} = 1$  if and only if  $\sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} \mathbb{1}_{\{[s,f] \subseteq [a,b]\}} = 1$ , by (2), we obtain

$$\begin{aligned}
J_{\boldsymbol{\mu}}^q(\mathbf{x}) &= \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \psi_{\boldsymbol{\mu},i,[s,f]}^q \left( \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} \mathbb{1}_{\{[s,f] \subseteq [a,b]\}} \right) \left[ r_{i,[s,f]} + J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x}) \right] \right\} + J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x}) \\
&\stackrel{(a)}{=} \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} \sum_{[s,f] \subseteq [a,b]} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \psi_{\boldsymbol{\mu},i,[s,f]}^q \left[ r_{i,[s,f]} + J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x}) \right] \right\} + J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x}) \\
&\stackrel{(b)}{=} \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} \sum_{[s,f] \subseteq [a,b]} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \psi_{\boldsymbol{\mu},i,[s,f]}^q \left[ r_{i,[s,f]} - V_{\boldsymbol{\mu},i}^{q+1}(a,b) + V_{\boldsymbol{\mu},i}^{q+1}(a,s-1) + V_{\boldsymbol{\mu},i}^{q+1}(f+1,b) \right] \right\} \\
&\quad + \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} V_{\boldsymbol{\mu},i}^{q+1}(a,b) \\
&\stackrel{(c)}{=} \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_i)} V_{\boldsymbol{\mu},i}^q(a,b),
\end{aligned}$$

where (a) follows by reordering the three sums on the left side of (a), (b) holds because (4) holds for all  $[a, b] \in \mathcal{I}(\mathbf{x}_i)$  and  $[s, f] \subseteq [a, b]$ , along with the induction hypothesis, and (c) holds by (3). ■

The rollout policy from the static policy  $\boldsymbol{\mu}$  makes its decisions by replacing  $\{J^q : q \in \mathcal{Q}\}$  in (1) with  $\{J_{\boldsymbol{\mu}}^q : q \in \mathcal{Q}\}$ . We discuss the rollout policy from the static policy  $\boldsymbol{\mu}$ .

### Rollout Policy Based on a Static Policy:

Given that the state of the system at time period  $q$  is  $\mathbf{x}$  and we have a booking request for interval  $[s, f]$ , the rollout policy from the static policy  $\boldsymbol{\mu}$  offers the assortment of rooms

$$S_{\boldsymbol{\mu},[s,f]}^{\text{Rollout},q}(\mathbf{x}) = \arg \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - J_{\boldsymbol{\mu}}^{q+1}(\mathbf{x}) \right] \right\}. \quad (5)$$

The problem above is identical to the maximization problem on the right side of (1) after replacing  $\{J^q : q \in \mathcal{Q}\}$  with  $\{J_{\boldsymbol{\mu}}^q : q \in \mathcal{Q}\}$ . By the same argument that we give just after (1), there exists an optimal solution  $S_{\boldsymbol{\mu},[s,f]}^{\text{Rollout},q}(\mathbf{x})$  to the problem above such that  $i \notin S_{\boldsymbol{\mu},[s,f]}^{\text{Rollout},q}(\mathbf{x})$  for each  $i \in \mathcal{N}$  with  $\prod_{\ell=s}^f x_{i,\ell} = 0$ . Thus, the rollout policy never offers an unavailable room for a booking request. It is a standard result that the total expected revenue obtained by the rollout policy from the static policy  $\boldsymbol{\mu}$  is at least as large as the total expected revenue obtained by the static policy  $\boldsymbol{\mu}$ , so the rollout policy from the static policy  $\boldsymbol{\mu}$  is guaranteed to be at least as good as the static policy  $\boldsymbol{\mu}$ ; see Section 6.4.1 in Bertsekas (2017). Thus, if we have a performance guarantee for a static policy, then the same performance guarantee holds for the rollout policy based on this static policy as well. Our computational experiments indicate that the rollout policy based on the static policy  $\boldsymbol{\mu}$  can perform significantly better than the static policy  $\boldsymbol{\mu}$  itself.

Next, we give static policies with performance guarantees. By the discussion above, these performance guarantees hold for the corresponding rollout policies.

## 4. Resource Based Static Policy with Linear Approximations

We develop a static policy based on linear approximations of the value functions. In particular, we use linear value function approximations  $\{\hat{J}_L^q : q \in \mathcal{Q}\}$  of the form

$$\hat{J}_L^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \eta_{i,\ell}^q x_{i,\ell}.$$

In the approximation above, we interpret the coefficient  $\eta_{i,\ell}^q$  as the opportunity cost of the capacity in room  $i$  on night  $\ell$  given that we are at time period  $q$ .

We use the following algorithm to compute the coefficients  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$ . For all  $i \in \mathcal{N}$  and  $t \in \mathcal{T}$ , we set  $\eta_{i,\ell}^{Q+1} = 0$ . For each  $q = Q, Q-1, \dots, 1$ , we execute the two steps.

- **Construct Ideal Assortments:** For each  $[s, f] \in \mathcal{F}$ , compute the ideal assortment to offer to a customer with a booking request for interval  $[s, f]$  as

$$A_{[s,f]}^q = \arg \max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right]. \quad (6)$$

- **Compute Opportunity Costs:** For all  $i \in \mathcal{N}$  and  $\ell \in \mathcal{T}$ , compute the opportunity cost of the capacity in room  $i$  on night  $\ell$  as

$$\eta_{i,\ell}^q = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] + \eta_{i,\ell}^{q+1}. \quad (7)$$

This algorithm fully specifies the parameters  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$ , which in turn specify our value function approximations. We give some intuition for the algorithm above.

In (6),  $\sum_{\ell=s}^f \eta_{i,\ell}^{q+1}$  is the total opportunity cost of the capacities consumed by booking room  $i$  for the interval  $[s, f]$  at time period  $q$ . Thus,  $r_{i,[s,f]} - \sum_{\ell=s}^f \eta_{i,\ell}^{q+1}$  gives the revenue from booking room  $i$  for interval  $[s, f]$ , net of the opportunity cost of the capacities consumed. In this case,  $A_{[s,f]}^q$  is the assortment that maximizes the net expected revenue from a customer making a booking request for interval  $[s, f]$  at time period  $q$ . In (7), we compute the opportunity costs of the capacity in room  $i$  on night  $\ell$  by using a recursion similar to that in (3). With probability  $\lambda_{[s,f]}^q$ , we have a customer arriving at time period  $q$  who makes a booking request for interval  $[s, f]$ . If we offer the ideal assortment  $A_{[s,f]}^q$ , then this customer chooses room  $i$  with probability  $\phi_i^q(A_{[s,f]}^q)$ . At time period  $q$ , if we book room  $i$  over the interval  $[s, f]$ , then we obtain a net revenue of  $r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}$ , but by doing so, we consume the capacities in room  $i$  on each night  $\ell \in \{s, \dots, f\}$ . Noting the fraction  $\frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1}$ , we spread the net revenue evenly over each night  $\ell \in \{s, \dots, f\}$ .

The discussion in the previous paragraph provides only rough intuition for the algorithm we use to construct our linear value function approximations. Nevertheless, we will be able to use

this algorithm to develop a static policy with a performance guarantee that we can precisely quantify. In particular, noting the ideal assortments  $\{A_{[s,f]}^q : [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  computed through (6), we consider a static policy that always offers the assortment  $A_{[s,f]}^q$  of rooms to a customer making a booking request for interval  $[s,f]$  at time period  $q$ . We refer to this static policy as the *resource based static policy* because our linear value function approximations have one component for each room and night combination. In revenue management vocabulary, each room and night combination within the context of our problem is referred to as a resource. In the next theorem, we give a performance guarantee for the resource based static policy. In this theorem and throughout the rest of the paper, we let  $D_{\max} = \max_{[s,f] \in \mathcal{F}} \{f - s + 1\}$ , which corresponds to the maximum stay duration for any possible booking request.

**Theorem 4.1 (Performance of Resource Based Static Policy)** *The resource based static policy obtains at least  $1/(2D_{\max})$  fraction of the optimal total expected revenue.*

We give a proof for the theorem above in Section 5. The proof explicitly uses the fact that the resource capacities are all one. When the resource capacities are not all one, it is unclear if we can use linear value function approximations to develop policies with performance guarantees. Being a static policy that offers each assortment of rooms with fixed probabilities, the resource based static policy may end up offering an unavailable room for a booking request, so it may not be appropriate to use this policy in practice. However, as discussed at the end of Section 3, the rollout policy from the resource based static policy never offers an unavailable room. Furthermore, the rollout policy from the resource based static policy is guaranteed to perform at least as well as the resource based static policy, so the rollout policy inherits the performance guarantee of  $1/(2D_{\max})$  from the resource based static policy.

The coefficients  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$  capture the opportunity costs of the capacities in different rooms on different nights, providing a natural interpretation of our linear value function approximations. Ma et al. (2020) use nonlinear value function approximations to give a policy with a performance guarantee of  $1/(1 + D_{\max})$ , which is stronger than the performance guarantee of  $1/(2D_{\max})$  in Theorem 4.1. In our computational experiments, the practical performance of our linear value function approximations turns out to be comparable to that of nonlinear value function approximations. The competitive practical performance of our linear value function approximations, coupled with their interpretability, can make them particularly appealing. In Section 6, we will use nonlinear value function approximations to give another static policy that improves the performance guarantee in Ma et al. (2020) even further.

In the next section, we give a proof for Theorem 4.1.

## 5. Performance Guarantee for Resource Based Static Policy

The proof of Theorem 4.1 is based on giving an upper bound on the performance of the optimal policy and a lower bound on the performance of the resource based static policy.

### 5.1 Preliminary Lemmas

We will need three lemmas. Room  $i$  is available for booking over interval  $[s, f]$  if and only if  $\prod_{\ell=s}^f x_{i,\ell} = 1$ . In the next lemma, we give a lower bound on  $\prod_{\ell=s}^f x_{i,\ell}$  that is linear in  $\mathbf{x}$ .

**Lemma 5.1 (Linear Proxy for Room Availability Condition)** *For each  $\mathbf{x} \in \{0, 1\}^{n \times T}$ ,  $i \in \mathcal{N}$ , and  $[s, f] \in \mathcal{F}$ , we have*

$$\prod_{\ell=s}^f x_{i,\ell} \geq \frac{(\sum_{\ell=s}^f x_{i,\ell}) - (f - s)}{1 + f - s}.$$

*Proof:* First, assume that  $\prod_{\ell=s}^f x_{i,\ell} = 1$ . Thus, we have  $x_{i,\ell} = 1$  for all  $\ell = s, \dots, f$ , so that  $\sum_{\ell=s}^f x_{i,\ell} = f - s + 1$ . In this case, the left side of the inequality in the lemma is one, whereas the right side is  $\frac{1}{1+f-s}$ . Because  $1 \geq \frac{1}{1+f-s}$ , the inequality holds whenever  $\prod_{\ell=s}^f x_{i,\ell} = 1$ . Second, assume that  $\prod_{\ell=s}^f x_{i,\ell} = 0$ . Thus, we have  $x_{i,\ell} = 0$  for some  $\ell = s, \dots, f$ , so that  $\sum_{\ell=s}^f x_{i,\ell} \leq f - s$ . In this case, noting that  $\sum_{\ell=s}^f x_{i,\ell} - (f - s) \leq 0$ , the left side of the inequality in the lemma is zero, whereas the right side is at most zero, so the inequality holds whenever  $\prod_{\ell=s}^f x_{i,\ell} = 0$ . ■

The above lemma requires every resource to have unit capacity; that is,  $\mathbf{x} \in \{0, 1\}^{n \times T}$ . The next lemma shows that the contribution of each room to the objective value in (6) is nonnegative.

**Lemma 5.2 (Nonnegative Contribution of Each Room)** *For each  $i \in \mathcal{N}$ ,  $[s, f] \in \mathcal{F}$ , and  $q \in \mathcal{Q}$ , we have  $\phi_i^q(A_{[s,f]}^q) [r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}] \geq 0$ .*

*Proof:* For notational brevity, fixing  $[s, f] \in \mathcal{F}$  and  $q \in \mathcal{Q}$ , let  $p_i = r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}$  for each  $i \in \mathcal{N}$ . Suppose on the contrary that we have  $\phi_k^q(A_{[s,f]}^q) p_k < 0$  for some  $k \in \mathcal{N}$ . Then, we have  $p_k < 0$  and  $\phi_k^q(A_{[s,f]}^q) > 0$ . Noting  $\phi_k^q(A_{[s,f]}^q) > 0$ , since the booking probability of a room that is not offered is zero, it must be the case that  $k \in A_{[s,f]}^q$ . We partition the assortment of rooms  $A_{[s,f]}^q$  into  $A^+ = \{j \in A_{[s,f]}^q : p_j \geq 0\}$  and  $A^- = \{j \in A_{[s,f]}^q : p_j < 0\}$ . Using the fact that we have  $k \in A^-$  and  $\phi_k^q(A_{[s,f]}^q) p_k < 0$ , we have the chain of inequalities

$$\sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) p_i = \sum_{i \in A^+} \phi_i^q(A_{[s,f]}^q) p_i + \sum_{i \in A^-} \phi_i^q(A_{[s,f]}^q) p_i < \sum_{i \in A^+} \phi_i^q(A_{[s,f]}^q) p_i \stackrel{(a)}{\leq} \sum_{i \in A^+} \phi_i^q(A^+) p_i,$$

where (a) uses the assumption that  $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$  for all  $i \in S$  and  $j \notin S$ . The chain of inequalities above contradicts the fact that  $A_{[s,f]}^q$  is an optimal solution to problem (6). ■

In the next lemma, we give an inequality that will become useful to lower bound the total expected revenue obtained by the resource based static policy.

**Lemma 5.3 (Upper Bound on Net Expected Revenue)** *Letting  $\{A_{[s,f]}^q : [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  and  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$  be computed through (6) and (7), we have*

$$\sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \frac{f-s}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \leq \frac{D_{\max} - 1}{D_{\max}} (\hat{J}_L^q(\mathbf{e}) - \hat{J}_L^{q+1}(\mathbf{e})).$$

*Proof:* Because  $x/(x+1)$  is increasing in  $x \in \mathbb{R}_+$  and  $f-s+1 \leq D_{\max}$ , we have  $\frac{f-s}{f-s+1} \leq \frac{D_{\max}-1}{D_{\max}}$ . Because  $\phi_i^q(A_{[s,f]}^q) [r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}] \geq 0$  for all  $i \in \mathcal{N}$  and  $[s,f] \in \mathcal{F}$  by Lemma 5.2, we get

$$\begin{aligned} & \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \frac{f-s}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\ & \leq \frac{D_{\max} - 1}{D_{\max}} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\ & \stackrel{(a)}{=} \frac{D_{\max} - 1}{D_{\max}} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\ & \stackrel{(b)}{=} \frac{D_{\max} - 1}{D_{\max}} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} (\eta_{i,\ell}^q - \eta_{i,\ell}^{q+1}) \stackrel{(c)}{=} \frac{D_{\max} - 1}{D_{\max}} (\hat{J}_L^q(\mathbf{e}) - \hat{J}_L^{q+1}(\mathbf{e})), \end{aligned}$$

where (a) holds by the identity  $\sum_{\ell \in \mathcal{T}} \frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} = 1$ , (b) follows by (7), and (c) follows because we have  $\hat{J}_L^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \eta_{i,\ell}^q x_{i,\ell}$ .

We focus on the first part of the proof of Theorem 4.1, which constructs an upper bound on the optimal total expected revenue.

## 5.2 Upper Bound on the Optimal Total Expected Revenue

Letting the value functions  $\{J^q : q \in \mathcal{Q}\}$  be computed through (1), it is a standard result that if, for all  $\mathbf{x} \in \{0,1\}^{n \times T}$  and  $q \in \mathcal{Q}$ , the value function approximations  $\{\tilde{J}^q : q \in \mathcal{Q}\}$  satisfy

$$\tilde{J}^q(\mathbf{x}) \geq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \tilde{J}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \tilde{J}^{q+1}(\mathbf{x}) \right] \right\} + \tilde{J}^{q+1}(\mathbf{x}), \quad (8)$$

then we have  $\tilde{J}^q(\mathbf{x}) \geq J^q(\mathbf{x})$  for all  $\mathbf{x} \in \{0,1\}^{n \times T}$  and  $q \in \mathcal{Q}$ ; see Section 5.3.3 in Bertsekas (2017). This result is known as the monotonicity of the dynamic programming operator. Note that the inequality above is the version of (1) in the greater than or equal to sense. Thus, if the value function approximations  $\{\tilde{J}_q : q \in \mathcal{Q}\}$  satisfy (1) in the greater than or equal to sense, then they form an upper bound on the optimal value functions  $\{J^q : q \in \mathcal{Q}\}$  that satisfy (1) in the equality sense. In the next proposition, we use this result to show that  $2\hat{J}_L^1(\mathbf{e})$  provides an upper bound on the optimal total expected revenue. Thus, we can use our linear value function approximations to come up with an upper bound on the optimal total expected revenue.



**Proposition 5.4 (Upper Bound on Optimal Performance)** *Noting that the optimal total expected revenue is  $J^1(\mathbf{e})$ , we have  $J^1(\mathbf{e}) \leq 2\hat{J}_L^1(\mathbf{e})$ .*

*Proof:* Using our linear value function approximations  $\{\hat{J}_L^q : q \in \mathcal{Q}\}$  computed through (6) and (7), let  $\tilde{J}^q(\mathbf{x}) = \hat{J}_L^q(\mathbf{e}) + \hat{J}_L^q(\mathbf{x})$ . We show that  $\{\tilde{J}^q : q \in \mathcal{Q}\}$  satisfies (8). In particular, we have

$$\begin{aligned}
& \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \tilde{J}^{q+1}(\mathbf{x}) - \tilde{J}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) \right] \right\} \\
&= \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \right\} \\
&\stackrel{(a)}{\leq} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \right\} \\
&\stackrel{(b)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\
&\stackrel{(c)}{=} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \frac{\sum_{\ell \in \mathcal{T}} \mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\
&= \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\
&\stackrel{(d)}{=} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} (\eta_{i,\ell}^q - \eta_{i,\ell}^{q+1}) \stackrel{(e)}{\leq} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} (\eta_{i,\ell}^q - \eta_{i,\ell}^{q+1}) (1 + x_{i,\ell}) = \tilde{J}^q(\mathbf{x}) - \tilde{J}^{q+1}(\mathbf{x}).
\end{aligned}$$

The inequality (a) holds because Lemma A.1 in Appendix A implies that if  $r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \leq 0$ , then there exists an optimal solution to the problem on the left side of (a) such that room  $i$  is not offered. So, if  $r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \leq 0$ , then we can drop room  $i$  from consideration in this problem, but if, on the other hand,  $r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} > 0$ , then we have  $(\prod_{\ell=s}^f x_{i,\ell}) [r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}] \leq r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}$ . Moreover, (b) follows from the definition of  $A_{[s,f]}^q$  in (6), (c) uses the fact that  $\sum_{\ell \in \mathcal{T}} \mathbb{1}_{\{\ell \in [s,f]\}} = f - s + 1$ , and (d) follows from the definition of  $\eta_{i,\ell}^q$  in (7). Lastly, (e) holds because  $\phi_i^q(A_{[s,f]}^q) [r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}] \geq 0$  by Lemma 5.2, which implies that  $\eta_{i,\ell}^q \geq \eta_{i,\ell}^{q+1}$  by (7).

The chain of inequalities above shows that  $\{\tilde{J}^q : q \in \mathcal{Q}\}$  satisfies (8). Thus, we have  $\tilde{J}^q(\mathbf{x}) \geq J^q(\mathbf{x})$  for all  $\mathbf{x} \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ , so  $J^1(\mathbf{e}) \leq \tilde{J}^1(\mathbf{e}) = 2\hat{J}_L^1(\mathbf{e})$ .  $\blacksquare$

We focus on the second part of the proof of Theorem 4.1, which constructs a lower bound on the total expected revenue of the resource based static policy.

### 5.3 Lower Bound on the Performance of the Static Policy

We give a dynamic program to compute the total expected revenue of the resource based static policy. Let  $U_L^q(\mathbf{x})$  be the total expected revenue obtained by the resource based static policy over

time periods  $\{q, \dots, Q\}$  given that the state of the system at time period  $q$  is  $\mathbf{x}$ . We can compute the value functions  $\{U_L^q : q \in \mathcal{Q}\}$  through the dynamic program

$$U_L^q(\mathbf{x}) = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + U_L^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - U_L^{q+1}(\mathbf{x}) \right] \right\} + U_L^{q+1}(\mathbf{x}), \quad (9)$$

with the boundary condition that  $U_L^{Q+1} = 0$ . The dynamic program above is similar to that in (2), but under the resource based static policy, a customer making a booking request for interval  $[s, f]$  at time period  $q$  chooses room  $i$  with probability  $\phi_i^q(A_{[s,f]}^q)$ . In the next proposition, we use the linear value function approximations  $\{\hat{J}_L^q : q \in \mathcal{Q}\}$  to give a lower bound on the performance of the resource based static policy. Lemma 5.1 plays an important role in the proof of the next proposition. Thus, the lower bound on the performance of the resource based static policy explicitly uses the fact that the resource capacities are all one.

**Proposition 5.5 (Lower Bound on Static Policy Performance)** *Letting the value functions  $\{U_L^q : q \in \mathcal{Q}\}$  be computed through (9), for each  $\mathbf{x} \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ , we have*

$$U_L^q(\mathbf{x}) \geq \hat{J}_L^q(\mathbf{x}) - \frac{D_{\max} - 1}{D_{\max}} \hat{J}_L^q(\mathbf{e}).$$

*Proof:* We give an inequality that will be useful later in the proof. Because  $\prod_{\ell=s}^f x_{i,\ell} \geq \frac{\sum_{\ell=s}^f x_{i,\ell} - (f-s)}{1+f-s}$  by Lemma 5.1 and  $\phi_i^q(A_{[s,f]}^q) [r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1}] \geq 0$  by Lemma 5.2, we have

$$\begin{aligned} & \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \hat{J}_L^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \hat{J}_L^{q+1}(\mathbf{x}) \right] \\ &= \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\ &\geq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \frac{\sum_{\ell \in \mathcal{T}} \mathbb{1}_{\{\ell \in [s,f]\}} x_{i,\ell} - (f-s)}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \\ &\stackrel{(a)}{\geq} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} x_{i,\ell} \left\{ \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \phi_i^q(A_{[s,f]}^q) \frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1} \left[ r_{i,[s,f]} - \sum_{h=s}^f \eta_{i,h}^{q+1} \right] \right\} - \frac{D_{\max} - 1}{D_{\max}} (\hat{J}_L^q(\mathbf{e}) - \hat{J}_L^{q+1}(\mathbf{e})) \\ &\stackrel{(b)}{=} \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} x_{i,\ell} (\eta_{i,\ell}^q - \eta_{i,\ell}^{q+1}) - \frac{D_{\max} - 1}{D_{\max}} (\hat{J}_L^q(\mathbf{e}) - \hat{J}_L^{q+1}(\mathbf{e})) \\ &\stackrel{(c)}{=} \hat{J}_L^q(\mathbf{x}) - \hat{J}_L^{q+1}(\mathbf{x}) - \frac{D_{\max} - 1}{D_{\max}} (\hat{J}_L^q(\mathbf{e}) - \hat{J}_L^{q+1}(\mathbf{e})), \end{aligned}$$

where (a) follows by arranging the terms on the left side of (a) and using Lemma 5.3, (b) uses (7), and (c) uses the fact that  $\hat{J}_L^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \eta_{i,\ell} x_{i,\ell}$ .

In the rest of the proof, we show the inequality in the proposition by using induction over the time periods. At time period  $Q+1$ , because we have  $U_L^{Q+1} = 0 = \hat{J}_L^{Q+1}$ , the inequality holds at time

period  $Q + 1$ . Assuming that the inequality holds at time period  $q + 1$ , we show that the inequality holds at time period  $q$  as well. Arranging the terms, the coefficient of  $U_{\mathcal{L}}^{q+1}(\mathbf{x})$  on the right side of (9) is  $1 - \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \prod_{\ell=s}^f x_{i,\ell}$ . Because  $\sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \leq 1$  and  $\sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \leq 1$ , this coefficient is nonnegative. Letting  $\alpha^q = \frac{D_{\max}-1}{D_{\max}} \hat{J}_{\mathcal{L}}^q(\mathbf{e})$  for notational brevity. By the induction assumption, we have  $U_{\mathcal{L}}^{q+1}(\mathbf{x}) \geq \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) - \alpha^{q+1}$  for all  $\mathbf{x} \in \{0, 1\}^{n \times T}$ . In this case, replacing  $U_{\mathcal{L}}^{q+1}(\mathbf{x})$  and  $U_{\mathcal{L}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]})$  on the right side of (9) with their corresponding lower bounds  $\hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) - \alpha^{q+1}$  and  $\hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \alpha^{q+1}$ , respectively, the right side of (9) gets smaller, so we get

$$\begin{aligned} U_{\mathcal{L}}^q(\mathbf{x}) &\geq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(A_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) \right] \right\} + \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) - \alpha^{q+1} \\ &\stackrel{(d)}{\geq} \hat{J}_{\mathcal{L}}^q(\mathbf{x}) - \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) - (\alpha^q - \alpha^{q+1}) + \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{x}) - \alpha^{q+1} = \hat{J}_{\mathcal{L}}^q(\mathbf{x}) - \alpha^q, \end{aligned}$$

where (d) uses the chain of inequalities earlier in the proof, along with  $\frac{D_{\max}-1}{D_{\max}} (\hat{J}_{\mathcal{L}}^q(\mathbf{e}) - \hat{J}_{\mathcal{L}}^{q+1}(\mathbf{e})) = \alpha^q - \alpha^{q+1}$ . Thus, the inequality in the proposition holds at time period  $q$ . ■

Finally, we can use Propositions 5.4 and 5.5 to give a proof for Theorem 4.1.

#### **Proof of Theorem 4.1:**

By Proposition 5.4, we have  $\hat{J}_{\mathcal{L}}^1(\mathbf{e}) \geq \frac{1}{2} J^1(\mathbf{e})$ . Using Proposition 5.5 with  $\mathbf{x} = \mathbf{e}$  and  $q = 1$ , we have  $U_{\mathcal{L}}^1(\mathbf{e}) \geq \hat{J}_{\mathcal{L}}^1(\mathbf{e}) - \frac{D_{\max}-1}{D_{\max}} \hat{J}_{\mathcal{L}}^1(\mathbf{e}) = \frac{1}{D_{\max}} \hat{J}_{\mathcal{L}}^1(\mathbf{e})$ , so we get  $U_{\mathcal{L}}^1(\mathbf{e}) \geq \frac{1}{D_{\max}} \hat{J}_{\mathcal{L}}^1(\mathbf{e}) \geq \frac{1}{2D_{\max}} J^1(\mathbf{e})$ . ■

When we compute the ideal assortments  $\{A_{[s,f]}^q : [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  through (6) and (7), we need to solve a problem of the form  $\max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) p_i$  for fixed  $\{p_i : i \in \mathcal{N}\}$ . Viewing  $p_i$  as the net revenue contribution obtained by booking room  $i$ , this problem finds an assortment of rooms to offer to a customer to maximize the expected net revenue contribution. Such an assortment optimization problem is of combinatorial nature, but there are a variety of choice models characterizing the choice probabilities  $\{\phi_i^q(S) : i \in S, S \subseteq \mathcal{N}, q \in \mathcal{Q}\}$ , under which we can give efficient algorithms to solve the assortment optimization problem. For example, there are polynomial time algorithms to solve this problem under the multinomial logit, mixture of independent and multinomial logit, nested logit, preference list, and Markov chain choice models; see Talluri and van Ryzin (2004), Davis et al. (2014), Blanchet et al. (2016), Aouad et al. (2020), and Cao et al. (2020). Using **Assort** to denote the number of operations needed to solve the assortment optimization problem and noting that  $O(|\mathcal{F}|) = O(D_{\max} T)$ , we can compute the ideal assortments through (6)-(7) in  $O(\text{Assort } D_{\max} T Q + n D_{\max}^2 T Q)$  operations, because in (7), the number of intervals containing  $\ell \in \mathcal{T}$  is  $O(D_{\max}^2)$ . Once we compute the ideal assortments, we can compute the value functions  $\{J_{\mu}^q : q \in \mathcal{Q}\}$  of the rollout policy through (3) in  $O(n D_{\max} T^3 Q)$  operations.

Next, we give a static policy using nonlinear approximations.

## 6. Itinerary Based Static Policy with Polynomial Approximations

We develop a static policy based on polynomial approximations of the value functions. In particular, we use polynomial value function approximations  $\{\hat{J}_p^q : q \in \mathcal{Q}\}$  of the form

$$\hat{J}_p^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \gamma_{i,[s,f]}^q \prod_{\ell=s}^f x_{i,\ell}.$$

Here, we have one component for each room and interval combination. The component for room  $i$  and interval  $[s, f]$  is a function of the capacities in room  $i$  over interval  $[s, f]$ . To choose the coefficients  $\{\gamma_{i,[s,f]}^q : i \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$ , we will use a succinct representation of whether two intervals overlap. We refer to the subset of nights  $C_{[s,f]} \subseteq \mathcal{T}$  as the *intersection preserving subset* for the interval  $[s, f]$  if it satisfies the following two properties. (i) We have  $C_{[s,f]} \subseteq [s, f]$ . (ii) For all  $[a, b] \in \mathcal{F}$ , if  $[s, f] \cap [a, b] \neq \emptyset$ , then we have  $C_{[s,f]} \cap [a, b] \neq \emptyset$ . By the first property, the subset  $C_{[s,f]}$  includes only nights in the interval  $[s, f]$ . By the second property, if the interval  $[s, f]$  overlaps with the interval  $[a, b]$ , then the subset of nights  $C_{[s,f]}$  preserves this relationship and overlaps with the interval  $[a, b]$  as well. A trivial way to construct the intersection preserving subset  $C_{[s,f]}$  is to set  $C_{[s,f]} = \{s, \dots, f\}$ . Using intersection preserving subsets with fewer elements will allow us to obtain better performance guarantees. Later in this section, we elaborate on finding the smallest intersection preserving subsets. Let  $\mathcal{C} = \{C_{[s,f]} : [s, f] \in \mathcal{F}\}$  be a collection that includes an intersection preserving subset for each interval  $[s, f] \in \mathcal{F}$ .

Using this collection, we compute the coefficients  $\{\gamma_{i,[s,f]}^q : i \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  as follows. For all  $i \in \mathcal{N}$  and  $[s, f] \in \mathcal{F}$ , set  $\gamma_{i,[s,f]}^{Q+1} = 0$ . For each  $q = Q, Q-1, \dots, 1$ , we execute the two steps.

- **Construct Ideal Assortments:** For all  $[s, f] \in \mathcal{F}$ , compute the ideal assortment to offer to a customer with a booking request for interval  $[s, f]$  as

$$B_{[s,f]}^q = \arg \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left[ r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} \left| [s, f] \cap C_{[a,b]} \right| \gamma_{i,[a,b]}^{q+1} \right] \right\}. \quad (10)$$

- **Compute Coefficients:** For all  $i \in \mathcal{N}$  and  $[s, f] \in \mathcal{F}$ , compute the coefficient corresponding to room  $i$  and interval  $[s, f]$  as

$$\gamma_{i,[s,f]}^q = \lambda_{[s,f]}^q \phi_i^q(B_{[s,f]}^q) \left[ r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} \left| [s, f] \cap C_{[a,b]} \right| \gamma_{i,[a,b]}^{q+1} \right] + \gamma_{i,[s,f]}^{q+1}. \quad (11)$$

The algorithm above fully specifies the coefficients  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$ , so it also specifies our value function approximations. We can give some intuition for the algorithm.

Given that we have capacity in all rooms on all nights at time period  $q$ ,  $J^q(\mathbf{e})$  corresponds to the optimal total expected revenue obtained from the booking requests arriving

over time periods  $\{q, \dots, Q\}$ . Our approximation of this optimal total expected revenue is given by  $\hat{J}_p^q(e) = \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \gamma_{i,[s,f]}^q$ . Thus, we interpret  $\gamma_{i,[s,f]}^q$  as an approximation of the optimal total expected revenue over time periods  $\{q, \dots, Q\}$  obtained from booking requests to stay in room  $i$  over interval  $[s, f]$ . In (10), by the definition of an intersection preserving subset, if  $[s, f] \cap [a, b] \neq \emptyset$ , then  $[s, f] \cap C_{[a,b]} \neq \emptyset$  as well. Furthermore, if  $[s, f] \cap [a, b] \neq \emptyset$ , then accepting a booking request in room  $i$  for the interval  $[s, f]$  prevents us from accepting a booking request in the same room for interval  $[a, b]$ . Therefore, the expression  $\sum_{[a,b] \in \mathcal{F}} \left| [s, f] \cap C_{[a,b]} \right| \gamma_{i,[s,f]}^{q+1}$  captures the opportunity cost of booking requests that we can no longer accept, by booking room  $i$  for interval  $[s, f]$ . In this case,  $B_{[s,f]}^q$  is the assortment that maximizes the net expected revenue from a customer making a booking request for interval  $[s, f]$  at time period  $q$ . In (11), we accumulate our approximation to the optimal total expected revenue obtained from the booking requests to stay in room  $i$  over interval  $[s, f]$ . On the right side of (11), the first term captures the net expected revenue obtained at time period  $q$  given that we offer the ideal assortment  $B_{[s,f]}^q$ , whereas the second term captures the total expected revenue over time periods  $\{q+1, \dots, Q\}$ .

Although the preceding discussion gives only rough intuition for the algorithm, we can use this algorithm to give a static policy with a performance guarantee, and this performance guarantee holds for any collection of intersection preserving subsets that we can possibly use in the algorithm. In particular, noting the ideal assortments  $\{B_{[s,f]}^q : [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  computed through (10), we consider a static policy that always offers the assortment  $B_{[s,f]}^q$  of rooms to a customer making a booking request for interval  $[s, f]$  at time period  $q$ . In revenue management vocabulary, each room and interval of stay combination that can be booked by a customer is referred to as an itinerary. Our polynomial value function approximations have one component for each room and interval of stay combination, so we call this static policy the *itinerary based static policy*. In the next theorem, we give a performance guarantee for the itinerary based static policy. For the collection of intersection preserving subsets  $\mathcal{C} = \{C_{[s,f]} : [s, f] \in \mathcal{F}\}$ , we define the norm of this collection as  $\|\mathcal{C}\| = \max_{[s,f] \in \mathcal{F}} |C_{[s,f]}|$ . The performance guarantee that we give for the itinerary based static policy depends on the norm of the collection of intersection preserving subsets that we use in the algorithm, and it favors using a collection with a smaller norm.

**Theorem 6.1 (Performance of Itinerary Based Static Policy)** *The itinerary based static policy obtains at least  $1/(1 + \|\mathcal{C}\|)$  fraction of the optimal total expected revenue.*

Similar to the proof of Theorem 4.1, the proof of the theorem above is based on using the value function approximations  $\{\hat{J}_p^q : q \in \mathcal{Q}\}$  to give an upper bound on the performance of the optimal policy and a lower bound on the performance of the itinerary based static policy, but

the specifics of proof uses the properties of intersection preserving subsets. We give the proof in Appendix B. Recall that setting  $C_{[s,f]} = [s, f]$  trivially yields an intersection preserving subset for the interval  $[s, f]$ . For this intersection preserving subset, we have  $|C_{[s,f]}| = f - s + 1 \leq D_{\max}$ , which implies that there exists a collection of intersection preserving subsets whose norm is at most  $D_{\max}$ . By Theorem 6.1, the itinerary based static policy that we obtain by using such a trivial collection of intersection preserving subsets has a performance guarantee of  $1/(1 + D_{\max})$ . In the next theorem, letting  $D_{\min} = \min_{[s,f] \in \mathcal{F}} \{f - s + 1\}$  capture the minimum stay duration for any possible booking request, we show that there exists a collection of intersection preserving subsets whose norm does not exceed  $1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$ , and we can find this collection by solving a linear program. In particular, noting that the norm of the collection  $\mathcal{C} = \{C_{[s,f]} : [s, f] \in \mathcal{F}\}$  is given by  $\|\mathcal{C}\| = \max_{[s,f] \in \mathcal{F}} |C_{[s,f]}|$ , using the decision variables  $\{C_{[s,f]} : [s, f] \in \mathcal{F}\}$  and  $t$ , to find the collection of intersection preserving subsets with the smallest norm, we can solve the problem

$$\begin{aligned} \min \left\{ t : t \geq |C_{[s,f]}| \quad \forall [s, f] \in \mathcal{F}, \right. \\ \left. |C_{[s,f]} \cap [a, b]| \geq 1 \quad \forall [s, f] \text{ and } [a, b] \in \mathcal{F} \text{ such that } [s, f] \cap [a, b] \neq \emptyset, \right. \\ \left. C_{[s,f]} \subseteq [s, f] \quad \forall [s, f] \in \mathcal{F}, \quad t \geq 0 \right\}. \end{aligned} \quad (12)$$

By the first constraint, we have  $t = \max_{[s,f] \in \mathcal{F}} |C_{[s,f]}|$  in an optimal solution to problem (12). The second and third constraints ensure that  $C_{[s,f]}$  is an intersection preserving subset.

**Theorem 6.2 (Smallest Norm Collection)** *The optimal objective value of problem (12) is at most  $1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$ . Furthermore, we can obtain an optimal solution to this problem by solving a linear program with  $O(D_{\max}^2 T)$  decision variables and  $O(D_{\max}^3 T)$  constraints.*

We give the proof of the above theorem in Appendix C. Thus, by Theorem 6.2, we can solve problem (12) to obtain a collection of intersection preserving subsets whose norm is at most  $1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$ , in which case, by Theorem 6.1, the corresponding itinerary based static policy has a performance guarantee of  $\frac{1}{2 + \lceil (D_{\max} - 1)/D_{\min} \rceil}$ . Ma et al. (2020) use nonlinear value function approximations to give a policy with a performance guarantee of  $1/(1 + D_{\max})$ . Because  $D_{\min} \geq 1$ , we have  $\frac{1}{2 + \lceil (D_{\max} - 1)/D_{\min} \rceil} \geq 1/(1 + D_{\max})$ , so the performance guarantee of the itinerary based static policy is at least as good as that of the policy proposed by Ma et al. (2020). As discussed in the introduction, Villa Mahal offers rooms with minimum length of stay requirements, in which case  $D_{\min}$  strictly exceeds one. The performance guarantee for the itinerary based static policy can be significantly better than  $1/(1 + D_{\max})$  when  $D_{\min}$  is larger than one. Closing this section, we note that the rollout policy from the itinerary based static policy inherits the performance guarantee of  $\frac{1}{2 + \lceil (D_{\max} - 1)/D_{\min} \rceil}$  from the itinerary based static policy.

## 7. Upper Bound on the Optimal Policy Performance

We give an approach to computing an upper bound on the optimal total expected revenue. Such an upper bound becomes useful for assessing the optimality gaps of different policies.

### 7.1 Room Based Problems Through Revenue Allocations

One approach to obtaining an upper bound on the optimal total expected revenue is based on solving a linear programming approximation that is formulated under the assumption that the arrivals and choices of the customers take on their expected values; see Gallego et al. (2004), Liu and van Ryzin (2008), and Kunnumkal and Topaloglu (2008). To formulate such an approximation, we use two sets of decision variables. First, we use the decision variable  $h_{[s,f]}^q(S)$  to capture the probability of offering the assortment  $S$  of rooms to a booking request for the interval  $[s, f]$  at time period  $q$ . Second, we use the decision variable  $y_{i,[s,f]}^q$  to capture the expected number of bookings in room  $i$  at time period  $q$  by a customer interested in making a booking for interval  $[s, f]$ . In this case, we consider the linear programming approximation

$$\begin{aligned}
Z_{\text{LP}}^* &= \max \sum_{q \in \mathcal{Q}} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} r_{i,[s,f]}^q y_{i,[s,f]}^q & (13) \\
&\text{st } \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \sum_{S \subseteq \mathcal{N}} \mathbb{1}_{\{\ell \in [s,f]\}} \phi_i^q(S) h_{[s,f]}^q(S) \leq 1 \quad \forall i \in \mathcal{N}, \ell \in \mathcal{T} \\
&\sum_{S \subseteq \mathcal{N}} h_{[s,f]}^q(S) = \lambda_{[s,f]}^q \quad \forall [s,f] \in \mathcal{F}, q \in \mathcal{Q} \\
&\sum_{S \subseteq \mathcal{N}} \phi_i^q(S) h_{[s,f]}^q(S) = y_{i,[s,f]}^q \quad \forall i \in \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q} \\
&h_{[s,f]}^q(S) \geq 0 \quad \forall [s,f] \in \mathcal{F}, S \subseteq \mathcal{N}, q \in \mathcal{Q},
\end{aligned}$$

where we do not explicitly impose a nonnegativity constraint on the decision variable  $y_{i,[s,f]}^q$  because the third constraint above already ensures this constraint.

In the objective function above, we accumulate the total expected revenue from the bookings. By the first constraint, noting that  $\sum_{S \subseteq \mathcal{N}} \phi_i^q(S) h_{[s,f]}^q(S)$  is the expected number of bookings in room  $i$  by a customer arriving at time period  $q$  interested in making a booking for interval  $[s, f]$ , the total expected capacity consumption of each room on each night does not exceed one. By the second constraint, the total probability that we offer an assortment to a customer arriving at time period  $q$  with an interest in making a booking for interval  $[s, f]$  is equal to the arrival probability of such a customer. By the third constraint, we compute the expected number of bookings of room  $i$  at time period  $q$  by a customer interested in making a booking for interval  $[s, f]$ . Linear programming approximations are commonly used in the revenue management literature to obtain

upper bounds on the optimal expected revenue. Such upper bounds are known to be asymptotically tight as the capacities of the resources and the expected number of customer arrivals increase linearly at the same rate. This kind of asymptotic tightness result is particularly relevant in airline network revenue management problems, in which the capacities of the flights and the volume of demand served are generally large. In our problem, however, the capacities of all resources are invariably one, even when the number of rooms under consideration is large. In our computational experiments, the upper bound provided by the linear programming approximation can indeed be quite loose. We give a different upper bound that is at least as tight as the one from the linear programming approximation. In our computational experiments, the new upper bound provided by our approach can improve that from the linear programming approximation by as much as 29%. Recall that we refer to each room and interval of stay combination as an itinerary. Thus, for each  $i \in \mathcal{N}$  and  $[s, f] \in \mathcal{F}$ , we have an itinerary  $(i, [s, f])$ . Our approach is based on allocating the revenue associated with an itinerary to the different rooms.

Let  $\beta_{i,[s,f] \rightarrow j}^q$  be the portion of the revenue associated with itinerary  $(i, [s, f])$  allocated to room  $j$  at time period  $q$ . We do not yet specify how we choose the revenue allocations, but if we add the revenue allocations of an itinerary over all rooms, then we should get the revenue of the itinerary, so the revenue allocations should satisfy  $\sum_{j \in \mathcal{N}} \beta_{i,[s,f] \rightarrow j}^q = r_{i,[s,f]}$ . In our approach, we solve a separate revenue management problem for each room and collect the value functions for the problem for each room to get an upper bound. In the revenue management problem for room  $j$ , we have limited capacity only in room  $j$ , but infinite capacity in all other rooms. Moreover, if we accept a booking request at time period  $q$  for itinerary  $(i, [s, f])$ , then the revenue we collect is the revenue allocation of this itinerary over room  $j$ , which is  $\beta_{i,[s,f] \rightarrow j}^q$ . We capture the state of room  $j$  by using the vector  $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,T}) \in \{0, 1\}^T$ , where  $x_{j,\ell} = 1$  if and only if room  $j$  is available for stay on night  $\ell$ . Noting that  $\mathbf{e}_{[s,f]} \in \{0, 1\}^T$  is the vector with ones in the components corresponding to nights  $\{s, \dots, f\}$ , we can find the optimal policy in the revenue management problem for room  $j$  by computing the value functions  $\{V_{\beta,j}^q : q \in \mathcal{Q}\}$  through the dynamic program

$$V_{\beta,j}^q(\mathbf{x}_j) = \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} + V_{\beta,j}^{q+1}(\mathbf{x}_j), \quad (14)$$

with the boundary condition that  $V_{\beta,j}^{Q+1} = 0$ . In the value functions  $\{V_{\beta,j}^q : q \in \mathcal{Q}\}$ , we make their dependence on  $\beta = \{\beta_{i,[s,f] \rightarrow j}^q : i, j \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  explicit.

To interpret the dynamic program above, at time period  $q$ , if we offer the assortment  $S$  of rooms to a customer making a booking request for interval  $[s, f]$ , then the customer chooses room  $j$



with probability  $\phi_j^q(S)$ . In this case, if we have capacity in room  $j$  to accommodate the booking request, then we make the revenue of  $\beta_{j,[s,f] \rightarrow j}^q$ , and the opportunity cost of the capacities that we consume is  $V_{\beta,j}^{q+1}(\mathbf{x}_j) - V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]})$ . Because we have infinite capacity in all rooms other than room  $j$ , if the customer chooses some other room  $i$ , then we make the revenue of  $\beta_{i,[s,f] \rightarrow j}^q$ , but the opportunity cost of the capacities that we consume is zero. In the next proposition, we show that we obtain an upper bound on the value functions  $\{J^q : q \in \mathcal{Q}\}$  in (1) by solving the dynamic program in (14). We defer the proofs of all results in this section to Appendix D.

**Proposition 7.1 (Decomposition Upper Bound)** *For any revenue allocations  $\beta$  that satisfy  $\sum_{j \in \mathcal{N}} \beta_{i,[s,f] \rightarrow j}^q = r_{i,[s,f]}$  for all  $i \in \mathcal{N}$  and  $[s, f] \in \mathcal{F}$ , letting the value functions  $\{V_{\beta,j}^q : q \in \mathcal{Q}\}$  be computed through (14), for each  $\mathbf{x} = (\mathbf{x}_j : j \in \mathcal{N}) \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ , we have*

$$\sum_{j \in \mathcal{N}} V_{\beta,j}^q(\mathbf{x}_j) \geq J^q(\mathbf{x}).$$

Next, we consider the question of choosing the revenue allocations such that our upper bound compares favorably against that from the linear programming approximation.

## 7.2 Choosing the Revenue Allocations

In problem (13), duplicating the decision variable  $h_{[s,f]}^q$  for each room  $j$  to obtain the decision variables  $\{h_{[s,f] \rightarrow j}^q : j \in \mathcal{N}\}$ , we consider a variant of this problem given by

$$\begin{aligned} \max \quad & \sum_{q \in \mathcal{Q}} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} r_{i,[s,f]}^q y_{i,[s,f]}^q & (15) \\ \text{st} \quad & \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \sum_{S \subseteq \mathcal{N}} \mathbf{1}_{\{\ell \in [s,f]\}} \phi_i^q(S) h_{[s,f] \rightarrow i}^q(S) \leq 1 \quad \forall i \in \mathcal{N}, \ell \in \mathcal{T} \\ & \sum_{S \subseteq \mathcal{N}} h_{[s,f] \rightarrow j}^q(S) = \lambda_{[s,f]}^q \quad \forall j \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q} \\ & \sum_{S \subseteq \mathcal{N}} \phi_i^q(S) h_{[s,f] \rightarrow j}^q(S) = y_{i,[s,f]}^q \quad \forall i, j \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q} \\ & h_{[s,f] \rightarrow j}^q(S) \geq 0 \quad \forall j \in \mathcal{N}, [s, f] \in \mathcal{F}, S \subseteq \mathcal{N}, q \in \mathcal{Q}. \end{aligned}$$

In Lemma D.1 in Appendix D, we show that the problem above has the same optimal objective value as the linear programming approximation in (13). Intuitively speaking, although problem (15) duplicates the decision variable  $h_{[s,f]}^q$  in problem (13) for each room  $j$ , it does not duplicate the decision variable  $y_{i,[s,f]}^q$ . Because the objective functions of both of these two problems depend only on the decision variables  $\{y_{i,[s,f]}^q : i \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$ , the two problems end up having the same optimal objective value. We use the dual solution to problem (15) to choose the revenue allocations. Associating the dual variables  $\{\beta_{i,[s,f] \rightarrow j}^q : i, j \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  with the third

constraint in problem (15), in the dual of problem (15), the constraint associated with the decision variable  $y_{i,[s,f]}^q$  is  $\sum_{j \in \mathcal{N}} \beta_{i,[s,f] \rightarrow j}^q = r_{i,[s,f]}$ . Thus, letting  $\hat{\beta} = \{\hat{\beta}_{i,[s,f] \rightarrow j}^q : i, j \in \mathcal{N}, [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  be the optimal values of the dual variables associated with the third constraint in problem (15), we use these optimal values as our revenue allocations. In the next theorem, we show that if we use these revenue allocations in the dynamic program in (14), then we obtain an upper bound on the optimal total expected revenue that is at least as tight as that from the linear program in (13). In this theorem, we use  $\mathbf{e}' \in \{0, 1\}^T$  to denote the vector of all ones.

**Theorem 7.2 (Choice of Revenue Allocations)** *Letting  $\hat{\beta}$  be the optimal values of the dual variables associated with the third constraint in problem (15) and the value functions  $\{V_{\hat{\beta},j}^q : q \in \mathcal{Q}\}$  be computed through (14) with the revenue allocations  $\hat{\beta}$ , we have*

$$\sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(\mathbf{e}') \leq Z_{\text{LP}}^*.$$

Noting Proposition 7.1, we get  $Z_{\text{LP}}^* \geq \sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(\mathbf{e}') \geq J^1(\mathbf{e})$ , so  $\sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(\mathbf{e}')$  is an upper bound on the optimal total expected revenue, and this upper bound is at least as tight as that provided by the optimal objective value of the linear programming approximation in (13).

### 7.3 Computing the Upper Bound

In (14), although we focus only on one room, the state variable is still high dimensional, so it is not clear whether we can solve this dynamic program efficiently. We will give an equivalent formulation for this dynamic program by using intervals of nights as the state variable, which we will be able to solve efficiently. We use  $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,T}) \in \{0, 1\}^T$  to denote the state of room  $j$ , where  $x_{j,\ell} = 1$  if and only if room  $j$  is available for stay on night  $\ell$ . We refer to the interval  $[a, b]$  as a *maximal unavailable interval* with respect to  $\mathbf{x}_j$  if and only if room  $j$  is unavailable on all nights in the interval  $[a, b]$ , but available on nights  $a - 1$  and  $b + 1$ ; that is,  $x_{j,\ell} = 0$  for all  $j \in \{a, \dots, b\}$ ,  $x_{j,a-1} = 1$ , and  $x_{j,b+1} = 1$ . In the revenue management problem for room  $j$ , given that we do not have capacity in room  $j$  over the maximal unavailable interval  $[a, b]$ , let  $\Gamma_{\beta,j}^q(a, b)$  be the total expected revenue obtained over time periods  $\{q, \dots, Q\}$  from the booking requests for intervals that start in the interval  $[a, b]$ . We can compute the value functions  $\{\Gamma_{\beta,j}^q : q \in \mathcal{Q}\}$  as

$$\Gamma_{\beta,j}^q(a, b) = \sum_{k=q}^Q \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a,b]\}} \lambda_{[s,f]}^k \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^k(S) \beta_{i,[s,f] \rightarrow j}^k \right\}. \quad (16)$$

To interpret (16), in the revenue management problem for room  $j$ , we have infinite capacity in all rooms other than room  $j$ . Therefore, given that we have no capacity in room  $j$  over the interval  $[a, b]$ , if a customer makes a booking request for an interval  $[s, f]$  that starts in the interval  $[a, b]$ ,

then we can offer only assortments of rooms that do not include room  $j$ . If the customer makes a booking in room  $i$ , then the revenue that we obtain is the revenue allocation of itinerary  $(i, [s, f])$  to room  $j$ . Because rooms other than room  $j$  have infinite capacity and we do not offer room  $j$ , we simply accumulate the expected revenue over time periods  $\{q, \dots, Q\}$ . On the other hand, recall that we refer to the interval  $[a, b]$  as a maximal available interval with respect to  $\mathbf{x}_j$  if and only if room  $j$  is available on all nights in the interval  $[a, b]$ , but unavailable on nights  $a - 1$  and  $b + 1$ . In the revenue management problem for room  $j$ , given that we do have capacity in room  $j$  over the maximal available interval  $[a, b]$ , let  $\Theta_{\beta,j}^q(a, b)$  be the total expected revenue obtained over time periods  $\{q, \dots, Q\}$  from the booking requests for intervals that start in the interval  $[a, b]$ . We can compute the value functions  $\{\Theta_{\beta,j}^q : q \in \mathcal{Q}\}$  through the dynamic program

$$\begin{aligned} \Theta_{\beta,j}^q(a, b) = & \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \mathbf{1}_{\{s \in [a,b], [s,f] \subseteq [a,b]\}} \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right. \\ & \left. + \phi_j^q(S) \left[ \beta_{j,[s,f] \rightarrow j}^q + \Theta_{\beta,j}^{q+1}(a, s-1) + \Theta_{\beta,j}^{q+1}(f+1, b) + \Gamma_{\beta,j}^{q+1}(s, f) - \Theta_{\beta,j}^{q+1}(a, b) \right] \right\} \\ & + \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \mathbf{1}_{\{s \in [a,b], [s,f] \not\subseteq [a,b]\}} \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} + \Theta_{\beta,j}^{q+1}(a, b). \quad (17) \end{aligned}$$

To interpret (17), given that we have capacity in room  $j$  over the interval  $[a, b]$ , if we have a booking request for an interval  $[s, f]$  that starts in  $[a, b]$  and the interval  $[s, f]$  is included in  $[a, b]$ , then we have capacity in room  $j$  to serve this booking request. If the customer books room  $i \neq j$ , then the revenue that we obtain is the revenue allocation of itinerary  $(i, [s, f])$  to room  $j$ , but because the rooms other than room  $j$  have infinite capacity, we do not account for the opportunity cost of the capacities that we lose. If, however, the customer books room  $j$ , then we lose the capacity on nights  $\{s, \dots, f\}$ , so the intervals  $[a, s - 1]$  and  $[f + 1, b]$  become maximal available intervals and the interval  $[s, f]$  becomes a maximal unavailable interval. Lastly, if the interval  $[s, f]$  is not included in  $[a, b]$ , then we do not have capacity in room  $j$  to serve a booking request for the interval  $[s, f]$ , so we can offer only an assortment of rooms not including room  $j$ .

Recall that  $\mathcal{I}(\mathbf{x}_j)$  is the collection of maximal available intervals with respect to  $\mathbf{x}_j$ . Let  $\mathcal{H}(\mathbf{x}_j)$  be the collection of maximal unavailable intervals with respect to  $\mathbf{x}_j$ . In the top portion of Figure 2, for example, for the value of  $\mathbf{x}_j$  in this figure, we have  $\mathcal{H}(\mathbf{x}_j) = \{[1, 1], [5, 5], [11, 12]\}$ . In the next theorem, we show that we can solve the dynamic program in (17) to get a solution for (14).

**Theorem 7.3 (Interval Formulation)** *Letting the value function  $\{V_{\beta,j}^q : q \in \mathcal{Q}\}$  be computed through (14), for each  $\mathbf{x}_j \in \{0, 1\}^T$  and  $q \in \mathcal{Q}$ , we have*

$$V_{\beta,j}^q(\mathbf{x}_j) = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^q(a, b) + \sum_{[a,b] \in \mathcal{H}(\mathbf{x}_j)} \Gamma_{\beta,j}^q(a, b).$$

Next, we give computational experiments to test the quality of our policies and bound.

## 8. Computational Experiments

We give two sets of computational experiments. The first set is on synthetic datasets that we generate. The second set is on a dataset from an actual boutique hotel.

### 8.1 Results on Synthetic Datasets

We use synthetically generated datasets to test the quality of the policies that we propose, as well as the tightness of our upper bound.

**Experimental Setup:** We generate our test problems as follows. We have  $n = 5$  rooms. The booking horizon is  $\mathcal{T} = \{1, \dots, 70\}$ , so the rooms are available for stay during a period of 10 weeks. Customers arrive over time periods  $\mathcal{Q} = \{1, \dots, 700\}$ . The set of possible intervals of stay is  $\mathcal{F} = \{[a, b] : a \leq b \leq a + D_{\max} - 1\}$ , where  $D_{\max}$  is the maximum duration of stay parameter that we vary. To come up with the revenues  $\{r_{i,[s,f]} : i \in \mathcal{N}, [s, f] \in \mathcal{F}\}$ , for each  $i \in \mathcal{N}$ , we sample a base revenue  $\psi_i$  from the uniform distribution over  $[0, 10]$ . Letting  $p_{i,\ell}$  be the price of stay in room  $i$  on night  $\ell$ , if night  $\ell$  is Friday, Saturday, or Sunday night, then we set  $p_{i,\ell} = \psi_i$ , whereas if night  $\ell$  is Monday, Tuesday, Wednesday, or Thursday night, then we set  $p_{i,\ell} = \delta \times \psi_i$ , where  $\delta$  is the discount parameter. The revenue associated with room  $i$  over interval  $[s, f]$  is  $r_{i,[s,f]} = \sum_{\ell=s}^f p_{i,\ell}$ .

To come up with the arrival probabilities  $\{\lambda_{[s,f]}^q : [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$ , for each  $[s, f] \in \mathcal{F}$  and  $q \in \mathcal{Q}$ , we sample a weight  $\beta_{[s,f]}^q$  from the uniform distribution over  $[0, \bar{U}_{[s,f]}^q]$  and normalize the weights by setting  $\gamma_{[s,f]}^q = \frac{\beta_{[s,f]}^q}{\sum_{[a,b] \in \mathcal{F}} \beta_{[a,b]}^q}$ . In this case, using  $\{\gamma_{[s,f]}^q : [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  as the arrival probabilities, if we always offer all rooms, then the total expected demand for the capacity in all rooms on all nights is  $\text{Demand} = \sum_{i \in \mathcal{N}} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \gamma_{[s,f]}^q \phi_i^q(\mathcal{N}) (f - s + 1)$ . Noting that the total capacity available in all rooms on all nights is  $nT$ , we set the arrival probability for a booking request for interval  $[s, f]$  at time period  $q$  as  $\lambda_{[s,f]}^q = \rho \gamma_{[s,f]}^q \frac{nT}{\text{Demand}}$ , where  $\rho$  is the load factor parameter that we vary. In this case, if we offer all rooms at all time periods, then the ratio between the total expected demand for the capacity and the total available capacity is given by  $\rho$ .

Splitting the 700 time periods into three roughly equal segments, we use  $\bar{U}_{[s,f]}^q = D_{\max} - (f - s)$  for  $q = 1, \dots, 233$ ,  $\bar{U}_{[s,f]}^q = 1$  for  $q = 234, \dots, 466$  and  $\bar{U}_{[s,f]}^q = f - s + 1$  for  $q = 467, \dots, 700$ . Thus, the requests for shorter intervals tend to have larger arrival probabilities at the earlier time periods, so we need to carefully reserve the capacity for the requests for longer intervals that tend to arrive later. In this way, we generate test problems that require carefully allocating the available capacity to obtain good performance. Choices of the customers are governed by the multinomial logit model. Thus, using  $v_i$  to denote the preference weight of room  $i$  and  $v_0$  to denote the preference weight of the no-purchase option, if we offer the assortment  $S$  of rooms, then a customer arriving at time period  $q$  chooses room  $i$  with probability  $\phi_i^q(S) = \frac{v_i}{v_0 + \sum_{j \in S} v_j}$ . To come up with the preference

weights, for each  $i \in \mathcal{N}$ , we sample  $v_i$  from the uniform distribution over  $[0, 1]$  and set  $v_0 = \frac{1}{9} \sum_{i \in \mathcal{N}} v_i$ , so if we offer all rooms, then a customer leaves without a booking with probability 0.1.

Varying the parameters  $D_{\max} \in \{6, 8, 10\}$ ,  $\rho \in \{1.2, 1.6, 2.0\}$ , and  $\delta \in \{0.7, 0.9\}$ , we obtain 18 parameter configurations for our test problems.

**Benchmark Policies:** Our benchmark policies are based on the linear and polynomial value function approximations, as well as the linear programming approximation.

*Linear Approximations* (LIN1, LIN5, LINR). We use three benchmark policies based on the linear value function approximations presented in Section 4. By Theorem 4.1, the resource based static policy has a performance guarantee of  $1/(2D_{\max})$ , but as a static policy, the resource based static policy may offer an unavailable room for a booking request, which may not be appropriate in practice. To overcome this difficulty, we use the greedy policy with respect to the linear approximations. Replacing  $J^{q+1}(\mathbf{x})$  on the right side of (1) with  $\hat{J}_L^{q+1}(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \eta_{i,\ell}^{q+1} x_{i,\ell}$ , if the state of the system at time period  $q$  is  $\mathbf{x}$  and there is a booking request for interval  $[s, f]$ , then the greedy policy with respect to the linear approximations offers the assortment of rooms  $\hat{S}_L^q(\mathbf{x}) = \arg \max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) (r_{i,[s,f]} - \sum_{\ell=s}^f \eta_{i,\ell}^{q+1})$ . By the discussion given immediately after (1), this policy does not offer an unavailable room. Using the same outline in the proof of Theorem 4.1, we can show that the greedy policy with respect to the linear approximations also has a performance guarantee of  $1/(2D_{\max})$ . The key is to observe that an analogue of the chain of inequalities in the proof of Proposition 5.5 still holds when we replace  $A_{[s,f]}^q$  with  $\hat{S}_L^q(\mathbf{x})$ .

We use LIN1 to refer to the greedy policy with respect to the linear approximations, where we emphasize the fact that this policy computes the opportunity costs  $\{\eta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$  once at the beginning of the selling horizon. We also use a version of LIN1 that splits the selling horizon into five equal segments to recompute the opportunity costs at the beginning of each segment based on the state of the system at the beginning of the segment. In particular, if the state of the system at the beginning of the segment is  $\mathbf{x}$ , then we set  $\eta_{i,\ell}^q = 0$  for all  $i \in \mathcal{N}$ ,  $\ell \in \mathcal{T}$ , and  $q \in \mathcal{Q}$  such that  $x_{i,\ell} = 0$ . We compute the other opportunity costs using the algorithm in Section 4. We refer to this policy as LIN5. Lastly, we use the rollout policy from the resource based static policy. We refer to this policy as LINR. By the discussion at the end of Section 3, this policy inherits the performance guarantee of  $1/(2D_{\max})$  from the resource based static policy. Furthermore, this policy never offers an unavailable room for a booking request. Observe that LIN1 and LIN5 are greedy policies with respect to linear approximations  $\{J_L^q : q \in \mathcal{Q}\}$ , whereas noting that  $\{J_\mu^q : q \in \mathcal{Q}\}$  in (5) is not separable, LINR does not necessarily use a nonseparable approximation.

*Polynomial Approximations* (POL1, POL5, POLR). The benchmark policies POL1, POL5, and POLR are the analogues of LIN1, LIN5, and LINR, but they use the polynomial value function

approximations in Section 6. As in LIN1, POL1 is the greedy policy with respect to the polynomial approximations. We can show that it shares the guarantee of  $\frac{1}{2+\lceil(D_{\max}-1)/D_{\min}\rceil}$  with the itinerary based static policy. In POL5, when recomputing the coefficients  $\{\gamma_{i,[s,f]}^q : i \in \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  at the beginning of a segment, if the state of the system is  $\mathbf{x}$ , then we set  $\gamma_{i,[s,f]}^q = 0$  for all  $i \in \mathcal{N}$ ,  $[s,f] \in \mathcal{F}$ , and  $q \in \mathcal{Q}$  such that  $\prod_{\ell=s}^f x_{i,\ell} = 0$ . We compute the other coefficients using the algorithm in Section 6. Lastly, POLR is the rollout of the itinerary based static policy.

*Linear Programming Approximation* (LP1, LP5, LPR). We use three benchmark policies based on the linear programming approximation in (13). The first constraint in problem (13) ensures that the total expected capacity consumption of each room on each night does not exceed the available capacity. Letting  $\{\theta_{i,\ell} : i \in \mathcal{N}, \ell \in \mathcal{T}\}$  be the optimal values of the dual variables associated with the first constraint, we use  $\theta_{i,\ell}$  as the opportunity cost of the capacity of room  $i$  on night  $\ell$ , yielding the value function approximations  $\{\hat{J}_{\text{LP}}^q : q \in \mathcal{Q}\}$  with  $\hat{J}_{\text{LP}}^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \theta_{i,\ell} x_{i,\ell}$ . Similar to LIN1, if the state of the system at time period  $q$  is  $\mathbf{x}$  and we have a booking request for interval  $[s,f]$ , then we offer the assortment of rooms  $\hat{S}_{\text{LP}}^q(\mathbf{x}) = \arg \max_{S \subseteq \mathcal{N}} \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) (r_{i,[s,f]} - \sum_{\ell=s}^f \theta_{i,\ell})$ . We refer to this policy as LP1. Thus, LP1 corresponds to the greedy policy with respect to the linear approximations  $\{\hat{J}_{\text{LP}}^q : q \in \mathcal{Q}\}$ , but the opportunity costs in these linear approximations are obtained from the dual solution to the linear programming approximation.

We also use a version of LP1 that splits the selling horizon into five equal segments to recompute the opportunity costs at the beginning of each segment. If the segment starts at time period  $\bar{q}$  with the state  $\mathbf{x}$ , then we solve problem (13) after replacing the set of time periods  $\mathcal{Q}$  with  $\{\bar{q}, \dots, Q\}$  and the right side of the first constraint with  $\{x_{i,\ell} : i \in \mathcal{N}, \ell \in \mathcal{T}\}$ . Letting  $\{\theta_{i,\ell} : i \in \mathcal{N}, \ell \in \mathcal{T}\}$  be the optimal values of the dual variables associated with the first constraint, we use  $\theta_{i,\ell}$  as the opportunity cost of the capacity of room  $i$  on night  $\ell$ . We use LP5 to refer to this policy. Lastly, we extract a static policy from problem (13) and use the rollout policy from this static policy. In our static policy, letting  $\{\hat{h}_{[s,f]}^q(S) : [s,f] \in \mathcal{F}, S \subseteq \mathcal{N}, q \in \mathcal{Q}\}$  and  $\{\hat{y}_{i,[s,f]}^q : i \in \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  be an optimal solution to problem (13), if we have a booking request for interval  $[s,f]$  at time period  $q$ , then we offer the assortment of rooms  $S$  with probability  $\hat{h}_{[s,f]}^q(S)/\lambda_{[s,f]}^q$ . We use the rollout policy from this static policy and refer to the resulting rollout policy as LPR.

*Dynamic Programming Decomposition* (DEC). This benchmark policy is the standard dynamic programming decomposition method in the revenue management literature; see Liu and van Ryzin (2004). In this policy, we heuristically decompose the dynamic programming formulation in (1) by the resources. We have one resource for each room and night combination. Furthermore, each resource has one unit of capacity. Therefore, this policy constructs linear value function approximations  $\{\hat{J}_{\text{Dec}}^q : q \in \mathcal{Q}\}$  of the form  $\hat{J}_{\text{Dec}}^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \zeta_{i,\ell}^q x_{i,\ell}$  for some opportunity costs

Params. ( $D_{\max}, \rho, \delta$ )	% Gap in Bounds	Params. ( $D_{\max}, \rho, \delta$ )	% Gap in Bounds	Params. ( $D_{\max}, \rho, \delta$ )	% Gap in Bounds
(6, 1.2, 0.9)	15.38	(8, 1.2, 0.9)	25.62	(10, 1.2, 0.9)	13.82
(6, 1.2, 0.7)	11.99	(8, 1.2, 0.7)	15.53	(10, 1.2, 0.7)	12.93
(6, 1.6, 0.9)	20.10	(8, 1.6, 0.9)	29.73	(10, 1.6, 0.9)	26.50
(6, 1.6, 0.7)	24.53	(8, 1.6, 0.7)	23.35	(10, 1.6, 0.7)	28.08
(6, 2.0, 0.9)	24.90	(8, 2.0, 0.9)	25.84	(10, 2.0, 0.9)	26.75
(6, 2.0, 0.7)	22.86	(8, 2.0, 0.7)	16.09	(10, 2.0, 0.7)	26.33
Avg.	19.96	Avg.	22.69	Avg.	22.40

**Table 1** Percent gap between the upper bounds for the synthetic dataset.

$\{\zeta_{i,\ell}^q : i \in \mathcal{N}, \ell \in \mathcal{T}, q \in \mathcal{Q}\}$ , in which case, we can use the greedy policy with respect to these linear approximations. We refer to this policy as DEC.

**Comparison of Upper Bounds:** The optimal objective value  $Z_{LP}^*$  of the linear programming approximation in (13) provides an upper bound on the optimal total expected revenue. By Proposition 7.1, we can solve the dynamic program in (14) to also obtain an upper bound on the optimal total expected revenue. Furthermore, by Theorem 7.2, we can use the optimal dual solution to problem (15) to choose the revenue allocations such that the upper bound we obtain is at least as tight as that provided by  $Z_{LP}^*$ . In particular, letting  $\hat{\beta}$  be the optimal values of the dual variables associated with the third constraint in problem (15),  $\sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(e')$  is an upper bound on the optimal total expected revenue, and this upper bound is at least as tight as  $Z_{LP}^*$ . In Table 1, we compare the upper bounds  $Z_{LP}^*$  and  $\sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(e')$ . In the table, the first column shows the parameter configuration for each test problem. The second column shows the percent gap  $100 \times \frac{Z_{LP}^* - \sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(e')}{\sum_{j \in \mathcal{N}} V_{\hat{\beta},j}^1(e')}$  between the two upper bounds.

Our results indicate that the upper bounds from our approach can dramatically improve those from the linear programming approximation. The gaps can reach 29.73%. In our approach, once we allocate the revenue associated with each itinerary over different rooms, the revenue management problem that we solve for each room incorporates the uncertainty in the customer arrivals and choices, giving our approach an edge over the linear programming approximation. It is known that DEC also provides an upper bound on the optimal total expected revenue. In our test problems, the upper bounds from DEC were virtually identical to those from the linear programming approximation. The upper bound from DEC is based on decomposing the problem by each room and night combination, whereas the upper bound from our approach is based on decomposing the problem by each room. Therefore, intuitively speaking, our approach captures the interactions between the different nights more faithfully than DEC.

**Policy Performance:** Considering the performance of the benchmark policies given earlier in this section, we estimate the total expected revenue obtained by each policy by simulating

Param. ( $D_{\max}, \rho, \delta$ )	Total Exp. Revenue									
	LIN1	POL1	LP1	LIN5	POL5	LP5	LINR	POLR	LPR	DEC
(6, 1.2, 0.9)	79.23	75.19	76.99	80.89	75.34	78.03	81.05	80.29	81.11	77.27
(6, 1.2, 0.7)	70.17	68.57	68.53	71.27	69.81	69.67	73.00	73.04	73.97	67.65
(6, 1.6, 0.9)	74.94	67.84	72.44	77.10	68.53	74.56	78.60	75.96	77.92	70.58
(6, 1.6, 0.7)	80.52	72.77	76.25	82.78	75.25	82.24	85.02	83.07	84.52	80.59
(6, 2.0, 0.9)	79.14	70.24	75.93	81.44	72.63	80.30	85.04	82.84	84.14	78.76
(6, 2.0, 0.7)	77.16	67.61	69.50	79.28	69.34	74.21	80.85	77.39	79.28	73.91
(8, 1.2, 0.9)	75.27	66.18	73.88	78.14	69.27	75.91	78.23	77.84	79.29	75.79
(8, 1.2, 0.7)	70.39	65.71	67.03	72.16	66.50	68.56	71.07	71.09	73.44	68.41
(8, 1.6, 0.9)	80.24	67.56	77.76	83.06	69.40	82.51	84.64	81.96	84.82	79.77
(8, 1.6, 0.7)	76.84	67.39	73.59	80.14	68.72	77.55	79.75	78.52	81.57	77.08
(8, 2.0, 0.9)	80.54	66.96	73.47	81.41	68.68	78.11	84.26	81.37	81.56	79.38
(8, 2.0, 0.7)	73.90	60.85	71.99	77.28	62.89	74.74	79.21	73.62	77.22	75.12
(10, 1.2, 0.9)	68.61	62.56	66.24	70.72	63.42	67.95	73.89	71.43	74.33	70.04
(10, 1.2, 0.7)	69.95	66.56	67.02	69.90	68.48	68.98	73.09	72.90	75.02	68.76
(10, 1.6, 0.9)	73.74	63.60	72.25	76.37	67.13	74.55	79.11	74.98	78.40	72.09
(10, 1.6, 0.7)	77.31	62.45	73.58	79.96	65.11	78.69	82.22	78.10	81.86	78.61
(10, 2.0, 0.9)	78.43	63.80	75.57	81.09	65.45	78.78	83.65	79.14	81.83	78.43
(10, 2.0, 0.7)	78.72	62.28	73.18	81.39	63.77	80.55	83.93	75.64	82.01	80.84
Avg.	75.84	66.56	72.51	78.02	68.32	75.88	79.81	77.18	79.57	75.17

**Table 2** Total expected revenues obtained by the benchmark policies for the synthetic datasets.

the decisions of the policy over 100 sample paths. In Table 2, we compare the total expected revenues obtained by the benchmark policies. In the table, the first column shows the parameter configuration. In the second to fourth columns, we focus on the performance of LIN1, POL1, and LP1, which compute the coefficients of the value function approximations once at the beginning of the selling horizon. In the fifth to seventh columns, we focus on the performance of LIN5, POL5, and LP5, which compute the coefficients of the value function approximations five times over the selling horizon. In the eighth to tenth columns, we focus on the performance of LINR, POLR, and LPR, which use a rollout policy from a static policy. In the eleventh column, we focus on the performance of DEC. For each benchmark policy, we express its total expected revenue as a percentage of the upper bound on the optimal total expected revenue. In other words, using  $Rev$  to denote the total expected revenue of a benchmark policy and  $UB$  to denote the upper bound, we report  $100 \times \frac{Rev}{UB}$ . We use the upper bound obtained from the dynamic program in (14), along with the revenue allocations provided by an optimal dual solution to problem (15).

Regarding the benchmark policies LIN1, POL1, and LP1, which compute the coefficients of the value function approximations once at the beginning of the selling horizon, our results indicate that LIN1 consistently provides better performance than the other two policies. When we compute the coefficients of the value function approximations five times over the selling horizon, on average, the total expected revenues obtained by LIN5, POL5, and LP5 improve those obtained by LIN1, POL1, and LP1, respectively, by 2.86%, 2.66%, and 4.61%. The performance of LIN5 is still noticeably better than that of POL5 and LP5 on a majority of our test problems. When we use the rollout policies,



on average, the total expected revenues obtained by LINR, POLR, and LPR improve those obtained by LIN5, POL5, and LP5, respectively, by 2.30%, 13.08%, and 4.96%. Noting the performance of improvement of POLR over POL5, even if the static policy that we use is not a superior policy, performing rollout on the static policy can dramatically improve the performance of the static policy. Overall, irrespective of whether the static policy is obtained by explicitly constructing value function approximations or by directly using an optimal solution to a linear programming approximation, our rollout approach can be quite effective in obtaining even better policies. The performance of DEC lags behind that of all rollout policies, which is quite encouraging, because dynamic programming decomposition methods are known to provide the strongest benchmarks for revenue management problems. Thus, the rollout policies, such as LINR, can perform noticeably better than the strongest benchmark, while also providing a performance guarantee.

Another useful feature of rollout policies is that we can perform rollout on a collection of static policies. Letting  $\{\mu_k : k = 1, \dots, K\}$  be a collection of  $K$  static policies, we define the value function approximations  $\{\hat{J}_{\max}^q : q \in \mathcal{Q}\}$  as  $\hat{J}_{\max}^q(\mathbf{x}) = \max_{k=1, \dots, K} J_{\mu_k}^q(\mathbf{x})$ , where  $\{J_{\mu_k}^q : q \in \mathcal{Q}\}$  are the value functions of the static policy  $\mu^k$  computed through (2). In this case, the total expected revenue from the greedy policy with respect to the value function approximations  $\{\hat{J}_{\max}^q : q \in \mathcal{Q}\}$  is at least as large as the total expected revenue from each of the static policies  $\{\mu_k : k = 1, \dots, K\}$ ; see Example 2.3.2 in Bertsekas (2012). Thus, we can obtain a policy that is at least as good as each of the static policies. We performed such an ensemble rollout on the collection of resource based static policy, itinerary based static policy, and the static policy from problem (13). In our test problems, the performance of the ensemble rollout policy was statistically indistinguishable from the best rollout policy in Table 2 for each test problem.

We carry out our computational experiments in Java 1.8.0 with 16 GB of RAM and 2.8 GHz Intel Core i7 CPU. For our largest test problems with  $D_{\max} = 10$ , the average CPU time to compute the linear and polynomial value function approximations are, respectively, 16.12 and 30.04 seconds. The average CPU time to compute our upper bound is 4.02 hours. The average CPU time to perform rollout on a static policy is 5.12 minutes.

## 8.2 Results on a Boutique Hotel Dataset

We test the performance of our benchmark policies and upper bounds by using the reservation data from an actual boutique hotel.

**Experimental Setup:** We have access to the booking data from a boutique hotel for reservations made between May 13, 2020 and September 8, 2020. The bookings are for stays

from June 1, 2020 to October 31, 2020. The hotel has six unique rooms. The price of each room and night pair is pre-fixed, but the hotel changes the availability of rooms based on real-time information on booked capacities. In total, the dataset contains 157 bookings. With the exception of three bookings, all bookings are for one to eight nights. We dropped those three bookings in our estimation procedure. We estimate the parameters of our model as follows.

There are 119 days between May 13 and September 8, so the customers arrive over a selling horizon of 119 days. We divide each day into 10 time periods. In this case, the set of time periods is  $\mathcal{Q} = \{1, \dots, 1190\}$ . For each  $q \in \mathcal{Q}$ , we use  $\text{day}(q)$  to denote the calendar date corresponding to time period  $q$ . On the other hand, there are 153 days between June 1 and October 31, so we use  $\mathcal{T} = \{1, \dots, 153\}$  to index the nights of stay. For each  $\ell \in \mathcal{T}$ , we use  $\overline{\text{day}}(\ell)$  to denote the calendar date corresponding to night  $\ell$ . To estimate the arrival probabilities for the booking requests, we proceed with the following two assumptions. First, at each time period, there is a fixed probability that we have a request for a booking for a certain number of days into the future. Second, given that we have a booking request, there is a fixed probability that the booking is for a certain length of stay. In particular, we let  $\theta_k$  be the probability that we have a booking request for  $k$  days into the future. Given that we have a booking request, we let  $\eta_d$  be the probability that this booking request is for  $d$  nights. In this case, the probability that we have a booking request for interval  $[s, f]$  at time period  $q$  is given by  $\lambda_{[s,f]}^q = \theta_{\overline{\text{day}}(s) - \text{day}(q)} \times \eta_{f-s+1}$ . Because there are 171 days between May 13 and October 31, a customer can make a booking as many as 171 days in advance. Thus, we need to estimate the parameter  $\theta_k$  for all  $k = 1, \dots, 171$ . Noting that the dataset contains only 157 bookings, we divide the interval  $\{1, \dots, 171\}$  into the seven subintervals  $[1, 3]$ ,  $[4, 7]$ ,  $[8, 14]$ ,  $[15, 28]$ ,  $[29, 42]$ ,  $[43, 56]$ , and  $[57, 171]$ , assuming that the value of  $\theta_k$  is constant when  $k$  takes values in each of these intervals. When estimating the parameters  $\{\theta_k : k = 1, \dots, 171\}$ , we impose the constraint that  $\sum_{k=1}^{171} \theta_k \leq 1$ , in which case we have  $\sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \leq 1$  for all  $q \in \mathcal{Q}$ . We use the multinomial logit model to capture the choice process among rooms. Thus, letting  $v_i$  be the preference weight of room  $i$  and normalizing the preference weight of the no-purchase option to one, the choice probability of room  $i$  within the assortment  $S$  is  $\phi_i^q(S) = \frac{v_i}{1 + \sum_{j \in S} v_j}$ .

We use maximum likelihood estimation to estimate the parameters of our model. In Appendix E, we give the details of our estimation procedure.

**Computational Results:** To broaden our experimental setup, we scaled the arrival probabilities  $\{\lambda_{[s,f]}^q : [s, f] \in \mathcal{F}, q \in \mathcal{Q}\}$  by a constant to obtain test problems with load factors taking values in  $\{0.8, 1.2, 1.6, 2.0\}$ . We measure the load factor in the same way we do for our test problems on the synthetic datasets. We give our results in Table 3. In this table, the first column shows the load factor. The second column shows the percent gap between the upper bounds obtained by using

Load Fact.	Gap in Bounds	Total Exp. Revenue									
		LIN1	POL1	LP1	LIN5	POL5	LP5	LINR	POLR	LPR	DEC
0.8	12.25	81.50	83.69	81.41	82.59	83.33	82.05	83.96	84.43	83.00	82.10
1.2	10.10	85.76	87.73	85.69	87.34	87.75	86.69	88.18	88.70	87.16	86.43
1.6	8.15	87.19	89.35	87.47	88.16	89.44	87.58	90.18	89.69	88.85	87.40
2.0	6.65	88.90	90.82	88.79	89.96	90.85	88.78	91.03	91.22	90.53	88.80
Avg.	9.29	85.84	87.90	85.84	87.01	87.84	86.28	88.34	88.51	87.38	86.18

**Table 3** Computational results for the boutique hotel datasets.

the dynamic program in (14) and the linear programming approximation in (13). The remaining columns give the total expected revenues obtained by our benchmark policies, each expressed as a percentage of the upper bound on the optimal total expected revenue. Our results indicate that the upper bounds provided by our approach significantly improve those from the linear programming approximation. The gaps in the upper bounds can exceed 12%. When compared with our results on the synthetic datasets, LIN5, POL5, and LP5 obtain more modest improvements in the total expected revenues by recomputing the coefficients of the value function approximations five times over the selling horizon, instead of once. On the other hand, similar to our results on the synthetic datasets, the rollout policies, especially LINR and POLR, perform noticeably better than DEC. There are test problems in which the total expected revenue improvements over DEC provided by our rollout policies can reach 3.08%. Thus, our rollout approach continues to provide effective policies for these datasets as well.

## 9. Conclusion

We developed policies with performance guarantees by exploiting two features of the underlying problem. First, the capacity of each resource is one. Second, the resources can be ordered such that each itinerary uses an interval of resources. We hope our study inspires other research on leveraging features of the underlying network. In our model, a customer makes a booking request for an interval of stay, in response to which we offer an assortment of rooms. This model is consistent with the booking systems of many boutique hotels, but it would be interesting to develop a model that also allows customers to specify a window of possible stay, along with the desired number of nights. In this case, the decision would be assortments of rooms to offer over different possible intervals of stay. Our analysis of the itinerary based static policy goes through, but our efficient rollout result does not readily extend and more work is needed. Lastly, the performance guarantee of  $1/(1 + \|\mathcal{C}\|)$  for the itinerary based static policy reveals a relationship between policy performance and pattern of resource usage. It would be interesting to explore such relationships in other problems.

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## Online Appendix

### Revenue Management for Boutique Hotels: Resources with Unit Capacities and Itineraries over Intervals of Resources

#### Appendix A: Offering Itineraries only with Positive Contributions

We establish a lemma that we use throughout the paper to argue that some of our policies never offer an unavailable room. In the next lemma, for fixed  $\{p_i : i \in \mathcal{N}\}$ , we focus on the problem

$$\max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) p_i \right\}. \quad (18)$$

**Lemma A.1 (Offering Only Positive Contributions)** *There exists an optimal solution  $S^*$  to problem (18) such that  $S^* \subseteq \{i \in \mathcal{N} : p_i > 0\}$ .*

*Proof:* Letting  $S^*$  be an optimal solution to problem (18), we define the assortment  $\hat{S} \subseteq S^*$  as  $\hat{S} = \{i \in S^* : p_i > 0\}$ . By the assumption that  $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$  for all  $i \in S$  and  $j \notin S$ , we have  $\phi_i^q(S^*) \leq \phi_i^q(\hat{S})$  for all  $i \in \hat{S}$ . In this case, using the fact that  $p_i \leq 0$  for all  $i \in S^* \setminus \hat{S}$ , and  $p_i > 0$  for all  $i \in \hat{S}$ , we obtain the chain of inequalities

$$\sum_{i \in S^*} \phi_i^q(S^*) p_i = \sum_{i \in \hat{S}} \phi_i^q(S^*) p_i + \sum_{i \in S^* \setminus \hat{S}} \phi_i^q(S^*) p_i \leq \sum_{i \in \hat{S}} \phi_i^q(S^*) p_i \leq \sum_{i \in \hat{S}} \phi_i^q(\hat{S}) p_i.$$

Thus, the chain of inequalities above shows that  $\hat{S}$  is also an optimal solution to problem (18). Furthermore, we have  $\hat{S} \subseteq \{i \in \mathcal{N} : p_i > 0\}$ . ■

#### Appendix B: Performance Guarantee for Itinerary Based Static Policy

In this section, we give a proof for Theorem 6.1. We need two preliminary lemmas. The next lemma is the analogue of Lemma 5.2. Its proof is identical to that of Lemma 5.2 and omitted.

**Lemma B.1 (Nonnegative Contribution of Each Room)** *For each  $i \in \mathcal{N}$ ,  $[s, f] \in \mathcal{F}$  and  $q \in \mathcal{Q}$ , we have  $\phi_i^q(B_{[s,f]}^q) [r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} | [s, f] \cap C_{[a,b]} | \gamma_{i,[a,b]}^{q+1}] \geq 0$ .*

In the next lemma, we give an upper bound on the opportunity cost of the capacities used by accepting a request for interval  $[s, f]$  in room  $i$  under the polynomial approximations.

**Lemma B.2 (Upper Bound on Opportunity Cost)** *For each  $i \in \mathcal{N}$ ,  $[s, f] \in \mathcal{F}$  and  $\mathbf{x} \in \{0, 1\}^{n \times T}$  such that  $\prod_{\ell=s}^f x_{i,\ell} = 1$ , we have  $\hat{J}_p^q(\mathbf{x}) - \hat{J}_p^q(\mathbf{x} - \mathbf{e}_{i,[s,f]}) \leq \sum_{[a,b] \in \mathcal{F}} | [s, f] \cap C_{[a,b]} | \gamma_{i,[a,b]}^{q+1}$ .*



*Proof:* Using the previous lemma, by (11), we have  $\gamma_{i,[s,f]}^q \geq 0$  for all  $i \in \mathcal{N}$  and  $[s, f] \in \mathcal{F}$ . Using the fact that  $\hat{J}_p^q(\mathbf{x}) = \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \gamma_{i,[s,f]}^q \prod_{\ell=s}^f x_{i,\ell}$ , the difference  $\hat{J}_p^q(\mathbf{x}) - \hat{J}_p^q(\mathbf{x} - \mathbf{e}_{i,[s,f]})$  is

$$\begin{aligned} \hat{J}_p^q(\mathbf{x}) - \hat{J}_p^q(\mathbf{x} - \mathbf{e}_{i,[s,f]}) &= \sum_{[a,b] \in \mathcal{F}} \gamma_{i,[a,b]}^q \mathbf{1}_{\{[a,b] \cap [s,f] \neq \emptyset\}} \left( \prod_{\ell=a}^b x_{i,\ell} \right) \\ &\leq \sum_{[a,b] \in \mathcal{F}} \gamma_{i,[a,b]}^q \mathbf{1}_{\{[a,b] \cap [s,f] \neq \emptyset\}} \leq \sum_{[a,b] \in \mathcal{F}} |[s, f] \cap C_{[a,b]}| \gamma_{i,[a,b]}^q, \end{aligned}$$

where the last inequality holds because  $C_{[a,b]}$  is intersection preserving, so if  $[a, b] \cap [s, f] \neq \emptyset$ , then  $[s, f] \cap C_{[a,b]} \neq \emptyset$ . In this case, we have  $\mathbf{1}_{\{[a,b] \cap [s,f] \neq \emptyset\}} \leq |[s, f] \cap C_{[a,b]}|$ . ■

In the next proposition, we show that we can use the value function approximations  $\{\hat{J}_p^q : q \in \mathcal{Q}\}$  to come up with an upper bound on the optimal total expected revenue.

**Proposition B.3 (Upper Bound on Optimal Performance)** *Noting that the optimal total expected revenue is  $J^1(\mathbf{e})$ , we have  $J^1(\mathbf{e}) \leq (1 + \|\mathcal{C}\|) \hat{J}_p^1(\mathbf{e})$ .*

*Proof:* Letting  $\beta_{i,\ell}^q = \sum_{[a,b] \in \mathcal{F}} \mathbf{1}_{\{\ell \in C_{[a,b]}\}} \gamma_{i,[a,b]}^q$  for notational brevity, we define the linear value function approximations  $\{\hat{V}_p^q : q \in \mathcal{Q}\}$  as  $\hat{V}_p^q(\mathbf{x}) = \hat{J}_p^q(\mathbf{e}) + \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \beta_{i,\ell}^q x_{i,\ell}$ . We have

$$\begin{aligned} &\sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \hat{V}_p^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \hat{V}_p^{q+1}(\mathbf{x}) \right] \right\} \\ &= \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} - \sum_{h=s}^f \beta_{i,h}^{q+1} \right] \right\} \\ &\stackrel{(a)}{\leq} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left[ r_{i,[s,f]} - \sum_{h=s}^f \beta_{i,h}^{q+1} \right] \right\} \\ &\stackrel{(b)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S) \left[ r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} |C_{[a,b]} \cap [s, f]| \gamma_{i,[a,b]}^{q+1} \right] \right\} \\ &\stackrel{(c)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{i \in \mathcal{N}} \phi_i^q(B_{[s,f]}^q) \left[ r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} |C_{[a,b]} \cap [s, f]| \gamma_{i,[a,b]}^{q+1} \right] \\ &\stackrel{(d)}{=} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} (\gamma_{i,[s,f]}^q - \gamma_{i,[s,f]}^{q+1}) = \hat{J}_p^q(\mathbf{e}) - \hat{J}_p^{q+1}(\mathbf{e}) \\ &\stackrel{(e)}{\leq} \hat{J}_p^q(\mathbf{e}) - \hat{J}_p^{q+1}(\mathbf{e}) + \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} (\beta_{i,\ell}^q - \beta_{i,\ell}^{q+1}) x_{i,\ell} = \hat{V}_p^q(\mathbf{x}) - \hat{V}_p^{q+1}(\mathbf{x}). \end{aligned}$$

Here, (a) follows from the same argument that we use to obtain the inequality (a) in the proof of Proposition 5.4. In (b), we use the definition of  $\beta_{i,\ell}^q$  and use the interchange of sums  $\sum_{h=s}^f \sum_{[a,b] \in \mathcal{F}} \mathbf{1}_{\{h \in C_{[a,b]}\}} = \sum_{[a,b] \in \mathcal{F}} \sum_{h=s}^f \mathbf{1}_{\{h \in C_{[a,b]}\}} = \sum_{[a,b] \in \mathcal{F}} |C_{[a,b]} \cap [s, f]|$ . Also, (c) follows from (10), and (d) follows from (11). To see that (e) holds, note that Lemma B.1, along with (11), implies that  $\gamma_{i,[s,f]}^q \geq \gamma_{i,[s,f]}^{q+1}$ , in which case, we also have  $\beta_{i,\ell}^q \geq \beta_{i,\ell}^{q+1}$ . The chain of inequalities above shows

that the value function approximations  $\{\hat{V}_p^q : q \in \mathcal{Q}\}$  satisfy (8), in which case, by the discussion that follows (8), we have  $\hat{V}_p^q(\mathbf{x}) \geq J^q(\mathbf{x})$  for all  $\mathbf{x} \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ . Thus, we have

$$\begin{aligned} J^1(\mathbf{e}) &\leq \hat{V}_p^1(\mathbf{e}) = \hat{J}_p^1(\mathbf{e}) + \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \beta_{i,\ell}^1 = \hat{J}_p^1(\mathbf{e}) + \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{F}} \sum_{\ell \in \mathcal{T}} \mathbf{1}_{\{\ell \in C_{[a,b]}\}} \gamma_{i,[a,b]}^1 \\ &= \hat{J}_p^1(\mathbf{e}) + \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{F}} |C_{[a,b]}| \gamma_{i,[a,b]}^1 \leq \hat{J}_p^1(\mathbf{e}) + \|\mathcal{C}\| \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{F}} \gamma_{i,[a,b]}^1 = (1 + \|\mathcal{C}\|) \hat{J}_p^1(\mathbf{e}), \end{aligned}$$

where the last inequality holds because  $\|\mathcal{C}\| = \max_{[a,b] \in \mathcal{F}} |C_{[a,b]}|$ , and the last equality follows from the fact that  $\hat{J}_p^q(\mathbf{e}) = \sum_{i \in \mathcal{N}} \sum_{[a,b] \in \mathcal{F}} \gamma_{i,[a,b]}^q$ .  $\blacksquare$

Let  $U_p^q(\mathbf{x})$  be the total expected revenue obtained by the itinerary based static policy over time periods  $\{q, \dots, Q\}$  given that the state of the system at time period  $q$  is  $\mathbf{x}$ . We can compute the value functions  $\{U_p^q : q \in \mathcal{Q}\}$  through the dynamic program in (9) after replacing the ideal assortments  $\{A_{[s,f]}^q : [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  with  $\{B_{[s,f]}^q : [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$ . In the next proposition, we lower bound the performance of the itinerary based static policy.

**Proposition B.4 (Lower Bound on Policy Performance)** *Letting  $\{U_p^q : q \in \mathcal{Q}\}$  be the value functions of the itinerary based static policy, for each  $\mathbf{x} \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ ,  $U_p^q(\mathbf{x}) \geq \hat{J}_p^q(\mathbf{x})$ .*

*Proof:* We show the result by using induction over the time periods. At time period  $Q+1$ , since we have  $U_p^{Q+1} = 0 = \hat{J}_p^{Q+1}$ , the inequality holds at time period  $Q+1$ . Assuming that the inequality holds at time period  $q+1$ , we show that the inequality holds at time period  $q$  as well. By the induction hypothesis,  $U_p^{q+1}(\mathbf{x})$  and  $U_p^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]})$  are, respectively, lower bounded by  $\hat{J}_p^{q+1}(\mathbf{x})$  and  $\hat{J}_p^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]})$ , so by (9), we get

$$\begin{aligned} U_p^q(\mathbf{x}) &\geq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(B_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} + \hat{J}_p^{q+1}(\mathbf{x} - \mathbf{e}_{i,[s,f]}) - \hat{J}_p^{q+1}(\mathbf{x}) \right] \right\} + \hat{J}_p^{q+1}(\mathbf{x}) \\ &\stackrel{(a)}{\geq} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(B_{[s,f]}^q) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \left[ r_{i,[s,f]} - \sum_{[a,b] \in \mathcal{F}} | [s,f] \cap C_{[a,b]} | \gamma_{i,[a,b]}^{q+1} \right] \right\} + \hat{J}_p^{q+1}(\mathbf{x}) \\ &\stackrel{(b)}{=} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} (\gamma_{i,[s,f]}^q - \gamma_{i,[s,f]}^{q+1}) \left( \prod_{\ell=s}^f x_{i,\ell} \right) + \hat{J}_p^{q+1}(\mathbf{x}) \stackrel{(c)}{=} [\hat{J}_p^q(\mathbf{x}) - \hat{J}_p^{q+1}(\mathbf{x})] + \hat{J}_p^{q+1}(\mathbf{x}) = \hat{J}_p^q(\mathbf{x}), \end{aligned}$$

where (a) follows from Lemma B.2, (b) holds by (11) and (c) holds by the definition of  $\hat{J}_p^q(\mathbf{x})$ . The chain of inequalities above establishes the result.  $\blacksquare$

We can use Propositions B.3 and B.4 to give a proof for Theorem 6.1.

### **Proof of Theorem 6.1:**

By Proposition B.3, we have  $\hat{J}_p^1(\mathbf{e}) \geq \frac{1}{1+\|\mathcal{C}\|} J^1(\mathbf{e})$ . Using Proposition B.4 with  $\mathbf{x} = \mathbf{e}$  and  $q = 1$ , we have  $U_p^1(\mathbf{e}) \geq \hat{J}_p^1(\mathbf{e})$ . So, we get  $U_p^1(\mathbf{e}) \geq \hat{J}_p^1(\mathbf{e}) \geq \frac{1}{1+\|\mathcal{C}\|} J^1(\mathbf{e})$ .  $\blacksquare$

### Appendix C: Norm of a Collection of Intersection Preserving Subsets

In this section, we give a proof for Theorem 6.2. To see that the first statement holds, we construct a feasible solution to problem (12) that provides an objective value of  $1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$ . In particular, we set  $\hat{t} = 1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$ . Furthermore, for each  $[s, f] \in \mathcal{F}$ , we set

$$\hat{C}_{[s,f]} = \left\{ s + k D_{\min} : k = 0, 1, 2, \dots, \left\lceil \frac{f-s}{D_{\min}} \right\rceil - 1 \right\} \cup \{f\}. \quad (19)$$

The solution  $\{\hat{C}_{[s,f]} : [s, f] \in \mathcal{F}\}$  and  $\hat{t}$  provides an objective value of  $\hat{t} = 1 + \lceil (D_{\max} - 1)/D_{\min} \rceil$  for problem (12). We proceed to arguing that this solution is feasible for problem (12). Using the fact that  $f - s + 1 \leq D_{\max}$  for all  $[s, f] \in \mathcal{F}$ , we have  $|\hat{C}_{[s,f]}| \leq \left\lceil \frac{f-s}{D_{\min}} \right\rceil + 1 \leq \left\lceil \frac{D_{\max}-1}{D_{\min}} \right\rceil + 1 = \hat{t}$ . Thus, the first constraint is satisfied. The smallest element of  $\hat{C}_{[s,f]}$  is  $s$ . Noting that  $s + \left( \left\lceil \frac{f-s}{D_{\min}} \right\rceil - 1 \right) D_{\min} \leq s + \frac{f-s}{D_{\min}} D_{\min} = f$ , none of the elements of  $\hat{C}_{[s,f]}$  exceeds  $f$ , so  $\hat{C}_{[s,f]} \subseteq [s, f]$ . Thus, the third constraint is satisfied. To check the second constraint, consider any  $[a, b] \in \mathcal{F}$  such that  $[s, f] \cap [a, b] \neq \emptyset$ . We show that  $\hat{C}_{[s,f]} \cap [a, b] \neq \emptyset$ . We consider three cases.

First, consider the case  $[s, f] \subseteq [a, b]$ . Because  $\hat{C}_{[s,f]} \subseteq [s, f]$  by the earlier discussion in the proof, we get  $\hat{C}_{[s,f]} \subseteq [s, f] \subseteq [a, b]$ , so  $\hat{C}_{[s,f]} \cap [a, b] \neq \emptyset$ , as desired.

Second, consider the case  $[s, f] \not\subseteq [a, b]$  and  $[s, f] \not\supseteq [a, b]$ . Because  $[s, f] \cap [a, b] \neq \emptyset$ , we must have  $f \in [a, b]$  or  $s \in [a, b]$ . Noting that  $s \in \hat{C}_{[s,f]}$  and  $f \in \hat{C}_{[s,f]}$ , we get  $\hat{C}_{[s,f]} \cap [a, b] \neq \emptyset$ , as desired.

Third, consider the case  $[s, f] \supseteq [a, b]$ . To get a contradiction, suppose, on the contrary, that  $\hat{C}_{[s,f]} \cap [a, b] = \emptyset$ . Since  $[s, f] \supseteq [a, b]$ , we have  $s \leq a \leq b \leq f$ , but noting that  $C_{[s,f]} \cap [a, b] = \emptyset$  and  $s, f \in \hat{C}_{[s,f]}$ , there are two successive nights  $c_1, c_2$  in  $\hat{C}_{[s,f]}$  such that  $c_1 < a \leq b < c_2$ . Thus, there are at least  $b - a + 1$  nights in between nights  $c_1$  and  $c_2$ . Because  $b - a + 1 \geq D_{\min}$ , there must be at least  $D_{\min}$  nights in between nights  $c_1$  and  $c_2$ . On the other hand, by our construction of  $\hat{C}_{[s,f]}$  in (19), there are at most  $D_{\min} - 1$  nights in between two successive nights in  $\hat{C}_{[s,f]}$ , which is a contradiction! Therefore, the first statement in the theorem holds.

To see that the second statement holds, we write the second constraint in problem (12) as  $|C_{[s,f]} \cap [a, b]| \geq \mathbb{1}_{\{[s,f] \cap [a,b] \neq \emptyset\}}$  for all  $[s, f], [a, b] \in \mathcal{F}$ . Define the constant  $Z_{[s,f]}^*$  as

$$Z_{[s,f]}^* = \min \left\{ |C_{[s,f]}| : |C_{[s,f]} \cap [a, b]| \geq \mathbb{1}_{\{[s,f] \cap [a,b] \neq \emptyset\}} \quad \forall [a, b] \in \mathcal{F}, C_{[s,f]} \subseteq [s, f] \right\}, \quad (20)$$

where the only decision variable is  $C_{[s,f]}$ . In this case, problem (12) becomes equivalent to  $\min\{t : t \geq Z_{[s,f]}^* \quad \forall [s, f] \in \mathcal{F}\}$ , which has the optimal objective value  $\max_{[s,f] \in \mathcal{F}} Z_{[s,f]}^*$ .

In the rest of the proof, we will show that we can compute  $Z_{[s,f]}^*$  in (20) by solving a minimization linear program with  $O(D_{\max})$  decision variables and  $O(D_{\max}^2)$  constraints. Thus, letting  $\chi^*$  be the

optimal objective value of problem (12), by the discussion in the previous paragraph, we have  $\chi^* = \max_{[s,f] \in \mathcal{F}} Z_{[s,f]}^*$ , so  $\chi^*$  is the maximum of the optimal objective values of  $|\mathcal{F}| = O(D_{\max} T)$  minimization linear programs, each having  $O(D_{\max})$  decision variables and  $O(D_{\max}^2)$  constraints. In this case, it immediately follows that we can compute the maximum of the optimal objective values of these linear programs by solving a single linear program with  $O(|\mathcal{F}| D_{\max}) = O(D_{\max}^2 T)$  decision variables and  $O(|\mathcal{F}| D_{\max}^2) = O(D_{\max}^3 T)$  constraints. To compute  $Z_{[s,f]}^*$  in (20) by solving a linear program, we use the decision variables  $\{x_\ell : \ell = s, \dots, f\} \in \{0, 1\}^{f-s+1}$ , where  $x_\ell = 1$  if and only if night  $\ell$  is included in the intersection preserving subset  $C_{[s,f]}$ . We write problem (20) as

$$\begin{aligned} \min \quad & \sum_{\ell=s}^f x_\ell \\ \text{st} \quad & \sum_{\ell=s}^f \mathbf{1}_{\{\ell \in [a,b]\}} x_\ell \geq \mathbf{1}_{\{[s,f] \cap [a,b] \neq \emptyset\}} \quad \forall [a,b] \in \mathcal{F} \\ & x_\ell \in \{0, 1\} \quad \forall \ell = s, \dots, f. \end{aligned}$$

Each row of the constraint matrix above includes only consecutive ones. Such a matrix is called an interval matrix and it is totally unimodular; see Corollary 2.10 in Chapter III.1 in Nemhauser and Wolsey (1988). Thus, we can relax the integrality requirements without an integrality gap. Also, the problem above has a covering constraint and the right side of the constraint never exceeds one, which implies that even if we did not have an upper bound of one on the decision variables, these decision variables would never take a value greater than one. Thus, we can drop the constraints  $x_\ell \leq 1$  for all  $\ell = s, \dots, f$ . Lastly, the right side of the constraint is nonzero for all  $[a,b] \in \mathcal{F}$  such that  $[s,f] \cap [a,b] \neq \emptyset$  and there are only  $O(D_{\max}^2)$  such constraints. Thus, the problem above actually has  $f - s + 1 = O(D_{\max})$  decision variables and  $O(D_{\max}^2)$  constraints.

## Appendix D: Upper Bound on the Optimal Policy Performance

In this section, we give the proofs of the three results in Section 7, along with Lemma D.1 that we use in that section. Here is the proof of Proposition 7.1.

### Proof of Proposition 7.1:

We will use a simple manipulation in the proof. In particular, for fixed values  $\{\varphi_i : i \in \mathcal{N}\}$ , we have the chain of equalities

$$\sum_{j \in \mathcal{N}} \varphi_j \sum_{i \in \mathcal{N} \setminus \{j\}} \beta_{j,[s,f] \rightarrow i}^q = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \varphi_j \beta_{j,[s,f] \rightarrow i}^q = \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N} \setminus \{j\}} \varphi_i \beta_{i,[s,f] \rightarrow j}^q, \quad (21)$$

where the first equality holds by interchanging the order of sums and the second equality holds by interchanging the roles of  $i$  and  $j$ . We show the proposition by using induction over the time

periods. At time period  $Q + 1$ , we have  $J^{Q+1} = 0 = \sum_{j \in \mathcal{N}} V_{\beta,j}^{Q+1}$ , so the result holds at time period  $Q + 1$ . Assuming that the result holds at time period  $q + 1$ , we show that the result holds at time period  $q$  as well. Fixing  $\mathbf{x} \in \{0, 1\}^{n \times T}$  and  $q \in \mathcal{Q}$ , let  $S_{[s,f]}^*$  be an optimal solution to the maximization problem on the right side of (1). By the induction hypothesis,  $J^{q+1}(\mathbf{x} - \mathbf{e}_{j,[s,f]}) \leq \sum_{i \in \mathcal{N} \setminus \{j\}} V_{\beta,i}^{q+1}(\mathbf{x}_i) + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]})$  and  $J^{q+1}(\mathbf{x}) \leq \sum_{j \in \mathcal{N}} V_{\beta,j}^{q+1}(\mathbf{x}_j)$ . Using (1), we get

$$\begin{aligned}
J^q(\mathbf{x}) &= \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{j \in \mathcal{N}} \phi_j^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ r_{j,[s,f]} + J^{q+1}(\mathbf{x} - \mathbf{e}_{j,[s,f]}) \right] + J^{q+1}(\mathbf{x}) \\
&\stackrel{(a)}{\leq} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sum_{j \in \mathcal{N}} \phi_j^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ r_{j,[s,f]} + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) \right] + \sum_{j \in \mathcal{N}} V_{\beta,j}^{q+1}(\mathbf{x}_j) \\
&\stackrel{(b)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{j \in \mathcal{N}} \phi_j^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] \right. \\
&\quad \left. + \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \beta_{i,[s,f] \rightarrow j}^q \right\} + \sum_{j \in \mathcal{N}} V_{\beta,j}^{q+1}(\mathbf{x}_j) \\
&\stackrel{(c)}{=} \sum_{j \in \mathcal{N}} \left\{ \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \phi_j^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] \right. \right. \\
&\quad \left. \left. + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S_{[s,f]}^*) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \beta_{i,[s,f] \rightarrow j}^q \right\} + V_{\beta,j}^{q+1}(\mathbf{x}_j) \right\} \\
&\leq \sum_{j \in \mathcal{N}} \left\{ \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] \right. \right. \\
&\quad \left. \left. + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \left( \prod_{\ell=s}^f x_{i,\ell} \right) \beta_{i,[s,f] \rightarrow j}^q \right\} + V_{\beta,j}^{q+1}(\mathbf{x}_j) \right\} \\
&\stackrel{(d)}{\leq} \sum_{j \in \mathcal{N}} \left\{ \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] \right. \right. \\
&\quad \left. \left. + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} + V_{\beta,j}^{q+1}(\mathbf{x}_j) \right\} \\
&\stackrel{(e)}{=} \sum_{j \in \mathcal{N}} V_{\beta,j}^q(\mathbf{x}_j).
\end{aligned}$$

In the chain of inequalities above, (a) uses the induction hypothesis. To see that (b) holds, we note that  $r_{j,[s,f]} = \sum_{i \in \mathcal{N}} \beta_{j,[s,f] \rightarrow i}^q = \beta_{j,[s,f] \rightarrow j}^q + \sum_{i \in \mathcal{N} \setminus \{j\}} \beta_{j,[s,f] \rightarrow i}^q$  and use (21) after identifying  $\varphi_j$  with  $\phi_j^q(S_{[s,f]}^*) \prod_{\ell=s}^f x_{j,\ell}$ . Also, (c) holds by rearranging the order of the sums. To get (d), we use the same argument that we use to obtain inequality (a) in the proof of Proposition 5.4. Lastly, (e) follows from (14). The chain of inequalities above completes the induction argument.  $\blacksquare$

Next, we state and prove Lemma D.1.

**Lemma D.1 (Equivalence of Linear Programs)** *Problems (13) and (15) have the same optimal objective value.*

*Proof:* We let  $\widehat{\mathbf{h}} = \left\{ \widehat{h}_{[s,f]}^q(S) : S \subseteq \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q} \right\}$  and  $\widehat{\mathbf{y}} = \{ \widehat{y}_{i,[s,f]} : i \in \mathcal{N}, [s,f] \in \mathcal{F} \}$  be an optimal solution to problem (13). For each  $j \in \mathcal{N}$ , we set  $\widetilde{h}_{[s,f] \rightarrow j}^q(S) = \widehat{h}_{[s,f]}^q(S)$ . In this case, we observe that the solution  $\widetilde{\mathbf{h}} = \left\{ \widetilde{h}_{[s,f] \rightarrow j}^q(S) : j \in \mathcal{N}, S \subseteq \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q} \right\}$  and  $\widehat{\mathbf{y}}$  is feasible to problem (15) and provides the same objective value as the optimal objective value of problem (13). Therefore, the optimal objective value of problem (15) is at least as large as that of problem (13). In the rest of the proof, we show that the reverse inequality also holds. We let  $\widetilde{\mathbf{h}} = \left\{ \widetilde{h}_{[s,f] \rightarrow j}^q(S) : j \in \mathcal{N}, S \subseteq \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q} \right\}$  and  $\widetilde{\mathbf{y}} = \{ \widetilde{y}_{i,[s,f]} : i \in \mathcal{N}, [s,f] \in \mathcal{F} \}$  be an optimal solution to problem (15). We define  $\widehat{h}_{[s,f]}^q(S)$  as

$$\widehat{h}_{[s,f]}^q(S) = \frac{1}{n} \sum_{j \in \mathcal{N}} \widetilde{h}_{[s,f] \rightarrow j}^q(S).$$

We establish that the solution  $\widehat{\mathbf{h}} = \{ \widehat{h}_{[s,f]}^q(S) : S \subseteq \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q} \}$  and  $\widetilde{\mathbf{y}}$  is feasible to problem (13). Using the definition of  $\widehat{h}_{[s,f]}^q(S)$  above, we have

$$\sum_{S \subseteq \mathcal{N}} \widehat{h}_{[s,f]}^q(S) = \frac{1}{n} \sum_{j \in \mathcal{N}} \sum_{S \subseteq \mathcal{N}} \widetilde{h}_{[s,f] \rightarrow j}^q(S) \stackrel{(a)}{=} \frac{1}{n} \sum_{j \in \mathcal{N}} \lambda_{[s,f]}^q = \lambda_{[s,f]}^q,$$

where (a) holds because the solution  $(\widetilde{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the second constraint in problem (15). Thus, the solution  $(\widehat{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the second constraint in problem (13).

To check that the solution  $(\widehat{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the third constraint in problem (13), using the definition of  $\widehat{h}_{[s,f]}^q(S)$  once more, we have

$$\sum_{S \subseteq \mathcal{N}} \phi_i^q(S) \widehat{h}_{[s,f]}^q(S) = \frac{1}{n} \sum_{j \in \mathcal{N}} \sum_{S \subseteq \mathcal{N}} \phi_i^q(S) \widetilde{h}_{[s,f] \rightarrow j}^q(S) \stackrel{(b)}{=} \frac{1}{n} \sum_{j \in \mathcal{N}} \widetilde{y}_{i,[s,f]}^q = \widetilde{y}_{i,[s,f]}^q, \quad (22)$$

where (b) holds because the solution  $(\widetilde{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the third constraint in problem (15). Thus, the solution  $(\widehat{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the third constraint in problem (13).

Lastly, we check that the solution  $(\widehat{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the first constraint in problem (13). In particular, we have the chain of inequalities

$$\begin{aligned} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \sum_{S \subseteq \mathcal{N}} \mathbf{1}_{\{\ell \in [s,f]\}} \phi_i^q(S) \widehat{h}_{[s,f]}^q(S) &= \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{\ell \in [s,f]\}} \sum_{S \subseteq \mathcal{N}} \phi_i^q(S) \widehat{h}_{[s,f]}^q(S) \\ &\stackrel{(c)}{=} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{\ell \in [s,f]\}} \widetilde{y}_{i,[s,f]}^q \stackrel{(d)}{=} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{\ell \in [s,f]\}} \sum_{S \subseteq \mathcal{N}} \phi_i^q(S) \widetilde{h}_{[s,f] \rightarrow i}^q(S) \stackrel{(e)}{\leq} 1, \end{aligned}$$

where (c) follows from (22), (d) follows from the fact that the solution  $(\widetilde{\mathbf{h}}, \widetilde{\mathbf{y}})$  satisfies the third constraint in problem (15) with  $i = j$ , and (e) holds because the solution  $(\widetilde{\mathbf{h}}, \widetilde{\mathbf{y}})$  also satisfies the first

constraint in problem (15). Therefore, the solution  $(\widehat{\mathbf{h}}, \widehat{\mathbf{y}})$  satisfies the first constraint in problem (13). Thus, the solution  $(\widehat{\mathbf{h}}, \widehat{\mathbf{y}})$  is feasible to problem (13). The solution  $(\widehat{\mathbf{h}}, \widehat{\mathbf{y}})$  provides the objective value  $\sum_{q \in \mathcal{Q}} \sum_{i \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} r_{i,[s,f]}^q \widehat{y}_{i,[s,f]}^q$ , which is the optimal objective value of problem (15). So, the optimal objective value of problem (13) is at least as large as that of problem (15). ■

To write the dual of problem (15), associating the dual variables  $\boldsymbol{\mu} = \{\mu_{i,\ell} : i \in \mathcal{N}, \ell \in \mathcal{T}\}$ ,  $\boldsymbol{\sigma} = \{\sigma_{[s,f] \rightarrow j}^q : j \in \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$  and  $\boldsymbol{\beta} = \{\beta_{i,[s,f] \rightarrow j}^q : i, j \in \mathcal{N}, [s,f] \in \mathcal{F}, q \in \mathcal{Q}\}$ , we have

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \mu_{i,\ell} + \sum_{q \in \mathcal{Q}} \sum_{j \in \mathcal{N}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \sigma_{[s,f] \rightarrow j}^q \quad (23) \\ \text{st} \quad & \sigma_{[s,f] \rightarrow j}^q \geq \sum_{i \in \mathcal{N}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q - \phi_j^q(S) \sum_{\ell=s}^f \mu_{j,\ell} \quad \forall j \in \mathcal{N}, [s,f] \in \mathcal{F}, S \subseteq \mathcal{N}, q \in \mathcal{Q} \\ & \sum_{j \in \mathcal{N}} \beta_{i,[s,f] \rightarrow j}^q = r_{i,[s,f]} \quad \forall i \in \mathcal{N}, [s,f] \in \mathcal{F} \\ & \boldsymbol{\mu} \geq \mathbf{0}, \boldsymbol{\sigma}, \boldsymbol{\beta} \text{ free.} \end{aligned}$$

It is simple to check that problem (15) is feasible and bounded, so by strong duality, problem (23) also has the optimal objective value  $Z_{\text{LP}}^*$ . We use problem (23) to give a proof for Theorem 7.2.

### **Proof of Theorem 7.2:**

Let  $(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\beta}})$  be an optimal solution to problem (23). For notational brevity, also letting  $\widehat{\alpha}_j^q = \sum_{k=q}^Q \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^k \widehat{\sigma}_{[s,f] \rightarrow j}^k$ , we will use induction over the time periods to establish that  $V_{\widehat{\boldsymbol{\beta}},j}^q(\mathbf{x}_j) \leq \widehat{\alpha}_j^q + \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} x_{j,\ell}$  for all  $\mathbf{x}_j \in \{0, 1\}^T$ ,  $j \in \mathcal{N}$  and  $q \in \mathcal{Q}$ . In this case, using this result with  $q = 1$  and  $\mathbf{x}_j = \mathbf{e}'$ , we obtain  $\sum_{j \in \mathcal{N}} V_{\widehat{\boldsymbol{\beta}},j}^1(\mathbf{e}') \leq \sum_{j \in \mathcal{N}} \widehat{\alpha}_j^1 + \sum_{j \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} = Z_{\text{LP}}^*$ , where the equality uses the fact that  $\sum_{j \in \mathcal{N}} \widehat{\alpha}_j^1 + \sum_{j \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} = \sum_{j \in \mathcal{N}} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \widehat{\sigma}_{[s,f] \rightarrow j}^q + \sum_{j \in \mathcal{N}} \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell}$  and the last quantity is the optimal objective value of problem (23), which is  $Z_{\text{LP}}^*$ . Thus, the desired result follows. In the rest of the proof, we focus on the induction argument. Noting that  $\widehat{\boldsymbol{\mu}} \geq \mathbf{0}$ , for each  $\mathbf{x}_j \in \{0, 1\}^T$ , at time period  $Q + 1$ , we have  $V_{\widehat{\boldsymbol{\beta}},j}^{Q+1}(\mathbf{x}_j) = 0 \leq \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} x_{j,\ell}$ , so the result holds at time period  $Q + 1$ . Assuming that the result holds at time period  $q + 1$ , we show that the result holds at time period  $q$  as well. By the induction hypothesis  $V_{\widehat{\boldsymbol{\beta}},j}^{q+1}(\mathbf{x}_j)$  and  $V_{\widehat{\boldsymbol{\beta}},j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]})$  are upper bounded by  $\widehat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} x_{j,\ell}$  and  $\widehat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} x_{j,\ell} - \sum_{\ell=s}^f \widehat{\mu}_{j,\ell}$ . So, by (14),

$$\begin{aligned} V_{\widehat{\boldsymbol{\beta}},j}^q(\mathbf{x}_j) \leq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{\ell=s}^f x_{j,\ell} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q - \sum_{\ell=s}^f \widehat{\mu}_{j,\ell} \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\ + \widehat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \widehat{\mu}_{j,\ell} x_{j,\ell}. \end{aligned}$$

Following the same argument that we used to obtain the inequality (a) in the proof of Proposition 5.4, we can drop the product  $\prod_{\ell=s}^f x_{j,\ell}$  on the right side above to make the right side of

the inequality even larger. After dropping the product  $\prod_{\ell=s}^f x_{j,\ell}$ , let  $S_{[s,f] \rightarrow j}^*$  be an optimal solution to the resulting maximization problem. In this case, using the inequality above, we get

$$\begin{aligned}
V_{\hat{\beta},j}^q(\mathbf{x}_j) &\leq \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left[ \beta_{j,[s,f] \rightarrow j}^q - \sum_{\ell=s}^f \hat{\mu}_{j,\ell} \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\
&\quad + \hat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \hat{\mu}_{j,\ell} x_{j,\ell} \\
&\stackrel{(a)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \phi_j^q(S_{[s,f] \rightarrow j}^*) \left[ \beta_{j,[s,f] \rightarrow j}^q - \sum_{\ell=s}^f \hat{\mu}_{j,\ell} \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S_{[s,f] \rightarrow j}^*) \beta_{i,[s,f] \rightarrow j}^q \right\} + \hat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \hat{\mu}_{j,\ell} x_{j,\ell} \\
&\stackrel{(b)}{=} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S_{[s,f] \rightarrow j}^*) \beta_{i,[s,f] \rightarrow j}^q - \phi_j^q(S_{[s,f] \rightarrow j}^*) \sum_{\ell=s}^f \hat{\mu}_{j,\ell} \right\} + \hat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \hat{\mu}_{j,\ell} x_{j,\ell} \\
&\stackrel{(c)}{\leq} \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \hat{\sigma}_{[s,f] \rightarrow j}^q + \hat{\alpha}_j^{q+1} + \sum_{\ell \in \mathcal{T}} \hat{\mu}_{j,\ell} x_{j,\ell} \stackrel{(d)}{=} \hat{\alpha}_j^q + \sum_{\ell \in \mathcal{T}} \hat{\mu}_{j,\ell} x_{j,\ell},
\end{aligned}$$

where (a) holds because  $S_{[s,f] \rightarrow j}^*$  is an optimal solution to the maximization problem on the left side of (a), (b) follows by arranging terms, (c) holds because  $(\hat{\mu}, \hat{\sigma}, \hat{\beta})$  satisfies the first constraint in (23), and (d) follows because  $\hat{\alpha}_j^q = \hat{\alpha}_j^{q+1} + \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \hat{\sigma}_{[s,f] \rightarrow j}^q$ . This completes the induction. ■

We focus on the proof of Theorem 7.3. To give an alternative representation of the value functions  $\{\Gamma_{\beta,j}^q : q \in \mathcal{Q}\}$  in (16), for each  $\ell \in \mathcal{T}$ , define the value function

$$\Psi_{\beta,j}^q(\ell) = \sum_{k=q}^Q \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^k \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^k(S) \beta_{i,[s,f] \rightarrow j}^k \right\}. \quad (24)$$

Directly by comparing (24) with (16), observe that the value functions  $\{\Gamma_{\beta,j}^q : q \in \mathcal{Q}\}$  and  $\{\Psi_{\beta,j}^q : q \in \mathcal{Q}\}$  satisfy the relationship  $\Gamma_{\beta,j}^q(a,b) = \sum_{\ell=a}^b \Psi_{\beta,j}^q(\ell)$  for each interval  $[a,b]$ . Given that the state of room  $j$  is  $\mathbf{x}_j \in \{0,1\}^T$ , let  $\mathcal{K}(\mathbf{x}_j)$  be the set of unavailable nights in this room; that is  $\ell \in \mathcal{K}(\mathbf{x}_j)$  if and only if  $x_{j,\ell} = 0$ . Note that  $\mathcal{K}(\mathbf{x}_j)$  is the union of the maximal unavailable intervals with respect to  $\mathbf{x}_j$ . In other words, we have  $\mathcal{K}(\mathbf{x}_j) = \cup_{[a,b] \in \mathcal{H}(\mathbf{x}_j)} [a,b]$ . In this case, by the discussion earlier in this paragraph, we obtain the identity

$$\sum_{[a,b] \in \mathcal{H}(\mathbf{x}_j)} \Gamma_{\beta,j}^q(a,b) = \sum_{[a,b] \in \mathcal{H}(\mathbf{x}_j)} \sum_{\ell \in [a,b]} \Psi_{\beta,j}^q(\ell) = \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^q(\ell),$$

where the last equality holds since  $\mathcal{K}(\mathbf{x}_j) = \cup_{[a,b] \in \mathcal{H}(\mathbf{x}_j)} [a,b]$ . Thus, to establish Theorem 7.3, it is enough to show that  $V_{\beta,j}^q(\mathbf{x}_j) = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^q(a,b) + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^q(\ell)$ .

### **Proof of Theorem 7.3:**

We will use the induction argument over the time periods to establish that  $V_{\beta,j}^q(\mathbf{x}_j) = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^q(a,b) + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^q(\ell)$  for all  $\mathbf{x}_j \in \{0,1\}^T$  and  $q \in \mathcal{Q}$ . At time period



$Q + 1$ , we have  $V_{\beta,j}^{Q+1} = \Theta_{\beta,j}^{Q+1} = \Psi_{\beta,j}^{Q+1} = 0$ , so the result holds at time period  $Q + 1$ . Assuming that the result holds at time period  $q + 1$ , we show that the result holds at time period  $q$  as well. For each  $\mathbf{x}_j \in \{0, 1\}^T$ , the collection of maximal available intervals  $\mathcal{I}(\mathbf{x}_j)$  and the set of unavailable nights  $\mathcal{K}(\mathbf{x}_j)$  collectively cover  $\mathcal{T}$ ; that is,  $\mathcal{T} = (\cup_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} [a, b]) \cup \mathcal{K}(\mathbf{x}_j)$ . Thus, for each  $s \in \mathcal{T}$ , we have  $\sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \mathbf{1}_{\{s \in [a,b]\}} + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \mathbf{1}_{\{s=\ell\}} = 1$ . In this case, by (14), we get

$$\begin{aligned}
V_{\beta,j}^q(\mathbf{x}_j) &= V_{\beta,j}^{q+1}(\mathbf{x}_j) + \sum_{[s,f] \in \mathcal{F}} \lambda_{[s,f]}^q \left( \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \mathbf{1}_{\{s \in [a,b]\}} + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \mathbf{1}_{\{s=\ell\}} \right) \times \\
&\quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\
&= \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \left( \Theta_{\beta,j}^{q+1}(a, b) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a,b]\}} \lambda_{[s,f]}^q \times \right. \\
&\quad \left. \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \right) \\
&+ \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \left( \Psi_{\beta,j}^{q+1}(\ell) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^q \times \right. \\
&\quad \left. \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \right),
\end{aligned}$$

where the second equality follows by using the induction hypothesis to note that  $V_{\beta,j}^{q+1}(\mathbf{x}_j) = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^{q+1}(a, b) + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^{q+1}(\ell)$  and rearranging the order of sums.

Noting the right side of the chain of equalities above, in the rest of the proof, for each  $[a, b] \in \mathcal{I}(\mathbf{x}_j)$  and  $\ell \in \mathcal{K}(\mathbf{x}_j)$ , we will establish the following two equalities

$$\begin{aligned}
\Theta_{\beta,j}^q(a, b) &= \Theta_{\beta,j}^{q+1}([a, b]) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a,b]\}} \lambda_{[s,f]}^q \times \\
&\quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
\Psi_{\beta,j}^q(\ell) &= \Psi_{\beta,j}^{q+1}(\ell) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^q \times \\
&\quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\}. \quad (26)
\end{aligned}$$

Then, by the first displayed equality in the proof,  $V_{\beta,j}^q(\mathbf{x}_j) = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^q(a, b) + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^q(\ell)$ , completing the induction argument.

First, we show that (25) holds for each  $[a, b] \in \mathcal{I}(\mathbf{x}_j)$ . So, throughout this portion of the discussion, we focus on the maximal available intervals  $[a, b] \in \mathcal{I}(\mathbf{x}_j)$ . In (25), we consider  $s \in [a, b]$ . Note that if

$s \in [a, b]$  and  $f \in [a, b]$ , then because  $[a, b] \in \mathcal{I}(\mathbf{x}_j)$ , we get  $x_{j,h} = 1$  for all  $h = s, \dots, f$ , so  $\prod_{h=s}^f x_{j,h} = 1$ . Furthermore, because  $s \in [a, b]$ , we have  $f \in [a, b]$  if and only if  $[s, f] \subseteq [a, b]$ . On the other hand, if  $s \in [a, b]$  and  $f \notin [a, b]$ , then since the interval  $[a, b]$  is a maximal available interval, there must be some  $h = s + 1, \dots, f$  such that  $x_{j,h} = 0$ , so  $\prod_{h=s}^f x_{j,h} = 0$ . Furthermore, because  $s \in [a, b]$ , we have  $f \notin [a, b]$  if and only if  $[s, f] \not\subseteq [a, b]$ . Thus, using  $\mathbf{1}_{\{s \in [a, b]\}} = \mathbf{1}_{\{s \in [a, b], [s, f] \subseteq [a, b]\}} + \mathbf{1}_{\{s \in [a, b], [s, f] \not\subseteq [a, b]\}}$ , we equivalently express the right side of (25) as

$$\begin{aligned}
& \Theta_{\beta,j}^{q+1}(a, b) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a, b], [s, f] \subseteq [a, b]\}} \lambda_{[s,f]}^q \times \\
& \quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\
& \quad + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a, b], [s, f] \not\subseteq [a, b]\}} \lambda_{[s,f]}^q \times \\
& \quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\
& \stackrel{(a)}{=} \Theta_{\beta,j}^{q+1}(a, b) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a, b], [s, f] \subseteq [a, b]\}} \lambda_{[s,f]}^q \times \\
& \quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\
& \quad + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a, b], [s, f] \not\subseteq [a, b]\}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \tag{27}
\end{aligned}$$

where (a) holds because if  $[s, f] \subseteq [a, b]$ , then  $\prod_{h=s}^f x_{j,h} = 1$ , whereas if  $s \in [a, b]$  and  $[s, f] \not\subseteq [a, b]$ , then  $\prod_{h=s}^f x_{j,h} = 0$ , as discussed right before the chain of equalities just above.

From Section 3, recall that  $\mathcal{I}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) = (\mathcal{I}(\mathbf{x}_j) \setminus [a, b]) \cup \{[a, s-1], [f+1, b]\}$  for  $[s, f] \subseteq [a, b]$ . By definition of  $\mathcal{K}(\mathbf{x}_j)$ ,  $\mathcal{K}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) = \mathcal{K}(\mathbf{x}_j) \cup \{s, \dots, f\}$ . So, by the induction hypothesis, we get

$$\begin{aligned}
& V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \\
& = \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j - \mathbf{e}_{[s,f]})} \Theta_{\beta,j}^{q+1}(a, b) + \sum_{\ell \in \mathcal{K}(\mathbf{x}_j - \mathbf{e}_{[s,f]})} \Psi_{\beta,j}^{q+1}(\ell) - \sum_{[a,b] \in \mathcal{I}(\mathbf{x}_j)} \Theta_{\beta,j}^{q+1}(a, b) - \sum_{\ell \in \mathcal{K}(\mathbf{x}_j)} \Psi_{\beta,j}^{q+1}(\ell) \\
& = \Theta_{\beta,j}^{q+1}(a, s-1) + \Theta_{\beta,j}^{q+1}(f+1, b) - \Theta_{\beta,j}^{q+1}(a, b) + \sum_{\ell=s}^f \Psi_{\beta,j}^{q+1}(\ell) \\
& \stackrel{(b)}{=} \Theta_{\beta,j}^{q+1}(a, s-1) + \Theta_{\beta,j}^{q+1}(f+1, b) - \Theta_{\beta,j}^{q+1}(a, b) + \Gamma_{\beta,j}^{q+1}(s, f), \tag{28}
\end{aligned}$$

where (b) follows from the discussion right before the proof of the theorem, which shows that the value functions  $\{\Gamma_{\beta,j}^q : q \in \mathcal{Q}\}$  and  $\{\Psi_{\beta,j}^q : q \in \mathcal{Q}\}$  computed, respectively, through (16) and (24)

satisfy the identity  $\Gamma_{\beta,j}^{q+1}(s, f) = \sum_{\ell=s}^f \Psi_{\beta,j}^{q+1}(\ell)$ . In this case, plugging (28) into (27), we equivalently express the right side of (25) as

$$\begin{aligned} & \Theta_{\beta,j}^{q+1}(a, b) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a,b], [s,f] \subseteq [a,b]\}} \lambda_{[s,f]}^q \times \\ & \quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left[ \beta_{j,[s,f] \rightarrow j}^q + \Theta_{\beta,j}^{q+1}(a, s-1) + \Theta_{\beta,j}^{q+1}(f+1, b) - \Theta_{\beta,j}^{q+1}(a, b) + \Gamma_{\beta,j}^{q+1}(s, f) \right] \right. \\ & \quad \quad \quad \left. + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\ & \quad + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s \in [a,b], [s,f] \not\subseteq [a,b]\}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\ & \stackrel{(c)}{=} \Theta_{\beta,j}^q(a, b), \end{aligned}$$

where (c) follows from (17). By the equality above, the right side of (25) is equal to  $\Theta_{\beta,j}^q(a, b)$ , establishing the equality in (25).

Second, we show that (26) holds for each  $\ell \in \mathcal{K}(\mathbf{x}_j)$ . Thus, we focus on the unavailable nights  $\ell \in \mathcal{K}(\mathbf{x}_j)$ . For  $\ell \in \mathcal{K}(\mathbf{x}_j)$ , we have  $x_{j,\ell} = 0$ , so  $\prod_{h=s}^f x_{j,h} = 0$  for  $s = \ell$ . So, the right side of (26) reads

$$\begin{aligned} & \Psi_{\beta,j}^{q+1}(\ell) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^q \times \\ & \quad \max_{S \subseteq \mathcal{N}} \left\{ \phi_j^q(S) \left( \prod_{h=s}^f x_{j,h} \right) \left[ \beta_{j,[s,f] \rightarrow j}^q + V_{\beta,j}^{q+1}(\mathbf{x}_j - \mathbf{e}_{[s,f]}) - V_{\beta,j}^{q+1}(\mathbf{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\ & = \Psi_{\beta,j}^{q+1}(\ell) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \\ & \stackrel{(d)}{=} \Psi_{\beta,j}^{q+1}(\ell) + \sum_{[s,f] \in \mathcal{F}} \mathbf{1}_{\{s=\ell\}} \lambda_{[s,f]}^q \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi_i^q(S) \beta_{i,[s,f] \rightarrow j}^q \right\} \stackrel{(e)}{=} \Psi_{\beta,j}^q(\ell), \end{aligned}$$

where (d) holds because room  $j$  does not appear in the objective function of the maximization problem on the left side of (d), and (e) follows by (24). Thus, the equality in (26) holds.  $\blacksquare$

## Appendix E: Estimating Model Parameters from the Boutique Hotel Dataset

We discuss the details of parameter estimates for the boutique hotel dataset. Recalling that the values of  $\{\theta_k : k = 1, \dots, 171\}$  are the same when  $k$  falls in each of the seven intervals  $[1, 3]$ ,  $[4, 7]$ ,  $[8, 14]$ ,  $[15, 28]$ ,  $[29, 42]$ ,  $[43, 56]$  and  $[57, 171]$ , we need to estimate seven parameters to come up with  $\{\theta_k : k = 1, \dots, 171\}$ . The length of stay for a customer ranges between one and eight nights, so we need to estimate eight parameters for  $\{\eta_d : d = 1, \dots, 8\}$ . Lastly, there are six rooms, which implies that we need to estimate the six preference weights  $\{v_i : i = 1, \dots, 6\}$ . Thus, the total number of

parameters that we need to estimate is 21. For each one of the time periods in the selling horizon, the dataset provides the assortment of rooms that was on offer to the customers and indicates whether there was a booking at the time period. If there was a booking, then the dataset also provides the room chosen in the booking and the interval of stay for the booking. Using the dataset, we use standard maximum likelihood estimation to estimate the parameters of our model. We provide summary statistics for the estimated values of the parameters. The largest values for  $\theta_k$  occur when  $k$  falls in one of the intervals  $[1, 3]$ ,  $[4, 7]$  and  $[8, 14]$ . More than 60% of the bookings requests are for intervals of three or fewer nights. Lastly, the largest and smallest preference weights for a room differ by a factor of 2.28.

We carry out five-fold cross-validation for our arrival probability and preference weight estimates. Recall that we have 1190 time periods in our selling horizon. Some time periods have bookings, some do not. We split the 1190 time periods in the selling horizon into five equal segments. After estimating the parameters of our model using four-fifths of the dataset, we validate the ability of our estimates to predict the arrivals and customer choices in the remaining one-fifth holdout dataset. To validate the estimated arrival probabilities and preference weights, we use our model parameters to predict the expected number of weekly bookings made within the assortments offered in the holdout dataset, and compare our predictions with the actual numbers of bookings. Over five holdout datasets, the average percent deviation is 23%.

For each time period with a reservation in the holdout dataset, we also order the offered rooms according to their choice probabilities and count the fraction of times that the booked room is one of the  $r$  rooms with the largest choice probabilities. We refer to this fraction as the  $r$ -hit rate. For example, 2-hit rate is the fraction of times that the booked room had one of the top two choice probabilities. The 1-hit and 2-hit rates, averaged over five holdout datasets, are, respectively, 0.58 and 0.82. Thus, more than 80% of the time, the booked room is one of the two options with the largest choice probabilities. More than 50% of the time, the booked room is indeed the one with the largest choice probability. The average rank of the booked room, averaged over all holdout datasets and bookings in each holdout dataset, is 1.75.