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LARGE DEVIATIONS FOR MINKOWSKI SUMS OF HEAVY-TAILED GENERALLY NON-CONVEX RANDOM COMPACT SETS

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Abstract

We prove large deviation results for Minkowski sums of iid random compact sets where we assume that the summands have a regularly varying distribution. The result confirms the heavy-tailed large deviation heuristics: "large" values of the sum are essentially due to the "largest" summand.

Keywords: Minkowski sum, random compact set, large deviation, regularly varying distribution

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1. Introduction

Preliminaries on random sets and Minkowski addition. The theory of random sets is summarized in the recent monograph [9]. For all definitions introduced below we refer to [9]. Let F be a separable Banach space with norm $\|\cdot\|$. For $A_1, A_2 \subseteq F$ and a real number λ, the Minkowski addition and scalar multiplication, respectively, are defined by

$$A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}, \quad \lambda A_1 = \{\lambda a_1 : a_1 \in A_1\}.$$

We denote by $\mathcal{K}(F)$ the class of all non-empty compact subsets of F. Note that this is not a vector space. However, it is well known that $\mathcal{K}(F)$ equipped with the Hausdorff distance

$$d(A_1, A_2) = \max\left\{\sup_{a_1\in A_1}\inf_{a_2\in A_2}\|a_1 - a_2\|, \sup_{a_2\in A_2}\inf_{a_1\in A_1}\|a_1 - a_2\|\right\}, \quad A_1, A_2 \in \mathcal{K}(F),$$

forms a complete separable metric space. The Hausdorff metric is subinvariant, i.e.,

$$d(A_1 + A, A_2 + A) \le d(A_1, A_2) \quad \text{for any } A_1, A_2, A \in \mathcal{K}(F). \tag{1}$$

For any subset $\mathcal{U}$ of $\mathcal{K}(F)$, a real number λ and a set $A \in \mathcal{K}(F)$ we use the notation $\lambda\mathcal{U} = \{\lambda C : C \in \mathcal{U}\}$ and $\mathcal{U} + A = \{C + A : C \in \mathcal{U}\}$.

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A random compact set X in F is a Borel measurable function from an abstract probability space $(\Omega, \mathcal{F}, \mathrm{P})$ into $\mathcal{K}(F)$. Since addition and scalar multiplication are defined for random compact sets it is natural to study the strong law of large numbers, the central limit theorem, large deviations, etc., for sequences of such random sets; see Chapter 3 in [9] for an overview of results obtained until 2005. A general Cramér-type large deviation result for Minkowski sums of iid random compact sets was proved in [2]. Cramér-type large deviations require exponential moments of the summands; see Chapter 8 in Valentin V. Petrov's classical monograph [13] for the case of sums of independent real-valued variables and [3] in the case of more general random structures. If such moments do not exist, then we are dealing with heavy-tailed random elements. Large deviations results for sums of heavy-tailed random elements significantly differ from Cramér-type results. In this case it is typical that only the largest summand determines the large deviation behavior; see the classical results by A. Nagaev [10, 11] for sums of iid random variables; cf. [12, 6]. It is the aim of this paper to prove large deviation results for sums of *heavy-tailed random compact sets.* In what follows, we make this notion precise by introducing regularly varying random sets.

Regularly varying random sets. A special element of $\mathcal{K}(F)$ is $A_0 = \{0\}$. In what follows, we say that $\mathcal{U} \subseteq \mathcal{K}(F)$ is bounded away from A_0 if $A_0 \notin \mathrm{cl}\,\mathcal{U}$, where $\mathrm{cl}\,\mathcal{U}$ stands for the closure of $\mathcal{U}$. We consider the subspace $\mathcal{K}_0(F) = \mathcal{K}(F) \setminus \{A_0\}$, which is a separable metric space in the relative topology. For any Borel set $\mathcal{U} \subseteq \mathcal{K}_0(F)$ and $\varepsilon > 0$, we write

$$\mathcal{U}^{\varepsilon} = \{A \in \mathcal{K}_0(F) : \, d(A, C) \le \varepsilon \text{ for some } C \in \mathcal{U}\}.$$

Furthermore, we define the norm $\|A\| = d(A, A_0) = \sup\{\|a\| : \, a \in A\}$ for $A \in \mathcal{K}(F)$, and denote $\mathcal{B}_r = \{A \in \mathcal{K}(F) : \, \|A\| \le r\}$. Let $M_0\big(\mathcal{K}_0(F)\big)$ be the collection of Borel measures on $\mathcal{K}_0(F)$ whose restriction to $\mathcal{K}(F) \setminus \mathcal{B}_r$ is finite for each $r > 0$. Let $\mathcal{C}_0$ denote the class of real-valued, bounded and continuous functions f on $\mathcal{K}_0(F)$ such that for each f there exists $r > 0$ and f vanishes on $\mathcal{B}_r$. The convergence $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ in $M_0\big(\mathcal{K}_0(F)\big)$ is defined to mean the convergence $\int f \, \mathrm{d}\mu_n \underset{n\to\infty}{\longrightarrow} \int f \, \mathrm{d}\mu$ for all $f \in \mathcal{C}_0$. By the portmanteau theorem ([5], Theorem 2.4), $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ in $M_0\big(\mathcal{K}_0(F)\big)$ if and only if $\mu_n(\mathcal{U}) \underset{n\to\infty}{\longrightarrow} \mu(\mathcal{U})$ for all Borel sets $\mathcal{U} \subseteq \mathcal{K}(F)$ which are bounded away from A_0 and satisfy $\mu(\partial\mathcal{U}) = 0$, where $\partial\mathcal{U}$ is the boundary of $\mathcal{U}$.

Following [5], for the general case of random elements with values in a separable metric space, we say that a random compact set X is *regularly varying* if there exist a non-null measure $\mu \in M_0\big(\mathcal{K}_0(F)\big)$ and a sequence $\{a_n\}_{n\ge1}$ of positive numbers such that

$$n\mathrm{P}(X \in a_n \cdot) \underset{n\to\infty}{\longrightarrow} \mu(\cdot) \quad \text{in } M_0\big(\mathcal{K}_0(F)\big). \tag{2}$$

The *tail measure* μ necessarily has the property $\mu(\lambda\mathcal{U}) = \lambda^{-\alpha}\mu(\mathcal{U})$ for some $\alpha > 0$ and all Borel sets $\mathcal{U}$ in $\mathcal{K}_0(F)$ and all $\lambda > 0$. We then also refer to *regular variation of X with index α.* From the definition of regular variation of X we get ([5], Theorem 3.1)

$$\left[\mathrm{P}(X \in t(\mathcal{K}(F) \setminus \mathcal{B}_1))\right]^{-1} \mathrm{P}(X \in t\cdot) \to c\mu(\cdot) \quad \text{in } M_0\big(\mathcal{K}_0(F)\big) \text{ as } t \to \infty, \tag{3}$$

for some $c > 0$. The sequence $\{a_n\}_{n\ge1}$ will always be chosen such that $n\mathrm{P}(X \in a_n(\mathcal{K}(F) \setminus \mathcal{B}_1)) \underset{n\to\infty}{\longrightarrow} 1$. With this choice of $\{a_n\}_{n\ge1}$, it follows that $c = 1$ in (3).

An important closed subset of $\mathcal{K}(F)$ is the family of non-empty *compact convex subsets* of F, denoted by $\operatorname{co}\mathcal{K}(F)$. Denote the topological dual of F by F^* and the unit ball of F^* by B^*, it is endowed with the weak-$*$ topology w^*. The *support function* h_A of a compact convex $A \in \operatorname{co}\mathcal{K}(F)$ is defined by (see [9])

$$h_A(u) = \sup\{u(x) : x \in A\}, \quad u \in B^*.$$

Since A is compact, $h_A(u) < \infty$ for all $u \in B^*$. The support function h_A is sublinear, i.e., it is subadditive ($h_A(u+v) \le h_A(u) + h_A(v)$ for all $u, v \in B^*$ with $u + v \in B^*$) and positively homogeneous ($h_A(cu) = ch_A(u)$ for all $c > 0$, $u \in B^*$ with $cu \in B^*$). Let $\mathcal{C}(B^*, w^*)$ be the set of continuous functions from B^* (endowed with the weak-$*$ topology) to $\mathbb{R}$ and consider the uniform norm $\|f\|_\infty = \sup_{u \in B^*} |f(u)|$, $f \in \mathcal{C}(B^*, w^*)$. The map $h : \operatorname{co}\mathcal{K}(F) \to \mathcal{C}(B^*, w^*)$ has the following properties

$$h_{A_1+A_2} = h_{A_1} + h_{A_2}, \quad h_{\lambda A_1} = \lambda h_{A_1}, \quad A_1, A_2 \in \operatorname{co}\mathcal{K}(F), \quad \lambda \ge 0,$$

which make it possible to convert the Minkowski sums and scalar multiplication, respectively, of convex sets into the arithmetic sums and scalar multiplication of the corresponding support functions. Furthermore,

$$d(A_1, A_2) = \|h_{A_1} - h_{A_2}\|_\infty.$$

Hence, the support function provides an isometric embedding of $\operatorname{co}\mathcal{K}(F)$ into $\mathcal{C}(B^*, w^*)$ with the uniform norm. If $\mathcal{G} = h\bigl(\operatorname{co}\mathcal{K}(F)\bigr)$, then $\mathcal{G}$ is a closed convex cone in $\mathcal{C}(B^*, w^*)$, and h is an isometry between $\operatorname{co}\mathcal{K}(F)$ and $\mathcal{G}$.

A random compact convex set X is a Borel measurable function from a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ into $\operatorname{co}\mathcal{K}(F)$, which we endow with the relative topology inherited from $\mathcal{K}(F)$. The support function of a random compact convex set is, clearly, a $\mathcal{C}(B^*, w^*)$-valued random variable taking values in $\mathcal{G}$.

The definition of a regularly varying random compact convex set parallels that of a regularly varying random compact set above, and we are using the same notation: a random compact convex set X is *regularly varying* if there exist a non-zero measure $\mu \in M_0\bigl(\operatorname{co}\mathcal{K}_0(F)\bigr)$ and a sequence $\{a_n\}_{n\ge1}$ of positive numbers such that

$$n\mathrm{P}(X \in a_n \cdot) \underset{n\to\infty}{\longrightarrow} \mu(\cdot) \ \text{ in } M_0\bigl(\operatorname{co}\mathcal{K}_0(F)\bigr). \tag{4}$$

Once again, the tail measure μ necessarily scales, leading to the notion of the index of regular variation.

The following lemma is elementary.

Lemma 1. (i) *A random compact convex set X is regularly varying in $\operatorname{co}\mathcal{K}(F)$ if and only if its support function h_X is regularly varying in $\mathcal{C}(B^*, w^*)$. Specifically, if* (4) *holds for some sequence $\{a_n\}$, then for the same sequence we have*

$$n\mathrm{P}(h_X \in a_n \cdot) \underset{n\to\infty}{\longrightarrow} \nu(\cdot) \ \text{ in } M_0\bigl(\mathcal{C}(B^*, w^*)\bigr), \tag{5}$$

where $\nu = \mu \circ h_X^{-1}$ (the "special element" of $\mathcal{C}(B^*, w^*)$ is, of course, the zero function). Conversely, if (5) *holds, then* (4) *holds as well with $\mu = \nu \circ h_X$. In particular, the exponents of regular variation of X and h_X are the same.*

(ii) *If a random compact set X is regularly varying in $\mathcal{K}(F)$ then its convex hull $\operatorname{co} X$ is a random compact convex set, that is regularly varying in $\operatorname{co}\mathcal{K}(F)$. Specifically, if* (2) *holds, then so does* (4)*, with the tail measure replaced by the image of the tail measure from* (2) *under the map $A \mapsto \operatorname{co} A$ from $\mathcal{K}(F)$ to $\operatorname{co}\mathcal{K}(F)$. In particular, X and $\operatorname{co} X$ have the same exponents of regular variation.*

Proof. Since isometry implies continuity, the support function is homogeneous of order 1, and assigns to the "special set" $\{0\}$ the "special element", the zero function, the statement of part (i) of the lemma follows from the mapping theorem (Theorem 2.5 in [5]). For part (ii) note that the map $A \mapsto \operatorname{co} A$ from $\mathcal{K}(F)$ to $\operatorname{co}\mathcal{K}(F)$ is a contraction in the Hausdorff distance, hence is continuous. It is also homogeneous of order 1. Since the "special set" $\{0\}$ is already convex, the statement follows once again from the mapping theorem.

For compact convex sets in $\mathbb{R}^d$, the intrinsic volumes V_j, $j = 0, \ldots, d$, play an important role. They can be introduced by means of the Steiner formula, see [9], Appendix F. In particular, V_d is the volume, V_{d-1} is half of the surface area, V_1 is a multiple of the mean width and $V_0 = 1$ is the Euler-Poincaré characteristic.

Corollary 1. *Let X be a random compact convex set which is regularly varying in $\operatorname{co}\mathcal{K}(\mathbb{R}^d)$ with index $\alpha > 0$ and tail measure μ. Then for $j \in \{1, \ldots, d\}$, $V_j(X)$ is a regularly varying non-negative random variable with index α/j and tail measure $\nu_j = \mu \circ V_j^{-1}$.*

Proof. Since V_j is continuous, homogeneous of order j and $V_j(A_0) = 0$, the continuous mapping theorem (Theorem 2.5 in [5]) yields that $V_j(X)$ is regularly varying with tail measure $\nu_j = \mu \circ V_j^{-1}$. Moreover, $\nu_j(\lambda\mathcal{U}) = \mu(V_j^{-1}(\lambda\mathcal{U})) = \mu(\lambda^{1/j} V_j^{-1}(\mathcal{U})) = \lambda^{-\alpha/j}\nu_j(\mathcal{U})$ for any measurable subset $\mathcal{U}$ of $\mathbb{R}^+$.

Organization of the paper. In Section 2 we consider various examples of regularly varying compact random sets. In Section 3 we prove large deviation results for Minkowski sums S_n of iid regularly varying random compact sets. We allow the random sets to be, generally, non-convex. To the best of our knowledge, such results are not available in the literature; they parallel those proved by A. and S. Nagaev [10, 11, 12] for sums of iid random variables. The price one has to pay for this generality is that the normalizations λ_n of the sums S_n have to exceed the level n. The situation with milder normalizations is more delicate. It is considered in [8]. Our main result there assumes that the random compact summands X_n are convex, but we also include partial results in the non-convex case.

2. Examples of regularly varying random sets

Example 1. (*Convex hull of random points.*) Let $k \geq 2$, and let $\xi_1, \ldots, \xi_k$ be iid regularly varying F-valued random elements with index $\alpha > 0$ and tail measure ν and let $X = \operatorname{co}\{\xi_1, \ldots, \xi_k\}$ be their convex hull. The mapping $g : (z_1, \ldots, z_k) \mapsto \operatorname{co}\{z_1, \ldots, z_k\}$ from F^k to $\operatorname{co}\mathcal{K}(F)$ is continuous and homogeneous of order 1. Moreover, this mapping sends the zero point in F^k to the "special element" A_0 of $\mathcal{K}(F)$. Since the random vector $\xi = (\xi_1, \ldots, \xi_k)$ is regularly varying in F^k, the continuous mapping theorem (Theorem 2.5 in [5]) yields that X is regularly varying with index α, and tail measure

$\tilde{\nu} \circ g^{-1}$ (as long as we are using the same sequence $\{a_n\}$ for each element ξ_i). Here

$$\tilde{\nu} = \sum_{i=1}^{k} \delta_0 \times \cdots \times \delta_0 \times \nu \times \delta_0 \times \cdots \times \delta_0$$

(with ν appearing at the ith place) is the tail measure of the vector $\xi = (\xi_1, \ldots, \xi_k)$. Clearly, the convex hull of k points, one of which is $x \in F$, and the rest are zero points, is the interval $[0, x] = \{y \in F : y = cx, 0 \le c \le 1\}$. Therefore, the convex hull X has tail measure $\mu = k\,\nu \circ T^{-1}$, where $T : F \to \operatorname{co}\mathcal{K}(F)$ is defined by the relation $T(x) = [0, x]$.

Example 2. (*Random zonotopes.*) As in the previous example, let $\xi_1, \ldots, \xi_k$ be iid regularly varying random elements in F with index $\alpha > 0$ and tail measure ν. Starting with the same ingredients, we construct a different convex compact subset of F. Consider the Minkowski sum of the random segments, $X = \sum_{i=1}^{k} [0, \xi_i]$, a so-called *zonotope*.

The function $g : (z_1, \ldots, z_k) \mapsto \sum_{i=1}^{k} [0, z_i]$ from F^k to $\operatorname{co}\mathcal{K}(F)$ is continuous, homogeneous of order 1, and maps the zero point in F^k to A_0. The same argument as in Example 1 shows that the random zonotope X is regularly varying with index α, and, if we use the same sequence $\{a_n\}$ as we used for each element ξ_i, has tail measure $\mu = k\,\nu \circ T^{-1}$, where $T : F \to \operatorname{co}\mathcal{K}(F)$ is as above.

Examples 1 and 2 construct different compact sets starting from a finite number of iid regularly varying random points in F, but the tail measures in the two cases turn out to be the same.

Example 3. (*Multiple of a deterministic set.*) Let $A \subseteq \mathcal{K}(F)$ be a deterministic compact set such that $\|A\| > 0$ and let R be a regularly varying random variable with index $\alpha > 0$, satisfying the tail balance condition

$$\frac{\mathrm{P}(R > x)}{\mathrm{P}(|R| > x)} \underset{x \to \infty}{\longrightarrow} p \quad \text{and} \quad \frac{\mathrm{P}(R \le -x)}{\mathrm{P}(|R| > x)} \underset{x \to \infty}{\longrightarrow} q.$$

Then the mapping $g : z \mapsto zA$ from $\mathbb{R}$ to $\mathcal{K}(F)$ is continuous and homogeneous of order 1, and it maps the origin in $\mathbb{R}$ into A_0. Therefore, $X = RA$ is regularly varying with index α. Recall that the tail measure of R has density $\big(p\mathbf{1}_{\{x>0\}} + q\mathbf{1}_{\{x<0\}}\big)|x|^{-(1+\alpha)}$ with respect to Lebesgue measure on $\mathbb{R}$. Using the sequence $\{a_n\}$ that defines the above tail measure on $\mathbb{R}$, we see that the tail measure μ of X can be written as

$$\mu(\mathcal{U}) = \int_0^\infty x^{-(1+\alpha)} \big(p\mathbf{1}_{\{xA \in \mathcal{U}\}} + q\mathbf{1}_{\{-xA \in \mathcal{U}\}}\big)\, \mathrm{d}x.$$

Example 4. (*Stable random compact convex set.*) A random compact convex set X has an α-stable distribution, $\alpha \in (0, 2)$, if for any $a, b > 0$ there are compact convex sets C and D such that

$$aX_1 + bX_2 + C \overset{d}{=} (a^\alpha + b^\alpha)^{1/\alpha} X + D,$$

where X_1, X_2 are independent copies of X; see [4] and [9], Section 2.3. By Theorem 2.2.14 in [9], the support function of an α-stable random compact convex set X is itself an α-stable random vector in $\mathcal{C}(B^*, w^*)$, hence is regularly varying in that space (see e.g. [7]). By Lemma 1, X is a regularly varying random compact convex set.

It follows from [4] that an α-stable random compact convex set for $\alpha \in [1, 2)$ must be of the form $X = K + \xi$, where $\xi \in F$ is an α-stable random element and $K \in \operatorname{co}\mathcal{K}(F)$ is deterministic.

3. A large deviation result for general random compact sets

In this section we consider an iid sequence $\{X_n\}_{n\ge1}$ of random compact sets which are not necessarily convex. We introduce the sequence of the corresponding Minkowski partial sums $S_n = X_1 + \cdots + X_n$, $n \ge 1$. Next we formulate our main result on the large deviations in this situation.

Theorem 1. *Let $\{X_n\}_{n\ge1}$ be an iid sequence of random compact sets which are regularly varying with index α and tail measure $\mu \in M_0\big(\mathcal{K}_0(F)\big)$. Let $\{a_n\}_{n\ge1}$ be the normalizing sequence in* (2)*. Consider a sequence $\lambda_n \nearrow \infty$ such that*

(i) $\lambda_n/a_n \underset{n\to\infty}{\longrightarrow} \infty$ *if* $\alpha < 1$,

(ii) $\lambda_n/n \underset{n\to\infty}{\longrightarrow} \infty$, $\lambda_n/a_n \underset{n\to\infty}{\longrightarrow} \infty$, $\frac{n}{\lambda_n}\mathbb{E}\|X_1\|\mathbf{1}_{\{\|X_1\|\le\lambda_n\}} \underset{n\to\infty}{\longrightarrow} 0$ *if* $\alpha = 1$,

(iii) $\lambda_n/n \underset{n\to\infty}{\longrightarrow} \infty$ *if* $\alpha > 1$.

Then, with $\gamma_n = [n\,\mathrm{P}(\|X_1\| > \lambda_n)]^{-1}$,

$$\gamma_n\,\mathrm{P}(S_n \in \lambda_n\cdot) \underset{n\to\infty}{\longrightarrow} \mu(\cdot) \quad \textit{in} \quad M_0\big(\mathcal{K}_0(F)\big).$$

Proof. First observe that our assumptions and an appeal to [14], Theorem 4.13, yield that

$$\lambda_n^{-1}(\|X_1\| + \cdots + \|X_n\|) \overset{\mathrm{P}}{\underset{n\to\infty}{\longrightarrow}} 0. \tag{6}$$

Let $\mathcal{U} \subseteq \mathcal{K}_0(F)$ be a μ-continuity set ($\mu(\partial\mathcal{U}) = 0$), bounded away from A_0. We have to prove that $\gamma_n \mathrm{P}(S_n \in \lambda_n\mathcal{U}) \underset{n\to\infty}{\longrightarrow} \mu(\mathcal{U})$. Following [6], Lemma 2.1, we start with an upper bound. For any $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{P}(S_n \in \lambda_n\mathcal{U}) &= \mathrm{P}\left(S_n \in \lambda_n\mathcal{U}, \cup_{i=1}^n\{X_i \in \lambda_n\mathcal{U}^\varepsilon\}\right) + \mathrm{P}\left(S_n \in \lambda_n\mathcal{U}, \cap_{i=1}^n\{X_i \notin \lambda_n\mathcal{U}^\varepsilon\}\right)\\
&\le n\mathrm{P}\big(X_1 \in \lambda_n\mathcal{U}^\varepsilon\big) + \mathrm{P}\left(\cap_{i=1}^n\{d(S_n, X_i) > \varepsilon\lambda_n\}\right)\\
&= I_1 + I_2.
\end{aligned}$$

Relation (3) implies that

$$\frac{\mathrm{P}(X_1 \in \lambda_n\cdot)}{\mathrm{P}(\|X_1\| > \lambda_n)} \underset{n\to\infty}{\longrightarrow} \mu(\cdot) \quad \text{in} \quad M_0\big(\mathcal{K}_0(F)\big).$$

Consequently, $\gamma_n I_1 \underset{n\to\infty}{\longrightarrow} \mu(\mathcal{U}^\varepsilon)$ whenever $\mathcal{U}^\varepsilon$ is a μ-continuity set. Since $\mu(\partial\mathcal{U}) = 0$, we have $\lim_{\varepsilon\searrow0}\mu(\mathcal{U}^\varepsilon) = \mu(\mathcal{U})$. Next we show that, for every $\varepsilon > 0$, $\gamma_n I_2 \underset{n\to\infty}{\longrightarrow} 0$. We consider the following disjoint partition of Ω for $\delta > 0$:

$$B_1 = \bigcup_{1\le i<j\le n}\{\|X_i\| > \delta\lambda_n, \|X_j\| > \delta\lambda_n\},$$

$$B_2 = \bigcup_{i=1}^{n} \{\|X_i\| > \delta\lambda_n, \|X_j\| \le \delta\lambda_n, j \ne i, j = 1, \dots, n\},$$
$$B_3 = \left\{ \max_{i=1,\dots,n} \|X_i\| \le \delta\lambda_n \right\}.$$

By regular variation of X_1 and definition of γ_n, we have $\gamma_n \mathrm{P}(B_1) \underset{n\to\infty}{\longrightarrow} 0$. As regards B_2, from (1) we get $d(S_n, X_n) \le \|S_{n-1}\|$ and accordingly,

$$\begin{aligned}
&\mathrm{P}\left(\cap_{i=1}^n \{d(S_n, X_i) > \varepsilon\lambda_n\} \cap B_2\right) \\
&\quad = \sum_{k=1}^{n} \mathrm{P}\left(\cap_{i=1}^n \{d(S_n, X_i) > \varepsilon\lambda_n\} \cap \{\|X_k\| > \delta\lambda_n, \|X_j\| \le \delta\lambda_n, j \ne k, j = 1, \dots, n\}\right) \\
&\quad \le \sum_{k=1}^{n} \mathrm{P}\big(d(S_n, X_k) > \varepsilon\lambda_n, \|X_k\| > \delta\lambda_n\big) \\
&\quad \le \mathrm{P}\big(\|S_{n-1}\| > \varepsilon\lambda_n\big)\big[n\mathrm{P}(\|X_1\| > \delta\lambda_n)\big].
\end{aligned}$$

Since X_1 is regularly varying and $\lambda_n^{-1}\|S_n\| \xrightarrow[n\to\infty]{\mathrm{P}} 0$,

$$\gamma_n n\mathrm{P}(\|X_1\| > \delta\lambda_n)\mathrm{P}(\|S_{n-1}\| > \varepsilon\lambda_n) \underset{n\to\infty}{\longrightarrow} 0.$$

For B_3, using again (1), we have

$$\begin{aligned}
&\mathrm{P}\big(\cap_{i=1}^n \{d(S_n, X_i) > \varepsilon\lambda_n\} \cap B_3\big) \le \mathrm{P}\big(\|S_{n-1}\| > \varepsilon\lambda_n, \max_{i=1,\dots,n-1} \|X_i\| \le \delta\lambda_n\big) \\
&\le \mathrm{P}\Big(\Big\|\sum_{i=1}^{n-1} X_i \mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}}\Big\| > \varepsilon\lambda_n\Big) \le \mathrm{P}\Big(\sum_{i=1}^{n-1} \|X_i\|\mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}} > \varepsilon\lambda_n\Big) \\
&\le \mathrm{P}\Big(\sum_{i=1}^{n-1} \big(\|X_i\|\mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}} - \mathbb{E}\|X_i\|\mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}}\big) > \varepsilon\lambda_n - (n-1)\mathbb{E}\|X_1\|\mathbf{1}_{\{\|X_1\|\le\delta\lambda_n\}}\Big).
\end{aligned}$$

By the Karamata theorem [1], for $\alpha < 1$,

$$\mathbb{E}\|X_1\|\mathbf{1}_{\{\|X_1\|\le\delta\lambda_n\}} = \int_0^{\delta\lambda_n} \mathrm{P}(\|X_1\| > x)\,\mathrm{d}x \sim \frac{\delta\lambda_n}{1-\alpha}\mathrm{P}(\|X_1\| > \delta\lambda_n).$$

Therefore,

$$(n-1)\lambda_n^{-1}\mathbb{E}\|X_1\|\mathbf{1}_{\{\|X_1\|\le\delta\lambda_n\}} \underset{n\to\infty}{\longrightarrow} 0. \tag{7}$$

If $\alpha \ge 1$, then (7) follows directly from the assumptions on $\{\lambda_n\}$. Taking into account (7), it suffices to show that

$$\gamma_n \mathrm{P}\left(\sum_{i=1}^{n-1} \big(\|X_i\|\mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}} - \mathbb{E}\|X_i\|\mathbf{1}_{\{\|X_i\|\le\delta\lambda_n\}}\big) > \frac{\varepsilon\lambda_n}{2}\right) \underset{n\to\infty}{\longrightarrow} 0,$$

which can be accomplished similarly as in the one-dimensional case by an application of the Fuk-Nagaev inequality ([14], p. 78) and the Karamata theorem. We conclude that

$$\limsup_{n\to\infty} \gamma_n \mathrm{P}(S_n \in \lambda_n \mathcal{U}) \le \mu(\mathcal{U}^\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} \mu(\mathcal{U})$$

for any μ-continuity set $\mathcal{U}$ bounded away from A_0.

To prove the corresponding lower bound, we consider a μ-continuity set $\mathcal{U} \subseteq \mathcal{K}_0(F)$ bounded away from A_0 with non-empty interior $\operatorname{int}\mathcal{U}$. Introduce the set $\mathcal{U}^{-\varepsilon} = ((\mathcal{U}^c)^\varepsilon)^c$, where $\mathcal{U}^c$ denotes the complement of $\mathcal{U}$. It is a non-empty μ-continuity set for a sequence of $\varepsilon > 0$ converging to zero. Notice that $(\mathcal{U}^{-\varepsilon})^\varepsilon \subseteq \mathcal{U}$. Then

$$\begin{aligned}
\mathrm{P}\big(S_n \in \lambda_n\mathcal{U}\big) &\geq \mathrm{P}\big(S_n \in \lambda_n\mathcal{U}, \cup_{i=1}^n \{X_i \in \lambda_n\mathcal{U}^{-\varepsilon}\}\big) \\
&\geq \mathrm{P}\big(\cup_{i=1}^n \{d(S_n, X_i) < \varepsilon\lambda_n, X_i \in \lambda_n\mathcal{U}^{-\varepsilon}\}\big) \\
&\geq n\mathrm{P}\big(X_1 \in \lambda_n\mathcal{U}^{-\varepsilon}\big)\mathrm{P}\big(\|S_{n-1}\| < \varepsilon\lambda_n\big) - \frac{n(n-1)}{2}\big[\mathrm{P}(X_1 \in \lambda_n\mathcal{U}^{-\varepsilon})\big]^2.
\end{aligned}$$

Since $\lambda_n^{-1}\|S_{n-1}\| \xrightarrow[n\to\infty]{\mathrm{P}} 0$ and $\mathcal{U}$ is a μ-continuity set,

$$\begin{aligned}
\liminf_{n\to\infty} \gamma_n \mathrm{P}(S_n \in \lambda_n\mathcal{U}) &\geq \lim_{n\to\infty}\left(\frac{\mathrm{P}(X_1 \in \lambda_n\mathcal{U}^{-\varepsilon})}{\mathrm{P}(\|X_1\| > \lambda_n)} - \frac{n-1}{2n\gamma_n}\frac{\big[\mathrm{P}(X_1 \in \lambda_n\mathcal{U}^{-\varepsilon})\big]^2}{\big[\mathrm{P}(\|X_1\| > \lambda_n)\big]^2}\right) \\
&= \mu(\mathcal{U}^{-\varepsilon}) \xrightarrow[\varepsilon\searrow 0]{} \mu(\mathcal{U}).
\end{aligned}$$

This completes the proof.

From Theorem 1 we get by the continuous mapping theorem (Theorem 2.5 in [5]) the following corollary concerning large deviations of the intrinsic volumes of random compact convex sets.

Corollary 2. *Let $\{X_n\}_{n\geq 1}$ be an iid sequence of random compact convex sets which are regularly varying with index α and tail measure $\mu \in M_0\big(\operatorname{co}\mathcal{K}_0(\mathbb{R}^d)\big)$. Under the same assumptions on the sequence $\{\lambda_n\}$ as in Theorem 1, we have*

$$\frac{\mathrm{P}(V_j(S_n)/\lambda_n^j \in \cdot)}{n\,\mathrm{P}(\|X_1\| > \lambda_n)} \xrightarrow[n\to\infty]{} \mu \circ V_j^{-1}(\cdot) \quad in \quad M_0(\mathbb{R}), \quad j = 1, \ldots, d.$$

Remark 1. For $j = 1$ we can also use the relation $V_1(X_1 + \cdots + X_n) = V_1(X_1) + \cdots + V_1(X_n)$, Corollary 1 and large deviations results for sums of iid random variables [10, 11] in order to obtain large deviations result for $V_1(S_n)$. However, for $j > 1$ similar results are not straightforward.

The assumptions of Theorem 1 imposed on the normalizing sequence $\{\lambda_n\}$ ensure that (6) holds, in particular, $\lambda_n/n \xrightarrow[n\to\infty]{} \infty$. If $\alpha > 1$, this is a rather strong assumption. In [8] we show that this condition can be weakened significantly if it is possible to introduce the notion of expectation of S_n. In particular, we will assume that the iid sequence $\{X_n\}$ consists of iid random convex compact sets which are regularly varying with index $\alpha \geq 1$ and $\mathbb{E}\|X_1\| < \infty$.

A personal remark of Thomas Mikosch. When I was a student of Valentin V. Petrov in the beginning of the 1980s I got familiar with large deviations by reading his monograph [13]. I remember the excitement when I read his proof of Cramér's theorem for sequences of independent variables: it is an example of extraordinary mathematical elegance and beauty. It was also then that I started getting interested in heavy-tail phenomena, in particular in distributions with power laws. The combination of regular variation and large deviations has fascinated me since then. I would like to thank Valentin Vladimirovich for opening the door to this exciting world.

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