

# HIGH LEVEL EXCURSION SET GEOMETRY FOR NON-GAUSSIAN INFINITELY DIVISIBLE RANDOM FIELDS

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We consider smooth, infinitely divisible random fields  $(X(t), t \in M)$ ,  $M \subset \mathbb{R}^d$ , with regularly varying Lévy measure, and are interested in the geometric characteristics of the excursion sets

$$A_u = \{t \in M : X(t) > u\}$$

over high levels  $u$ .

For a large class of such random fields we compute the  $u \rightarrow \infty$  asymptotic joint distribution of the numbers of critical points, of various types, of  $X$  in  $A_u$ , conditional on  $A_u$  being non-empty. This allows us, for example, to obtain the asymptotic conditional distribution of the Euler characteristic of the excursion set.

In a significant departure from the Gaussian situation, the high level excursion sets for these random fields can have quite a complicated geometry. Whereas in the Gaussian case non-empty excursion sets are, with high probability, roughly ellipsoidal, in the more general infinitely divisible setting almost any shape is possible.

**1. Introduction.** Let  $(X(t), t \in M)$ , where  $M$  is a compact set in  $\mathbb{R}^d$  of a kind to be specified later, be a smooth infinitely divisible random field. We shall assume, in a sense that we shall make precise later, that  $X$  has regularly varying tails. Note that this means that the tails of  $X$  are heavier than exponential and, in particular, heavier than those of a Gaussian random field. Nevertheless, the model we are considering allows both heavy tails (e.g.

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infinite mean or variance) and light tails, in the sense of the existence of finite moments of arbitrary given order.

We are interested in studying the excursions of the random field over levels  $u > 0$ , particularly when the level  $u$  becomes high. Writing

$$(1) \quad A_u \equiv A_u(X, M) \triangleq \{t \in M : X(t) > u\}$$

for the excursion set of  $X$  over the level  $u$ , we shall study the geometric characteristics of  $A_u$  under the condition that it is not empty, i.e. under the condition that the level  $u$  is, in fact, exceeded. In particular, we shall be interested in computing the conditional limit distribution of the Euler characteristic of  $A_u$  as  $u \rightarrow \infty$ . We refer the reader to [1] for a recent detailed exposition of the geometric theory of the excursion sets of smooth Gaussian and related random fields, and to [2] for applications of the theory.

In a significant departure from the well understood Gaussian situation, the excursion sets over high levels for the random fields in this paper can have quite a complicated geometry. In the Gaussian case excursion sets, unless they are empty, tend, with high probability, to contain a single component which is almost ellipsoidal in shape, and so have an Euler characteristic equal to one. In contrast, the Euler characteristics of the excursion sets in our fields can have highly non-degenerate conditional distributions. As a consequence, these models are sufficiently flexible to open the possibility of fitting empirically observed excursion sets with widely different geometric characteristics. This, more statistical, problem is something we plan to tackle in the future.

The main result of the paper is Theorem 4.1. While it is rather too technical to summarise here in full, here is the beginning of a special case. Suppose that  $N_X(i, u)$  is the number of critical points of  $X$  in  $A_u$  of index  $i$ . Thus, if  $d = 2$ ,  $N_X(0, u)$  is the number of local minima of  $X$  above the level  $u$  in the interior of  $M$ ,  $N_X(1, u)$  the number of saddle points and  $N_X(2, u)$  the number of local maxima, all above the level  $u$ . Then Theorem 4.1 gives an explicit expression for the limiting joint distribution

$$(2) \quad \lim_{u \rightarrow \infty} \mathbb{P} \left\{ N_X(i, u) = n_i, \quad i = 0, \dots, d, \quad \middle| \quad A_u \neq \emptyset \right\}.$$

In fact, Theorem 4.1 goes far beyond this, since it includes not only these critical points, but also the critical points of  $X$  restricted to the boundary of  $M$ . The importance of this result lies in the fact that Morse theory shows how to use the full collection of these critical points to describe much of the geometry of  $A_u$ , whether this geometry be algebraic, integral, or differential.

Furthermore, Theorem 4.1 can also be exploited to describe a very simple stochastic model for high level excursion sets, as well as to develop a simple algorithm for simulating them.

The remainder of the paper begins in Section 2 with a description of the parameter spaces  $M$  with which we work, and is then organised as follows: In Section 3 we define our model, discuss the smoothness assumptions we are imposing, as well as those related to the regular variation of the tails. Section 4 contains the main result of the paper, on the joint distribution of the numbers of high level critical points of infinitely divisible random fields. This is followed with one of its main consequences, the distribution of the Euler characteristic of high level excursion sets, in Section 5. In Section 6 we introduce a class of moving average infinitely divisible random fields and derive conditions under which the main result of the Section 4 applies to them. We also provide examples to show that, by choosing appropriately the parameters of the model, one can make the geometric structure of the high level excursion sets either ‘Gaussian-like’ or ‘non-Gaussian-like’. Finally, Section 7 contains the proof of the main theorem for the special case of  $M$  the unit cube in  $\mathbb{R}^d$ , while Section 8 discusses how to extend this to more general parameter spaces.

Throughout the paper  $C$  stands for finite positive constants whose precise value is irrelevant and which may change from line to line.

**2. The parameter space  $M$ .** Throughout this paper we shall assume that  $M$  is a compact,  $C^2$ , Whitney stratified manifold embedded in  $\mathbb{R}^d$ . These are basically sets that can be partitioned into the disjoint union of  $C^2$  manifolds, so that we can write  $M$  as the disjoint union

$$(3) \quad M = \bigsqcup_{j=0}^{\dim M} \partial_j M,$$

where each stratum,  $\partial_j M$ ,  $0 \leq j \leq \dim(M)$ , is itself a disjoint union of a number of bounded  $j$ -dimensional manifolds. We shall also assume that each of these manifolds has bounded curvature. The ‘‘Whitney’’ part of the definition relates to rules as to how the  $\partial_j M$  are glued together.

The simplest example of such a stratified manifold would be a simple manifold, in which there would be only a single term, of maximal dimension  $\dim(M)$ , in (3). A  $C^2$  domain  $M \subset \mathbb{R}^d$  is not much more complicated, and could be stratified into its interior, of dimension  $d$ , and boundary, of dimension  $d - 1$ .

A familiar example of a stratified manifold which has components of all dimensions is the  $d$ -dimensional cube  $I_d \triangleq [-1, 1]^d$ . In this case  $\partial_d M$  would

be the interior of the cube,  $\partial_{d-1}M$  the union of the interiors of the  $2d$  ‘sides’ of dimension  $(d-1)$ , and so on, with  $\partial_0M$  being made up of the  $2^d$  vertices.

For further details on Whitney stratified manifolds see, for example, Chapter 8 of [1]. However, if you want to avoid this level of generality, you will lose little (beyond many applications) by taking  $M = I_d$  throughout the paper. In fact, the main proofs in Section 7 treat only this case in full, with only a brief description, in Section 8 of how to extend the result to general  $M$ .

**3. Smooth infinitely divisible random fields and regular variation.** In this section we shall define the random fields of interest to us, describe their distributional structure, and then specify the smoothness assumptions necessary for studying the geometry of their excursion sets.

A reader familiar with the theory of infinitely divisible processes will note that the route we take goes back to first principles to some extent. (For example, it would be more standard, nowadays, to start with the function space Lévy measure  $\lambda_X$  of Section 3.3 rather than invest a couple of pages in defining it.) The need for this, as should become clear below, is to be able to carefully define random fields, along with their first and second order partial derivatives, on a common probability space.

*3.1. Probabilistic structure of infinitely divisible random fields.* As a first step, we shall need to define our random fields on a region slightly larger than the basic parameter space  $M$ , and so we take  $\widetilde{M}$  be a bounded open set in  $\mathbb{R}^d$ , with  $M \subset \widetilde{M}$ .

We now consider infinitely divisible random fields of the form

$$(4) \quad X(t) = \int_S f(s; t) \mu(ds), \quad t \in \widetilde{M},$$

where  $(S, \mathcal{S})$  is a measurable space and  $\mu$  is an infinitely divisible random measure on  $S$  with characteristics defined below. (We refer you to [11] for more information on infinitely divisible random measures and stochastic integrals with respect to these measures. See also the forthcoming monograph [17] for a more complete account.)

The infinitely divisible random measure  $\mu$ , which we shall define in a moment, is characterised by its ‘generating triple’  $(\gamma, F, \beta)$ . Here  $\gamma$  is a  $\sigma$ -finite measure on  $(S, \mathcal{S})$ , and plays the role of the variance measure for the Gaussian part of  $\mu$ . More important for us is the Lévy measure  $F$ , which is a  $\sigma$ -finite measure on  $S \times (\mathbb{R} \setminus \{0\})$ , equipped with the product  $\sigma$ -field. Finally,  $\beta$  is a signed measure on  $(S, \mathcal{S})$ , which plays the role of the shift

measure for  $\mu$ . Denote by  $\mathcal{S}_0$  the collection of sets  $B$  in  $\mathcal{S}$  for which

$$\gamma(B) + \|\beta\|(B) + \int_{\mathbb{R} \setminus \{0\}} \llbracket x \rrbracket^2 F(B, dx) < \infty,$$

where

$$\llbracket x \rrbracket = \begin{cases} x & \text{if } |x| \leq 1, \\ \text{sign}(x) & \text{otherwise,} \end{cases}$$

and  $\|\beta\|$  is the total variation norm of  $\beta$ .

With all elements of the triple defined, we can now define the infinitely divisible random measure  $(\mu(B), B \in \mathcal{S}_0)$  as a stochastic process for which, for every sequence of disjoint  $\mathcal{S}_0$ -sets  $B_1, B_2, \dots$ , the random variables  $\mu(B_1), \mu(B_2), \dots$  are independent (that is,  $\mu$  is independently scattered) and if, in addition,  $\bigcup_n B_n \in \mathcal{S}_0$ , then  $\mu(\bigcup_n B_n) = \sum_n \mu(B_n)$  a.s. (that is,  $\mu$  is  $\sigma$ -additive). Finally, for every  $B \in \mathcal{S}_0$ ,  $\mu(B)$  is an infinitely divisible random variable with characteristic function given by

$$\begin{aligned} & \mathbb{E} \left\{ e^{i\theta\mu(B)} \right\} \\ &= \exp \left\{ -\frac{1}{2}\gamma(B)\theta^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{i\theta x} - 1 - i\theta \llbracket x \rrbracket) F(B, dx) + i\theta\beta(B) \right\}, \end{aligned}$$

for  $\theta \in \mathbb{R}$ . The monograph [19] can be consulted for information on infinitely divisible random variables.

We shall assume (without loss of generality) that the Lévy measure  $F$  has the form

$$(5) \quad F(A) = \int_S \rho(s; A_s) m(ds),$$

for each measurable  $A \subset S \times (\mathbb{R} \setminus \{0\})$ , where  $A_s = \{x \in \mathbb{R} \setminus \{0\} : (s, x) \in A\}$  is the  $s$ -section of the set  $A$ . In (5),  $m$  is a  $\sigma$ -finite measure on  $(S, \mathcal{S})$  (the *control measure* of  $\mu$ ), and the measures  $(\rho(s; \cdot))$  (the *local Lévy measures*) form a family of Lévy measures on  $\mathbb{R}$  such that for every Borel set  $C \subset \mathbb{R} \setminus \{0\}$ ,  $s \rightarrow \rho(s; C)$  is a measurable function on  $S$ . We can, and shall, choose the control measure  $m$  in (5) in such a way that  $\|\beta\|$  is absolutely continuous with respect to  $m$ , and define the Radon-Nikodym derivative  $b = d\beta/dm$ . The local Lévy measures  $\rho$ , which, intuitively, control the Poisson structure of the random measure  $\mu$  around different points of the space  $S$ , will play a central role in all that follows.

Note that while it is possible, and common, to choose  $m$  in with the added feature that  $\gamma$  is also absolutely continuous with respect to  $m$ , and

that  $\rho(s; \mathbb{R} \setminus \{0\}) > 0$  on a set of  $s \in S$  of full measure  $m$ , we shall not require this and so shall not do so.

Finally, we assume that the kernel  $f(s; t)$ ,  $s \in S$ ,  $t \in \widetilde{M}$  in (4) is deterministic and real, such that, for every  $t \in \widetilde{M}$ , the mapping  $f(\cdot; t) : S \rightarrow \mathbb{R}$  is measurable, and that the following three inequalities hold:

$$(6) \quad \int_S f(s; t)^2 \gamma(ds) < \infty,$$

$$(7) \quad \int_S \int_{\mathbb{R} \setminus \{0\}} \llbracket xf(s; t) \rrbracket^2 F(ds, dx) < \infty,$$

and

$$(8) \quad \int_S \left| b(s)f(s; t) + \int_{\mathbb{R} \setminus \{0\}} (\llbracket xf(s; t) \rrbracket - \llbracket x \rrbracket f(s; t)) \rho(s; dx) \right| m(ds) < \infty.$$

These conditions guarantee that the random field  $(X(t), t \in \widetilde{M})$  in (4) is well defined.

A particularly simple, but rather useful, example of this setup is studied in Section 6 below, when  $X$  is a moving average random field. In this example both  $\gamma$  and  $\beta$  components of the generating triple vanish, so, in particular, the random field has no Gaussian component. Furthermore,  $S = \mathbb{R}^d$ , the control measure  $m$  is Lebesgue, and the local Lévy measures  $\rho(s, \cdot)$  are independent of  $s$ . Finally, the kernel function  $f$  is of the form  $f(s, t) = g(s + t)$  for some suitable  $g$ , and so the random field is given by

$$(9) \quad X(t) = \int_{\mathbb{R}^d} g(s + t) \mu(ds), \quad t \in \widetilde{M} \subset \mathbb{R}^d.$$

The random measure  $\mu$  has, in this case, the stationarity property  $\mu(A) \stackrel{\mathcal{L}}{=} \mu(t + A)$  for all Borel  $A$  of a finite Lebesgue measure and  $t \in \mathbb{R}^d$ , which immediately implies that a moving average random field is stationary. An impatient reader, who already wants to see results without wading through technicalities, might want to now skip directly to Section 6.2 to see what our results have to say for moving averages.

Returning to the general model (4), note that it has been defined in considerable generality, so as to allow for as wide range of applications as possible. For example, we retain the Gaussian component of the random field  $X$ . However, the tail assumptions imposed below will have the effect of ensuring that the Gaussian component will not play a role in the geometric structure of high level excursion sets.

3.2. *Regularity properties.* We shall require that the sample paths of  $X$  satisfy a number of regularity properties for the theory we are developing to hold. The main assumption will be that the paths of  $X$  are a.s.  $C^2$ , for which good sufficient conditions exist. The secondary assumptions require a little more regularity, centered around the notion of *Morse functions*. For more details see Chapter 9 in [1].

Recall that a function  $f : M \rightarrow \mathbb{R}$  is called a Morse function on the stratified manifold  $M$  if it satisfies the following two conditions on each stratum  $\partial_k M$ ,  $k = 0, \dots, \dim(M)$ .

- (i)  $f|_{\partial_k M}$  it is non-degenerate on  $\partial_k M$ , in the sense that the determinant of the Hessian of  $f|_{\partial_k M}$  at its critical points does not vanish.
- (ii) The restriction of  $f$  to  $\overline{\partial_k M} = \bigcup_{j=0}^k \partial_j M$  has no critical points on  $\bigcup_{j=0}^{k-1} \partial_j M$ .

Here is our first, and henceforth ubiquitous, assumption.

ASSUMPTION 3.1. *On an event of probability 1, the random field  $X$  has  $C^2$  sample paths on  $\widetilde{M}$  and is a Morse function on  $M$ .*

Sufficient conditions for Assumption 3.1 to hold are not hard to come by. As far as the  $C^2$  assumption is concerned, it suffices to treat the Gaussian and non-Gaussian components of  $X$  separately. For the Gaussian part, there is a rich theory of necessary and sufficient conditions for continuity, which are easy, at least as far as sufficiency is concerned, to translate into conditions that ensure the  $C^2$  property also holds. For details, see Section 1.4.2 of [1].

Necessary and sufficient conditions for the  $C^2$  assumption on the non-Gaussian component are not known, but a number of sufficient conditions exist. It is not our goal in this paper to develop the best possible conditions of this sort, so we restrict ourselves to one situation that covers, nonetheless, a wide range of random fields. Specifically, we will assume that the  $\gamma$  and the  $\beta$  components in the generating triple of the infinitely divisible random measure  $M$  vanish, and that the local Lévy measures  $\rho$  in (5) are symmetric; i.e.  $\rho(s; -A) = \rho(s; A)$  for each  $s \in S$  and each Borel  $A \in \mathbb{R} \setminus \{0\}$ . That is,  $M$  is a symmetric infinitely divisible random measure without a Gaussian component.

The following result gives sufficient conditions for a symmetric infinitely divisible random field without a Gaussian component to have sample functions in  $C^2$ . (The conditions are also necessary after a slight tightening of the assumptions on the null sets involved. cf. Theorem 5.1 of [4].)

**THEOREM 3.2.** *For a symmetric random field of the form (4), with  $M$  an infinitely divisible random measure without a Gaussian component, suppose that the kernel  $f : S \times \widetilde{M} \rightarrow \mathbb{R}$  is (product)-measurable. Assume that for every  $s \in S$  outside of set of zero  $m$ -measure the function  $f(s; \cdot) : \widetilde{M} \rightarrow \mathbb{R}$  is  $C^2$ . Furthermore, assume that the partial derivatives*

$$f_i(s; t) = \frac{\partial f}{\partial t_i}(s; t), \quad i = 1, \dots, d, \quad f_{ij}(s; t) = \frac{\partial^2 f}{\partial t_i \partial t_j}(s; t), \quad i, j = 1, \dots, d$$

satisfy the following conditions.

- (i) *The integrability condition (7) holds when the kernel  $f(s; t)$  there is replaced by any of the  $f_i(s; t)$  or  $f_{ij}(s; t)$ .*
- (ii) *The random fields*

$$(10) \quad X_{ij}(t) = \int_S f_{ij}(s; t) \mu(ds), \quad t \in \widetilde{M},$$

*$i, j = 1, \dots, d$ , are all sample continuous.*

*Then the random field  $(X(t), t \in \widetilde{M})$  has (a version with) sample functions in  $C^2$ .*

**PROOF.** For  $i = 1, \dots, d$ , define the random fields

$$X_i(t) = \int_S f_i(s; t) \mu(ds), \quad t \in \widetilde{M},$$

where  $t = (t_1, \dots, t_d) \in \widetilde{M}$ . Furthermore, defining for  $j = 1, \dots, d$

$$\ell_j(t) = \inf \left\{ s : (t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_d) \in \widetilde{M} \right\},$$

note that

$$\begin{aligned} X_i(t) &= X_i(t_1, \dots, t_{j-1}, \ell_j(t), t_{j+1}, \dots, t_d) \\ &\quad + \int_{\ell_j(t)}^{t_j} X_{ij}(t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_d) ds. \end{aligned}$$

(cf. Theorem 5.1 of [4].) Since each  $X_{ij}$  was assumed to be sample continuous, it follows that the  $X_i$  are  $C^1$ .

Repeating the argument gives that

$$\begin{aligned} X(t) &= X(t_1, \dots, t_{i-1}, \ell_j(t), t_{i+1}, \dots, t_d) \\ &\quad + \int_{\ell_j(t)}^{t_j} X_i(t_1, \dots, t_{j-1}, s, t_{j+1}, \dots, t_d) ds, \end{aligned}$$

which shows that  $X$  is  $C^2$ . □

Thus, in searching for sufficient conditions for the a.s. second order differentiability of  $X$ , it suffices to establish the continuity of the random fields of (10). While there are no known necessary and sufficient conditions for sample continuity of general infinitely divisible random fields, various sufficient conditions are available. See, for example, Chapter 10 of [18] for the special case of stable random fields, or [9] for certain other classes of infinitely divisible random fields.

This is as far as we will go at the moment discussing the issue of differentiability in Assumption 3.1. Conditions sufficient for  $X$  to be a Morse function, also required in this assumption, are, in principle, available as well. For example, it follows from the arguments of Section 11.3 of [1] (cf. Theorem 11.3.1 there) that a  $C^2$  field  $X$  will also be, a.s., a Morse function on the unit cube  $I_d$  if the following two conditions are satisfied, for each face  $J$  of  $I_d$ , and for all  $t \in J$ .

- (i) The marginal densities  $p_t(x)$  of  $\nabla X_{|J}(t)$  are continuous at 0, uniformly in  $t$ .
- (ii) The conditional densities  $p_t(z|x)$  of  $Z = \det \nabla^2 X_{|J}(t)$  given  $\nabla X_{|J}(t) = x$  are continuous in  $(z, x)$  in a neighbourhood of 0, uniformly in  $t$ .

Related conditions are given in Theorem 11.3.4 there for the case of  $M$  a stratified manifold.

It does not seem to be trivial to translate the above conditions into general conditions on the kernel  $f$  and the triple  $(\gamma, F, \beta)$ , and we shall not attempt to do so in this paper. On the other hand, given a specific kernel and triple, they are generally not too hard to check. In the purely Gaussian case, simple sufficient conditions are provided by Corollary 11.3.2 of [1], but it is the more involved infinitely divisible case that is at the heart of the current paper. If the latter random field is, actually a so-called type- $G$  random field (see [15]) (symmetric  $\alpha$ -stable random fields,  $0 < \alpha < 2$  are a special case of type- $G$  random fields), then these fields can be represented as mixtures of centered Gaussian random fields, and Corollary 11.3.2 in [1] may be helpful once again.

We close this section with a remark and a further assumption.

REMARK 3.3. Unless  $X$  is Gaussian, Assumption 3.1 implies that it is possible to modify the kernel  $f$  in (4), without changing the finite dimensional distributions of  $X$ , in such a way that  $f(s, \cdot)$  is  $C^2$  for every  $s \in S$ ; see Theorem 4 of [13]. For simplicity we shall therefore assume throughout that  $f$  has such  $C^2$  sections. This ensures, in particular, measurability of functions of the type  $\sup_{t \in M} |f(s, t)|$ ,  $s \in S$ , which we shall take as given in what follows.

ASSUMPTION 3.4. *The kernel  $f(s, t)$ ,  $s \in S$ ,  $t \in \widetilde{M}$ , along with its first and second order spatial partial derivatives  $f_i$  and  $f_{ij}$  are (uniformly) bounded and, for every  $s \in S$ , the function  $f(s, \cdot)$  is a Morse function on  $M$ .*

3.3. *The function space Lévy measure.* Although the infinitely divisible random fields we are studying in this paper were constructed above via stochastic integrals (4) and, as such, are characterised by the triple  $(\gamma, F, \beta)$  of the random measure  $\mu$  and the kernel  $f$ , in what follows the most important characteristic of the infinitely divisible random field (4) will be its function space Lévy measure. This is a measure on the cylinder sets of  $\mathbb{R}^{\widetilde{M}}$ , related to the parameters in the integral representation of the field by the formula

$$(11) \quad \lambda_X = F \circ T_f^{-1},$$

where  $F$  is the Lévy measure of the infinitely divisible random measure  $\mu$  and  $T_f : S \times (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}^{\widetilde{M}}$  is given by

$$(12) \quad T_f(s, x) = xf(s, \cdot), \quad s \in S, \quad x \in \mathbb{R} \setminus \{0\},$$

cf. [11]. Thus the finite dimensional distributions of  $X$  are given via the joint characteristic function

$$(13) \quad \mathbb{E} \left\{ \exp \left\{ i \sum_{j=1}^k \gamma_j X(t_j) \right\} \right\} = \exp \left\{ -Q(\gamma_1, \dots, \gamma_k) \right. \\ \left. + \int_{\mathbb{R}^{\widetilde{M}}} \left[ \exp \left( i \sum_{j=1}^k \gamma_j x(t_j) \right) - 1 - i \sum_{j=1}^k \gamma_j \llbracket x(t_j) \rrbracket \right] \lambda_X(dx) + iL(\gamma_1, \dots, \gamma_k) \right\},$$

for  $k \geq 1$ ,  $t_1, \dots, t_k \in \widetilde{M}$ , and real numbers  $\gamma_1, \dots, \gamma_k$ , where  $Q$  is a quadratic function (corresponding to the Gaussian part of  $X$ ), and  $L$  is a linear function (corresponding to the shift). Their exact forms are not important for us at the moment.

Note that the Lévy measures of the first and second order partial derivatives  $X_i$  and  $X_{ij}$  are similarly (cf. Theorem 5.1, [4]) given by

$$(14) \quad \lambda_{X_i} = F \circ T_{f_i}^{-1}, \quad \lambda_{X_{ij}} = F \circ T_{f_{ij}}^{-1}, \quad i, j = 1, \dots, d.$$

3.4. *Regular variation.* We now turn to the final set of technical assumptions on our infinitely divisible random fields, these being related to the regular variation of their Lévy measures, and which we formulate in

terms of the local Lévy measures of (5). While these may be our final set of assumptions, our main results hinge on these assumptions.

Recall that a function  $f$  is regularly varying at infinity, with exponent  $\alpha$ , if

$$(15) \quad \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha \quad \text{for all } \lambda > 0.$$

ASSUMPTION 3.5. *There exists a  $H : (0, \infty) \rightarrow (0, \infty)$  that is regularly varying at infinity with exponent  $-\alpha$ ,  $\alpha > 0$ , and non-negative measurable functions  $w_+$  and  $w_-$  on  $S$  such that*

$$(16) \quad \lim_{u \rightarrow \infty} \frac{\rho(s; (u, \infty))}{H(u)} = w_+(s), \quad \lim_{u \rightarrow \infty} \frac{\rho(s; (-\infty, -u))}{H(u)} = w_-(s)$$

for all  $s \in S$ . Furthermore, the convergence is uniform in the sense there is  $u_0 > 0$  such that, for all  $u > u_0$  and all  $s \in S$ ,

$$\frac{\rho(s; (u, \infty))}{H(u)} \leq 2w_+(s), \quad \frac{\rho(s; (-\infty, -u))}{H(u)} \leq 2w_-(s).$$

The following simple lemma relates Assumption 3.5 to the corresponding behaviour of the Lévy measure  $\lambda_X$  on a set of crucial importance to us. We adopt the standard notation  $a_+ = \max(a, 0)$  and  $a_- = (-a)_+$  for the positive and negative parts of a real.

LEMMA 3.6. *Let Assumption 3.5 hold.*

(i) *Assume that the kernel  $f(s, t)$ ,  $t \in \widetilde{M}$  is uniformly (in  $s \in S$ ) bounded, and that for some  $\epsilon > 0$ ,*

$$(17) \quad \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f(s, t)|^{\alpha - \epsilon} m(ds) < \infty.$$

Then

$$(18) \quad \begin{aligned} & \lim_{u \rightarrow \infty} \frac{\mathbb{P} \{ \sup_{t \in M} X(t) > u \}}{H(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\lambda_X \{ g : \sup_{t \in M} g(t) > u \}}{H(u)} \\ &= \int_S \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \right] m(ds), \end{aligned}$$

where  $M$  can be replaced with  $\widetilde{M}$  throughout. Furthermore,

$$\begin{aligned}
(19) \quad & \lim_{u \rightarrow \infty} \frac{\mathbb{P} \left\{ \sup_{t \in \widetilde{M}} |X(t)| > u \right\}}{H(u)} \\
&= \lim_{u \rightarrow \infty} \frac{\lambda_X \left\{ g : \sup_{t \in \widetilde{M}} |g(t)| > u \right\}}{H(u)} \\
&= \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f(s, t)|^\alpha m(ds).
\end{aligned}$$

(ii) Assume that the first order partial derivatives  $f_i(s, t)$ ,  $t \in \widetilde{M}$ ,  $i = 1, \dots, d$  are uniformly (in  $s \in S$ ) bounded, and that for some  $\epsilon > 0$ ,

$$(20) \quad \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f_i(s, t)|^{\alpha - \epsilon} m(ds) < \infty.$$

Then

$$\begin{aligned}
(21) \quad & \lim_{u \rightarrow \infty} \frac{\lambda_{X_i} \left\{ g : \sup_{t \in \widetilde{M}} |g(t)| > u \right\}}{H(u)} \\
&= \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f_i(s, t)|^\alpha m(ds).
\end{aligned}$$

(iii) Assume that the second order partial derivatives  $f_{ij}(s, t)$ ,  $t \in \widetilde{M}$ ,  $i, j = 1, \dots, d$  are uniformly (in  $s \in S$ ) bounded, and that for some  $\epsilon > 0$ ,

$$(22) \quad \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f_{ij}(s, t)|^{\alpha - \epsilon} m(ds) < \infty.$$

Then

$$\begin{aligned}
(23) \quad & \lim_{u \rightarrow \infty} \frac{\lambda_{X_{ij}} \left\{ g : \sup_{t \in \widetilde{M}} |g(t)| > u \right\}}{H(u)} \\
&= \int_S (w_+(s) + w_-(s)) \sup_{t \in \widetilde{M}} |f_{ij}(s, t)|^\alpha m(ds).
\end{aligned}$$

PROOF. The first equality in (18) follows from the second equality there by Theorem 2.1 in [16]. As for the second equality in (18), it follows from (11) and (5) that

$$\begin{aligned}
\lambda_X \left\{ g : \sup_{t \in M} g(t) > u \right\} &= \int_S \left[ \rho \left( s; \left( \frac{u}{\sup_{t \in M} f(s, t)_+}, \infty \right) \right) \right. \\
&\quad \left. + \rho \left( s; \left( -\infty, \frac{-u}{\sup_{t \in M} f(s, t)_-} \right) \right) \right] m(ds).
\end{aligned}$$

Using the uniform boundedness of the kernel and Potter's bounds (cf. [10] or [3], Theorem 1.5.6) we see that for any  $\epsilon > 0$  there is  $C > 0$  such that for all  $u > 1$ ,

$$\frac{\rho\left(s; \left(\frac{u}{\sup_{t \in M} f(s, t)_+}, \infty\right)\right)}{H(u)} \leq C \sup_{t \in M} f(s, t)_+^{\alpha - \epsilon},$$

and

$$\frac{\rho\left(s; \left(-\infty, -\frac{u}{\sup_{t \in M} f(s, t)_-}\right)\right)}{H(u)} \leq C \sup_{t \in M} f(s, t)_-^{\alpha - \epsilon}.$$

The limit (18) now follows from Assumption 3.5 via (17), regular variation, and dominated convergence. The proof of (19) is identical, as are the proofs of (ii) and (iii).  $\square$

**REMARK 3.7.** The assumption of uniform boundedness of the kernel  $f$  in (4) and its partial derivatives will be kept throughout the paper (it is already a part of Assumption 3.4), but the only place it is used is in Lemma 3.6. It is not difficult to see that this assumption can be removed at the expense of appropriate assumptions on the behaviour near the origin of the local Lévy measures in (5) and of slightly modifying the integrability condition (17). Given that this paper is already rather heavy on notation, we shall continue to work with uniform integrability, which helps keep things comparatively tidy. Note that it is also clear that, for the purpose of proving (18) alone, the integrability assumption (17) could be relaxed.

**4. Limiting distributions for critical points.** Our initial aim, as described in the Introduction, was to obtain information about the distribution of the Euler characteristic of the excursion sets of (1). As is known from Morse critical point theory, Euler characteristics of excursion sets are closely related to the critical points above fixed levels. We shall describe this connection in the following section and, for the moment, concentrate on the critical points of  $X$ , which are also of intrinsic interest.

Recall the partition (3) of the stratified manifold  $M$  into collections  $\partial_k M$  of manifolds of dimension  $k$ . Let  $J$  denote one such manifold, of dimension  $0 \leq k \leq d = \dim(M)$ .

Now let  $g$  be a  $C^2$  function on  $\widetilde{M}$ , and for  $i = 0, 1, \dots, \dim(J)$ , let  $\mathcal{C}_g(J; i)$  be the set of *critical points of index  $i$*  of  $g|_J$ . These are the points for which  $\nabla g(t)$  is normal to the tangent plane to  $J$  at  $t$ , and for which the index of the Hessian of  $g|_J$ , computed with respect to any orthonormal basis

for the tangent plane, when considered as a matrix, has index  $i$ . (Recall that the index of a matrix is the number of its negative eigenvalues.) For a full discussion of the critical points of functions on stratified manifolds see Chapter 9 of [1]. Let

$$N_g(J; i) = \text{Card}(\mathcal{C}_g(J; i)),$$

and, for real  $u$ ,

$$N_g(J; i : u) = \text{Card}(\mathcal{C}_g(J; i) \cap \{t : g(t) > u\}),$$

be the overall number of the critical points of different types of  $g$ , and the number of these critical points above the level  $u$ , correspondingly. Since  $g$  is a Morse function, it is standard fare that all of the above numbers are finite. (e.g. [1])

Just a little more notation is required for the main theorem. Let  $f$  be the kernel in the integral representation (4) of an infinitely divisible random field. For  $k = 0, 1, \dots, d$ , a manifold  $J$  and  $i = 0, 1, \dots, \dim(J)$ , let

$$c_i(J; s) = N_{f(s; \cdot)}(J; i)$$

be the number of the critical points of the  $s$ -section of  $f$  of the appropriate type, well defined since by Assumption 3.4 the sections are Morse functions.

Furthermore, let  $(t_l(J; i; s), l = 1, \dots, c_i(J; s))$  be an enumeration of these critical points, and, for  $1 \leq m \leq c_i(J; s)$  let

$$f_{[m]}^{(J; i; +)}(s), \quad f_{[m]}^{(J; i; -)}(s),$$

be, correspondingly, the  $m$ -th largest of the positive parts  $(f(s; t_l(J; i; s)))_+$ ,  $l = 1, \dots, c_i(J; s)$ , and the  $m$ -th largest of the negative parts  $(f(s; t_l(J; i; s)))_-$ ,

$l = 1, \dots, c_i(J; s)$ . (Both of these quantities are set to be equal zero if  $m > c_i(J; s)$ ).

Finally, extend these definitions to  $m = 0$  by setting

$$f_{[0]}^{(J; i; +)}(s) = \sup_{t \in M} (f(s; t))_+, \quad f_{[0]}^{(J; i; -)}(s) = \sup_{t \in M} (f(s; t))_-.$$

The following theorem is the main result of this paper. It describes the limiting, conditional, joint distribution of the number of critical points of all possible types of a infinitely divisible random field over the level  $u$ , as  $u \rightarrow \infty$ , given that the random field actually exceeds level  $u$  at some point.

**THEOREM 4.1.** *Let  $(X(t), t \in \widetilde{M})$  be an infinitely divisible random field with representation (4), satisfying Assumptions 3.1, 3.4 and 3.5. Assume, furthermore, that (17) holds for some  $\epsilon > 0$ . Then, for any collection  $\mathcal{J}$  of manifolds in the various strata  $\partial_k M$ ,  $k \in \{0, 1, \dots, d\}$ , of  $M$ , and any collection of non-negative integers  $\{n(J; i) = 0, 1, \dots, i = 0, 1, \dots, \dim(J), J \in \mathcal{J}\}$ ,*

$$(24) \quad \mathbb{P} \left\{ N_X(J; i : u) \geq n(J; i), J \in \mathcal{J}, i = 0, 1, \dots, \dim(J) \mid \sup_{t \in M} X(t) > u \right\} \\ \rightarrow \frac{\int_S \left[ w_+(s) \left( \min_{J,i} f_{[n(J;i)]}^{(J;i;+)}(s) \right)^\alpha + w_-(s) \left( \min_{J,i} f_{[n(J;i)]}^{(J;k-i;-)}(s) \right)^\alpha \right] m(ds)}{\int_S \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \right] m(ds)},$$

as  $u \rightarrow \infty$ .

The proof of the theorem appears in Section 7.

**REMARK 4.2.** While the structure of (24) might be rather forbidding at first sight, its meaning is actually rather simple. The main point of Theorem 4.1 is that, once the random field reaches a high level, its behaviour above that level is very similar to that of the much simpler random field,

$$(25) \quad Z(t) = Vf(W, t), \quad t \in \widetilde{M},$$

where  $(V, W) \in (\mathbb{R} \setminus \{0\}) \times S$  is a random pair, the joint law of which is the finite restriction of the Lévy measure  $F$  to the set

$$\left\{ (x, s) \in (\mathbb{R} \setminus \{0\}) \times S : \sup_{t \in \widetilde{M}} |xf(s; t)| > 1 \right\},$$

normalized to be a probability measure on that set.

**REMARK 4.3.** In fact, one can go much further than in the previous remark, and interpret the limit (24) as showing that limiting conditional joint distribution of critical points is a mixture distribution, that can be described as follows. Set

$$H = \int_S \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \right] m(ds).$$

1. Select a random point  $W \in S$  according to the probability law  $\eta$  on  $S$  with

$$\frac{d\eta}{dm}(s) = H^{-1} \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \right], \quad s \in S.$$

2. Given  $W = s$ , select a random value  $I \in \{-1, 1\}$  according to the law

$$P(I = 1 | W = s) = \frac{w_+(s) \sup_{t \in M} f(s, t)_+^\alpha}{w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha}.$$

3. Let  $V_\alpha$  be a random variable independent of  $W$  and  $I$ , with  $\mathbb{P}\{V_\alpha \leq x\} = x^\alpha$  for  $0 \leq x \leq 1$ . Then the critical points  $(N_X(J; i : u), J \in \mathcal{J}$ , given that  $\sup X > u$ , have, as  $u \rightarrow \infty$ , the same distribution as the critical points of the random field

$$(26) \quad \left( \frac{f(W, t)_+}{\sup_{r \in M} f(W, r)_+}, t \in M \right)$$

above the level  $V_\alpha$  if  $I = 1$ , and the critical points of the field

$$(27) \quad \left( \frac{f(W, t)_-}{\sup_{r \in M} f(W, r)_-}, t \in M \right)$$

above the level  $V_\alpha$  if  $I = -1$ .

REMARK 4.4. While Theorem 4.1 counts critical points classified by their indices, there are also other properties of critical points that are of topological importance. For example, in [1] considerable emphasis was laid on the so-called ‘extended outward critical points’, these being the critical points  $t \in M$  for which  $\nabla f(t) \in N_t(M)$ , where  $N_t(M)$  is the normal cone of  $M$  at  $t$ .

Extended outward critical points play a major role in Morse theory, in terms of defining the Euler characteristics of excursion sets. It will be easy to see from the proof of Theorem 4.1 that its the statement remains true if one replaces critical points by extended outward critical points. This will be used in certain applications of Theorem 4.1 below.

**5. The Euler characteristic of excursion sets.** One application of Theorem 4.1 is to the Euler characteristic  $\varphi(A_u)$  of the excursion set  $A_u$  over a high level  $u$ . We shall not define the Euler characteristic here, but rather send you to [1] for details. The Euler characteristic of an excursion set of a Morse function is equal to the alternating sum of the numbers of extended outward critical points of the function over the level. This leads to the following result, an immediate corollary of Theorem 4.1, (9.4.1) in [1], and Remarks 4.3 and 4.4 above.

**COROLLARY 5.1.** *Under the conditions of Theorem 4.1, the conditional distribution of the Euler characteristic of the excursion set of an infinitely divisible random field computed with its limiting conditional distribution given that the level is exceeded, is given by the mixture of the Euler characteristics of the random fields (26) and (27), with the mixing distribution as described in Remark 4.3. In particular, the expected Euler characteristic of the excursion set of the limiting (conditional) random field is given by*

$$H^{-1} \int_S \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha \mathbb{E}\{C_+(s)\} + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \mathbb{E}\{C_-(s)\} \right] m(ds).$$

Here, for  $s \in S$ ,  $C_\pm(s)$  is the Euler characteristic of the excursion set of the field  $(f(s, t)_\pm / \sup_{r \in M} f(s, r)_\pm, t \in M)$  above the level  $V_\alpha$ .

**6. An example: Moving average fields.** The power and variety of the results of the previous two sections can already be seen in a relatively simple but application rich class of random fields, the moving average fields that were already introduced at (9). Recall that these are stationary fields with representation

$$(28) \quad X(t) = \int_{\mathbb{R}^d} g(s+t) \mu(ds), \quad t \in \mathbb{R}^d.$$

In this section we will consider a moving average random field on  $M = I_d = [-1, 1]^d$ , and the ambient manifold is simply  $\bar{M} = \mathbb{R}^d$ .

Our basic assumptions, that will hold throughout this section, are

- (i) The function  $g$  is  $C^2$  on  $\mathbb{R}^d$  and satisfies (7) and (8).
- (ii)  $\mu$  is an infinitely divisible random measure on  $\mathbb{R}^d$ , for which the Gaussian and shift components in the generating triple,  $\gamma$  and  $\beta$ , vanish.
- (iii) The control measure  $m$  in (5) is  $d$ -dimensional Lebesgue measure.
- (iv) The local Lévy measures  $\rho(s, \cdot) = \rho(\cdot)$  are independent of  $s \in \mathbb{R}^d$ .

By choosing different kernels  $g$  we shall see that quite different types of high level excursion sets arise, as opposed to the Gaussian case, in which ellipsoidal sets are, with high probability, ubiquitous,

**6.1. Checking the conditions of Theorem 4.1: type  $G$  moving averages.** In this subsection we exhibit a broad family of moving average random fields (28) for which we will verify the conditions required by the main result of Section 4. These are the so-called type  $G$  random fields. We emphasize that the applicability of our main results is not restricted to type  $G$  random fields. For the latter we can use standard tools to check the assumptions of Theorem 4.1, which is why they are presented here. The main result of this subsection is

**THEOREM 6.1.** *A moving average infinitely divisible random field  $X$  satisfying Conditions 6.2 and 6.3 below also satisfies the assumptions of Theorem 4.1.*

**CONDITION 6.2.** *The local Lévy measure  $\rho$  is a symmetric measure of the form*

$$(29) \quad \rho(B) = \mathbb{E} \left\{ \rho_0(Z^{-1}B) \right\},$$

where  $B$  is a Borel set,  $Z$  is a standard normal random variable, and  $\rho_0$  is a symmetric Lévy measure on  $\mathbb{R}$ . Furthermore, the function  $\rho_0((u, \infty))$ ,  $u > 0$  is regularly varying at infinity with exponent  $-\alpha$ ,  $\alpha > 1$ , and there is  $\beta \in [1, 2)$  such that

$$\rho_0((u, \infty)) \leq au^{-\beta},$$

for all  $0 < u < 1$ , for some  $0 < a < \infty$ .

In fact, for any Lévy measure  $\rho_0$  on  $\mathbb{R}$ , (29) defines a Lévy measure on  $\mathbb{R}$ ; see e.g. Proposition 2.2 in [8]. Furthermore, it is simple to check that the behaviour of the measures  $\rho$  and  $\rho_0$  are similar at zero and infinity. Specifically,

$$(30) \quad \lim_{u \rightarrow \infty} \frac{\rho((u, \infty))}{\rho_0((u, \infty))} = \mathbb{E} \{ Z_+^\alpha \},$$

and

$$(31) \quad \rho((u, \infty)) \leq \frac{1}{2} \mathbb{E} \left\{ \max(|Z|^\beta, 1) au^{-\beta} \right\}$$

for  $0 < u < 1$ . In particular, a moving average infinitely divisible random field satisfying Condition 6.2 automatically also satisfies Assumption 3.5. It suffices to choose  $H(u) = \rho_0((u, \infty))$ ,  $u > 0$  and  $w_+(s) = w_-(s) = \mathbb{E} \{ Z_+^\alpha \}$ .

It is condition 6.2 that makes the random field a “type  $G$  random field”. It implies that the random field  $X$  can be represented as a certain mixture of stationary Gaussian fields, cf. [8]. Under the conditions we impose, each one of the latter is a.s. a Morse function, which will tell us that the moving average itself has sample functions which are, with probability 1, Morse functions.

If it is known from other considerations that the sample functions of a specific infinitely divisible random field are, with probability 1, Morse functions, then (29) is not needed, and only the assumptions on the behaviour of the tails of the Lévy measure  $\rho((u, \infty))$  as  $u \rightarrow 0$  or  $u \rightarrow \infty$  are required. In the present form of Condition 6.2 these assumptions become the conclusions (30) and (31) from the corresponding assumptions on the Lévy measure  $\rho_0$ .

CONDITION 6.3. *The kernel  $g$  is in  $C^3$ , and its restriction to any bounded hypercube is a Morse function. Assume that the first and the second derivatives  $g_i$ ,  $i = 1, \dots, d$  and  $g_{ij}$ ,  $i, j = 1, \dots, d$  satisfy (7) and (8). Assume, further, that for almost every  $s \in \mathbb{R}^d$  there is no subspace of dimension strictly less than  $(d^2 + 3d)/2$  to which the vectors  $(g_i(s), i = 1, \dots, d, g_{ij}(s), i, j = 1, \dots, d, i \leq j)$  belong.*

*Finally, assume that the function*

$$T_g(s) = \sup_{t \in [-1,1]^d} |g(s+t)|, \quad s \in \mathbb{R}^d$$

*satisfies  $T_g \in L^{\alpha-\varepsilon}(\mathbb{R}^d)$ , while the function*

$$\tilde{T}_g(s) = \max_{i,j \in 1, \dots, d} |g_{ij}(s)| + \sup_{t \in [-1,1]^d, i,j,k \in 1, \dots, d} |g_{ijk}(s+t)|, \quad s \in \mathbb{R}^d$$

*satisfies  $\tilde{T}_g \in L^{\alpha-\varepsilon}(\mathbb{R}^d) \cap L^\beta(\mathbb{R}^d)$  for some  $\varepsilon > 0$  and for the  $\alpha$  and  $\beta$  for which Condition 6.2 holds.*

Since these are assumptions on the kernel  $g$  in the integral representation (28) of the random field, and the kernel is often explicitly given, the above conditions are, generally, easy to apply. See the examples below.

Clearly, a moving average infinitely divisible random field satisfying Condition 6.3 will also satisfy Assumption 3.4. It also satisfies (17). Theorem 6.1 is then an immediate consequence of the following two lemmas.

LEMMA 6.4. *A moving average infinitely divisible random field satisfying Condition 6.3 has sample paths in  $C^2$ .*

PROOF. The only assumption of Theorem 3.2 that is not stated explicitly above is the sample continuity of the second order derivative random fields. To establish that we use Theorem 3.1 in [9]. The conditions of the latter theorem are easily seen to follow from the simple facts that, firstly, for any  $C^1$  function  $h$  on  $\mathbb{R}^d$ ,

$$\sup_{\substack{t, r \in I_d, \\ r \neq t}} \frac{h(t) - h(r)}{\|t - r\|} \leq d \sup_{t \in I_d, i=1, \dots, d} |h_i(t)|,$$

and, secondly, that the metric entropy condition in Remark 2.1 in [9] trivially holds for any  $q > 2$  for the hypercube  $I_d = [-1, 1]^d$  and Euclidian distance on that hypercube.  $\square$

To complete the proof of Theorem 6.1 we need to check that a moving average satisfying Conditions 6.2 and 6.3 has sample functions that are, with probability 1, Morse functions. As mentioned above, we shall accomplish this by representing the random field  $X$  as a mixture of zero mean Gaussian random fields, each one of which will have, with probability 1, sample functions that are Morse functions.

LEMMA 6.5. *A moving average infinitely divisible random field satisfying Conditions 6.2 and 6.3 has sample functions that are, with probability 1, Morse functions.*

PROOF. Let  $\nu$  and  $\tilde{\nu}$  be probability measures on  $\mathbb{R}$  and  $\mathbb{R}^d$  absolutely continuous with respect to the Lévy measure  $\rho_0$  and to  $d$ -dimensional Lebesgue measure  $\lambda_d$ , respectively. Let

$$\psi(x) = \frac{d\nu}{d\rho_0}(x), \quad x \in \mathbb{R}, \quad \varphi(s) = \frac{d\tilde{\nu}}{d\lambda_d}(s), \quad s \in \mathbb{R}^d.$$

Then the random field  $X$  has a representation as an infinite sum of the form

$$(32) \quad X(t) = \sum_{k=1}^{\infty} Z_k V_k g(t + H_k) \mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1), \quad t \in \mathbb{R}^d,$$

where  $(Z_1, Z_2, \dots)$  are i.i.d. standard normal random variables,  $(V_1, V_2, \dots)$  are i.i.d. random variables with a common law  $\nu$ ,  $(H_1, H_2, \dots)$  are i.i.d. random vectors in  $\mathbb{R}^d$  with a common law  $\tilde{\nu}$ , and  $(\Gamma_1, \Gamma_2, \dots)$  are the points of a unit rate Poisson process on  $(0, \infty)$ . All four sequences are independent. See Section 5 in [8] for details. Furthermore, by Lemma 6.4 and Theorem 3.2, the first and second order partial derivatives of  $X$  are also moving average random fields, and so have a corresponding series representation. In particular,

$$(33) \quad X_i(t) = \sum_{k=1}^{\infty} Z_k V_k g_i(t + H_k) \mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1), \quad t \in \mathbb{R}^d$$

for each  $i = 1, \dots, d$  and

$$(34) \quad X_{ij}(t) = \sum_{k=1}^{\infty} Z_k V_k g_{ij}(t + H_k) \mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1), \quad t \in \mathbb{R}^d$$

for each  $i, j = 1, \dots, d$ .

We may assume, without loss of generality, that the standard Gaussian sequence  $(Z_1, Z_2, \dots)$  is defined on a probability space  $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ , and

the remaining random variables on the right hand sides of (32) - (34) are defined on a different probability space  $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ , so that the random fields defined by the series are defined on the product probability space  $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$ . Thus, for every fixed  $\omega_2 \in \Omega_2$ , the conditional random field  $X((\omega_1, \omega_2))$ ,  $\omega_1 \in \Omega_1$ , is a centered Gaussian random field. We now apply to this random field Corollary 11.3.2 in [1].

Firstly we check the condition on the incremental variance of the second order partial derivatives there. In obvious notation, for every  $i, j = 1, \dots, d$ , and  $t, s \in M$ ,

$$\begin{aligned} & \mathbb{E}_1 \left\{ (X_{ij}(t) - X_{ij}(s))^2 \right\} \\ &= \sum_{k=1}^{\infty} V_k^2 (g_{ij}(t + H_k) - g_{ij}(s + H_k))^2 \mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1). \end{aligned}$$

Bounding the Hölder constant of a function by the largest value of its partial derivatives, as in the proof of Lemma 6.4, we obtain

$$\begin{aligned} & \mathbb{E}_1 \left\{ (X_{ij}(t) - X_{ij}(s))^2 \right\} \\ & \leq d^2 \|t - s\|^2 \sum_{k=1}^{\infty} V_k^2 \tilde{T}_g^2(H_k) \mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1), \end{aligned}$$

and, hence, the incremental variance condition will follow once we check that the infinite sum above converges. For this, we need to check (see [14]) that

$$\int_0^{\infty} \mathbb{E}_2 \left\{ \min \left[ 1, V^2 \tilde{T}_g(H)^2 \mathbb{1}(\psi(V)\varphi(H)x \leq 1) \right] \right\} dx < \infty.$$

(The random variables without a subscript represent generic members of the appropriate sequences.) By the definition of the derivatives  $\psi$  and  $\varphi$ , this reduces to checking that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \min \left[ 1, y^2 \tilde{T}_g(s)^2 \right] ds dy < \infty,$$

which is an elementary consequence of the integrability assumptions imposed on  $\tilde{T}_g$  in Condition 6.3, and of the assumptions imposed on the Lévy measure  $\rho_0$  in Condition 6.2.

It remains to check that the joint distribution under  $\mathbb{P}_1$  of the random vectors of partial derivatives  $(X_i, X_{ij})$  is non-degenerate for  $\mathbb{P}_2$ -almost every  $\omega_2$ . This, however, follows from the representation (33) and (34) of the derivatives and the part of Condition 6.3 that rules out the possibility that the derivatives of the kernel  $g$  belong to a lower dimensional subspace.  $\square$

6.2. *Examples: how the shape of a kernel can affect the geometry of the excursion set.* In the examples below Condition 6.2 is a standing assumption, and will not be mentioned explicitly. Our first example is of an infinitely divisible moving average random field whose high level excursion sets have a similar geometric structure to those of Gaussian random fields.

EXAMPLE 6.6. Let  $g$  be a non-negative kernel satisfying Condition 6.3, that is also rotationally invariant and radially decreasing. Formally,  $g(t) = g_r(\|t\|)$  for some non-negative and decreasing function  $g_r$  on  $[0, \infty)$ . An example is the Gaussian kernel  $g(t) = \exp\{-a\|t\|^2\}$ ,  $a > 0$ . It is trivial to check that the restrictions in Condition 6.3 on the various partial derivatives of  $g$  hold in this case.

Corollary 5.1 tells us that the Euler characteristic of the excursion set over a high level, given that the level is exceeded, is asymptotically that of the field

$$(35) \quad \left(\sup_{r \in I_d} g(s+r)\right)^{-1} g(s+t), \quad t \in I_d,$$

with a randomly chosen  $s \in \mathbb{R}^d$  and over a random level  $V_\alpha$ .

The assumption of rotational invariance and radial monotonicity on the kernel  $g$  implies that, in this case, the excursion set of the random field is the intersection of a Euclidian ball centered at the point  $-s$  and the cube  $I_d$ . This is a convex set and, hence, has Euler characteristic equal to 1, regardless of the point  $s \in \mathbb{R}^d$  or the random level  $V_\alpha$ .

In this case the limiting conditional distribution of the Euler characteristic is degenerate at the point 1. Furthermore, the excursion set has, with high probability, a “ball-like shape”, as is the case for smooth Gaussian random fields.

In spite of the ‘Gaussian-like’ conclusion in the previous example, it is easy to modify it to make the high level excursion sets of an infinitely divisible random field behave quite differently. Here is a simple example.

EXAMPLE 6.7. We modify the kernel  $g$  of the previous example by adding to it oscillations, while preserving its smoothness and integrability properties. For example, take

$$g(t) = (1 + \cos\langle\theta, t\rangle)e^{-a\|t\|^2}, \quad t \in \mathbb{R}^d$$

for fixed  $\theta \in \mathbb{R}^d$ .

Then, depending on the random choice of the point  $s$  in (35), the structure of the excursion sets in  $I_d$  could be quite varied, as it depends on the shape of

$g$  in the translated cube  $I_d^{(-s)}$ . Thus, depending on the random level  $V_\alpha$ , the shape of the excursion set may be quite different from a ball-like shape. In particular, its Euler characteristic will have a non-degenerate distribution.

6.3. *The bottom line.* The bottom line, of course, is that the shape of the excursion sets is determined, to a large extent, by the shape of the kernel in the integral representation of the random field or, alternatively, by the geometric properties of the functions on which the Lévy measure of the random field is supported. By choosing appropriate parameters for the random field one can generate quite different distributions for the Euler and other geometric characteristics of high level excursion sets.

Our hope is that this fact will generate greater flexibility in applications, allowing the practitioner to choose models with pre-determined excursion set shapes.

Furthermore, the description of the limiting conditional *distribution* (and not only the expected value) of the numbers of critical points and so the Euler characteristic should allow one to devise better statistical tests based on the observed excursion sets. Precisely how this should be done, however, is something we shall leave for the future.

**7. Proof of Theorem 4.1 for the case  $M = [0, 1]^d$ .** The proof is rather long and rather technical, although the basic idea is not difficult.

The basic idea, which is common to many proofs involving infinitely divisible random fields  $X$ , is to write  $X$  as a sum of two parts, one of which has a much higher chance of being large than the second one. When the random field reaches a high level, it the former, larger piece that actually accounts for the limit in Theorem 4.1. This piece behaves distributionally as a Poisson sum of random, but explicit functions with random multipliers, is comparatively simple to handle. One then needs to show that the second, smaller piece can be ignored in the  $u \rightarrow \infty$  limit. In the argument that follows this is somewhat more difficult than is usually the case, since even if the small part is small in magnitude it can, in principle, have a major effect on variables such as the number of critical points of the sum. (Think of a constant function,  $f(t) \equiv \lambda$ , to which is added  $g(t) = \epsilon \cos(\langle \theta, t \rangle)$ . No matter how large  $\lambda$  might be, nor how small  $\epsilon$  might be, the critical points of  $f + g$  are determined by  $g$ , not  $f$ .)

Due to the length of the ensuing proof, we shall do our best to signpost it as it progresses.

(i) *Some notation for the parameter space and for critical points.* As mentioned earlier, in this section we shall take as our parameter space the cube

$I_d$ . The first step is to develop notation for describing its stratification.

Let  $\mathcal{J}_k$  be the collection of the  $2^{d-k} \binom{d}{k}$  faces of  $I_d$  of dimension  $k$ ,  $k = 0, \dots, d$ , and let  $\mathcal{J} = \bigcup_k \mathcal{J}_k$ . For each face  $J \in \mathcal{J}_k$  there is a corresponding set  $\sigma(J) \subseteq \{1, \dots, d\}$  of cardinality  $k$  and a sequence  $\epsilon(J) \in \{-1, 1\}^{\sigma(J)}$  such that

$$J = \{t = (t_1, \dots, t_d) \in I_d : t_j = \epsilon_j \text{ if } j \notin \sigma(J) \text{ and } 0 < t_j < 1 \text{ if } j \in \sigma(J)\}.$$

Let  $g$  be a  $C^2$  function on an open set  $\widetilde{M}$  containing  $I_d$ . For  $J \in \mathcal{J}_k$  and  $i = 0, 1, \dots, k$ , let  $\mathcal{C}_g(J; i)$  be the set of points  $t \in J$  satisfying the following two conditions.

$$(36) \quad \frac{\partial g}{\partial t_j}(t) = 0 \text{ for each } j \in \sigma(J),$$

$$(37) \quad \text{the matrix } \left( \frac{\partial^2 g(t)}{\partial t_m \partial t_n} \right)_{m, n \in \sigma(J)} \text{ has non-zero determinant}$$

and its index is equal to  $k - i$ .

Now define  $N_g(J; i)$  and  $N_g(J; i : u)$  in terms of  $\mathcal{C}_g(J; i)$  as in Section 4.

(ii) *Splitting  $X$  into large and small components.*

By Assumption 3.1,  $X$  and its first and second order partial derivatives are a.s. bounded on  $\widetilde{M}$ , and, by (11), the Lévy measure of  $X$  is concentrated on  $C^2$  functions. Defining

$$S_L = \left\{ g \in C^2 : \max \left[ \sup_{t \in \widetilde{M}} |g(t)|, \sup_{t \in \widetilde{M}, i=1, \dots, d} |g_i(t)|, \sup_{t \in \widetilde{M}, i, j=1, \dots, d} |g_{ij}(t)| \right] > 1 \right\},$$

the sample boundedness of  $X$ , along with (14) and general properties of Lévy measures on Banach spaces (e.g. [7]) imply that

$$(38) \quad \theta \triangleq \lambda_X \{S_L\} < \infty.$$

We are now ready to decompose the infinitely divisible random field  $X$  into a sum of two independent infinitely divisible components by writing

$$(39) \quad X(t) = X^L(t) + Y(t), \quad t \in \widetilde{M},$$

where  $X^L$  is a compound Poisson random field with characteristic functions, which, for  $k \geq 1$ ,  $t_1, \dots, t_k \in \widetilde{M}$ , and real numbers  $\gamma_1, \dots, \gamma_k$ , are given by

$$(40) \quad \mathbb{E} \left\{ \exp \left\{ i \sum_{j=1}^k \gamma_j X_L(t_j) \right\} \right\} = \exp \left\{ \int_{S_L} \left( \exp \left\{ i \sum_{j=1}^k \gamma_j x(t_j) \right\} - 1 \right) \lambda_X(dx) \right\}.$$

The second, or ‘residual’, component  $Y$  has characteristic functions

$$\begin{aligned} \mathbb{E} \left\{ \exp \left\{ i \sum_{j=1}^k \gamma_j Y(t_j) \right\} \right\} &= \exp \left\{ -Q(\gamma_1, \dots, \gamma_k) \right. \\ &\left. + \int_{\mathbb{R}^{\tilde{M}} \setminus S_L} \left( \exp \left\{ i \sum_{j=1}^k \gamma_j x(t_j) \right\} - 1 - i \sum_{j=1}^k \gamma_j \llbracket x(t_j) \rrbracket \right) \lambda_X(dx) + iL_1(\gamma_1, \dots, \gamma_k) \right\} \end{aligned}$$

where we are using the notation of (13), and

$$L_1(\gamma_1, \dots, \gamma_k) = L(\gamma_1, \dots, \gamma_k) - \int_{S_L} \sum_{j=1}^k \gamma_j \llbracket x(t_j) \rrbracket \lambda_X(dx).$$

We shall ultimately show that the limiting behaviour of the critical points of  $X$  depends only on the component  $X^L$ , so we study it first.

(iii) *A limit theorem for the critical points of  $X^L$ .* We start by noting that it follows from the form of the characteristic function (40) and the definition (11) that  $X^L$  can, in law, be written as

$$(41) \quad X^L(t) = \sum_{m=1}^N X_m f(S_m, t),$$

where  $N$  is a Poisson random variable with mean  $\theta$  given by (38), independent of an i.i.d. sequence of random pairs  $((X_m, S_m))$ ,  $m = 1, 2, \dots$  taking values in  $(\mathbb{R} \setminus \{0\}) \times S$  with the common law  $\theta^{-1}F$  restricted to the set

$$\left\{ (s, x) \in (\mathbb{R} \setminus \{0\}) \times S : \max \left( \sup_{t \in \tilde{M}} |xf(s; t)|, \sup_{t \in \tilde{M}, i=1, \dots, d} |xf_i(s; t)|, \sup_{t \in \tilde{M}, i, j=1, \dots, d} |xf_{ij}(s; t)| > 1 \right) \right\}.$$

Recall that  $F$  is the Lévy measure of the infinitely divisible random measure  $M$  in (4).

Since the sum in (41) is a.s. finite, and the kernel  $f$  has bounded  $C^2$  sections  $f(s; \cdot)$  for all  $s \in S$ , it follows that  $X^L$  is bounded and  $C^2$  on  $\tilde{M}$ .

We now decompose the compound Poisson term  $X^L$  itself into a sum of two independent pieces, the stochastically larger of which will be responsible for the limiting behaviour of the critical points of  $X$ . For  $u > 0$  and  $1/2 < \beta < 1$  define the sequence of independent events

$$\begin{aligned} A_m(u) = \left\{ \max \left[ \sup_{t \in \tilde{M}} |X_m f(S_m; t)|, \sup_{t \in \tilde{M}, i=1, \dots, d} |X_m f_i(S_m; t)|, \right. \right. \\ \left. \left. \sup_{t \in \tilde{M}, i, j=1, \dots, d} |X_m f_{ij}(S_m; t)| \right] > u^\beta \right\}, \end{aligned}$$

and write

$$(42) \quad X^L(t) = \sum_{m=1}^N X_m f(S_m; t) \mathbb{1}_{A_m(u)} + \sum_{m=1}^N X_m f(S_m; t) \mathbb{1}_{A_m(u)^c} \\ \triangleq X^{(L,1)}(t) + X^{(L,2)}(t).$$

In Lemma 7.1 we shall show that  $X^{(L,2)}$  and its partial derivatives have suprema the tail probabilities of which decay faster than the function  $H$ , and so are unlikely to affect the critical points of  $X$ . We shall return to this point later.

Now, however, we shall concentrate on the critical points over high levels of  $X^{(L,1)}$ . Define two new events

$$(43) \quad B_1(u) = \left\{ \sum_{m=1}^N \mathbb{1}(A_m(u)) = 1 \right\}, \quad B_2(u) = \left\{ \sum_{m=1}^N \mathbb{1}(A_m(u)) \geq 2 \right\}.$$

The first of these occurs when there is a single large term in the Poisson sum (41), the second when there are more. On the event  $B_1(u)$  we define the random variable  $K(u)$  to be the index of large term, and otherwise allow it to be arbitrarily

In the notation of Section 4 in general and Theorem 4.1 in particular, it follows that, on the event  $B_1(u)$ , the following representation holds for the numbers of the critical points of  $X^{(L,1)}$  over the level  $u$ . For  $k = 0, 1, \dots, d$ , a face  $J \in \mathcal{J}_k$  and  $i = 0, 1, \dots, k$ ,

$$N_{X^{(L,1)}}(J; i : u) \\ = \mathbb{1}(X_{K(u)} > 0) \sum_{l=0}^{c_i(J; S_{K(u)})} \mathbb{1}\left(X_{K(u)} f(S_{K(u)}; t_l(J; i; S_{K(u)})) > u\right) \\ + \mathbb{1}(X_{K(u)} < 0) \sum_{l=0}^{c_{k-i}(J; S_{K(u)})} \mathbb{1}\left(X_{K(u)} f(S_{K(u)}; t_l(J; k-i; S_{K(u)})) > u\right).$$

Therefore, for any number  $r = 1, 2, \dots$ , on the event  $B_1(u)$ , we have  $N_{X^{(L,1)}}(J; i : u) \geq r$  if, and only if,

$$X_{K(u)} > (f_{[r]}^{(J; i; +)}(S_{K(u)}))^{-1} u \quad \text{or} \quad X_{K(u)} < -(f_{[r]}^{(J; k-i; -)}(S_{K(u)}))^{-1} u.$$

We conclude that for any numbers  $n(J; i) = 1, 2, \dots$ , for all  $J \in \mathcal{J}_k$ , and for

all  $k = 0, 1, \dots, d$  and  $i = 0, 1, \dots, k$ ,

$$(44) \quad \mathbb{P}\left\{\{N_{X(L,1)}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i\} \cap B_1(u)\right\} \\ = \mathbb{P}\left\{\left[X_{K(u)} > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(S_{K(u)}))^{-1}u \text{ or} \right. \right. \\ \left. \left. X_{K(u)} < - \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;k-i:-)}(S_{K(u)}))^{-1}u\right] \cap B_1(u)\right\}.$$

Write  $\mathcal{E}$  for the union of sets  $(-\infty, -\max]$  and  $[\max, \infty)$ , where the ‘max’ come from the preceding lines. Then

$$(45) \quad \mathbb{P}\left\{\{X_{K(u)} \in \mathcal{E}\} \cap B_1(u)\right\} \\ = \mathbb{P}\left\{\bigcup_{m=1}^N \left(A_m(u) \cap \bigcap_{m_1 \neq m} A_{m_1}(u)^c \cap \{X_m \in \mathcal{E}\}\right)\right\} \\ = e^{-\theta} \sum_{n=0}^{\infty} \frac{\theta^n}{n!} n \mathbb{P}\left\{A_1(u) \cap \bigcap_{m_1=2}^n A_{m_1}(u)^c \cap \{X_1 \in \mathcal{E}\}\right\} \\ = \theta \mathbb{P}\left\{A_1(u) \cap \{X_1 \in \mathcal{E}\}\right\} - \mathbb{P}\left\{\{X_{K(u)} \in \mathcal{E}\} \cap B_2(u)\right\}.$$

Applying this to the right hand side of (44) and using part (iii) of Lemma 7.1 yields

$$(46) \quad \mathbb{P}\left\{\{N_{X(L,1)}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i\} \cap B_1(u)\right\} \\ = \theta \mathbb{P}\left\{A_1(u) \cap \left\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(S_1))^{-1}u\right\}\right\} \\ + \theta \mathbb{P}\left\{A_1(u) \cap \left\{X_1 < - \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;k-i:-)}(S_1))^{-1}u\right\}\right\} \\ - Q_{\text{small}}(u),$$

where  $Q_{\text{small}}(u)/H(u) \rightarrow 0$  as  $u \rightarrow \infty$ .

Assume for the moment that all the  $n(J; i)$  are strictly positive. Since the parameter  $\beta$  in the definition of the event  $A_1(u)$  is less than 1, it follows that, as  $u \rightarrow \infty$ ,

$$\mathbb{P}\left\{A_1(u) \cap \left\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(S_1))^{-1}u\right\}\right\} \\ \sim \mathbb{P}\left\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(S_1))^{-1}u\right\} \\ = \frac{1}{\theta} \int_S \rho\left(s; \left(\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(s))^{-1}u, \infty\right)\right) m(ds).$$

In the last step we used the law of  $X_1$  introduced after (41) and the decomposition (5) of the measure  $F$ , and in the middle one the asymptotic equivalence means that the two ratio of the two probabilities tends to 1 as  $u \rightarrow \infty$ . Since a similar asymptotic expression can be written for the second term in the right hand side of (46), we obtain

$$\begin{aligned} & \lim_{u \rightarrow \infty} \frac{\mathbb{P}\left\{\left\{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i\right\} \cap B_1(u)\right\}}{H(u)} \\ &= \lim_{u \rightarrow \infty} H(u)^{-1} \int_S \left[ \rho\left(s; \left(\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(s))^{-1} u, \infty\right)\right) \right. \\ & \quad \left. + \rho\left(s; \left(-\infty, -\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;k-i:-)}(s))^{-1} u\right)\right) \right] m(ds), \end{aligned}$$

provided the last limit exists. Applying (17) and Potter's bounds, as in Lemma 3.6, to justify an interchange of limit and integration, and noting Assumption 3.5 relating  $\rho$ ,  $\omega$  and  $H$ , we have

$$\begin{aligned} & \lim_{u \rightarrow \infty} \frac{\mathbb{P}\left\{\left\{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i\right\} \cap B_1(u)\right\}}{H(u)} \\ &= \int_S \left[ w_+(s) \min_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;i:+)}(s))^\alpha \right. \\ & \quad \left. + w_-(s) \min_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} (f_{[n(J;i)]}^{(J;k-i:-)}(s))^\alpha \right] m(ds) \\ & \triangleq I_c. \end{aligned}$$

Finally, since by part (iii) of Lemma 7.1, the event  $B_2(u)$  has a probability of a smaller order, we conclude that

$$(47) \quad \lim_{u \rightarrow \infty} \frac{\mathbb{P}\left\{\left\{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i\right\}\right\}}{H(u)} = I_c$$

as well.

In view of (18), we can rewrite this as

$$(48) \quad \begin{aligned} & \lim_{u \rightarrow \infty} \mathbb{P}\left\{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \text{ for all } J \text{ and } i \mid \sup_{t \in M} X_t \geq u\right\} \\ &= \frac{I_c}{\int_S \left[ w_+(s) \sup_{t \in M} f(s, t)_+^\alpha + w_-(s) \sup_{t \in M} f(s, t)_-^\alpha \right] m(ds)}. \end{aligned}$$

This will complete the proof of the theorem, at least for the case of strictly positive  $n(J; i)$ , once we show that the lighter-tailed random fields  $Y$  of (39)

and  $X^{(L,2)}$  of (42) do not change the asymptotic distribution of the numbers of critical points of  $X$ . This will take us a while to show, and makes up the remainder of the proof.

Before we do this, note that handling situations in which some or all of the numbers  $n(J; i)$  are zero is actually only an issue of semantics, once we recall our convention regarding the 0-th order statistic introduced prior to the statement of the theorem. For example, in the case when *all* the  $n(J; i)$  are zero, the event on the left hand side of (48) should be interpreted as stating that  $X^{(L,1)}$  has crossed the level  $u$ , given that it has done so. Not surprisingly, the resulting limit, and the right hand side, turn out to be 1. Similar reductions work when only some of the  $n(J; i)$  are zero.

(iv) *An outline of what remains to do.* It follows from what we have done so far that

$$(49) \quad X(t) = X^{(L,1)}(t) + X^{(L,2)}(t) + Y(t), \quad t \in \widetilde{M},$$

or, equivalently, that

$$(50) \quad X^{(L,1)}(t) = X(t) - X^{(L,2)}(t) - Y(t), \quad t \in \widetilde{M}.$$

What we plan to show is that when either  $X$  or  $X^{(L,1)}$  reaches a high level  $u$ , then the lighter-tailed random fields  $Y$  and  $X^{(L,2)}$  can be thought of as small perturbations, both in terms of their absolute values, and those of their first and second order partial derivatives. This will imply that the asymptotic conditional joint distributions of the number of the critical points of the random fields  $X$  and  $X^{(L,1)}$  are not affected by the lighter tailed fields and, hence, coincide.

In fact, what we establish is that near every critical point of one of the random fields  $X$  and  $X^{(L,1)}$  there is a critical point, of the same index, of the other. Equation (49) allows us to do this in one direction, and (50) will give us the other direction. The two equations are of the same type, and the fact that the terms in the right hand side of (49) are independent, while the terms in the right hand side of (50) are not, will play no role in the argument. Therefore, we shall treat in detail only one of the two directions, and describe only briefly the additional steps needed for the other. The first steps in this program involve collecting some probabilistic bounds on the closeness of critical points and the behaviour of Hessians there.

(v) *Bounds on critical points and Hessians.* We start by introducing a function  $D : S \rightarrow (0, \infty]$  that describes what we think of as the degree of non-degeneracy of the critical points of an  $s$ -section of the kernel  $f$ . This includes

the minimal Euclidian distance between two distinct critical points of an  $s$ -section of the kernel  $f$  and the smallest absolute value of an eigenvalue of the Hessian matrices of the section evaluated at critical points. Specifically, starting with critical points, and recalling the definition of the  $t_l(J; i; s)$  as the critical points of index  $i$  on the face  $J$  for the  $s$ -section of  $X$ , define

$$D_1(s) = \min \left\{ \|t_{l_1}(J_1; i_1; s) - t_{l_2}(J_2; i_2; s)\| : J_j \in \mathcal{J}_{k_j}, \right. \\ \left. 0 \leq k_1, k_2 \leq d, 0 \leq i_j \leq k_j, 0 \leq l_j \leq c_{i_j}(J_j; s), j = 1, 2 \right\},$$

where the minimum is taken over distinct points. Furthermore, define

$$D_2(s) = \min \left\{ |\lambda| : \lambda \text{ is an eigenvalue of } (f_{mn}(s; t_l(J; i; s)))_{m,n \in \sigma(J)} ; \right. \\ \left. J \in \mathcal{J}_k, 0 \leq k \leq d, 0 \leq i \leq k, 1 \leq l \leq c_i(J; s) \right\}.$$

As usual, both minima are defined to be equal to  $+\infty$  if taken over an empty set.

Now set

$$(51) \quad D(s) = \min(D_1(s), D_2(s)).$$

Note that, by Assumption 3.4,  $D$  is a strictly positive function, so that for any any  $S$ -valued random variable  $W$  one has  $\lim_{\tau \rightarrow 0} \mathbb{P}\{D(W) \leq \tau\} = 0$ . Choose  $W$  to have the law  $N_W$  given by

$$(52) \quad \frac{dN_W}{dm}(s) = c_*(w_+(s) + w_-(s)) \sup_{t \in I_d} |f(s, t)|^\alpha, \quad s \in S,$$

where  $c_*$  is a normalising constant. That this is possible is a consequence of (17). For  $\varepsilon > 0$ , choose  $\tau_0 > 0$  so small that  $\mathbb{P}\{D(W) \leq \tau_0\} \leq \varepsilon$ . With the random variable  $K(u)$  as before, Lemma 7.2 gives us that

$$(53) \quad \limsup_{u \rightarrow \infty} \frac{\mathbb{P} \left\{ \left\{ D(S_{K(u)}) \leq \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u \right\} \cap B_1(u) \right\}}{H(u)} \leq c_*^{-1} \varepsilon,$$

where  $B_1(u)$  was defined at (43) and indicates that there was only one ‘large’ component in the decomposition of  $X$ .

Note that, since the event

$$\left\{ \sup_{t \in I_d} |X^{(L,1)}(t)| > u \right\} \cap B_1(u)$$

is a subset of  $B_1(u)$ , on this event  $X^{(L,1)}(t) = X_{K(u)}f(S_{K(u)}; t)$  for all  $t \in \widetilde{M}$ . Thus, again on this event, since the supremum of this field over  $I_d$  exceeds  $u$ , while the kernel  $f$  is uniformly bounded, we conclude that  $|X_{K(u)}| > u/\|f\|_\infty$ . Therefore, on the event

$$\{D(S_{K(u)}) > \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)$$

the smallest eigenvalue length

$$(54) \quad D_{\min} \triangleq \min \left\{ |\lambda| : \lambda \text{ is an eigenvalue of } \left( X_{mn}^{(L,1)}(t) \right)_{m,n \in \sigma(J)} ; \right. \\ \left. J \in \mathcal{J}_k, 0 \leq k \leq d, t \text{ is a critical point on } J \right\}$$

satisfies  $D_{\min} > (\tau_0/\|f\|_\infty)u$ .

We now combine (53) with (47) as follows. Introduce the event  $\widetilde{\Omega}_\tau(u)$  that occurs whenever the minimal Euclidian distance between two distinct critical points of the random field  $(X^{(L,1)}(t), t \in I_d)$  is at least  $\tau > 0$ , while the smallest eigenvalue length of the Hessian evaluated at the critical points satisfies  $D_{\min} > (\tau/\|f\|_\infty)u$ . Thus we have

$$(55) \quad \liminf_{u \rightarrow \infty} \frac{\mathbb{P} \left\{ \{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \forall J, i\} \cap \widetilde{\Omega}_{\tau_0}(u) \right\}}{H(u)} \geq I_c - c_*^{-1}\varepsilon,$$

where  $I_c$  is as in (47). We can, furthermore, ‘sacrifice’ another  $\varepsilon$  in the right hand side of (55) to add to the event  $\widetilde{\Omega}_\tau(u)$  a requirement that the *largest* eigenvalue of the Hessian evaluated at the critical points, which we denote by  $D_{\max}$ , satisfies  $D_{\max} \leq Mu$  for some positive  $M = M(\varepsilon)$ . This is possible because  $D_{\max}$  is bounded from above by the largest absolute value of the elements of the Hessian, which we bound from above by  $Mu$  with a large enough  $M$ . For the same reason, we can also bound from above the largest value of  $\|\nabla X^{(L,1)}(t)\|$  over  $I_d$  by  $Mu$ .

Denoting the resulting event by  $\Omega_\tau(u)$ , we obtain

$$(56) \quad \liminf_{u \rightarrow \infty} \frac{\mathbb{P} \left\{ \{N_{X^{(L,1)}}(J; i : u) \geq n(J; i) \forall J, i\} \cap \Omega_{\tau_0}(u) \right\}}{H(u)} \geq I_c - 2c_*^{-1}\varepsilon.$$

Now note that since, as stated above,  $X^L$  is bounded and  $C^2$  on  $\widetilde{M}$ , and the same is true for  $X$  by Assumption 3.1, it follows that the ‘remainder’  $Y$  in (39) is also a.s. bounded and  $C^2$ . Furthermore, by construction,  $Y$  and its first and second order partial derivatives have Lévy measures that are supported on uniformly bounded functions. Consequently, the tail of

their absolute suprema decays exponentially fast; see [5]. In particular, for  $i, j = 1, \dots, d$ ,

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |Y(t)| > u\}}{H(u)} &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |Y_i(t)| > u\}}{H(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |Y_{ij}(t)| > u\}}{H(u)} \\ &= 0. \end{aligned}$$

It follows from this, part (ii) of Lemma 7.1, and the regular variation of  $H$ , that there is a function  $l(u) \uparrow \infty$  such that  $l(u)/u \rightarrow 0$  as  $u \rightarrow \infty$  and

$$\begin{aligned} (57) \quad \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |Y(t)| > l(u)\}}{H(u)} &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |Y_{ij}(t)| > l(u)\}}{H(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X^{L,2}(t)| > l(u)\}}{H(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X_{ij}^{L,2}(t)| > l(u)\}}{H(u)} \\ &= 0, \end{aligned}$$

for  $i, j = 1, \dots, d$ ,

We now combine (56) and (57) in the following way. Let  $\Omega_\tau^{(1)}(u)$  be the intersection of the event  $\Omega_\tau(u)$  with the complements of all 4 events whose probabilities are displayed in (57) and set

$$\Omega_{cr}(u) = \{N_{X^{(L,1)}}(J; i : (1 + \tau_2)u) \geq n(J; i) \forall J, i\} \cap \Omega_{\tau_1}^{(1)}(u).$$

Then, given  $0 < \varepsilon_1 < 1$ , and using the regular variation of  $H$ , we can find  $\tau_1, \tau_2 > 0$  such that

$$(58) \quad \liminf_{u \rightarrow \infty} \frac{\mathbb{P}\{\Omega_{cr}(u)\}}{H(u)} \geq (1 - \varepsilon_1)I_c.$$

(vi) *The (almost) end of the proof.* Continuing with the above notation, we now claim that, on the event  $\Omega_{cr}(u)$ , for  $u$  large enough so that

$$(59) \quad \frac{u}{l(u)} \geq \max\left(\frac{8k\|f\|_\infty}{\tau_1}, \frac{4}{\tau_2}\right),$$

we also have

$$(60) \quad N_X(J; i : u) \geq n(J; i), \quad J \in \mathcal{J}_k, \quad k = 0, 1, \dots, d, \quad i = 0, 1, \dots, k.$$

Note that, once this is established, we shall have

$$\liminf_{u \rightarrow \infty} \frac{\mathbb{P}\{N_X(J; i : u) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u)} \geq (1 - \varepsilon_1)I_c,$$

and, since this holds for all  $0 < \varepsilon_1 < 1$ , we also have

$$(61) \quad \liminf_{u \rightarrow \infty} \frac{\mathbb{P}\{N_X(J; i : u) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u)} \geq I_c.$$

Combining this with (18) gives Theorem 4.1, albeit with an inequality rather than an equality in (24).

To obtain the opposite inequality assume that, to the contrary, for some numbers  $n(J; i)$ ,

$$(62) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{P}\{N_X(J; i : u_n) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u_n)} > I_c$$

along some sequence  $u_n \uparrow \infty$ .

Now proceed by repeating the steps performed above and, this time using (50) rather than (49), and so demonstrate the existence of a critical point of  $X^{(L,1)}$  near each one of  $X$ . Thus (62) also holds with  $X$  replaced by  $X^{(L,1)}$ , viz.

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}\{N_{X^{(L,1)}}(J; i : u_n) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u_n)} > I_c.$$

Since this contradicts (47), (62) cannot be true, we have the required lower bound, and the proof of Theorem 4.1 is complete, modulo the need to establish the claim (60).

(vii) *Establishing (60) to finish the proof.* In order to establish (60), we shall show that, on the event  $\Omega_{cr}(u)$ , to every critical point above the level  $(1 + \tau_2)u$  of the random field  $X^{(L,1)}$  we can associate a critical point above the level  $u$  of  $X$  which is in the same face and of the same type.

To this end, let  $t_0$  be a critical point above the level  $(1 + \tau_2)u$  of  $X^{(L,1)}$  that belongs to a face  $J \in \mathcal{J}_k$  for some  $0 \leq k \leq d$ , and which is of the type  $i$  for some  $0 \leq i \leq k$ . Let  $(e_1, \dots, e_k)$  be an orthonormal basis of  $\mathbb{R}^k$  consisting of normalised eigenvectors of the Hessian matrix

$$(63) \quad \mathcal{H}^{(L,1)}(t_0) = \left( X_{mn}^{(L,1)}(t_0) \right)_{m,n \in \sigma(J)},$$

and let  $\lambda_1, \dots, \lambda_k$  be the corresponding eigenvalues. Note that, by the definition of the event  $\Omega_{cr}(u)$ , we have  $|\lambda_n| > (\tau_1/\|f\|_\infty)u$  for  $n = 1, \dots, k$ . We naturally embed the vectors  $(e_1, \dots, e_k)$  into the face  $J$  and make them  $d$ -dimensional vectors by appending to them the  $d - k$  fixed coordinates of the face  $J$ . (We shall continue to denote these vectors by  $(e_1, \dots, e_k)$ .) Note that for small real numbers  $\epsilon_1, \dots, \epsilon_k$  we have

$$(64) \quad \nabla X^{(L,1)}(t_0 + \sum_{j=1}^k \epsilon_j e_j) = \sum_{j=1}^k \epsilon_j \lambda_j e_j + o(\max(|\epsilon_1|, \dots, |\epsilon_k|)).$$

In particular, the directional derivatives

$$g_j^{(L,1)}(t) \triangleq \langle \nabla X^{(L,1)}(t), e_j \rangle, \quad j = 1, \dots, k,$$

satisfy

$$(65) \quad g_j^{(L,1)}(t_0 + \sum_{j=1}^k \epsilon_j e_j) = \epsilon_j \lambda_j + o(\max(|\epsilon_1|, \dots, |\epsilon_k|)).$$

In what follows we shall work with a small positive number  $\epsilon > 0$ , placing more and more conditions on it as we progress, to clarify precisely how small it will need to be. As a first step, take  $\epsilon < \tau_1/2$ , where  $\tau_1$  is as in (58).

Consider a  $k$ -dimensional cube (which is a subset of the face  $J$ ) defined by

$$C_\epsilon = \left\{ t_0 + \sum_{j=1}^k \theta_j e_j, |\theta_j| \leq \epsilon, j = 1, \dots, k \right\},$$

along with its  $(k - 1)$ -dimensional faces

$$F_n^\pm = \left\{ t_0 + \sum_{j=1}^k \theta_j e_j, \theta_n = \pm\epsilon, |\theta_j| \leq \epsilon, 1 \leq j \leq k, j \neq n \right\},$$

where  $n = 1, \dots, k$ . It follows from (65) that, for  $\epsilon > 0$  small enough,  $u > 1$ , and, as above,  $M$  large enough, we have

$$(66) \quad 2M\epsilon u \geq 2\epsilon|\lambda_n| \geq |g_n^{(L,1)}(t)| \geq \frac{\epsilon|\lambda_n|}{2} \geq \frac{\tau_1\epsilon}{2\|f\|_\infty}u$$

for all  $t \in F_n^\pm$ ,  $n = 1, \dots, k$ . The assumption that  $\epsilon$  be small enough now entails that (66) holds for all critical points and for all relevant  $n$ . Since the number of critical points is finite, this requirement is easy to satisfy.

Similarly, the continuity of the eigenvalues of a quadratic matrix in its components (see e.g. Section 7.2. and Corollary 2 in Section 7.4 of [6]) shows that, for all  $\epsilon > 0$  small enough, the eigenvalues of the matrix of the second order partial derivatives  $(X_{mn}(t))_{m,n \in \sigma(J)}$  have all absolute values satisfying  $|\lambda_n| > (\tau/2\|f\|_\infty)u$  for  $n = 1, \dots, k$  and  $t \in C_\epsilon$ . Finally, we require that  $\epsilon$  be small enough that this lower bounds holds for all critical points  $t_0$  considered above. In particular, this implies that the signs of these eigenvalues throughout  $C_\epsilon$  are the same as those at the point  $t_0$ .

Next, for a non-empty  $I \subset \{1, \dots, k\}$  and  $p \in \{-1, 1\}^k$  consider the vector

$$(67) \quad x(I, p) = \sum_{i \in I} p_i e_i.$$

Consider a point  $t$  that belongs to the (relative to the face  $J$ ) boundary of the cube  $C_\epsilon$  and, more specifically, belongs to the face of that cube defined by

$$(68) \quad \left( \bigcap_{i \in I, p_i=1} F_i^+ \right) \cap \left( \bigcap_{i \in I, p_i=-1} F_i^- \right),$$

and to no other  $(k-1)$ -dimensional face of  $C_\epsilon$ . Define a function  $h^{(L,1)} : C_\epsilon \rightarrow \mathbb{R}$  by

$$h^{(L,1)}(t) = \sum_{i=1}^k (g_i^{(L,1)}(t))^2.$$

This is a  $C^1$ -function, and its gradient (within the face  $J$ ) is given by

$$\nabla h^{(L,1)}(t) = 2 \sum_{i=1}^k g_i^{(L,1)}(t) \nabla g_i^{(L,1)}(t) = 2 \sum_{i=1}^k g_i^{(L,1)}(t) \mathcal{H}^{(L,1)}(t) e_i^T.$$

Note also that for all  $I$  and  $p$  as above,

$$\left\langle \nabla g_i^{(L,1)}(t_0), x(I, p) \right\rangle = \begin{cases} \lambda_i p_i & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

In particular, we can write for any  $t$  belonging to the face of  $C_\epsilon$  defined by

(68),

$$\begin{aligned}
(69) \quad & \langle \nabla h^{(L,1)}(t), x(I, p) \rangle \\
&= 2 \sum_{i=1}^k g_i^{(L,1)}(t) \langle \nabla g_i^{(L,1)}(t), x(I, p) \rangle \\
&= 2 \sum_{i \in I} \lambda_i p_i g_i^{(L,1)}(t) \\
&\quad + 2 \sum_{i=1}^k g_i^{(L,1)}(t) \langle (\nabla g_i^{(L,1)}(t) - \nabla g_i^{(L,1)}(t_0)), x(I, p) \rangle.
\end{aligned}$$

It follows from (65) and (66) that

$$\begin{aligned}
(70) \quad & g_n^{(L,1)}(t) > 0 \text{ for } t \in F_n^+ \text{ if } \lambda_n > 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n < 0, \\
& g_n^{(L,1)}(t) < 0 \text{ for } t \in F_n^+ \text{ if } \lambda_n < 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n > 0.
\end{aligned}$$

Consequently, we can conclude, by (70) and (66), that the first term in the right hand side of (69) is negative and, more specifically, does not exceed

$$-2 \text{Card}(I) D_{\min} \frac{\tau_1 \epsilon}{2 \|f\|_{\infty}} u \leq -(\tau_1 / \|f\|_{\infty})^2 \epsilon u^2.$$

We can bound the absolute value of the second term in the right hand side of (69) from above by

$$2k \sum_{i=1}^k |g_i^{(L,1)}(t)| \cdot \|\nabla g_i^{(L,1)}(t) - \nabla g_i^{(L,1)}(t_0)\| \leq 2k^2 M^2 \epsilon u^2,$$

by the definition of the event  $\Omega_{cr}(u)$ . This, obviously, indicates that, for  $\epsilon > 0$  small enough,

$$\langle \nabla h^{(L,1)}(t), x(I, p) \rangle \leq -C \epsilon u^2,$$

where  $C$  is a finite positive constant determined by the parameters in the event  $\Omega_{cr}(u)$ . If

$$g_j(t) = \langle \nabla X(t), e_j \rangle, \quad j = 1, \dots, k,$$

and we define

$$(71) \quad h(t) = \sum_{i=1}^k (g_i(t))^2,$$

then, on the event  $\Omega_{cr}(u)$ ,

$$\langle \nabla h(t), x(I, p) \rangle \leq -C\epsilon u^2 + kl(u)^2.$$

Taking into account that  $l(u)/u \rightarrow 0$  as  $u \rightarrow \infty$ , where  $l$  is given by (57), we see that for  $u$  large enough it is possible to choose  $\epsilon > 0$  small enough such that

$$(72) \quad \langle \nabla h(t), x(I, p) \rangle < 0,$$

for any  $t$  belonging to the face of  $C_\epsilon$  defined by (68). The final requirement on  $\epsilon$  is that (72) holds.

Similarly, since by the definition of the event  $\Omega_{cr}(u)$ , the first order partial derivatives of  $X^{(L,2)}$  and  $Y$  are bounded by  $l(u) = o(u)$  in absolute value, we have that (70) and (59) give us

$$(73) \quad \begin{aligned} g_n(t) &> 0 \text{ for } t \in F_n^+ \text{ if } \lambda_n > 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n < 0, \\ g_n(t) &< 0 \text{ for } t \in F_n^+ \text{ if } \lambda_n < 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n > 0 \end{aligned}$$

as well.

In order to complete the proof and establish (60) it is enough to prove that, on  $\Omega_{cr}(u)$ ,  $X$  has a critical point in the cube  $C_\epsilon$ . In fact, if such a critical point exists, Lemma 7.3 below implies that it will be above the level  $u$  and of the same type as  $t_0$ . Furthermore, these critical points of  $X$  will all be distinct.

To establish the existence of this critical point, note that, by the continuity of  $\nabla X$  and the compactness of  $C_\epsilon$ , there is a point  $t_1$  in  $C_\epsilon$  at which the norm of the vector function  $g(t) = (g_1(t), \dots, g_k(t))$  achieves its minimum over  $C_\epsilon$ . We shall prove that, in fact,  $g(t_1) = 0$ . By the linear independence of the basis vectors  $e_1, \dots, e_k$ , this will imply that  $g_j(t_1) = 0$  for  $j = 1, \dots, k$ , and so  $t_1$  is, indeed, a critical point.

Suppose that, to the contrary,  $g(t_1) \neq 0$ , and consider firstly the possibility that the point  $t_1$  belongs to the (relative to the face  $J$ ) interior of  $C_\epsilon$ . Note that the Jacobian of the transformation  $g : C_\epsilon \rightarrow \mathbb{R}^k$  is given by

$$J_g(t) = \mathcal{E}\mathcal{H}(t),$$

where  $\mathcal{H}(t) = (X_{mn}(t))_{m,n \in \sigma(J)}$  is the Hessian of  $X$ , and  $\mathcal{E}$  is a  $k \times k$  matrix with rows  $e_1, \dots, e_k$ . We have already established above that, on the event  $\Omega_{cr}(u)$ ,  $\mathcal{H}$  is non-degenerate throughout  $C_\epsilon$ . Since the vectors  $e_1, \dots, e_k$  are linearly independent, we conclude that the matrix  $\mathcal{E}$  is non-degenerate as

well. Since the vector  $g(t_1)$  does not vanish, it has a non-vanishing component. Without loss of generality, we can assume that  $g_1(t_1) \neq 0$ . Choose a vector  $x \in \mathbb{R}^k$  for which

$$J_g(t_1)x' = (1, 0, \dots, 0)'.$$

Then for  $\delta \in \mathbb{R}$ , with  $|\delta|$  small,

$$\begin{aligned} g(t_1 + \delta x) &= g(t_1) + \delta J_g(t_1)x^T + o(|\delta|) \\ &= \begin{pmatrix} g_1(t_1) + \delta \\ g_2(t_1) \\ \vdots \\ g_k(t_1) \end{pmatrix} + o(|\delta|), \end{aligned}$$

and so

$$\begin{aligned} \|g(t_1 + \delta x)\|^2 &= \sum_{j=1}^k g_j(t_1)^2 + 2\delta g_1(t_1) + o(|\delta|) \\ &< \sum_{j=1}^k g_j(t_1)^2 = \|g(t_1)\|^2 \end{aligned}$$

for  $\delta$  with  $|\delta|$  small enough and such that  $\delta g_1(t_1) < 0$ . This contradicts the assumed minimality of  $\|g(t_1)\|$  and so we must have  $g(t_1) = 0$ , as required, for this case.

It remains to consider the case  $g(t_1) \neq 0$ , but the point  $t_1$  belongs to the boundary of the cube  $C_\epsilon$ . Let  $g(t_1)$  belong to the face of the cube defined by (68). With the function  $h$  defined in (71), we have, for  $\delta > 0$  small,

$$h(t_1 + \delta x(I, p)) = h(t_1) + \delta(\nabla h(t), x(I, p)) + o(\delta).$$

By (72), this last expression is smaller than  $h(t_1)$  if  $\delta > 0$  is small enough. However, by the definition of the vector  $x(I, p)$ , the point  $t_1 + \delta x(I, p)$  belongs to  $C_\epsilon$  for  $\delta > 0$  small. Once again, this contradicts the assumed minimality of  $\|g(t_1)\|$ .

Thus we have established (60) and, therefore, (61), and so the theorem, modulo the need to prove the following three lemmas.  $\square$

LEMMA 7.1. *The following three results hold:*

(i) *The random fields  $X^{(L,1)}$  and  $X^{(L,2)}$  on the right hand side of the decomposition (42) are independent.*

(ii) The random field  $X^{(L,2)}$  has  $C^2$  sample functions and satisfies

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X^{(L,2)}(t)| > u\}}{H(u)} &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X_i^{(L,2)}(t)| > u\}}{H(u)} \\ &= \lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X_{ij}^{(L,2)}(t)| > u\}}{H(u)} = 0, \end{aligned}$$

$i, j = 1, \dots, d$ , where  $H$  is the regularly varying function of Assumption 3.5.

(iii) The number of terms in the sum defining  $X^{(L,1)}$  satisfies

$$\lim_{u \rightarrow \infty} \frac{\mathbb{P}\{\sum_{m=1}^N \mathbb{1}(A_m(u)) \geq 2\}}{H(u)} = 0.$$

PROOF. The claim (i) follows from the fact that a Poisson random measure, when restricted to disjoint measurable sets, forms independent Poisson random measures on these sets (see e.g. [12]). Since the sum defining the random field  $X^{(L,2)}$  is a.s. finite, the fact that it has sample functions in  $C^2$  follows from Assumption 3.1. Furthermore, for  $\epsilon > 0$ , choose  $n_\epsilon > 0$  so large that  $\mathbb{P}\{N > n_\epsilon\} \leq \epsilon$ . The above discussion implies that the number  $K(u, \epsilon)$  of the terms in the sum defining  $X^{(L,2)}$  in (42) that satisfy

$$\sup_{t \in \tilde{M}} |X_m f(S_m; t)| > \frac{u}{2n_\epsilon}$$

is Poisson with the mean less or equal to

$$\begin{aligned} F\left\{(s, x) \in (\mathbb{R} \setminus \{0\}) \times S : \sup_{t \in \tilde{M}} |xf(s; t)| > u/(2n_\epsilon)\right\} \\ = \lambda_X \left\{g : \sup_{t \in \tilde{M}} |g(t)| > u/(2n_\epsilon)\right\} \sim CH(u) \end{aligned}$$

as  $u \rightarrow \infty$ , where we have used (11) and Lemma 3.6. Therefore, for large  $u$

$$\mathbb{P}\{\sup_{t \in \tilde{M}} |X^{(L,2)}(t)| > u\} \leq \mathbb{P}\{N > n_\epsilon\} + \mathbb{P}\{K(u, \epsilon) \geq 2\} \leq \epsilon + CH(u)^2,$$

and so

$$\limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \in \tilde{M}} |X^{(L,2)}(t)| > u\}}{H(u)} \leq \epsilon.$$

Letting  $\epsilon \rightarrow 0$  completes the proof of the first limit in part (ii) of the lemma, and the other limits are established in the same way. Part (iii) of the lemma can be proven similarly.  $\square$

LEMMA 7.2. *The random variables  $D_m$  in (51) satisfy*

$$\limsup_{u \rightarrow \infty} \frac{\mathbb{P}\left\{\{D(S_{K(u)}) \leq \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)\right\}}{H(u)} \leq c_*^{-1} \varepsilon,$$

where  $c_*$  is as in (52).

PROOF. We use a decomposition as in (45) to obtain

$$\begin{aligned} & \mathbb{P}\left\{\{D(S_{K(u)}) \leq \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)\right\} \\ & \leq \theta \mathbb{P}\left\{A_1(u) \cap \bigcap_{m_1=2}^n A_{m_1}(u)^c \cap \left\{D(S_1) \leq \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u\right\}\right\}. \end{aligned}$$

Since on the event  $A_m(u)^c$  one has  $\sup_{t \in \tilde{M}} |X_m f(S_m; t)| \leq u^\beta$ , it follows that the latter probability can be asymptotically bounded by

$$\begin{aligned} & \theta \mathbb{P}\left\{D(S_1) \leq \tau_0, \sup_{t \in I_d} |X_1 f(S_1; t)| > u\right\} \\ & = \int_S \int_{\mathbb{R} \setminus \{0\}} \mathbb{1}(D(s) \leq \tau_0, |x| \sup_{t \in I_d} |f(s; t)| > u) F(ds, dx) \\ & = \int_S \mathbb{1}(D(s) \leq \tau_0) \left( \int_{\mathbb{R} \setminus \{0\}} \mathbb{1}\left(|x| > u \left(\sup_{t \in I_d} |f(s; t)|\right)^{-1}\right) \rho(s; dx) \right) m(ds) \\ & \sim H(u) \int_S \mathbb{1}(D(s) \leq \tau_0) (w_+(s) + w_-(s)) \sup_{t \in I_d} |f(s; t)|^\alpha m(ds), \end{aligned}$$

where we used Assumptions 3.5, 3.4 and (17). The lemma now follows from the choice of  $\tau_0$ .  $\square$

LEMMA 7.3. *Suppose that for every critical point  $t_0$  of the random field  $(X^{L,1}(t), t \in I_d)$ , the random field  $(X(t), t \in I_d)$  has, on the event  $\Omega_{cr}(u)$ , a critical point in the cube  $C_\epsilon$ . Then the critical points of  $X$  in  $I_d$  correspond to distinct critical points of  $X^{L,1}$ , are themselves distinct, are all above the level  $u$ , and each of them is of the same type as the corresponding critical point of  $X^{L,1}$ .*

PROOF. The fact that the critical points of  $X$  corresponding to distinct critical points of the field  $X^{L,1}$  are all distinct follows from the lower bound on the distance between two distinct critical points of  $X^{L,1}$  in the definition of the event  $\Omega_{cr}(u)$  and the choice of  $\epsilon$ . The fact that all the critical points are above the level  $u$  follows from the lower bounds on the values of  $X^{L,1}$  at

its critical points in the definition of  $\Omega_{cr}(u)$  and, once again, the choice of  $\epsilon$ . It remains, therefore, to prove that a critical point in the cube  $C_\epsilon$  of  $X$  is of the same type as the critical point  $t_0$  of  $X^{L,1}$ .

To this end, note that the absolute values of the eigenvalues of the matrix of the second order partial derivatives  $(X_{mn}^{L,1}(t) + Y_{mn}(t))_{m,n \in \sigma(J)}$  are, on the event  $\Omega_{cr}(u)$ , bounded from above by  $2kl(u)$ . Using continuity of the eigenvalues of a quadratic matrix in its components (see, once again, Section 7.2 and Corollary 2 in Section 7.4 of [6]) we see that the Euclidian distance between an eigenvalue of  $(X_{mn}(t))_{m,n \in \sigma(J)}$  and the corresponding eigenvalue of  $(X_{mn}^{L,1}(t))_{m,n \in \sigma(J)}$  is bounded from above by  $2kl(u)$ . Using the choice of  $\epsilon$  then shows that the numbers of the negative eigenvalues of the two Hessians are identical, as required.  $\square$

**8. Proof of Theorem 4.1 for the general case.** If you carefully followed the proof of Theorem 4.1 for the case  $M = I_d$ , then you will have noticed that there were two main components to the arguments. In the first, global bounds on suprema of random fields, derivatives, etc, played a crucial role. These arguments are no different for cubes than they are for other compact sets, including compact stratified manifolds, and so throughout we could have worked with general  $M$  rather than the special case  $M = I_d$  when handling these arguments.

The second component relied on showing that the critical points above high levels of the random fields  $X$  and  $X^{L,1}$  were in one–one correspondence, in terms of their (approximate) positions, their (approximate) heights, and their type. The arguments here were of a purely local nature, and took into account, for example, on what face of  $I_d$  the critical points occurred. Transferring these arguments to the case of  $M$  a stratified manifold is not trivial, but it is also not too hard. The main step lies in obtaining an analogue to the linearisation (64) of  $\nabla X^{L,1}$  (the random field itself is defined precisely as in the simpler case), as well as an analogue of the Hessian (63).

The natural place to do this, of course, is on the tangent spaces at the critical points of  $X^{L,1}$ . In small neighbourhoods of each such critical point, both  $X$  and  $X^{L,1}$  can be pushed forward to the tangent space via the exponential map. Once this is done, we are back in a simple Euclidean setting and can argue as before, but now with respect to the push forwards. Pulling the results back to  $M$  itself, again via the exponential map, is straightforward, as long as the mapping is smooth enough. However, looking again at the case  $M = I_d$ , it is easy to see that all that ‘smooth enough’ really requires is a universal bound on the second order partial derivatives of  $X^{L,1}$  and that the pullbacks can be done in a uniform fashion. It is to ensure this this is

indeed the case that the assumption that the component manifolds  $\partial_j M$  of  $M$  have bounded curvature, made already in Section 2, was introduced.

We leave the details to the interested reader. If you are uninterested, or unprepared to take our word for the fact that all that remains is, in essence, to repeat the above argument with heavier notation, you have the choice of believing only the case  $M = I_d$ . There is enough new in this case to make the basic results of this paper interesting and useful.

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