Subadditivity Re–Examined: the Case for Value–at–Risk*

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Abstract

This paper explores the potential for violations of VaR subadditivity both theoretically and by simulations, and finds that for most practical applications VaR is subadditive. Hence, there is no reason to choose a more complicated risk measure than VaR, solely for reasons of subadditivity.

KEY WORDS: Value–at–Risk, subadditivity, regular variation, tail index, heavy tailed distribution.

JEL Classification: G00, G18

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1 Introduction

Value–at–risk (VaR) has become a central plank in banking regulations and internal risk management in banks. While superior to volatility as a measure of risk, VaR is often criticized for lack of subadditivity. VaR is much easier to implement operationally than most other measures of risk, and is likely to retain its preeminent practical status. Our objective is to explore VaR subadditivity, to analyze which asset classes are likely to suffer from violations of subadditivity, and examine the asymptotic and finite sample properties of the VaR risk measure with respect to subadditivity.

Our main result is that the problem of subadditivity violations is not much important for assets that meet the stylized facts of returns. Indeed, for most assets and applications, we do not expect to see any violations of subadditivity. There are two main, but rare, exceptions to this: assets with super fat tails, and probability levels that are in the interior of return distributions. As a consequence, worries about subadditivity are in general not pertinent for risk management applications relying on VaR. This applies especially to stress tests, which rely on probabilities deep in tails.

Following the 1996 'amendment to incorporate market risk', to the Basel Accord, 99% VaR has become the primary risk measurement tool for determining capital charges against market risk. VaR has also become central to internal risk management systems in banks. Its role is likely to remain unchanged under the Basel II Accord. Artzner et al. (1999) criticize VaR on the grounds that it is not *subadditive*, i.e., that VaR of a portfolio can be higher than the sum of VaRs of the individual assets in the portfolio. In other words, VaR is not a "coherent" measure of risk. This problem is caused by the fact that VaR is a quantile on the distribution of profit and loss and not an expectation, so that the shape of the tail before and after the VaR probability need not have any bearing on the actual VaR number.

Unrecognized violations of VaR subadditivity can have serious consequences for risk models. First, they can provide a false sense of security, so that a financial institution may not be adequately hedged. Second, it can lead a financial institution to make a suboptimal investment choice, if VaR, or a change in VaR, is used for identifying the risk in alternative investment choices.

In the specific case of normality of returns, a property at odds with stylized facts of financial returns, VaR is known to be coherent below the mean. However, it has been known at least since Mandelbrot (1963) and Fama (1965) that returns are "fat tailed", and in that case, it has hitherto not been generally known when subadditivity is violated. We demonstrate that VaR is subadditive for the tails of all fat tailed distributions, provided the tails are not super fat. For most asset classes we do not have to worry about violations of subadditivity. The main exception are assets that are so fat that the first moment is not defined, such as those that follow the Cauchy distribution.

Central to our analysis is a precise notion of what constitutes *fat tails*. Common usage might suggests that kurtosis in excess of three indicates fat tails. Kurtosis captures the mass of the distribution in the center relative to the tails, which may be thin. Indeed, it is straightforward to construct a distribution with truncated tails, and hence thin tails, which exhibit high kurtosis. An alternative, more formal, definition of a fat tailed distribution requires the tails to be *regularly varying*.[1] The thickness of the tails is indicated by the *tail index*. The lower the tail index, the fatter the tails. For most financial assets the tail index is between three and five, (see e.g. Jansen and de Vries, 1991; Daníelsson and de Vries, 1997, 2000).

We demonstrate below that for all assets with (jointly) regularly varying non–degenerate tails, subadditivity holds in the tail region provided the tail index is larger than 1. This includes, in particular, situations where different assets are affected by different independent sources of fat tailed randomness, which are market–wide, industry–wide, or idiosyncratic to different assets.

We are most likely to observe super fat tails for assets that most of the time change very little in price, or at all, but are subject to the occasional jumps. For example, an exchange rate that is usually pegged, but subject to occasional devaluations, is likely to have super fat tails. Other assets might be the short term higher risk bonds, which either give a steady income stream or default. Options can also be constructed in a way to give super fat tails. In such cases, subadditivity violations are likely to be a matter of serious concern.

Our results only apply to the tail region, so that for probabilities in the interior of the distributions we may see violations of VaR subadditivity. However, risk management, and especially stress tests, generally focus only on tail probabilities so this is not likely to be a serious concern. We explore this particular issue by means of Monte Carlo simulations. The simulation results confirm the theoretical results. For very large sample sizes we do not observe any violations of subadditivity when the tail index is less than or equal to one, and for small sample sizes the number of violations is very small.

[1] A function is regularly varying if it has a Pareto distribution–like power expansion at infinity.

2 Subadditivity

Artzner et al. (1999) propose a classification scheme for risk measures whereby a risk measure $\rho(\cdot)$ is said to be "coherent" if it satisfies certain conditions. Let X and Y be two financial assets. A risk measure $\rho(\cdot)$ is coherent if the following four axioms hold:

Subadditivity $\rho(X+Y) \leq \rho(X) + \rho(Y)$

Homogeneity For any number $\alpha > 0, \rho(\alpha X) = \alpha\rho(X)$

Monotonicity $\rho(Y) \geq \rho(X)$ if $X \leq Y$

Risk Free Condition $\rho(X+k) = \rho(X) - k$ for any constant k.

Value–at–Risk[2] (VaR) satisfies all but subadditivity. Artzner et al. (1999) argue that subadditivity is a desirable property for a risk measure e.g. because "...a merger does not create extra risk". Indeed, in most cases subadditivity is a desirable property for a risk measure, even if exceptions exist, as discussed below. Subadditivity ensures that the diversification principle of modern portfolio theory holds since a subadditive measure would always generate a lower risk measure for a diversified portfolio than a non–diversified portfolio. In terms of internal risk management, subadditivity also implies that the overall risk of a financial firm is equal to or less than the sum of the risks of individual departments of the firm.

A key question when assessing the relative importance of subadditivity is whether a violation of subadditivity is an artifact of the chosen risk measure, or caused by a particular combination of assets.

Violations of subadditivity can cause a number of problems for financial institutions. Suppose a financial institution is employing a VaR measure without realizing it is actually violating subadditivity, e.g. by using VaR to rank investment choices or impose limits on traders. In this case, the financial institution is likely to assume too much risk, or not hedge when needed.

From the point of view of financial regulations, subadditivity violations might lead financial institutions to hold less capital than desired.

In response to the lack of coherence for the VaR risk measure, several alternatives have been proposed. Of these the most common is expected shortfall,

[2] Let X be the return, then for the probability p, VaR is the loss level such that $\Pr(X \leq -\text{VaR}) \leq p$.

proposed by Acerbi et al. (2001), the tail conditional expectation and worst conditional expectation proposed by Artzner et al. (1999). These risk measures are often considered superior to VaR because they are coherent, but they have not gained much traction, not the least because of problems with back testing. See e.g. Yamai and Yoshiba (2001) for more on the practical problems with alternative risk measures.

However, while coherence is an appealing mathematical property of risk measures, from a financial theoretic point of view it is less attractive. Dhaene et al. (2003) argue that "imposing subadditivity for all risks (including dependent risks) is not in line with what could be called best practice". They further state that the axioms of coherence lead to a very restrictive set of risk measures that cannot be used in practical situations. The measure of global risk may not be *a priori* smaller than the sum total of local risks such that diversification does not necessarily lead to a reduction in the global risk. From this point of view subadditivity may not depict the complex nature of the financial markets.

2.1 Statistical Violations of Subadditivity

It is easy to demonstrate that VaR violates subadditivity.[3] A simple example is:

Example 1 *Consider two assets X and Y that are usually normally distributed, but subject to the occasional independent shocks:*

$$X = \epsilon + \eta, \quad \epsilon \sim \text{IID}\,\mathcal{N}(0,1), \qquad \eta = \begin{cases} 0 & \text{with probability} \quad 0.991 \\ -10 & \text{with probability} \quad 0.009 \end{cases}$$

In this case the 1% VaR is 3.1, since the probability that η is -10 is less than 1%. Suppose that Y has the same distribution independently from X, and that we formulate an equally weighted portfolio of X and Y. In that case, the 1% portfolio VaR is 9.8, because for $(X+Y)$ the probability of getting the -10 draw for either X or Y is much higher.

$$\text{VaR}(X+Y) = 9.8 > \text{VaR}(X) + \text{VaR}(Y) = 3.1 + 3.1 = 6.2$$

[3]See e.g. Artzner et al. (1999); Acerbi and Tasche (2001); Acerbi et al. (2001).

3 Heavy Tailed Asset Returns and Regular Variation

Empirical studies have established that the distribution of speculative asset returns tends to have heavier tails than the normal distribution, at least since Mandelbrot (1963) and Fama (1965). Heavy tailed distributions are often defined in terms of higher than normal kurtosis. However, the kurtosis of a distribution may be high if either the tails of the cumulative distribution function (cdf) are heavier than the normal or if the center is more peaked, or both. Further, it is not only the higher than normal kurtosis, but also failure of higher moments that defines heavy tails.

Informally, heavy tails are characterized as distributions where it is not possible to calculate one or more moments of order m (> 0) or higher. Such distributions have tails which exhibit a power type behavior like the Pareto distribution, as commonly observed in finance. Mathematically, a certain tail regularity is also required, formally defined as:

Definition 1 *Definition: A cdf $F(x)$ varies regularly*[4] *at minus infinity with tail index $\alpha > 0$ if*

$$\lim_{t\to\infty} \frac{F(-tx)}{F(-t)} = x^{-\alpha} \quad \forall x > 0 \tag{1}$$

This implies that, to a first order approximation, a regular varying distribution has a tail of the form

$$F(-x) = x^{-\alpha} L(x)[1 + o(1)],\ x > 0, \quad \text{for } \alpha > 0$$

where L is a *slowly varying function* (e.g. a logarithm). An often used particular class of such distributions has a tail comparable to the Pareto distribution:

$$F(-x) = Ax^{-\alpha}[1 + o(1)],\ x > 0, \quad \text{for } \alpha > 0 \text{ and } A > 0\,. \tag{2}$$

A regularly varying density implies Pareto type tails. If (2) holds, then for large x,

$$f(-x) \approx \alpha A x^{-\alpha-1} \quad x > 0, \quad \text{for } \alpha > 0 \text{ and } A > 0\,. \tag{3}$$

This means that the density declines at a power rate $x^{-\alpha-1}$ far to the left of the centre of the distribution which contrasts with the exponentially fast

[4]For an encyclopedic treatment of regular variation, see Bingham et al. (1987); Resnick (1987).

declining tails of the Gaussian distribution. The power is outweighed by the explosion of x^m in the computation of moments of order $m \geq \alpha$. Thus, moments of order $m \geq \alpha$ are unbounded. The power α is called the *tail index* it determines the number of bounded moments; A is the *scale coefficient.* It is readily verified that Student–t distributions, vary regularly at infinity, have degrees of freedom equal to the tail index and satisfy the above approximation. Likewise, the stationary distribution of the popular GARCH(1,1) (Bollerslev, 1986), process has regularly varying tails, see de Haan et al. (1989).

4 Subadditivity of VaR in the Tail

In order to address whether the VaR risk measure is subadditive, we focus only on the case where returns are fat tailed, i.e. are regularly varying. If returns are normal we know subadditivity holds, so it is sufficient to focus on the fat tailed case. We focus only on the lower tail, but a short selling agent would focus on the other tail, and the theoretical results apply equally to the upper tail since we can turn into the other by multiplying returns with minus one. In general, the lower and upper tails of return distributions may have different tail thickness, but is irrelevant for the analysis below.

As before, let X and Y be two asset returns, each having a regularly varying tail with the same tail index $\alpha > 0$. We consider the effect of combining the assets into one portfolio, which requires estimating the simultaneous (joint) tails. The corresponding mathematical definition of jointly regularly varying tails is given in the Appendix. The non–degeneracy assumption in our main result below means, that in the extreme region the two returns are not *deterministically proportional* to each other. This is, of course, the case for most models of interest. One such model is in (4) below. The following proposition, which is our main result, allows arbitrary dependence between the returns. If the tail indices of the two assets are different, a sligthly weaker form of subadditivity holds; see the Appendix.

Proposition 1 *Suppose that X and Y are two asset returns having jointly regularly varying non–degenerate tails with tail index $\alpha > 1$. Then VaR is subadditive in the tail region.*

Proof. See the Appendix

Thus the result says that at sufficiently low probability levels, the VaR of a portfolio position is lower than the sum of the VaRs of the individual positions, if the return distribution exhibits fat tails. For example, this applies to a multivariate Student–t distribution with degrees of freedom larger than 1. Note that X and Y are allowed to be dependent. This further implies that diversification will not work for super fat tails, i.e. $\alpha < 1$, a result already established by Fama and Miller (1972, page 270).

Data falling into this category would be characterized by a large number of very small outcomes interdispersed with very large outcomes. While such assets do exist they are hard to find. An example could be a pegged exchange rate subject to the occasional devaluation, or a portfolio of fixed income assets providing a steady income stream most of the time, but with potential bond defaults, resulting in large negative outcomes. Such anomalous cases are easy to identify and require special treatment in risk models.

Example 2 *As an example, suppose X and Y have independent unit Pareto loss distributions,* $\Pr\{X < -x\} = \Pr\{Y < -x\} = x^{-\alpha}, x \geq 1$. *By inversion,* $\mathrm{VaR}_p(X) = \mathrm{VaR}_p(Y) = p^{-1/\alpha}$. *Using Feller's convolution theorem (Feller, 1971, page 278), we have*

$$p = \Pr\{X + Y \leq \mathrm{VaR}_p(X+Y)\} \approx 2[\mathrm{VaR}_p(X+Y)]^{-\alpha}.$$

Hence

$$\mathrm{VaR}_p(X+Y) - [\mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)] \approx p^{-1/\alpha}[2^{\frac{1}{\alpha}} - 2] < 0.$$

4.1 More on Dependent Returns

Proposition 1 establishes that subadditivity is not violated for fat tailed data regardless of dependent structure. We can illustrate this result by an example of assets with linear dependence, via a factor structure.

Suppose that X_1 and X_2 are two assets, which are dependent via a common market factor R:

$$X_i = \beta_i \ R + Q_i, \ \ i = 1, 2 \tag{4}$$

where R denotes the return of the market portfolio, β_i the market risk and Q_i the idiosyncratic risk of asset X_i. Q_i's and R are independent of each other; further, individual Q_is are independent of each other. Thus, the only source of cross–sectional dependence between X_1 and X_2 is the common market risk. The security specific risks Q_i are independent of each other and therefore can be diversified away.

Since R and Q_i are independent, we can use Feller (1971)'s convolution theorem to approximate the tails of X_1 and X_2, depending upon the tail behaviour of R, Q_1 and Q_2. We can further use it to approximate the tail of $X_1 + X_2$. Thus, under such a model, we can proceed in a similar manner as in the case of independent asset returns. To illustrate this, we present below one particular case, viz., the case where R, Q_1 and Q_2 have regularly varying tails with the same tail index α, but with different tail coefficients A, see (2).

Suppose R, Q_1 and Q_2 have Pareto-like tails with the same tail index α, but with potentially different scale coefficients. In that case, the following corollary follows from Proposition 1:

Corollary 1 *Suppose that asset returns X_1 and X_2 can be modelled by the single index market model, where R, Q_1 and Q_2 all have Pareto–like tails with tail index $\alpha > 1$, and scale coefficients $A_r > 0$, $A_1 > 0$ and $A_2 > 0$ respectively, as in (2). Then VaR is subadditive in the tail region.*

In general, the single index market model (4) may not describe the true nature of the dependence between X_1 and X_2 since Q_i's may not be cross sectionally independent, although each one of them may be independent from the common market factor R. For example, apart from the market risk, the assets X_1 and X_2 may be dependent on an industry specific risk, depicted by the movement of an industry specific index S, also known as "sectoral index" in finance. Such industry specific factors may lead to dependence between Q_1 and Q_2. We may model cross sectional dependence by generalising the model (4) by incorporating a sector specific factor S.

$$X_i = \beta_i \ R + \tau_i \ S + Q_i, \ i = 1, 2 \tag{5}$$

where R is the market factor, S is the industry specific factor and Q_i is the idiosyncratic risk of the asset X_i. In this model Q_i is independent of R and S. Further, Q_i is cross sectionally independent. In this model, τ_i is the industry specific risk of the asset X_i. If S has Pareto–like tails with scale coefficient A_s and tail index α, then under the assumption of Proposition 1:

$$\mathrm{VaR}_p(X_1 + X_2) \leq \mathrm{VaR}_p(X_1) + \mathrm{VaR}_p(X_2)$$

5 Simulation Results

The theoretical results presented above can only be expected to hold in the tails, leaving open the question of whether these results hold in practice. In order to establish the finite sample properties of these results, we simulate from three categories of bivariate distributions, across a range of probabilities and sample sizes.

The distributions chosen are Student–t for both super fat tails as well as tails usually obtained from return distributions. The second class of models is jump processes. In both cases we consider both correlated and non–correlated draws. Finally, we estimate a bivariate GARCH model in order to provide parameters so that the GARCH model can be simulated.

The number of simulations is chosen to represent both very large sample sizes expected to give asymptotic results as well as a smaller sample representing typical applications. The larger sample size is 10^6 and a smaller is 10^3. The probability levels are chosen to capture those typically used in practice, i.e., 5% and 1%. Finally, the number of simulations is 10^3 for the larger sample size and 10^4 for the smaller sample size.

In the simulations, p denotes the significance level of VaR and n denotes the relative number of cases where subadditivity is violated, i.e. number of violations/number of simulations, expressed as a percentage.

5.1 Student–t Distribution

The Student–t distribution has widespread applications in risk modelling. It has a number of desirable properties, and for our purpose the fact that it's degrees of freedom equal the tail index, and is regularly varying, makes it especially convenient for the purpose of establishing a small sample properties of subadditivity.

We consider bivariate cases where the degrees of freedom ν are in the range of one to six, and consider both cases where the degrees of freedom of the first random variable X_1, ν_1 equals that of the second random variable, and also allow for the case where the degrees of freedom of the second distribution $\nu_1 < \nu_2$ are larger.

Furthermore, consider the case where both random samples are independent and are linearly correlated with correlation coefficient 0.5.

The results from simulating this data are presented in Tables 1 and 2.

As shown in these tables, n, the number of times when subadditivity fails is very high when $\nu_1 = 1$. In these cases the first moments are not well defined.

When the degrees of freedom is higher than 1, then the first moment is well defined. As shown in Tables 1 and 2 for well defined first moment, the violation of subadditivity is negligible or zero.

5.2 Jump Process

In addition to the student–t, jump processes has seen widespread applications. As a consequence, our second case is a jump process whereby with probability $1 - p$ the random variable X_1 is IID standard normal, and with probability p it experiences a negative jump.

First we let the second random variable X_2 have the same distribution, but independently.

$X_i, i = 1, 2$ is drawn from :

$$X_i \sim \begin{cases} \text{IID}\, \mathcal{N}(0,1) & \text{with probability} \quad p \\ b - c, \quad c \sim \mathcal{U}(0, d) & \text{with probability} \quad 1 - p \end{cases}$$

in our case

$$X_i \sim \begin{cases} \text{IID}\, \mathcal{N}(0,1) & \text{with probability} \quad 0.995 \\ -10 - c, \quad c \sim \mathcal{U}(0, 0.2) & \text{with probability} \quad 0.005 \end{cases}$$

Finally, we allow for dependent probabilities for jumps, i.e. the *joint event probability* denoted by q, which is the probability that on days when the X_1 jumps, X_2 also jumps.

Table 3 presents the results of the simulation from the jump process. They confirm the results in Example 1.

5.3 BEKK GARCH

The final simulation is based on estimating a bivariate volatility model and use that to generate random samples. In particular, we use the bivariate BEKK model (see Engle and Kroner, 1995), which is one of the most common multivariate volatility models. The data we use to estimate the BEKK model is daily return data from Microsoft and Goldman Sachs over June 1, 1999 to December 31, 2003, or 1155 observations.

We then simulate from this model. Using these simulated results, we estimate VaRs of the individual returns and their sum. In Table 4 we present the results from the simulation. It is seen from this table that the number of subadditivity failure is close to zero.

6 Conclusion

VaR has been criticized because of its lack of subadditivity. We take a fresh look at the issue of subadditivity of VaR, focusing on the tails of heavy tailed assets, that are most commonly observed in financial applications. We find that for such distributions, VaR is subadditive in the tails, at probabilities that are most relevant for practical applications. We further identify the specifications when VaR may fail subadditivity. The results suggest that there is a strong case for using VaR, and it is usually not necessary to consider other risk measures, solely for reasons of coherence.

Appendix

Proposition 1 deals with left tails, but for notational simplicity the argument below treats right tails. Recall the following definition.

Definition 2 *A random vector* (X, Y) *has regularly varying right tails with tail index* α *if there is a function* $a(t) > 0$ *that is regularly varying at infinity with exponent* $1/\alpha$ *and a nonzero measure* μ *on* $(0, \infty)^2 \setminus \{0\}$ *such that*

$$t\,P\left((X, Y) \in a(t)\cdot\right) \to \mu \tag{6}$$

as $t \to \infty$ *vaguely in* $(0, \infty]^2 \setminus \{0\}$ *(see e.g. Resnick, 1986).*

The measure μ has a scaling property

$$\mu(cA) = c^{-\alpha}\mu(A) \tag{7}$$

for any $c > 0$ and Borel set A. The non-degeneracy assumption in Proposition 1 means that the measure μ is not concentrated on a straight line $\{ax = by\}$ for some $a, b \geq 0$.

Proof of Proposition 1.

For $p > 0$ small,

$$\mathrm{VaR}_p(X) \sim \left(\mu\Big\{(1, \infty) \times (0, \infty)\Big\}\right)^{1/\alpha} a\left(\frac{1}{p}\right),$$

$$\mathrm{VaR}_p(Y) \sim \left(\mu\Big\{(0, \infty) \times (1, \infty)\Big\}\right)^{1/\alpha} a\left(\frac{1}{p}\right)$$

and

$$\mathrm{VaR}_p(X + Y) \sim \left(\mu\Big\{x \geq 0,\, y \geq 0 :\, x + y > 1\Big\}\right)^{1/\alpha} a\left(\frac{1}{p}\right)$$

as $p \to 0$.

The scaling property (7) means that there is a finite measure η on $B_1 = \{x \geq 0,\, y \geq 0 :\, x + y = 1\}$ such that

$$\mu(A) = \int_{B_1}\int_0^\infty \mathbf{1}((u, v)r \in A)\, \alpha r^{-(1+\alpha)}\, dr\, \eta(du, dv)\,. \tag{8}$$

Then

$$\mu\Big\{(1, \infty) \times (0, \infty)\Big\} = \int_{B_1} u^\alpha\, \eta(du, dv)\,,$$

$$\mu\Big\{(0,\infty)\times(1,\infty)\Big\} = \int_{B_1} v^\alpha\, \eta(du,dv)\,,$$

and

$$\mu\Big\{x\geq 0,\, y\geq 0:\, x+y>1\Big\} = \int_{B_1} (u+v)^\alpha\, \eta(du,dv)\,.$$

Since by the triangle inequality in $L^\alpha(\eta)$

$$\left(\int_{B_1} (u+v)^\alpha\, \eta(du,dv)\right)^{1/\alpha}$$

$$< \left(\int_{B_1} (u)^\alpha\, \eta(du,dv)\right)^{1/\alpha} + \left(\int_{B_1} (v)^\alpha\, \eta(du,dv)\right)^{1/\alpha}\,,$$

with the strict inequality under the non-degeneracy assumption, we conclude that

$$\mathrm{VaR}_p(X+Y) - \mathrm{VaR}_p(X) - \mathrm{VaR}_p(Y) < 0$$

holds for all $p>0$ small enough.

■

Remark 1 *From the proof above we see that, even without the non-degeneracy assumptions (and, in particular, if the two assets have different tail indices) we still have*

$$\limsup_{p\to 0} \frac{\mathrm{VaR}_p(X+Y)}{\mathrm{VaR}_p(X)+\mathrm{VaR}_p(Y)} \leq 1\,,$$

which is a weaker form of subadditivity in the tails.

Table 1: Simulation from Student's t-distribution. 1000 observations, 10,000 simulations

This is 10,000 draws, of vectors of size 1000 from Student–t, first vector with degrees of freedom ν_1, the other with ν_2. ρ is the linear correlation coefficient. There are 2 VaR probability levels 1% and 5%. Last 2 columns record the percentage number of subadditivity violations.

Degrees of freedom		Correlation	Fraction of Subadditivity violations for VaR probability	
ν_1	ν_2	ρ	1%	5%
1	1	0.0	40.9%	46.1%
1	1	0.5	43.1%	45.8%
2	1	0.0	6.1%	0.8%
2	1	0.5	13.6%	12.6%
2	2	0.0	0.9%	0.0%
2	2	0.5	8.7%	1.3%
3	3	0.0	0.0%	0.0%
3	3	0.5	1.7%	0.1%
4	4	0.0	0.0%	0.0%
4	4	0.5	0.5%	0.0%

Table 2: Simulation from Student's t-distribution. 1,000,000 observations, 1,000 simulations

This is 1,000 draws, of vectors of size 1,000,000 from Student–t, first vector with degrees of freedom ν_1, the other with ν_2. ρ is the linear correlation coefficient. There are 2 VaR probability levels 1% and 5%. Last 2 columns record the percentage number of subadditivity violations.

Degrees of freedom		Correlation	Fraction of Subadditivity violations for VaR probability	
ν_1	ν_2	ρ	1%	5%
1	1	0.0	51.2%	51%
1	1	0.5	50.8%	48.5%
2	1	0.0	0.0%	0.0%
2	1	0.5	0.0%	0.0%
2	2	0.0	0.0%	0.0%
2	2	0.5	0.0%	0.0%
3	3	0.0	0.0%	0.0%
3	3	0.5	0.0%	0.0%
4	4	0.0	0.0%	0.0%
4	4	0.5	0.0%	0.0%

Table 3: Simulation from Jump processes

This is simulations from a Jump process, see Section 5.2. There are 2 VaR probability levels 1% and 5%, and both independent and dependent jumps. Last 2 columns record the percentage number of subadditivity violations.

(a) 1000 observations, 10,000 simulations

Joint event probability	Fraction of Subadditivity violations for VaR probability	
q	1%	5%
q		
0.00	25%	0.0%
0.05	48.1%	0.0%

(b) 1,000,000 observations, 1,000 simulations

Joint event probability	Fraction of Subadditivity violations for VaR probability	
q	1%	5%
q		
0.0	43.8%	0.0%
0.05	0.2%	0.0%

Table 4: Simulation from a BEKK bivariate GARCH processes

This is simulations from a BEKK bivariate GARCH, parameters estimated with daily return data from Microsoft and Goldman Sachs over June 1, 1999 to December 31, 2003, or 1155 observations. process, see Section 5.3. There are 2 VaR probability levels 1% and 5%. Last 2 columns record the percentage number of subadditivity violations.

Observations	Simulations	Fraction of Subadditivity violations for VaR probability	
N	S	1%	5%
1000	10,000	0.0%	0.0%
1,000,000	1000	0.0%	0.0%

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