

A NEW SHAPE OF EXTREMAL CLUSTERS FOR CERTAIN STATIONARY SEMI-EXPONENTIAL PROCESSES WITH MODERATE LONG RANGE DEPENDENCE

ZAOLI CHEN AND GENNADY SAMORODNITSKY

ABSTRACT. Extremal clusters of stationary processes with long memory can be quite intricate. For certain stationary infinitely divisible processes with subexponential tails, including both power-like tails and certain lighter tails, e.g. lognormal-like tails, such clusters may take the shape of stable regenerative sets. In this paper we show that for semi-exponential tails, which are even lighter, a new shape of extremal clusters arises. In this case each stable regenerative set supports a random panoply of varying extremes.

1. INTRODUCTION

In this paper, we study the extremes of certain stationary infinitely divisible processes with long range dependence. The marginal log-tails are, roughly, of the order $-x^{-\alpha}$ for some $0 < \alpha < 1$. We call such tails semi-exponential, and the exact definition and the assumptions are given in Section 3. Such marginal tails are in the Gumbel maximum domain of attraction, and the assumptions we impose will also guarantee that these tails are subexponential.

Extremal limit theorems for such processes are interesting from several points of view. First of all, how do the extreme values of such processes cluster? Second, to what extent does the “single large jump” heuristic hold for such processes? This principle usually governs both extreme values and large deviations of weakly dependent subexponential stochastic systems. Extremal clusters appear when the values of the process at distinct time points are tail (asymptotically) dependent; see Resnick (2007). The size of extremal clusters is a subject of long-standing interest, and the notion of extremal index due to Davis (1982) and Leadbetter (1983) is specifically designed to quantify this size. When the stationary process has long memory affecting the extreme values of the process in the sense of Samorodnitsky (2016), the extremal clusters may become so large that scaling is necessary to obtain a finite limit, and then it becomes possible to talk about the limiting shape of an extremal cluster. For certain classes of stationary infinitely divisible processes with subexponential tails, this limiting shape is a random fractal, specifically a stable regenerative set, supporting one extreme value. This has been shown by Lacaux and Samorodnitsky (2016) in the case when the process has regularly varying tails (which are, of course, in the Fréchet maximum domain of attraction) and by Chen and Samorodnitsky (2020) in the case when the process has certain marginal tails in the Gumbel maximum domain of attraction. The results of the latter paper required, however, that these marginal tails were not too light.

2000 *Mathematics Subject Classification.* Primary 60G70, 60F17; Secondary 60G57.

Key words and phrases. Extreme value theory, long range dependence, random sup-measure, stable regenerative set, subexponential distributions, semi-exponential distributions, Gumbel maximum domain of attraction.

This research was partially supported by the ARO grant W911NF-18 -10318 and the NSF grant DMS-1506783 at Cornell University.

In particular, Chen and Samorodnitsky (2020) allowed lognormal-like marginal tails but excluded semi-exponential marginal tails, whose limiting shape of the extremal clusters remained unknown. In this paper, we solve this problem and characterize the limiting shape of the extremal clusters. While the clusters are still supported by stable regenerative sets, a new random structure appears, in which related but different extreme values are placed in randomly chosen locations over each stable regenerative set.

The new shape of the extremal clusters is related to a failure of the “single large jump” principle: the extreme values of the process are caused by multiple large values of the underlying Poisson random measure. Recall that a distribution H on $[0, \infty)$ is called subexponential if in the usual notation for distributional tails,

$$\lim_{x \rightarrow \infty} \frac{\overline{H * H}(x)}{\overline{H}(x)} = 2. \quad (1.1)$$

“Single large jump” is a widely used heuristic for processes with subexponential tails; see *e.g.* Foss et al. (2007). This principle already fails in the case of heavier tails in the Gumbel maximum domain of attraction, see Chen and Samorodnitsky (2020). But in the case of semi-exponential tails, this failure is even more dramatic. In one of our limit theorems we obtain a new limiting process of two parameters. It can be viewed as a bridge between the standard Gumbel extremal process of Resnick and Rubinovitch (1973) and the time-changed extremal process of Chen and Samorodnitsky (2020). When the parameters of the new process tend to some of their boundary values, either of the latter two processes can be recovered.

The paper is organized as follows. Section 2 reviews the main notions and tools we will use throughout the paper: random closed sets, null recurrent Markov chains, distributions in the Gumbel domain of attraction and random sup-measures. Section 3 describes the stationary infinitely divisible process whose extremes are to be analyzed. We state the two main extremal limit theorems, establishing weak convergence in the spaces of sup-measures and càdlàg functions respectively. This section also contains most of the proofs. Several auxiliary proofs are postponed to the two Appendices.

The adjective “moderate” decorating the term “long range dependence” in the title of the paper is due to the restricted range $\beta \in (0, 1/2)$ of the parameter responsible for the long memory. What happens if memory becomes even longer, *i.e.* $\beta \in (1/2, 1)$, remains a subject of future investigations, and we expect to get limit theorems with non-Gumbel limits. When the marginal tails are regularly varying, non-Fréchet limits in this range of β are established in Samorodnitsky and Wang (2019).

The following notation will be used throughout the paper. We denote the set of natural numbers $\{1, 2, \dots\}$ by \mathbb{N} . For a nondecreasing function H on \mathbb{R} , the inverse of H is defined by $H^{\leftarrow}(x) = \inf\{s : H(s) \geq x\}$, with the usual convention $\inf \emptyset = \infty$. Further, the tail of a measure ν on \mathbb{R} is $\bar{\nu}(x) = \nu(x, \infty)$. In particular $\bar{F}(x)$ is the tail of a distribution F . We will use the following symbols when comparing positive sequences.

- (a) $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$,
- (b) $a_n \lesssim b_n$ if there exist $C > 0$ such that $a_n \leq Cb_n$ for large enough n , and analogously with $a_n \gtrsim b_n$,
- (c) $a_n \asymp b_n$ if both $a_n \lesssim b_n$ and $a_n \gtrsim b_n$.

If $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ are two sequences of positive random variables, we write

- (a) $A_n = o_P(B_n)$ if $A_n/B_n \rightarrow 0$ in probability,
- (b) $A_n \lesssim_P B_n$ if (A_n/B_n) is tight, and analogously with $A_n \gtrsim_P B_n$.

2. PRELIMINARIES

2.1. Random Closed Sets. Random closed sets play a key role in many parts of this paper, particularly in the description of the main limiting objects. This section is an overview of mostly well-known facts about random closed sets. Unless stated otherwise, these facts are taken from Molchanov (2017).

We work with an underlying space E , which will be either $[0, 1]$ or \mathbb{R}_+ . We write $\mathcal{G}, \mathcal{F}, \mathcal{F}', \mathcal{K}$ and \mathcal{K}' for the family of open, closed, nonempty closed, compact and nonempty compact sets in E , respectively. If we want to emphasize the choice of E , we will use a notation of the type $\mathcal{F}([0, 1])$. For any $A \subset E$, we define

$$\mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \emptyset\}, \quad (2.1)$$

$$\mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \emptyset\}. \quad (2.2)$$

The *Fell* topology on \mathcal{F} is generated by the sub-basis consisting of $\{\mathcal{F}_G : G \in \mathcal{G}\}$ and $\{\mathcal{F}^K : K \in \mathcal{K}\}$, under which the space \mathcal{F} is compact and metrizable. In the case $E = [0, 1]$, $\mathcal{F} = \mathcal{K}$ and the Fell topology on \mathcal{F} agrees with the so-called *myopic* topology on \mathcal{K} . In particular, \mathcal{F}' , with the subspace topology, is metrizable by the Hausdorff metric

$$\rho_H(F_1, F_2) := \max \left\{ \sup_{x \in F_1} \rho(x, F_2), \sup_{x \in F_2} \rho(F_1, x) \right\}, \quad F_1, F_2 \in \mathcal{F}', \quad (2.3)$$

where $\rho(\cdot, \cdot)$ is the standard distance function

$$\rho(x, F) = \min\{|x - y| : y \in F\}. \quad (2.4)$$

We will also use another common distance function

$$\rho(F_1, F_2) = \min\{|x - y| : x \in F_1, y \in F_2\}. \quad (2.5)$$

For either choice of E , a *random closed set* is a measurable mapping from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. A sequence of random closed sets $\{F_n\}_{n \in \mathbb{N}}$ weakly converges to F if

$$\lim_{n \rightarrow \infty} \mathbb{P}\{F_n \cap A \neq \emptyset\} = \mathbb{P}\{F \cap A \neq \emptyset\}, \quad \text{for any } A \in \mathcal{A} \cap \mathfrak{S}_F. \quad (2.6)$$

Here, \mathcal{A} is the collection of all finite unions of open intervals, and \mathfrak{S}_F is the collection of all continuity sets of F :

$$\left\{ B \in \mathcal{B} : B \text{ is relatively compact and } \mathbb{P}\{F \cap \text{cl } B \neq \emptyset\} = \mathbb{P}\{F \cap \text{int } B \neq \emptyset\} \right\}. \quad (2.7)$$

We now introduce the random closed sets of our primary concern. For a $\beta \in (0, 1/2)$, let $\{Z(t)\}_{t \in \mathbb{R}_+}$ be a standard β -stable subordinator, which is an increasing Lévy process with the Laplace transform

$$\mathbb{E} \exp\{-\theta Z(t)\} = \exp\{-t\theta^\beta\}, \quad \theta \in \mathbb{R}_+. \quad (2.8)$$

The *β -stable regenerative set* R is the closure of the range of $\{Z(t)\}_{t \in \mathbb{R}_+}$,

$$R = \text{cl} \{Z(t) : t \in \mathbb{R}_+\}. \quad (2.9)$$

Next, take a random variable $Z^*(0)$ independent of $\{Z(t)\}_{t \in \mathbb{R}_+}$ with the distribution

$$\mathbb{P}\{Z^*(0) \leq x\} = x^{1-\beta}, \quad x \in [0, 1]. \quad (2.10)$$

We define the process

$$Z^*(t) = Z^*(0) + Z(t), \quad t \in \mathbb{R}_+, \quad (2.11)$$

whose range induces another random closed set

$$R^* := \text{cl} \{Z^*(t) : t \in \mathbb{R}_+\} = Z^*(0) + R. \quad (2.12)$$

Let m^ϕ be the measure associated with the dimension (or gauge) function

$$\phi(x) = x^\beta (\log|\log x|)^{1-\beta}. \quad (2.13)$$

According to Theorem 1 in Taylor and Wendel (1966), there is a finite positive constant c_β such that on an event of probability 1,

$$m^\phi(R \cap [0, t]) = c_\beta Z^\leftarrow(t) \text{ for all } t \in \mathbb{R}_+. \quad (2.14)$$

Note that $Z^\leftarrow(\cdot)$ is a standard Mittag-Leffler process, which is self-similar with exponent β and has continuous sample paths. An immediate consequence of (2.14) is that on the same event of probability 1,

$$m^\phi(R^* \cap [0, t]) = c_\beta Z^{*\leftarrow}(t) \text{ for all } t \in \mathbb{R}_+. \quad (2.15)$$

In the sequel, we will be mostly interested in the restriction

$$\overline{R^*} := R^* \cap [0, 1] \quad (2.16)$$

of R^* to the unit interval. Furthermore, we will need to sample points from this restriction according to the normalized measure m^ϕ on it. We now set up a technical framework for doing so. Assuming the random set R^* is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{U_i\}_{i \geq 1}$ be a sequence of *i.i.d.* standard uniform random variables defined on another probability space, say, $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. For $\omega \in \Omega$ define

$$\eta_\omega : [0, 1] \rightarrow [0, 1], \quad t \mapsto \frac{m^\phi(R^*(\omega) \cap [0, t])}{m^\phi(R^*(\omega) \cap [0, 1])} \quad (2.17)$$

if the denominator is positive, while setting $\eta_\omega(t) \equiv t$ for the ω in the zero probability event that the denominator vanishes. We define

$$J_i(\omega, \omega_1) = \eta_\omega^\leftarrow(U_i(\omega_1)), \quad i = 1, 2, \dots, (\omega, \omega_1) \in (\Omega \times \Omega_1), \quad (2.18)$$

and view $(\{Z^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+}, \overline{R^*}, \{J_i\} : i \in \mathbb{N})$ as a random element of the space $C[0, \infty) \times \mathcal{F}([0, 1]) \times (\mathcal{F}([0, 1]))^\infty$, defined on the product probability space $(\Omega \times \Omega_1, \mathcal{F} \times \mathcal{F}_1, \mathbb{P} \times \mathbb{P}_1)$. The law of this random element will be very important in the sequel. It follows from (2.15) that for $\mathbb{P} - a.s.$ $\omega \in \Omega$

$$\mathbb{P}_1 \{ \cup_{i \geq 1} \{J_i(\omega, \omega_1)\} \subset \overline{R^*} \} = \mathbb{P}_1 \{ \text{cl}(\cup_{i \geq 1} \{J_i(\omega, \omega_1)\}) = \overline{R^*} \} = 1, \quad (2.19)$$

Therefore also

$$\mathbb{P} \{ \cup_{i \geq 1} \{J_i(\omega, \omega_1)\} \subset \overline{R^*} \} = \mathbb{P} \{ \text{cl}(\cup_{i \geq 1} \{J_i(\omega, \omega_1)\}) = \overline{R^*} \} = 1. \quad (2.20)$$

2.2. Null Recurrent Markov Chains. We introduce certain null recurrent Markov chains from an ergodic theoretic perspective. More details can be found in Aaronson (1997).

Let $\{Y_t\}_{t \in \mathbb{Z}}$ be an irreducible, aperiodic, and null recurrent Markov chain on \mathbb{Z} . We specify a unique invariant measure $(\pi_i)_{i \in \mathbb{Z}}$ by taking $\pi_0 = 1$. Let (E, \mathcal{E}) be the path space $(\mathbb{Z}^\mathbb{Z}, \mathcal{B}(\mathbb{Z}^\mathbb{Z}))$ and $\mathbb{P}_i(\cdot)$ the law $\mathbb{P} \{ \cdot \mid Y_0 = i \}$ induced by the trajectories of the Markov chain on E . Setting

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i \mathbb{P}_i(\cdot), \quad (2.21)$$

$$\theta : E \rightarrow E, \quad (\cdots, y_0, y_1, y_2, \cdots) \mapsto (\cdots, y_1, y_2, y_3, \cdots), \quad (2.22)$$

makes $(E, \mathcal{E}, \mu, \theta)$ a measure preserving, conservative and ergodic dynamical system, see Harris and Robbins (1953).

We consider the *first visit time* to 0,

$$\varphi(y) = \inf\{t \geq 1 : y_t = 0\}, \quad y \in E, \quad (2.23)$$

and adopt the following assumption.

Assumption 2.1. For some $\beta \in (0, 1/2)$ and slowly varying function L ,

$$\bar{F}(n) := \mathbb{P}_0\{\varphi > n\} = n^{-\beta} L(n), \quad (2.24)$$

$$\sup_{n \geq 0} \frac{n \mathbb{P}_0\{\varphi = n\}}{\bar{F}(n)} < \infty. \quad (2.25)$$

Remark 2.2. The main results of the paper require the assumption (2.25), see Theorem B in Doney (1997).

The wandering rate sequence $\{w_n\}_{n \geq 0}$ is defined by

$$w_n = \mu\{\cup_{k=0}^n A_k\} \quad \text{with } A_n = \{y \in E : y_n = 0\}, \quad n \in \mathbb{N}_0. \quad (2.26)$$

Under (2.24) it follows from Lemma 3.3 in Resnick et al. (2000) that

$$w_n \sim n^{1-\beta} L(n) / (1 - \beta). \quad (2.27)$$

The extreme value analysis in this paper requires certain additional delicate details hidden in $(E, \mathcal{E}, \mu, \theta)$. For each $n \in \mathbb{N}_0$, we define a probability measure on E by

$$\mu_n(\cdot) = \mu\left(\cdot \cap \bigcup_{k=0}^n A_k\right) / w_n. \quad (2.28)$$

Let $\{Y^{(k;n)}\}_{k \in \mathbb{N}_0}$ be *i.i.d.* random elements in E with law μ_n . We are interested in the (random) zero sets

$$I_{k;n} = \{0 \leq t \leq n : Y_t^{(k;n)} = 0\}, \quad k \in \mathbb{N}_0, \quad (2.29)$$

and their intersections. For fixed $n, k \in \mathbb{N}$ we define

$$j_{k,1;n} = \inf\{j > k : I_{j;n} \cap I_{k;n} \neq \emptyset\} \quad (2.30)$$

and continue inductively by setting for $i \geq 2$,

$$j_{k,i;n} = \inf\{j > j_{k,i-1;n} : I_{j;n} \cap I_{k;n} \neq \emptyset\} \quad (2.31)$$

if $j_{k,i-1;n} < \infty$ and $j_{k,i;n} = \infty$ otherwise. For $i \geq 1$, on the event $\{j_{k,i;n} < \infty\}$ we define

$$I_{k,i;n} = I_{k;n} \cap I_{j_{k,i;n};n}. \quad (2.32)$$

For $k \in \mathbb{N}$, consider the random probability

$$\bar{p}_{k;n} := \mathbb{P}\{I_{k;n} \cap I_{0;n} \neq \emptyset \mid I_{k;n}\}, \quad (2.33)$$

and note that, conditionally on $I_{k;n}$, $j_{k,1;n} - k$ is geometrically distributed with success probability $\bar{p}_{k;n}$. The following theorem is interesting on its own right. We precede it with some notation. Let $(\{Z_k^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+}, \bar{R}_k^*, \{J_{k,i}\} : i \in \mathbb{N})$, $k = 1, 2, \dots$ be *i.i.d.* copies of the random element $(\{Z^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+}, \bar{R}^*, \{J_i\} : i \in \mathbb{N})$ in $C[0, \infty) \times \mathcal{F}([0, 1]) \times (\mathcal{F}([0, 1]))^\infty$ constructed in Section 2.1. Let Γ_k , $k = 1, 2, \dots$ be an independent of them *i.i.d.* sequence of unit rate Poisson processes on $(0, \infty)$. That is, each $\Gamma_k = \{\Gamma_{k,i}\}_{i \in \mathbb{N}}$ consists of the arrival times of a unit rate Poisson processes on $(0, \infty)$ (listing the points of Γ_k in the increasing order).

Theorem 2.3. *Under Assumption 2.1 there is a constant $c_\infty \in (0, 1)$ such that for any $K, m \in \mathbb{N}$*

$$\begin{aligned} & \left(\frac{w_n}{\vartheta_n} \bar{p}_{k;n}, (j_{k,i;n} \bar{p}_{k;n})_{1 \leq i \leq m}, \frac{1}{n} I_{k;n}, (I_{k,i;n}/n)_{1 \leq i \leq m} \right)_{1 \leq k \leq K} \\ & \Rightarrow \left(c_\infty Z_k^{*\leftarrow}(1), (\Gamma_{k,i})_{1 \leq i \leq m}, \bar{R}_k^*, \{J_{k,i}\}_{1 \leq i \leq m} \right)_{1 \leq k \leq K} \end{aligned} \quad (2.34)$$

as $n \rightarrow \infty$, weakly in the space $\left(\mathbb{R}_+^{1+m} \times (\mathcal{F}([0, 1]))^{1+m} \right)^K$, where

$$\vartheta_n = \frac{(2 - \beta)n^\beta}{\beta L(n)}, \quad n \in \mathbb{N}. \quad (2.35)$$

This theorem is proved in Appendix A, and so is the following proposition that establishes an exponential integrability of $\#I_{1,1;n}$ in both annealed and quenched situations.

Proposition 2.4. (i) *Let $c_\infty \in (0, 1)$ be the constant in Theorem 2.3. Then*

$$\mathbb{P} \{ I_{1,1;n} \geq m \} \leq (1 - c_\infty)^m, \quad m = 1, 2, \dots \quad (2.36)$$

(ii) *Let $I_{1;n}$ be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, $C = C(\epsilon) > 0$, and an event $\Omega_n \subset \Omega$ satisfying $\mathbb{P} \{ \Omega_n \} \geq 1 - \epsilon$ such that*

$$\sup_{\omega \in \Omega_n} \mathbb{E}_\omega e^{\delta \cdot \#I_{1,1;n}} \leq C, \quad (2.37)$$

where \mathbb{E}_ω denotes the conditional expectation $\mathbb{E} \{ \cdot \mid I_{1,n}(\omega) \}$.

The last proposition of this subsection is adapted from (A.3) and (A.9) of Chen and Samorodnitsky (2020).

Proposition 2.5. *For any $p > 0$ there is $\mu_p < \infty$ such that*

$$\sup_{n \geq 1} \mathbb{E} (\bar{F}(n) \#I_{1;n})^p \leq \mu_p. \quad (2.38)$$

Further, for any $C > 0$ there is $c > 0$ so that for all $n \in \mathbb{N}$,

$$\mathbb{P} \left\{ \#I_{1;n} \geq \frac{c \log n}{\bar{F}(n)} \right\} \leq n^{-C}. \quad (2.39)$$

2.3. Distributions in the Gumbel maximum domain of attraction. Recall that a distribution H , with an unbounded support on the right, is in the Gumbel maximum domain of attraction if and only if there exist $x_0 \in \mathbb{R}$ and $c(x) \rightarrow c > 0$ as $x \rightarrow \infty$ such that for $x_0 < x < \infty$

$$\bar{H}(x) = c(x) \exp \left\{ - \int_{x_0}^x \frac{1}{h(u)} du \right\} \quad (2.40)$$

where h (the so-called auxiliary function) is an absolutely continuous positive function on (x_0, ∞) with density h' satisfying $\lim_{u \rightarrow \infty} h'(u) = 0$; we refer the reader to Resnick (1987) and Goldie and Resnick (1988) for more details. The function h must satisfy $h(x) = o(x)$ as $x \rightarrow \infty$; if the distribution H is also subexponential, then its support is unbounded on the right and $\lim_{u \rightarrow \infty} h(u) = \infty$.

For a distribution H satisfying (2.40), the centering and scaling required for convergence in the extremal limit theorem can be chosen as

$$b_n = \left(\frac{1}{1 - H} \right)^{\leftarrow}(n), \quad a_n = h(b_n).$$

We will often use the following fact: if one replaces the function $c(\cdot)$ in (2.40) by an asymptotically equivalent function, and denotes the new normalizing sequences by (\tilde{a}_n) and (\tilde{b}_n) , then

$$\lim_{n \rightarrow \infty} \frac{b_n - \tilde{b}_n}{a_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{\tilde{a}_n}{a_n} = 1. \quad (2.41)$$

2.4. Random Sup-Measures. We will deal with sup-measures taking values in $\overline{\mathbb{R}} = [-\infty, \infty]$. The main reference is O'Brien et al. (1990).

A sup-measure is a mapping $m : \mathcal{G} \rightarrow \overline{\mathbb{R}}$ such that $m(\emptyset) = -\infty$ and $m(\cup_\alpha G_\alpha) = \vee_\alpha m(G_\alpha)$ for an arbitrary collection (G_α) of open sets. The sup-derivative $d^\vee m$ of m is

$$d^\vee m(t) = \bigwedge_{t \in G} m(G);$$

it is automatically an upper semicontinuous $\overline{\mathbb{R}}$ -valued function of t . Given any $\overline{\mathbb{R}}$ -valued function f , the sup-integral of f

$$i^\vee f(G) = \bigvee_{t \in G} f(t), \quad G \in \mathcal{G}$$

is a sup-measure. The domain of a sup-measure can be extended to all Borel sets via

$$m(B) = \bigvee_{t \in B} d^\vee m(t), \quad B \text{ Borel.}$$

The collection SM of sup-measures admits a natural metrizable sup-vague topology with its corresponding Borel measurability, which allows one to talk about random sup-measures. In particular, if $\{\mathcal{M}_n\}_{n \geq 1}$ and \mathcal{M} are random sup-measures, then $\mathcal{M}_n \Rightarrow \mathcal{M}$ if and only if

$$(\mathcal{M}_n(I_1), \dots, \mathcal{M}_n(I_m)) \Rightarrow (\mathcal{M}(I_1), \dots, \mathcal{M}(I_m)) \quad (2.42)$$

for arbitrarily disjoint open intervals I_1, \dots, I_m such that $\mathbb{P}\{\mathcal{M}(I_i) = \mathcal{M}(\overline{I_i})\} = 1$ for all $i = 1, \dots, m$.

A stochastic process $\{X_t\}_{t \in \mathbb{N}}$ induces a family of random sup-measures $\{\mathcal{M}_n(\cdot)\}_{n \geq 1}$ via

$$\mathcal{M}_n(B) := \max_{t \in nB} X_t, \quad B \in \mathcal{B}(E), \quad (2.43)$$

We now describe the limiting random sup-measures appearing in our main results. The construction is doubly stochastic.

First, let $\alpha \in (0, 1)$ and $\beta \in (0, 1/2)$, and denote by P_β the law of the closed range R of the β -stable subordinator in (2.9). Consider a Poisson point process \mathcal{N} on $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure

$$e^{-x} dx \times (1 - \beta)y^{-\beta} dy \times dP_\beta,$$

defined on some probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and let $(X_k, Y_k, R_k)_{k \in \mathbb{N}}$ be a measurable enumeration of its points. For each point $(X_k, Y_k, R_k) = (X_k(\omega_1), Y_k(\omega_1), R_k(\omega_1))$ the dimension function ϕ in (2.13) produces a (random) measure

$$H_k(t, \omega_1) = m^\phi \left\{ (Y_k(\omega_1) + R_k(\omega_1)) \cap [0, t] \right\}, \quad t \in \mathbb{R}_+ \quad (2.44)$$

on \mathbb{R}_+ . The second level of randomness is now introduced, conditionally on this point $(X_k(\omega_1), Y_k(\omega_1), R_k(\omega_1))$, via a Poisson point process $\mathbb{C}_k = \mathbb{C}_k(\omega_1)$ on $\mathbb{R} \times \mathbb{R}_+$ with the mean measure

$$\frac{C_{\alpha, \beta}}{c_\beta} \exp \left\{ -C_{\alpha, \beta}(\lambda - X_k(\omega_1)) \right\} d\lambda \times dH_k(\cdot, \omega_1),$$

where

$$C_{\alpha,\beta} := \left(\frac{1-\beta}{\beta} \right)^{\frac{1}{\alpha}-1} > 1, \quad (2.45)$$

and c_β is the constant in (2.14). We assume that the point processes (\mathbb{C}_k) live on some other probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ and, conditionally on \mathcal{N} , are independent of each other. The overall probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.

For $k \in \mathbb{N}$, let $\{\Lambda_{k,i}, W_{k,i}\}_{i \in \mathbb{N}}$ be a measurable enumeration of the points of \mathbb{C}_k . We note that the points $\{W_{k,i}\}_{i \in \mathbb{N}}$ belong to the set $Y_k + R_k$ with probability 1. Consider a random sup-measure \mathcal{M} on \mathbb{R}_+ given by

$$\mathcal{M}(B) = \sup_{k,i \in \mathbb{N}} \{\Lambda_{k,i} : W_{k,i} \in B\}, \quad B \in \mathbb{R}_+. \quad (2.46)$$

This sup-measure has certain invariance properties. First of all, it is clear that \mathcal{M} can be represented in the form $\mathcal{M} = \varphi(\mathcal{N}(\omega_1), \omega_2)$ for some measurable function φ , and the mean measure of \mathcal{N} is invariant under positive shifts of the closed sets. Hence the stationarity of the \mathcal{M} :

$$\mathcal{M}(r + \cdot) \stackrel{d}{=} \mathcal{M}(\cdot) \text{ for any } r \geq 0;$$

cf. Proposition 4.3 in Lacaux and Samorodnitsky (2016). Next, \mathcal{M} has a self-affiness property: for $a > 0$,

$$\mathcal{M}(a \cdot) \stackrel{d}{=} \mathcal{M}(\cdot) + (1 - \beta + \beta/C_{\alpha,\beta}) \log a,$$

which will be established in Remark 2.7 below.

The random sup-measure \mathcal{M} naturally induces a stochastic process of independent interest by

$$\mathbb{M}(t) = \mathcal{M}([0, t]), \quad t \in (0, \infty). \quad (2.47)$$

This process is clearly nondecreasing, continuous in probability, and the sample paths are in $D(0, \infty) = \cap_{\epsilon > 0} D[\epsilon, \infty)$. The finite-dimensional distributions can be read off the following proposition, which implies that $\mathbb{M}(t) \rightarrow -\infty$ as $t \downarrow 0$.

Proposition 2.6. *Let $0 = t_0 < t_1 < \dots < t_k < \infty$. Then for any $x_i \in \mathbb{R}$, $i = 1, \dots, k$,*

$$\mathbb{P}(\mathbb{M}((t_{i-1}, t_i]) \leq x_i, i = 1, \dots, k) = \exp \left\{ -\Gamma(1 - 1/C_{\alpha,\beta}) \int_0^\infty (1 - \beta)y^{-\beta} \mathbb{E} \left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i} [Z^{\leftarrow}((t_i - y)_+) - Z^{\leftarrow}((t_{i-1} - y)_+)] \right]^{1/C_{\alpha,\beta}} dy \right\}. \quad (2.48)$$

In particular, for any $t > 0$ and $x \in \mathbb{R}$ we have

$$\mathbb{P}\{\mathbb{M}(t) \leq x\} = \exp \left\{ -K(\alpha, \beta) t^{1-\beta+\beta C_{\alpha,\beta}^{-1}} e^{-x} \right\}, \quad (2.49)$$

where

$$K(\alpha, \beta) = (1 - \beta)\Gamma(1 - 1/C_{\alpha,\beta})B(1 - \beta, 1 + \beta/C_{\alpha,\beta})\mathbb{E}(Z^{\leftarrow}(1))^{1/C_{\alpha,\beta}}.$$

Here $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ are, respectively, the Gamma function and the Beta function.

Remark 2.7. Letting $t \downarrow 0$ in (2.49) shows that $\mathbb{M}(t) \rightarrow -\infty$ as $t \downarrow 0$. Next, let $a > 0$. By (2.48) and the self-similarity of Z^\leftarrow ,

$$\begin{aligned}
& \mathbb{P}(\mathbb{M}((at_{i-1}, at_i]) \leq x_i, i = 1, \dots, k) \\
&= \exp \left\{ -a^{\beta/C_{\alpha,\beta}} \Gamma(1 - 1/C_{\alpha,\beta}) \int_0^\infty (1 - \beta)y^{-\beta} \right. \\
& \quad \left. \mathbb{E}_1 \left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i} [Z^\leftarrow((t_i - y/a)_+ - Z^\leftarrow((t_{i-1} - y/a)_+))] \right]^{1/C_{\alpha,\beta}} dy \right\} \\
&= \exp \left\{ -a^{1-\beta+\beta/C_{\alpha,\beta}} \Gamma(1 - 1/C_{\alpha,\beta}) \int_0^\infty (1 - \beta)y^{-\beta} \right. \\
& \quad \left. \mathbb{E}_1 \left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i} [Z^\leftarrow((t_i - y)_+ - Z^\leftarrow((t_{i-1} - y)_+))] \right]^{1/C_{\alpha,\beta}} dy \right\} \\
&= \mathbb{P}(\mathbb{M}((t_{i-1}, t_i]) + (1 - \beta + \beta/C_{\alpha,\beta}) \log a \leq x_i, i = 1, \dots, k).
\end{aligned}$$

Hence the self-affiness of \mathbb{M} .

Proof of Proposition 2.6. We view the collection $\{\Lambda_{k,i}, W_{k,i}\}_{k,i \in \mathbb{N}}$ is a function of the points of the Poisson process \mathcal{N} and their *i.i.d.* marks defined on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Since a marked Poisson process is still Poisson, and the event described in the left hand side of (2.48) is the event that the marked Poisson process has no points in a part of its domain, we see that

$$\begin{aligned}
& \mathbb{P}(\mathbb{M}((t_{i-1}, t_i]) \leq x_i, i = 1, \dots, k) \\
&= \exp \left\{ - \int_{-\infty}^\infty e^{-z} dz \int_0^\infty (1 - \beta)y^{-\beta} dy \right. \\
& \quad \left. \mathbb{E}_1 [\mathbb{P}_2 \{ \mathbb{C}_{z,y}(\cup_{i=1}^k (x_i, \infty) \times (t_{i-1}, t_i]) > 0 \}] \right\},
\end{aligned}$$

where, given a defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ β -stable regenerative set R , the process $\mathbb{C}_{z,y}$ is a defined on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ Poisson process on $\mathbb{R} \times \mathbb{R}_+$ with mean measure

$$\frac{C_{\alpha,\beta}}{c_\beta} \exp \{ -C_{\alpha,\beta}(\lambda - z) \} d\lambda \times dm^\phi \{ (y + R) \cap [0, \cdot] \}.$$

Therefore,

$$\begin{aligned}
& \mathbb{P}_2 \{ \mathbb{C}_{z,y}(\cup_{i=1}^k (x_i, \infty) \times (t_{i-1}, t_i]) > 0 \} \\
&= 1 - \mathbb{P}_2 \{ \mathbb{C}_{z,y}(\cup_{i=1}^k (x_i, \infty) \times (t_{i-1}, t_i]) = 0 \} \\
&= 1 - \exp \left\{ -c_\beta^{-1} \sum_{i=1}^k e^{-C_{\alpha,\beta}(x_i - z)} m^\phi \{ (y + R) \cap (t_{i-1}, t_i] \} \right\},
\end{aligned}$$

and we conclude by (2.14) that

$$\begin{aligned}
& \mathbb{E}_1 [\mathbb{P}_2 \{ \mathbb{C}_{z,y}(\cup_{i=1}^k (x_i, \infty) \times (t_{i-1}, t_i]) > 0 \} \\
&= \mathbb{E}_1 \left(1 - \exp \left\{ - \sum_{i=1}^k e^{-C_{\alpha,\beta}(x_i - z)} [Z^\leftarrow((t_i - y)_+ - Z^\leftarrow((t_{i-1} - y)_+))] \right\} \right),
\end{aligned}$$

and (2.48) follows by simple integration. Finally, using (2.48) with $k = 1$ and $t_1 = t$, $x_1 = x$ we obtain

$$\mathbb{P}\{\mathbb{M}(t) \leq x\} = \exp\left\{-\Gamma(1 - 1/C_{\alpha,\beta})e^{-x} \int_0^t (1 - \beta)y^{-\beta} \mathbb{E}Z^{\leftarrow}(t - y)^{1/C_{\alpha,\beta}} dy\right\},$$

and (2.49) follows by the self-similarity of Z^{\leftarrow} and simple integration. \blacksquare

We can obtain an explicit representation of the restriction of the sup-measure \mathcal{M} to $[0, 1]$.

First, the restriction of the Poisson point process $(X_k, Y_k + R_k)_{k \in \mathbb{N}}$ to $\mathbb{R} \times \mathcal{K}'([0, 1])$ (we only need to look at nonempty compact sets) can be represented as a Poisson point process \mathcal{N}_0 on \mathbb{R} with the mean measure $e^{-x} dx$, marked by *i.i.d.* copies of the random closed set $\overline{R^*}$ in (2.16). The markings are independent of \mathcal{N}_0 . The k^{th} copy $\overline{R_k^*}$ is associated via (2.15) with a shifted stable subordinator (Z_k^*) satisfying for $t \in [0, 1]$,

$$m^\phi(\overline{R_k^*} \cap [0, t]) = c_\beta Z_k^{*\leftarrow}(t).$$

Furthermore, the process \mathcal{N}_0 itself can be represented by the points $-\log \Gamma_k$, $k = 1, 2, \dots$, where (Γ_k) form a standard unit rate Poisson process on \mathbb{R}_+ .

Second, for a fixed k the mean measure of the Poisson point process \mathbb{C}_k can be rewritten in the form

$$\begin{aligned} & \frac{C_{\alpha,\beta}}{c_\beta} m^\phi(\overline{R_k^*} \cap [0, 1]) \exp\{-C_{\alpha,\beta}(\lambda - X_k(\omega_1))\} d\lambda \times d\eta_k(\cdot, \omega_1) \\ &= C_{\alpha,\beta} Z_k^{*\leftarrow}(1) \exp\{-C_{\alpha,\beta}(\lambda - X_k(\omega_1))\} d\lambda \times d\eta_k(\cdot, \omega_1), \end{aligned}$$

where $\eta_k(\cdot, \omega_1)$ is the (ω_1) -dependent probability measure (2.17) associated with (Z_k^*) . Therefore, we can choose a measurable enumeration of the points of \mathbb{C}_k by first selecting a measurable enumeration $\{\Lambda_{k,i}\}_{i \in \mathbb{N}}$ of the points of the Poisson point process on \mathbb{R} with the mean measure

$$C_{\alpha,\beta} Z_k^{*\leftarrow}(1) \exp\{-C_{\alpha,\beta}(\lambda - X_k(\omega_1))\} d\lambda$$

and then attaching to these points independent *i.i.d.* marks with the common law $\eta_k(\cdot, \omega_1)$. Since the former Poisson process is, once again, easily generated as a transformation of a unit rate Poisson process on \mathbb{R}_+ (say, $(\Gamma_{k,i})$), we conclude that we can choose a measurable enumeration of the points of \mathbb{C}_k so that, in law,

$$(\Lambda_{k,i}, W_{k,i})_{k,i \in \mathbb{N}} = \left(-\log \Gamma_k + \frac{1}{C_{\alpha,\beta}} \left(-\log \Gamma_{k,i} + \log Z_k^{*\leftarrow}(1) \right), J_{k,i} \right)_{k,i \in \mathbb{N}}, \quad (2.50)$$

where $\{\Gamma_k\}_{k \in \mathbb{N}}$ is a unit rate Poisson process on \mathbb{R} independent of the rest random elements that are defined in Theorem 2.3.

3. EXTREMAL LIMIT THEOREMS FOR STATIONARY SEMI-EXPONENTIAL PROCESSES

We focus on stationary infinitely divisible processes of form

$$X_t = \int_E 1_{A_0} \circ \theta^t(x) M(dx), \quad t \in \mathbb{Z}, \quad (3.1)$$

where $(E, \mathcal{E}, \mu, \theta)$ is the dynamical system described in (2.21)-(2.22), $A_0 = \{y \in E : y_0 = 0\}$, and M is an infinitely divisible random measure on (E, \mathcal{E}) with control measure μ and with constant local characteristic triple (σ^2, ν, b) ; see Samorodnitsky (2016). We note that the choice of the indicator function in (3.1) is mainly for convenience, and more general integrands could be considered.

The processes we consider have the marginal tails that are both subexponential and in the Gumbel maximum domain of attraction. They will also have a certain semi-exponential decay. Correspondingly, we choose the auxiliary function h in (2.40) to be of a specific type. The assumptions are imposed through the local Lévy measure ν of the infinitely divisible random measure.

Assumption 3.1. For some $\alpha \in (0, 1)$ and $\gamma > 0$,

$$\bar{\nu}(x) := \nu((x, \infty)) \sim \gamma \bar{H}(x) := \gamma \exp \left\{ - \int_1^x \frac{du}{u^{1-\alpha} L_\alpha(u)} \right\}, \quad x \rightarrow \infty \quad (3.2)$$

for a differentiable slowly varying function L_α such that

$$L'_\alpha(x) = o(L_\alpha(x)/x). \quad (3.3)$$

For large values of the argument both $\bar{\nu}$ and \bar{H} can be viewed as distributional tails. Automatically, these distributions are in the Gumbel maximum domain of attraction. Additionally, by Theorem 2 in Pitman (1980), these distributions are also subexponential. We conclude that the distribution of $X_0 \in D(\Lambda) \cap \mathcal{S}$ since $\mathbb{P}\{X_0 > x\} \sim \bar{\nu}(x)$, see Theorem 1 in Embrechts et al. (1979).

To state the main extremal limit theorems we define two functions by

$$V(x) = (1/\bar{\nu})^{\leftarrow}(x), \quad h(x) = x^{1-\alpha} L_\alpha(x), \quad x \geq 1. \quad (3.4)$$

Some properties of these and other important functions are described in Proposition B.1 in the Appendix B.

The normalizing constants in the extremal limit theorems are given by

$$b_n = V(w_n) + V(c_\infty \vartheta_n), \quad a_n = h \circ V(w_n), \quad (3.5)$$

where $\{w_n\}_{n \geq 1}$ is the wandering rate sequence in (2.26), $\{\vartheta_n\}_{n \geq 1}$ is the sequence in (2.35), and c_∞ is the constant in Theorem 2.3, given explicitly in (A.12).

The first extremal limit theorem establishes convergence in the space of random sup-measures.

Theorem 3.2. Let $\{X_t\}_{t \in \mathbb{Z}}$ be the stationary infinitely divisible process defined by (3.1), such that (3.2) and (3.3) hold. Assume also that the dynamical system $(E, \mathcal{E}, \mu, \theta)$ satisfies the Assumptions 2.1. Then the random sup-measures defined by (2.43) satisfy

$$\frac{\mathcal{M}_n(\cdot) - b_n}{a_n} \Rightarrow \mathcal{M}(\cdot) \quad \text{in } SM([0, \infty)), \quad (3.6)$$

where \mathcal{M} is given in (2.46).

The second extremal limit theorem establishes convergence in the functional space $D(0, \infty)$.

Theorem 3.3. Under the assumptions of Theorem 3.2, let $\mathbb{M}_n(t) = \max_{i \leq nt} X_i$, $t \in \mathbb{R}_+$, $n = 1, 2, \dots$. Then

$$\left\{ \frac{\mathbb{M}_n(t) - b_n}{a_n} \right\}_{t > 0} \Rightarrow \{\mathbb{M}(t)\}_{t > 0} \quad \text{in } (D(0, \infty), J_1), \quad (3.7)$$

where $\{\mathbb{M}(t)\}_{t > 0}$ is the stochastic process in (2.47).

Remark 3.4. It follows from (2.49) that the limiting process in Theorem 3.3 has one dimensional marginal distributions equal to the one dimensional marginal distributions

of the process $(X_G(t^{1-\beta+\beta/C_{\alpha,\beta}}), t \geq 0)$, where $(X_G(t), t \geq 0)$ is a shifted Gumbel extremal process, i.e. a nondecreasing process satisfying

$$\mathbb{P}(X_G(t_i) \leq x_i, i = 1, \dots, k) = \exp \left\{ -K(\alpha, \beta) \sum_{i=1}^k (t_i - t_{i-1}) e^{-x_i} \right\} \quad (3.8)$$

for $0 = t_0 < t_1 < \dots < t_k$ and $x_1 \leq x_2 \leq \dots \leq x_k$.

We recall that in Chen and Samorodnitsky (2020), similar extremal limit theorems are proved for stationary processes with marginal tails heavier than the ones in Theorem 3.3. The limiting processes therein are, in distribution, the power time changes of the standard Gumbel extremal process. Analogous time-changed results for Fréchet extremal processes are showed in Owada and Samorodnitsky (2015) and Lacaux and Samorodnitsky (2016).

Interestingly, the law of the limiting process in Theorem 3.3 is different from the law of the power time change of the Gumbel extremal process in (3.8). Indeed, if the two processes had the same law, we would have by (2.48) and (2.49), for $0 < t_1 < t_2$ and $x_1 \leq x_2$,

$$\begin{aligned} & \exp \left\{ -K(\alpha, \beta) \left[t_1^{1-\beta+\beta/C_{\alpha,\beta}} e^{-x_1} + (t_2^{1-\beta+\beta/C_{\alpha,\beta}} - t_1^{1-\beta+\beta/C_{\alpha,\beta}}) e^{-x_2} \right] \right\} \\ &= \mathbb{P} \left(X_G(t_1^{1-\beta+\beta/C_{\alpha,\beta}}) \leq x_1, X_G(t_2^{1-\beta+\beta/C_{\alpha,\beta}}) \leq x_2 \right) \\ &= \mathbb{P} (M(t_1) \leq x_1, M((t_1, t_2] \leq x_2) \\ &= \exp \left\{ -\Gamma(1 - 1/C_{\alpha,\beta}) \int_0^\infty (1 - \beta) y^{-\beta} dy \mathbb{E} \left[e^{-C_{\alpha,\beta} x_1} Z^\leftarrow((t_1 - y))_+ \right. \right. \right. \\ & \quad \left. \left. \left. + e^{-C_{\alpha,\beta} x_2} \left[Z^\leftarrow((t_2 - y))_+ - Z^\leftarrow(t_1 - y)_+ \right] \right]^{1/C_{\alpha,\beta}} \right\}, \end{aligned}$$

and due to the connection between the constants in (2.48) and (2.49) this reduces to

$$\begin{aligned} & \mathbb{E} \int_0^\infty (1 - \beta) y^{-\beta} dy \left(e^{-C_{\alpha,\beta} x_1} Z^\leftarrow((t_1 - y))_+ \right)^{1/C_{\alpha,\beta}} \\ &+ \mathbb{E} \int_0^\infty (1 - \beta) y^{-\beta} dy \left[\left(e^{-C_{\alpha,\beta} x_2} Z^\leftarrow((t_2 - y))_+ \right)^{1/C_{\alpha,\beta}} \right. \\ & \quad \left. - \left(e^{-C_{\alpha,\beta} x_2} Z^\leftarrow((t_1 - y))_+ \right)^{1/C_{\alpha,\beta}} \right] \\ &= \mathbb{E} \int_0^\infty (1 - \beta) y^{-\beta} dy \left[e^{-C_{\alpha,\beta} x_1} Z^\leftarrow((t_1 - y))_+ \right. \\ & \quad \left. + e^{-C_{\alpha,\beta} x_2} \left[Z^\leftarrow((t_2 - y))_+ - Z^\leftarrow(t_1 - y)_+ \right] \right]^{1/C_{\alpha,\beta}}. \end{aligned}$$

We argue that this is impossible since $C_{\alpha,\beta} > 1$. Indeed, for every $t_1 > 0$ and $t_2, t_3 > t_1$, a simple convexity argument shows that for $C > 1$ we have

$$t_2^C - t_1^C + t_3^C < (t_2 - t_1 + t_3)^C. \quad (3.9)$$

Now apply (3.9) with

$$\begin{aligned} t_1 &= e^{-x_2} \left[Z^\leftarrow((t_1 - y))_+ \right]^{1/C_{\alpha,\beta}}, \quad t_2 = e^{-x_1} \left[Z^\leftarrow((t_1 - y))_+ \right]^{1/C_{\alpha,\beta}}, \\ t_3 &= e^{-x_2} \left[Z^\leftarrow((t_2 - y))_+ \right]^{1/C_{\alpha,\beta}}. \end{aligned}$$

Remark 3.5. We will prove both theorems with the time domain restricted to the interval $[0, 1]$. The general case is only notationally different.

As it is often done when analyzing the extremes of subexponential processes, we start by decomposing the process in (3.1) into a sum of two independent processes. One will collect the large Poissonian contributions of the original process and the other will collect the small such contributions. Note that, by (3.3), we can choose $x_0 > 0$ (which we assume to be 1 for notational simplicity) satisfying

$$\left(\frac{1}{x^{1-\alpha} L_\alpha(x)} \right)' < 0 \quad \text{for all } x > x_0, \quad (3.10)$$

and we split the random measure M in (3.1) into a sum $M \stackrel{d}{=} M^{(1)} + M^{(2)}$ of two independent infinitely divisible random measures $M^{(1)}$ and $M^{(2)}$ with the same control measure as M and with constant local characteristic $(0, [\nu]_{(x_0, \infty)}, 0)$ and $(\sigma^2, [\nu]_{(-\infty, x_0]}, b)$, respectively. We define two independent stationary infinitely divisible processes by

$$X_t^{(i)} = \int_E 1_{A_0} \circ \theta^t(x) M^{(i)}(dx), \quad t \in \mathbb{Z}, i = 1, 2. \quad (3.11)$$

This gives us a desired decomposition

$$\{X_t\}_{t \in \mathbb{Z}} \stackrel{d}{=} \{X_t^{(1)}\}_{t \in \mathbb{Z}} + \{X_t^{(2)}\}_{t \in \mathbb{Z}}. \quad (3.12)$$

By construction, the random variables $\{X_t^{(1)}\}_{t \in \mathbb{Z}}$ are compound Poisson. For each $n \in \mathbb{N}$, it is convenient to take a series representation of $\{X_t^{(1)}\}_{0 \leq t \leq n}$, which arranges the Poissonian jumps in the decreasing order. The representation uses crucially the zero sets $(I_{k;n})$ defined in (2.29). It follows from Corollary 3.4.2 in Samorodnitsky (2016) (see also (4.12) in Chen and Samorodnitsky (2020)) that

$$\left(X_t^{(1)} \right)_{0 \leq t \leq n} \stackrel{d}{=} \left(\sum_{j \geq 1} V_1(w_n / \Gamma_j) 1_{\{t \in I_{j;n}\}} \right)_{0 \leq t \leq n}. \quad (3.13)$$

Here (Γ_j) are the ordered arrival times of a unit rate Poisson process on \mathbb{R}_+ independent of the *i.i.d.* zero sets $(I_{j;n})$. Furthermore, V_1 is a truncated function V in (3.4):

$$V_1(y) := \begin{cases} V(y), & \text{if } y > 1/\bar{\nu}(x_0) \\ 0, & \text{otherwise} \end{cases}.$$

For notational simplicity, we will hereafter use (3.13) with V instead of V_1 . We keep in mind that this function vanishes in a neighborhood of 0.

In the sequel, we will view $\{X_t^{(1)}\}_{t \in \mathbb{Z}}$ as defined by the series in (3.13) and write

$$\mathcal{M}_n(B) = \max_{t \in nB} \left\{ X_t^{(2)} + \sum_{j \geq 1} V(w_n / \Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}, \quad B \in \mathcal{B}([0, 1]). \quad (3.14)$$

The proofs of Theorems 3.2 and 3.3 use a number of random sup-measures related to (3.14) and we list them below. They use the random sets defined in (2.32). We also use the random sets

$$\begin{aligned} \widehat{I}_{k;n} &= I_{k;n} \setminus \cup_{j=1}^{k-1} I_{j;n}, \quad k \geq 1, \\ \widehat{I}_{k,i;n} &= I_{k,i;n} \setminus \cup_{j=1}^{i-1} I_{k,j;n}, \quad k, i \geq 1. \end{aligned} \quad (3.15)$$

For $B \subset [0, 1]$ and $n \in \mathbb{N}$, we define for $k, i, K \in \mathbb{N}$,

$$\mathcal{M}_{k,i;n}(B) = \max_{t \in \widehat{I}_{k,i;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j \geq 1} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}; \quad (3.16)$$

$$\mathcal{M}_{k;n}(B) = \max_{t \in \widehat{I}_{k;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j \geq 1} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}; \quad (3.17)$$

$$\mathcal{M}_{[K];n}(B) = \bigvee_{k=1}^K \mathcal{M}_{k;n}(B). \quad (3.18)$$

Furthermore, referring to the random sup-measure \mathcal{M} in (2.46) we also define

$$\mathcal{M}_{k,i}(B) = \begin{cases} \Lambda_{k,i}, & W_{k,i} \in B \\ -\infty, & W_{k,i} \notin B \end{cases}, \quad k, i \in \mathbb{N}; \quad (3.19)$$

$$\mathcal{M}_k(B) = \bigvee_{i=1}^{\infty} \mathcal{M}_{k,i}(B), \quad k \in \mathbb{N}; \quad (3.20)$$

$$\mathcal{M}_{[K]}(B) = \bigvee_{k=1}^K \mathcal{M}_k(B), \quad K \in \mathbb{N}. \quad (3.21)$$

The proofs of Theorems 3.2 and 3.3 rely heavily on Theorem 2.3. By the Skorohod embedding we may assume that all random elements appearing in (3.14), (3.16), (3.17), and (3.18) are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the convergence in Theorem 2.3 holds as the *a.s.* convergence for these random elements. Furthermore, the random elements appearing as the limit in the right hand side of (2.34) are used to construct the points of the point processes (\mathbb{C}_k) via (2.50). The random variables (Γ_j) are already naturally coupled via (3.14) and (2.50). This way the random sup-measure \mathcal{M} is coupled to the random sup-measures (\mathcal{M}_n) . This setup will be in force for the duration of this section, and it follows from (2.42) that the following proposition suffices to prove Theorem 3.2.

Proposition 3.6. *For each open interval $B \subset [0, 1]$,*

$$\frac{\mathcal{M}_n(B) - b_n}{a_n} \xrightarrow{P} \mathcal{M}(B). \quad (3.22)$$

This proposition is a consequence of the three statements below.

Proposition 3.7. *For each $k, i \in \mathbb{N}$ and each open interval $B \subset [0, 1]$,*

$$\frac{\mathcal{M}_{k,i;n}(B) - b_n}{a_n} \xrightarrow{P} \mathcal{M}_{k,i}(B). \quad (3.23)$$

In the next two propositions, B is an open subset of $[0, 1]$ and we use the notation $\Omega_B = \{\overline{R}_1^* \cap B \neq \emptyset\}$,

Proposition 3.8. *Let $\ell_n := \lfloor \rho \log n \rfloor$ with $\rho > 0$. If ρ is small enough, then*

$$\lim_{\ell_0 \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{1;n}(B) > \bigvee_{i=2^{\ell_0}}^{2^{\ell_n-1}} \mathcal{M}_{1,i;n}([0, 1]) \mid \Omega_B \right\} = 1 \quad (3.24)$$

and

$$\lim_{\ell_0 \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_n(B) > \bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{k;n}([0, 1]) \right\} = 1. \quad (3.25)$$

Proposition 3.9. For any $\rho > 0$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_n(B) > \bigvee_{k \geq n^\rho} \mathcal{M}_{k;n}([0, 1]) \right\} = 1 \quad (3.26)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{1;n}(B) > \bigvee_{i \geq n^\rho} \mathcal{M}_{1,i;n}([0, 1]) \mid \Omega_B \right\} = 1. \quad (3.27)$$

We start by showing how Proposition 3.6 follows from the three statements above.

Proof of Proposition 3.6. First, we note that by (3.24) and (3.27)

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{1;n}(B) = \bigvee_{i=1}^m \mathcal{M}_{1,i;n}(B) \mid \Omega_B \right\} = 1.$$

Second, for almost every $\omega \in \Omega_B^c$, we also have $\overline{R_1^*} \cap \overline{B} = \emptyset$ because the stable regenerative set does not hit fixed points. Due to $I_{1;n}/n \rightarrow \overline{R_1^*}$ a.s., we therefore see that for almost every $\omega \in \Omega_B^c$, $I_{1;n} \cap nB = \emptyset$ for all n large enough, and hence, $\mathcal{M}_{1;n}(B) = \mathcal{M}_{1,i;n}(B) = -\infty$ for all $i \in \mathbb{N}$. We deduce that

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{1;n}(B) = \bigvee_{i=1}^m \mathcal{M}_{1,i;n}(B) \right\} = 1.$$

This identity obviously remains valid if $\mathcal{M}_{1;n}$ and $\mathcal{M}_{1,i;n}$ are replaced by $\mathcal{M}_{k;n}$ and $\mathcal{M}_{k,i;n}$ respectively. Thus for any $K \in \mathbb{N}$,

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{k;n}(B) = \bigvee_{i=1}^m \mathcal{M}_{k,i;n}(B), 1 \leq k \leq K \right\} = 1.$$

We proceed to note from (3.25) and (3.26) that

$$\lim_{K \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_n(B) = \mathcal{M}_{[K];n}(B) \right\} = 1,$$

and conclude that

$$\lim_{\substack{K \rightarrow \infty \\ m \rightarrow \infty}} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \bigvee_{\substack{1 \leq k \leq K \\ 1 \leq i \leq m}} \mathcal{M}_{k,i;n}(B) = \mathcal{M}_n(B) \right\} = 1. \quad (3.28)$$

Finally, we note that

$$\lim_{\substack{K \rightarrow \infty \\ m \rightarrow \infty}} \bigvee_{\substack{1 \leq k \leq K \\ 1 \leq i \leq m}} \mathcal{M}_{k,i}(B) = \mathcal{M}(B) \text{ a.s.} \quad (3.29)$$

By the standard ‘‘convergent together’’ argument, (3.22) follows from Proposition 3.7, (3.28) and (3.29). \blacksquare

The proof of Theorem 3.2 is, therefore, complete apart from proving Propositions 3.7, 3.8 and 3.9 which we now commence.

Proof of Proposition 3.7. We consider $k = 1$. Recall that for any i , a.s.,

$$\frac{1}{n}(I_{1,1;n}, \dots, I_{1,i;n}) \rightarrow (\{J_{1,1}\}, \dots, \{J_{1,i}\})$$

and $J_{1,1}, \dots, J_{1,i}$ are distinct points. Hence, the sets $I_{1,1;n}, \dots, I_{1,i;n}$ are disjoint for all sufficiently large n , so it is enough to consider the case $i = 1$.

Once again, since $I_{1,1;n}/n \rightarrow \{J_{1,1}\}$ a.s., for almost every $\omega \in \{J_{1,1} \notin B\}$ we have $I_{1,1;n} \cap nB = \emptyset$ for all n large enough. Hence for such ω both sides of (3.23) are equal (to $-\infty$), and so we only need to show that

$$\frac{\mathcal{M}_{1,1;n}(B) - b_n}{a_n} \xrightarrow{P} \Lambda_{1,1} \text{ on } \{J_{1,1} \in B\}. \quad (3.30)$$

To this end, observe that on the event $\{I_{1,1;n} \cap nB \neq \emptyset\}$,

$$\begin{aligned} \mathcal{M}_{1,1;n}(B) &= V(w_n/\Gamma_1) + V(w_n/\Gamma_{j_{1,1;n}}) \\ &\quad + \max_{t \in I_{1,1;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j > j_{1,1;n}} V(w_n/\Gamma_j) \mathbf{1}_{\{t \in I_{j;n}\}} \right\}. \end{aligned}$$

First, it follows from (B.7) that as $n \rightarrow \infty$,

$$\frac{V(w_n/\Gamma_1) - V(w_n)}{a_n} \rightarrow -\log \Gamma_1.$$

Second, by the strong law of large numbers,

$$V(w_n/\Gamma_{j_{1,1;n}}) - V(w_n/j_{1,1;n}) = o(h \circ V(w_n/j_{1,1;n})) \leq o(h \circ V(w_n)).$$

By Theorem 2.3

$$\begin{aligned} V(w_n/j_{1,1;n}) - V(w_n \bar{p}_{1;n}/\Gamma_{1,1}) &= o(h \circ V(w_n \bar{p}_{1;n}/\Gamma_{1,1})) \\ &\leq o(h \circ V(w_n)), \\ V(w_n \bar{p}_{1;n}/\Gamma_{1,1}) - V(c_\infty \vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1}) &= o(h \circ V(c_\infty \vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1})) \\ &\leq o(h \circ V(w_n)). \end{aligned}$$

Apply (B.7) again to get

$$\frac{V(c_\infty \vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1}) - V(c_\infty \vartheta_n)}{h \circ V(c_\infty \vartheta_n)} = \log Z_1^{*\leftarrow}(1) - \log \Gamma_{1,1} + o(1).$$

Due to (B.6), we have

$$\frac{h \circ V(c_\infty \vartheta_n)}{h \circ V(w_n)} \sim \left(\frac{\log w_n}{\log \vartheta_n} \right)^{1/\alpha-1} \rightarrow \frac{1}{C_{\alpha,\beta}}.$$

This implies that

$$\frac{V(w_n/\Gamma_{j_{1,1;n}}) - V(c_\infty \vartheta_n)}{a_n} \rightarrow C_{\alpha,\beta} (-\log \Gamma_{1,1} + \log Z_1^{*\leftarrow}(1)), \quad (3.31)$$

and so,

$$\begin{aligned} &\frac{V(w_n/\Gamma_1) + V(w_n/\Gamma_{j_{1,1;n}}) - b_n}{a_n} \\ &\rightarrow -\log \Gamma_1 + C_{\alpha,\beta}^{-1} (-\log \Gamma_{1,1} + \log Z_1^{*\leftarrow}(1)) = \Lambda_{1,1} \end{aligned}$$

by (2.50).

Finally, we notice that the cardinality $\#I_{1,1;n}$ is tight by Proposition 2.4 (i). Because a_n grows to infinity, we see that

$$\frac{\max_{t \in I_{1,1;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j > j_{1,1;n}} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}}{a_n} \xrightarrow{P} 0.$$

This proves (3.23). \blacksquare

Proof of Proposition 3.8. It is convenient to assume that the random sets $\{I_{1;n}, n \in \mathbb{N}\}$ and \overline{R}_1^* are defined a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while the remaining random elements are defined on another probability space, $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, so that the overall probability space is the product space. Clearly, Ω_B can be viewed as an element of \mathcal{F} .

Observe that, for $\omega \in \Omega_B$ the random variables $(J_{1,i})$ are *i.i.d.* on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and each has a positive \mathbb{P}_1 -probability to be in B . By Theorem 2.3, the \mathbb{P}_1 -probability that each $I_{1,i;n}$ intersects nB is bounded away from zero for all large n . Therefore, we can choose K_1 so large that with

$$A_{B;n}^{(1)} = \{I_{1,i;n} \cap nB \neq \emptyset \text{ for some } 1 \leq i \leq K_1\}$$

we have

$$\liminf_{n \rightarrow \infty} \mathbb{P}\{A_{B;n}^{(1)} \mid \Omega_B\} \geq 1 - \epsilon/2.$$

It is clear that on $A_{B;n}^{(1)}$,

$$\mathcal{M}_{1;n}(B) \geq V(w_n/\Gamma_1) + V(w_n/\Gamma_{j_{1,K_1;n}}) + O_P(1). \quad (3.32)$$

Using once again (B.7) and arguing as in the proof of Proposition 3.7, if we choose $c_1 > 0$ sufficiently large, then we can make the probability

$$\mathbb{P}\{V(w_n/\Gamma_{j_{1,K_1;n}}) \geq V(\vartheta_n) - c_1 h \circ V(\vartheta_n)\}$$

arbitrarily close to 1 as $n \rightarrow \infty$. Hence for some large $c_1 > 0$, there is a sequence of subsets of $A_{B;n}^{(2)} \subseteq A_{B;n}^{(1)}$ that satisfy

$$\liminf_{n \rightarrow \infty} \mathbb{P}\{A_{B;n}^{(2)} \mid \Omega_B\} \geq 1 - \epsilon, \quad (3.33)$$

$$\liminf_{n \rightarrow \infty} \inf_{\omega \in A_{B;n}^{(2)}} \mathbb{P}_\omega \left\{ V(w_n/\Gamma_{j_{1,K_1;n}}) \geq V(\vartheta_n) - c_1 h \circ V(\vartheta_n) \right\} \geq 1 - \epsilon, \quad (3.34)$$

with $\mathbb{P}_\omega(\cdot) = \mathbb{P}\{\cdot \mid I_{1;n}, n \in \mathbb{N}, \overline{R}_1^*\}$.

Next, for any ℓ ,

$$\begin{aligned} \bigvee_{i=2^\ell}^{2^{\ell+1}-1} \mathcal{M}_{1,i;n}([0, 1]) &\leq V(w_n/\Gamma_1) + V(w_n/\Gamma_{j_{1,2^\ell;n}}) \\ &\quad + \max \left\{ X_t^{(2)} + \sum_{j > j_{1,2^\ell;n}} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} : t \in \cup_{i=2^\ell}^{2^{\ell+1}-1} \widehat{I}_{1,i;n} \right\} \\ &\leq_{st} V(w_n/\Gamma_1) + V(w_n/\Gamma_{j_{1,2^\ell;n}}) + \max \left\{ X_t^{(0)} : 1 \leq t \leq \sum_{i=2^\ell}^{2^{\ell+1}-1} \#I_{1,i;n} \right\}, \end{aligned}$$

where $\{X_t^{(0)}\}_{t \in \mathbb{Z}}$ are *i.i.d.* with $X_0^{(0)} \stackrel{d}{=} X_0$, which are also independent of the rest two random variables on the right hand side. It suffices to derive suitable upper bounds for the last two terms.

Take $A_{B;n}^{(3)} = \Omega_n$ as defined in Proposition 2.4 (ii). We then note that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left\{ A_{B;n}^{(3)} \mid \Omega_B \right\} \geq 1 - \epsilon \quad (3.35)$$

and we can choose $c_2 = c_2(\epsilon)$ so that

$$\lim_{\ell_0 \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\omega \in A_{B;n}^{(3)}} \mathbb{P}_\omega \left\{ \frac{\sum_{i=2^\ell}^{2^{\ell+1}-1} \#I_{1,i;n}(\omega)}{2^\ell} \leq c_2, \ell_0 \leq \ell \leq \ell_n \right\} = 1. \quad (3.36)$$

Write $v_\ell = V(c_2 2^\ell)$. From the facts that $\mathbb{P}\{X_0 > x\} \sim \bar{\nu}(x)$ and that V is the inverse of $1/\bar{\nu}$, we have for any positive constant c_3 ,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq t \leq c_2 2^\ell} X_t^{(0)} > v_\ell + c_3 \ell h(v_\ell) \right\} \leq c_2 2^\ell \cdot \mathbb{P} \{X_0 \geq v_\ell + c_3 \ell h(v_\ell)\} \\ & \lesssim \frac{\bar{H}(v_\ell + c_3 \ell h(v_\ell))}{\bar{H}(v_\ell)} = \exp \left\{ - \int_0^1 \frac{c_3 \ell h(v_\ell) \cdot du}{h(v_\ell + c_3 \ell h(v_\ell) \cdot u)} \right\}, \quad \text{as } \ell \rightarrow \infty. \end{aligned}$$

By (3.3), h is eventually increasing, and we use (B.5) and (B.6) to verify that for large ℓ the integral in the exponent is at least

$$\frac{h(v_\ell) c_3 \ell}{h(v_\ell + c_3 \ell h(v_\ell))} \sim c_3 \left(\frac{\alpha \log 2}{\alpha \log 2 + c_3} \right)^{1-\alpha} \ell.$$

It follows that

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P} \left\{ \max_{1 \leq t \leq c_2 2^\ell} X_t^{(0)} > v_\ell + c_3 \ell h(v_\ell) \right\} = 0. \quad (3.37)$$

Recalling the Skorohod embedding of the convergence in Theorem 2.3, we see that we can choose an event $A_B^{(4)}$ and some $c_4 = c_4(\epsilon) > 0$ such that

$$\mathbb{P} \left\{ A_B^{(4)} \mid \Omega_B \right\} \geq 1 - \epsilon, \quad \sup_{n \geq 1} \sup_{\omega \in A_B^{(4)}} \frac{w_n}{\vartheta_n} \bar{p}_{1;n}(\omega) \leq c_4. \quad (3.38)$$

The upper bound on $\bar{p}_{1;n}$ in (3.38) guarantees the uniform convergence

$$\sup_{\omega \in A_B^{(4)}} \sup_{\lambda \leq 1/2} \left| \mathbb{E}_\omega e^{\lambda j_{1,1;n} \bar{p}_{1;n}} - \mathbb{E} e^{\lambda \Gamma_{1,1}} \right| \rightarrow 0,$$

as in the argument for (A.21). Since under \mathbb{P}_ω the product $j_{1,2^\ell;n} \cdot \bar{p}_{1;n} - 1$ is the sum of 2^ℓ independent copies of $j_{1,1;n} \cdot \bar{p}_{1;n} - 1$, the exponential Markov inequality tells us that

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\omega \in A_B^{(4)}} \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P}_\omega \left\{ j_{1,2^\ell;n} \cdot \bar{p}_{1;n} \leq 2^{\ell-1} \right\} = 0. \quad (3.39)$$

Combining (3.38) with (3.39) gives us

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\omega \in A_{B;n}^{(4)}} \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P}_\omega \left\{ w_n / j_{1,2^\ell;n} \geq c_4 \vartheta_n / 2^{\ell-1} \right\} = 0 \quad (3.40)$$

and, therefore,

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \inf_{\omega \in A_B^{(4)}} \mathbb{P}_\omega \left\{ V(w_n / \Gamma_{1,2^\ell;n}) \leq V(c_4 \vartheta_n / 2^\ell), \ell_0 \leq \ell \leq \ell_n \right\} = 1. \quad (3.41)$$

Set

$$A_{B;n} = A_{B;n}^{(2)} \cap A_{B;n}^{(3)} \cap A_B^{(4)}, \quad n \in \mathbb{N}.$$

so that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \{A_{B;n} \mid \Omega_B\} \geq 1 - 3\epsilon, . \quad (3.42)$$

Using the constants defined above, we set for $\ell = \ell_0, \dots, \ell_n$

$$\begin{aligned} T_{1,\ell} &:= V(\vartheta_n) - V(c_4 \vartheta_n 2^{-\ell}), \\ T_{2,\ell} &:= c_1 h \circ V(\vartheta_n) + v_\ell + c_3 \ell h(v_\ell). \end{aligned}$$

It is elementary to check that

$$\begin{aligned} & \lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \inf_{\omega \in A_{B;n}} \mathbb{P}_\omega \left\{ \mathcal{M}_{1,n}(B) > \bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{1,i;n}([0,1]) \right\} \\ & \geq -\epsilon + \lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{1}(T_{1,\ell} > T_{2,\ell} \text{ for all } \ell = \ell_0, \dots, \ell_n). \end{aligned} \quad (3.43)$$

However, by (B.4), (B.2), (B.6), once we take $0 < \rho < \beta$, we see that, uniformly in ℓ ,

$$\begin{aligned} T_{1,\ell} &= G(\vartheta_n) - G(c_4 \vartheta_n 2^{-\ell}) - o(h \circ G(\vartheta_n)) \\ &= \int_{c_4 2^{-\ell}}^1 \frac{h \circ G(\vartheta_n u)}{u} du - o(h \circ G(\vartheta_n)) \gtrsim \ell (\log n)^{1/\alpha-1} \mathcal{L}(\log n). \end{aligned}$$

On the other hand, by (B.4) and (B.6), uniformly in ℓ ,

$$T_{2,\ell} \lesssim \left(\ell^{1/\alpha} + (\log n)^{1/\alpha-1} \right) \mathcal{L}(\log n).$$

As long as ρ is small enough, we see that the indicator function in the right hand side of (3.43) is equal to 1 for all n large enough, and so

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \inf_{\omega \in A_{B;n}} \mathbb{P}_\omega \left\{ \mathcal{M}_{1,n}(B) > \bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{1,i;n}([0,1]) \right\} \geq 1 - \epsilon. \quad (3.44)$$

It follows from (3.44) and (3.42) that

$$\lim_{\ell_0 \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_{1;n}(B) > \bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{1,i;n}([0,1]) \mid \Omega_B \right\} \geq 1 - 4\epsilon.$$

Letting $\epsilon \rightarrow 0$ establishes (3.24).

We proceed now to prove (3.25). Since $\lim_{n \rightarrow \infty} \mathbb{P}\{I_{1;n} \cap nB \neq \emptyset\} = \mathbb{P}\{\overline{R_1^*} \cap B \neq \emptyset\} > 0$, it follows that

$$\lim_{K \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P} \{I_{k;n} \cap nB \neq \emptyset \text{ for some } 1 \leq k \leq K\} = 1,$$

Therefore, repeating the argument used to find a lower bound on $\mathcal{M}_{1;n}(B)$ in the proof of (3.24) shows that for any $\epsilon > 0$ we have, outside of an event of probability ϵ ,

$$\mathcal{M}_n(B) \geq V(w_n/\Gamma_1) + V(\vartheta_n) - O_P(h \circ V(\vartheta_n)). \quad (3.45)$$

Next, continuing to use the notation of the proof of (3.24),

$$\mathcal{M}_{k;n}([0,1]) \leq_{\text{st}} V(w_n/\Gamma_k) + \max \left\{ X_t^{(0)} : t \in I_{k;n} \right\}, \quad k \in \mathbb{N},$$

with the process $\{X_t^{(0)}\}$ depending on k , even though our notation does not show it. For a large constant $c_5 > 0$, let $v_{\ell;n} = V(c_5 2^\ell / \bar{F}(n))$, where F is the law of the first hitting time φ in (2.24). For a small positive constant c_6 , we thus have

$$\begin{aligned} & \mathbb{P} \left\{ \bigcup_{\ell=\ell_0}^{\ell_n} \left\{ \bigvee_{k=2^\ell}^{2^{\ell+1}-1} \mathcal{M}_{k;n}([0,1]) \geq V(w_n/2^{\ell-1}) + v_{\ell;n} + c_6 \ell h(v_{\ell;n}) \right\} \right\} \\ & \leq \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P} \left\{ \Gamma_{2^\ell} < 2^{\ell-1} \right\} + \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P} \left\{ 2^{-\ell} \sum_{k=2^\ell}^{2^{\ell+1}-1} \#I_{k;n} > c_5 / \bar{F}(n) \right\} \\ & \quad + \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P} \left\{ \max_{1 \leq t \leq c_5 2^\ell / \bar{F}(n)} X_t^{(0)} \geq v_{\ell;n} + c_6 \ell h(v_{\ell;n}) \right\} =: S_{1,n} + S_{2,n} + S_{3,n}. \end{aligned}$$

We emphasize that the process $\{X_t^{(0)}\}$ in $S_{3,n}$ is a concatenation of 3 different processes with the same marginal distribution. Only the marginal distribution is relevant in the subsequent calculation.

An exponential Markov inequality immediately shows that

$$\lim_{\ell_0 \rightarrow \infty} \lim_{n \rightarrow \infty} S_{1,n} = 0. \quad (3.46)$$

Next, in the notation of (2.38), choosing $c_5 \geq 2\mu_1$ we have by the Chebyshev inequality

$$\begin{aligned} & \mathbb{P} \left\{ 2^{-\ell} \sum_{k=2^\ell}^{2^{\ell+1}-1} \#I_{k;n} > c_5 / \bar{F}(n) \right\} \\ & \leq \mathbb{P} \left\{ 2^{-\ell} \sum_{k=2^\ell}^{2^{\ell+1}-1} \left(\#I_{k;n} \bar{F}(n) - \mathbb{E}(\#I_{1;n} \bar{F}(n)) \right) > c_5 / 2 \right\} \\ & \leq c 2^{-\ell} \text{Var}(\#I_{1;n} \bar{F}(n)) \leq c 2^{-\ell} \mu_2, \end{aligned}$$

which implies that

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} S_{2,n} = 0. \quad (3.47)$$

Finally, the statement

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} S_{3,n} = 0. \quad (3.48)$$

follows the same way as in the proof of (3.37). By (3.46), (3.47) and (3.48), we note that

$$\lim_{\ell_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \bigcap_{\ell=\ell_0}^{\ell_n} \left\{ \bigvee_{k=2^\ell}^{2^{\ell+1}-1} \mathcal{M}_{k;n}([0,1]) \leq V(w_n/2^{\ell-1}) + v_{\ell;n} + c_6 \ell h(v_{\ell;n}) \right\} \right\} = 1.$$

For $\ell_0 \leq \ell \leq \ell_n$ let

$$\begin{aligned} T_{1,\ell} & := V(w_n/\Gamma_1) - V(w_n 2^{-\ell+1}), \\ T_{2,\ell} & := v_{\ell;n} + 2c_7 \ell h(v_{\ell;n}) - V(\vartheta_n), \end{aligned}$$

so that for any $\epsilon > 0$

$$\mathbb{P} \left\{ \mathcal{M}_n(B) > \bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{k;n}([0,1]) \right\} \geq \mathbf{1}(T_{1,\ell} > T_{2,\ell} \text{ for all } \ell = \ell_0, \dots, \ell_n) - \epsilon$$

for all large n . As in the proof of (3.24), for any $\epsilon > 0$, on an event of probability converging to 1, the following two inequalities holds uniformly in ℓ .

$$\begin{aligned} T_{1,\ell} &\geq \frac{\log 2(1 - \beta - \rho \log 2)^{1/\alpha-1}}{\alpha} \ell (\log n)^{1/\alpha-1} \mathcal{L}(\log n), \\ T_{2,\ell} &\leq \frac{(1 + \epsilon)(\log 2 + c_7)((\beta + \rho \log 2))^{1/\alpha-1}}{\alpha} \ell (\log n)^{1/\alpha-1} \mathcal{L}(\log n). \end{aligned}$$

If c_7 , ρ and ϵ are chosen to be small enough, we have $T_{1,\ell} > T_{2,\ell}$ uniformly in ℓ for large n . Thus

$$\lim_{\ell_0 \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \mathcal{M}_n(B) > \bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{k;n}([0, 1]) \right\} \geq 1 - \epsilon,$$

and (3.25) follows by letting $\epsilon \rightarrow 0$. \blacksquare

Proof of Proposition 3.9. Due to the lower bound (3.45), the claim (3.26) will follow once we show that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ V(w_n/\Gamma_1) + V(\vartheta_n) - o(V(w_n)) > \bigvee_{\Gamma_k > n^\rho} \mathcal{M}_{k;n}([0, 1]) \right\} = 1. \quad (3.49)$$

It suffices to consider the case $\rho < r_1$ as defined in Lemma B.2 (i). Removing an event of probability ϵ ensures that Γ_1 is bounded from above by a constant. Modifying, if necessary, $o(V(w_n))$ shows that (3.49) will follow once we check that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ V(w_n) + V(\vartheta_n) - o(V(w_n)) > \bigvee_{\Gamma_k > n^\rho} \mathcal{M}_{k;n}([0, 1]) \right\} = 1.$$

Furthermore, since the support of the marginal Lévy measure of the process $X_0^{(2)}$ is bounded on the right, the process has marginal distributional tails lighter than exponentially light; see Section 26 in Sato (2013). It follows that

$$\max_{0 \leq t \leq n} X_t^{(2)} = o_P(\log n) = o_P(V(w_n))$$

as $n \rightarrow \infty$. Therefore, it is enough to prove that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \bigvee_{\Gamma_k > n^\rho} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} < V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} = 1. \quad (3.50)$$

We prove (3.50) through a series of steps.

Let $\tilde{\psi}$ be the function defined in Lemma B.2. We start by proving that for any $0 < r < 1 - \beta$ and any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P} \left\{ \sum_{\Gamma_j > n^r} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} + V(w_n/n^r) \right. \\ \left. > V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \leq \exp \left\{ (-\tilde{\psi}(r) + \epsilon) \log n \right\} \end{aligned} \quad (3.51)$$

for all large n .

For this purpose denote

$$\begin{aligned} z_n &= z_n(r) = V(w_n) + V(\vartheta_n) - V(w_n/n^r) - o(V(w_n)), \\ \bar{z}_n &= \bar{z}_n(r) = V(w_n/n^r). \end{aligned}$$

Recalling the partition $0 = r_0 < r_2 < \dots$ of the interval $(0, 1 - \beta)$ defined in Lemma B.2, we see that $r \in (r_m, r_{m+1}]$ for some $m = 0, 1, 2, \dots$, so by the same Lemma B.2, for large n , the probability in (3.51) is

$$\begin{aligned} & \mathbb{P} \left\{ \sum_{\Gamma_j > n^r} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} > z_n(r) \right\} \\ & \leq \bar{z}_n(r)^{\gamma_m} (\bar{H}(\bar{z}_n(r)))^m \bar{H}(z_n(r) - m\bar{z}_n(r)). \end{aligned} \quad (3.52)$$

It follows from (B.5) that

$$\bar{z}_n(r)^{\gamma_m} \lesssim \left((\log n)^{1/\alpha} \mathcal{L}(\log n) \right)^{\gamma_m}. \quad (3.53)$$

Next, by Karamata's theorem, (B.5) and (B.6)

$$\begin{aligned} \int_1^{\bar{z}_n(r)} \frac{du}{u^{1-\alpha} L_\alpha(x)} & \sim \frac{\bar{z}_n(r)}{\alpha h(\bar{z}_n(r))} \\ & = \frac{1}{\alpha} \cdot \frac{V(w_n/n^r)}{h \circ V(w_n/n^r)} \sim \log(w_n/n^r) \sim (1 - \beta - r) \log n. \end{aligned} \quad (3.54)$$

Let

$$\theta = \left((1 - \beta)^{1/\alpha} + \beta^{1/\alpha} - (m + 1)(1 - \beta - r)^{1/\alpha} \right)^\alpha;$$

due to the range of r this is a well defined power of a positive number. It follows by (B.5) that

$$z_n - m\bar{z}_n \sim V(n^\theta),$$

Therefore, by (B.5) and (B.6),

$$\int_1^{z_n - m\bar{z}_n} \frac{du}{u^{1-\alpha} L_\alpha(u)} \sim \frac{z_n - m\bar{z}_n}{\alpha h(z_n - m\bar{z}_n)} \sim \frac{V(n^\theta)}{\alpha h \circ V(n^\theta)} \sim \theta \log n. \quad (3.55)$$

Putting (3.52), (3.53), (3.54) and (3.55) together establishes (3.51).

Next, we prove that (3.50) holds if $\rho \in (r_m, r_{m+1}]$ for any sufficiently large m . Indeed, by (3.51),

$$\begin{aligned} & \mathbb{P} \left\{ \bigvee_{\Gamma_k > n^\rho} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) 1_{\{t \in I_{j;n}\}} \geq V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \\ & \leq n \mathbb{P} \left\{ \sum_{\Gamma_j > n^r} V\left(\frac{w_n}{\Gamma_j}\right) 1_{\{0 \in I_{j;n}\}} + V(w_n/n^r) \geq V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \\ & \leq n \exp \left\{ (-\tilde{\psi}(r) + \epsilon) \log n \right\} \end{aligned}$$

for large n . Since $\tilde{\psi}(r) \rightarrow \infty$ as $r \rightarrow 1 - \beta$, the claim follows.

Given that (3.50) holds if $\rho \in (r_m, r_{m+1}]$ for any sufficiently large m , the fact that (3.50) also holds for any $0 < \rho < r_1$ will follow once we prove that for any such ρ and any

m ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \bigvee_{n^{\rho} < \Gamma_k \leq n^{r_m}} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) \mathbf{1}_{\{t \in I_{j;n}\}} \right. \\ \left. < V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} = 1.$$

We will prove the above statement by constructing two increasing sequences,

$$\{i_s\}_{0 \leq s \leq m} \subseteq \mathbb{N} \quad \text{and} \quad \{\rho_{i_s}\}_{0 \leq s \leq m} \subseteq \mathbb{R}_+,$$

with $i_0 := 0$, $\rho_0 := \rho$, $\rho_{i_s} \in (r_s, r_{s+1})$ for $s = 1, \dots, m$ such that for every such s ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \bigvee_{n^{\rho_{i_{s-1}}} < \Gamma_k \leq n^{\rho_{i_s}}} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) \mathbf{1}_{\{t \in I_{j;n}\}} \right. \\ \left. \geq V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} = 0. \quad (3.56)$$

We will describe the construction in the case $m = 1$. The case of a general m is similar.

Let $\delta_0 = \tilde{\psi}(\rho_0) - (\rho_0 + \beta) > 0$, by Lemma B.2 (ii). So $\tilde{\psi}(\rho_0) > \rho_0 + \beta + 2\delta_0/3$, and so using (3.51) and the notation that follows it,

$$\mathbb{P} \left\{ \sum_{\Gamma_j > n^{\rho_0}} V(w_n/\Gamma_j) \mathbf{1}_{\{0 \in I_{j;n}\}} \geq z_n(\rho_0) \right\} = o\left(n^{-\rho_0 - \beta - 2\delta_0/3}\right). \quad (3.57)$$

Take $\rho_1 = \rho_0 + \delta_0/3$, then for any $c > 0$,

$$\mathbb{P} \left\{ \bigvee_{n^{\rho_0} < \Gamma_k \leq n^{\rho_1}} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) \mathbf{1}_{\{t \in I_{j;n}\}} \geq V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \\ \leq \mathbb{P} \left\{ \#\{k : n^{\rho_0} < \Gamma_k \leq n^{\rho_1}\} > 2n^{\rho_1} \right\} \\ + \mathbb{P} \left\{ \#I_{k;n} \geq \frac{cn^{\beta} \log n}{L(n)} \text{ for some } k \text{ with } n^{\rho_0} < \Gamma_k \leq n^{\rho_1} \right\} \\ + 2n^{\rho_1} \frac{cn^{\beta} \log n}{L(n)} \cdot \mathbb{P} \left\{ \sum_{\Gamma_j > n^{\rho_0}} V(w_n/\Gamma_j) \mathbf{1}_{\{0 \in I_{j;n}\}} > z_n(\rho_0) \right\}.$$

The first term in the right hand side vanishes in the limit by the law of large numbers. If c is large enough, by Proposition 2.5 so does the second term. The same is true for the third term by (3.57). Notice that, if $\rho_1 > r_1$, we set $i_1 = 1$ and replace ρ_1 with $\min(\rho_1, r_2)$ to complete the construction of the two sequences. Otherwise, we inductively define, for as long as $\rho_i \leq r_1$,

$$\delta_i = \tilde{\psi}(\rho_i) - (\rho_i + \beta), \quad \rho_{i+1} = \rho_i + \frac{i+1}{i+3} \delta_{i+1}, \quad i \geq 1. \quad (3.58)$$

The same argument as above then shows that for every i ,

$$\mathbb{P} \left\{ \bigvee_{n^{\rho_i} < \Gamma_k \leq n^{\rho_{i+1}}} \max_{t \in I_{k;n}} \sum_{j \geq k} V(w_n/\Gamma_j) \mathbf{1}_{\{t \in I_{j;n}\}} \geq V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \rightarrow 0.$$

We claim that this inductive procedure must end after finitely many steps. That is, $\rho_{i_1} > r_1$ for some $i_1 \in \mathbb{N}$, in which case we replace ρ_{i_1} with $\min(\rho_{i_1}, r_2)$ and, once again, complete the construction.

Indeed, if the inductive procedure did not terminate, it would follow that $\delta_i \rightarrow 0$, so $\rho_i \uparrow \rho_* \leq r_1$ satisfying $\tilde{\psi}(\rho_* -) = \rho_* + \beta$. Since $\tilde{\psi}$ is constant on $(0, r_1)$, this is easily seen to contradict the property $r_1 < 1/2 - \beta$ established in Lemma B.2 (i).

This completes the proof of (3.26).

We proceed to prove (3.27). Let $\epsilon > 0$. For $c, C > 0$ we set $\Omega_n = \Omega_n^{(1)} \cap \Omega_n^{(2)}$, where

$$\begin{aligned}\Omega_n^{(1)} &= \left\{ \omega \in \Omega : \min(\Gamma_{j_{1,k;n}}(\omega), j_{1,k;n}(\omega)) \geq cw_n k / \vartheta_n \text{ for all } k \in \mathbb{N} \right\}, \quad n \in \mathbb{N}; \\ \Omega_n^{(2)} &= \left\{ \#I_{1,k;n}(\omega) < C \log n \text{ for all } 1 \leq k \leq \vartheta_n / c \right\}, \quad n \in \mathbb{N}.\end{aligned}$$

If $\delta > 0$ is such that V vanishes on $(0, \delta]$, then on $\Omega_n^{(1)}$ we have

$$\#\{k : \Gamma_{j_{1,k;n}} \leq w_n / \delta\} \leq \vartheta_n / (c\delta).$$

We claim that we can choose $c = c(\epsilon)$ small enough and $C = C(\epsilon)$ large enough so that

$$\liminf_{n \rightarrow \infty} \mathbb{P}\{\Omega_n^{(i)}\} \geq 1 - \epsilon, \quad i = 1, 2. \quad (3.59)$$

The fact that this is true for $i = 1$ if c is small enough follows as in (3.40) and the law of large numbers, for $i = 2$ it follows for large enough C by Proposition 2.4 (ii).

With c chosen above we denote

$$\begin{aligned}\overline{\mathcal{M}}_{1,k;n} &= \max_{t \in I_{1,k;n}} \left\{ V(w_n / \Gamma_{j_{1,k;n}}) + \sum_{j \geq j_{1,k;n} + 1} V(w_n / \Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}, \quad k \in \mathbb{N}; \\ \mathcal{M}'_{1,k;n} &= \max_{t \in I_{1,k;n}} \left\{ V(\vartheta_n / (ck)) + \sum_{\Gamma_j > ckw_n / \vartheta_n} V(w_n / \Gamma_j) 1_{\{t \in I_{j;n}\}} \right\}, \quad k \in \mathbb{N},\end{aligned}$$

so that on each event Ω_n , we have $\overline{\mathcal{M}}_{1,k;n} \leq \mathcal{M}'_{1,k;n}$, $k \geq 1$. Since $c_\infty < 1$, it follows from (3.31) that on the event Ω_B ,

$$V(w_n / \Gamma_{j_{1,1;n}}) \geq V(\vartheta_n) - o_P(V(\vartheta_n)).$$

Since $\epsilon > 0$ can be taken as small as we wish, (3.27) will follow from the following claim:

$$\lim_{n \rightarrow \infty} \mathbb{P}\left\{ \max_{\lfloor n^\rho \rfloor \leq k \leq \lceil \vartheta_n / (c\delta) \rceil} \mathcal{M}'_{1,k;n} < V(\vartheta_n) - o(V(\vartheta_n)) \mid \Omega_n \right\} = 1. \quad (3.60)$$

We start by constructing an increasing sequence $\rho_i \uparrow \beta$, $i \in \mathbb{N}_0$ with $\rho_0 = \rho$ such that for every i

$$\mathbb{P}\left\{ \max_{\lfloor n^{\rho_i} \rfloor \leq k \leq \lceil n^{\rho_{i+1}} \rceil} \mathcal{M}'_{1,k;n} \geq V(\vartheta_n) - o(V(\vartheta_n)) \mid \Omega_n \right\} \rightarrow 0, \quad n \rightarrow \infty. \quad (3.61)$$

To this end, we define inductively for $i = 0, 1, \dots$

$$\delta_i = \left(\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha} \right)^\alpha - \rho_i, \quad \rho_{i+1} = \rho_i + \frac{i+1}{i+3} \delta_i; \quad (3.62)$$

note that $\delta_i > 0$ for all i . We have

$$\begin{aligned} & \mathbb{P} \left\{ V(\vartheta_n/(cn^{\rho_i})) + \sum_{\Gamma_j > cn^{\rho_i} w_n/\vartheta_n} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} \geq V(\vartheta_n) - o(V(\vartheta_n)) \right\} \\ & \leq \mathbb{P} \left\{ \sum_{j=0}^{\infty} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} \geq V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n/(cn^{\rho_i})) \right\} \\ & \lesssim \overline{H} \left(V(\vartheta_n) - V(\vartheta_n/(cn^{\rho_i})) - o(V(\vartheta_n)) \right), \end{aligned}$$

and by (B.5),

$$\log \left[\overline{H} \left(V(\vartheta_n) - V(\vartheta_n/(cn^{\rho_i})) - o(V(\vartheta_n)) \right) \right] \sim - \left(\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha} \right) \log n.$$

Therefore, by the first part of (3.62), for large n we have

$$\overline{H} \left(V(\vartheta_n) - V(\vartheta_n/(cn^{\rho_i})) - o(V(\vartheta_n)) \right) \leq \exp \left\{ - \left(\rho_i + \frac{i+2}{i+3} \delta_i \right) \log n \right\}$$

and, hence, since $\Omega_n \supseteq \Omega_n^{(2)}$,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{\lfloor n^{\rho_i} \rfloor \leq k \leq \lceil n^{\rho_i+1} \rceil} \mathcal{M}'_{1,k;n} \geq V(\vartheta_n) - o(V(\vartheta_n)) \mid \Omega_n \right\} \\ & \lesssim n^{\rho_i+1} \cdot C \log n \cdot \exp \left\{ - \left(\rho_i + \frac{i+2}{i+3} \delta_i \right) \log n \right\} \\ & = C \log n \cdot \exp \left\{ - \frac{\delta_i}{i+3} \log n \right\} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore, (3.61) follows, and (3.60) will be established once we show that for all i large enough,

$$\mathbb{P} \left\{ \max_{\lfloor n^{\rho_i} \rfloor \leq k \leq \lceil \vartheta_n/c \rceil} \mathcal{M}'_{1,k;n} \geq V(\vartheta_n) - o(V(\vartheta_n)) \mid \Omega_n \right\} \rightarrow 0, \quad n \rightarrow \infty. \quad (3.63)$$

Note that

$$\sum_{\Gamma_j > cn^{\rho_i} w_n/\vartheta_n} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} \stackrel{d}{=} \sum_{j=1}^{N_n} \overline{\xi}_j,$$

where N_n is a Poisson random variable with mean $\overline{\nu}(x_0) - cn^{\rho_i}/\vartheta_n$, and $\{\overline{\xi}_i\}_{i \geq 1}$ is a family of *i.i.d.* random variables independent of N_n , and the law of $\overline{\xi}_1$ is the restriction of ν to $(1, V(\vartheta_n c^{-1} n^{-\rho_i}))$, normalized to be a probability measure. Here we have used the fact that we assume $x_0 = 1$ in (3.10).

Let

$$m = m(i) = \left\lceil \frac{V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1} n^{-\rho_i})}{V(\vartheta_n c^{-1} n^{-\rho_i})} \right\rceil;$$

$m(i)$ also depends on n , but by (B.5) it is bounded as a function of n . By Lemma B.3 (i) we have for some large numbers $\gamma_1(i)$, $\gamma_2(i)$,

$$\begin{aligned} & \mathbb{P} \left\{ \sum_{\Gamma_j > cn^{\rho_i} w_n / \vartheta_n} V(w_n / \Gamma_j) 1_{\{0 \in I_{j,n}\}} > V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1} n^{-\rho_i}) \right\} \\ & \leq \sum_{s=m}^{\infty} \frac{1}{s!} \mathbb{P} \left\{ \sum_{j=1}^s \bar{\xi}_j \geq V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1} n^{-\rho_i}) \right\} \\ & \leq \sum_{s=m}^{\infty} \frac{1}{s!} \gamma_1(i)^s (V(\vartheta_n c^{-1} n^{-\rho_i}))^{\gamma_2(i)} (\bar{H}(V(\vartheta_n c^{-1} n^{-\rho_i})))^{m(i)} \\ & \quad \cdot \bar{H}(V(\vartheta_n) - o(V(\vartheta_n)) - mV(\vartheta_n c^{-1} n^{-\rho_i})) \\ & \lesssim (V(\vartheta_n c^{-1} n^{-\rho_i}))^{\gamma_2(i)} (\bar{H}(V(\vartheta_n c^{-1} n^{-\rho_i})))^{m(i)} \lesssim (\log n)^{\gamma_2(i)/\alpha} (\vartheta_n n^{-\rho_i})^{-m(i)}, \end{aligned}$$

where we have once again used (B.5). Furthermore,

$$m(i) \rightarrow \frac{\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha}}{(\beta - \rho_i)^{1/\alpha}}$$

as $n \rightarrow \infty$, so that for large n , $(\vartheta_n n^{-\rho_i})^{-m(i)} \leq n^{-C_1}$ for

$$C_1 = (\beta - \rho_i) \frac{\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha}}{(\beta - \rho_i)^{1/\alpha}}.$$

Since $\rho_i \rightarrow \beta$ as $i \rightarrow \infty$ and $\alpha < 1$, C_1 will become arbitrarily large for large i . Taking into account the bound on the cardinality of $I_{t,k;n}$ on the event Ω_n , we see that for i such that $C_1 > \beta$ and for large n ,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{\lfloor n^{\rho_i} \rfloor \leq k \leq \lceil \vartheta_n / c \rceil} \mathcal{M}'_{1,k;n} \geq V(\vartheta_n) - o(V(\vartheta_n)) \mid \Omega_n \right\} \\ & \lesssim \vartheta_n \log n \cdot n^{-C_1} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This establishes (3.63) and, hence, completes the proof. \blacksquare

Proof of Theorem 3.3. Recall that the random sup-measure \mathcal{M} is coupled to the random sup-measures (\mathcal{M}_n) as described when proving Theorem 3.2. Therefore, it suffices to show that for any fixed $0 < T_1 < T_2 \leq 1$,

$$\left\{ \frac{\mathbb{M}_n(t) - b_n}{a_n} \right\}_{t \in [T_1, T_2]} \xrightarrow{P} \{\mathbb{M}(t)\}_{t \in [T_1, T_2]} \quad \text{in } (D[T_1, T_2], J_1). \quad (3.64)$$

For the duration of the proof we fix a small $\epsilon > 0$.

We start by observing that by the construction of $\mathbb{M}(t)$ in (2.46), (2.50) and (2.47), there exists $N = N(\epsilon) > 0$ such that the event

$$\Omega^{(1)} := \left\{ \mathbb{M}(t) = \max_{1 \leq k, i \leq N} \{\Lambda_{k,i} : W_{k,i} \in [0, t]\} \text{ for } t \in [T_1, T_2] \right\} \quad (3.65)$$

satisfies $\mathbb{P}\{\Omega^{(1)}\} \geq 1 - \epsilon$. Consider the random finite set $\mathcal{T} = \{W_{k,i} : 1 \leq k, i \leq N\} \cup \{0, T_1, T_2\}$. Since the random measure η_ω in (2.17) is a.s. atomless, it follows that there is $\delta > 0$ such that the event

$$\Omega^{(2)} := \{|t_1 - t_2| \geq \delta \text{ for all distinct } t_1, t_2 \in \mathcal{T}\} \quad (3.66)$$

satisfies $\mathbb{P}\{\Omega^{(2)}\} \geq 1 - \epsilon$. Then on $\Omega^{(1)} \cap \Omega^{(2)}$ the process $\{\mathbb{M}(t) : t \in [T_1, T_2]\}$ has nondecreasing piecewise constant sample paths, with at most N^2 jumps, and each two jump points are separated by at least δ . Let $T_1 = W_{k_0, i_0} < W_{k_1, i_1} < \dots < W_{k_\ell, i_\ell} \leq T_2$ be the enumeration of the jumps.

We saw when establishing (3.30) with $B = [0, 1]$, that for any $k, i \in \mathbb{N}$,

$$\max_{t \in I_{k, i, n}} \left| \frac{X(t) - b_n}{a_n} - \Lambda_{k, i} \right| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Therefore by decreasing δ if necessary, the event

$$\Omega_n^{(3)} := \left\{ \left| \frac{\mathbb{M}_n([0, T_i]) - b_n}{a_n} - \mathbb{M}(T_i) \right| \leq \delta, i = 1, 2, \right. \\ \left. \bigvee_{k, i=1}^N \max_{t \in I_{k, i, n}} \left| \frac{X(t) - b_n}{a_n} - \Lambda_{k, i} \right| \leq \delta \right\}$$

satisfies $\mathbb{P}\{\Omega_n^{(3)}\} \geq 1 - \epsilon$ for all large n . Furthermore, by Proposition 3.8 and Proposition 3.9, for N large enough, the event

$$\Omega_n^{(4)} := \left\{ \mathbb{M}_n(t) = \max_{1 \leq k, i \leq N} \max_{t \in I_{k, i, n} \cap [0, nt]} X(t) \text{ for all } t \in [T_1, T_2] \right\} \quad (3.67)$$

satisfies $\mathbb{P}\{\Omega_n^{(4)}\} \geq 1 - \epsilon$ for all large n . It is clear that we can select the same N in (3.65) and (3.67). Finally, since for every k, i

$$I_{k, i, n}/n \longrightarrow \{W_{k, i}\} \text{ a.s.}$$

with respect to the Hausdorff metric as $n \rightarrow \infty$, the event

$$\Omega_n^{(5)} := \left\{ \max_{1 \leq k, i \leq N} \max_{t \in I_{k, i, n}} |t/n - W_{k, i}| \leq \delta/2 \right\} \quad (3.68)$$

satisfies $\mathbb{P}\{\Omega_n^{(5)}\} \geq 1 - \epsilon$ for all large n .

Fix $\omega \in \Omega^{(1)} \cap \Omega^{(2)} \cap \Omega_n^{(3)} \cap \Omega_n^{(4)} \cap \Omega_n^{(5)}$. We construct a function $e_n : [T_1, T_2] \rightarrow [T_1, T_2]$ as follows. First set

$$e_n(t) = \begin{cases} T_1 & t = T_1 \\ W_{k_s, i_s} & t = \min(I_{k_s, i_s, n}/n) \text{ for } s = 1, \dots, \ell \\ T_2, & t = T_2 \end{cases}$$

and note that due to the choice of ω this an increasing function. We extend it to a continuous increasing map from $[T_1, T_2]$ onto $[T_1, T_2]$ by linear interpolation. Then

$$\sup_{t \in [T_1, T_2]} |e_n(t) - \text{id}(t)| = \bigvee_{s=1}^{\ell} |\min(I_{k_s, i_s, n}/n) - W_{k_s, i_s}| \leq \delta/2$$

so that

$$\begin{aligned}
& d_{J_1} \left(\left\{ \frac{\mathbb{M}_n(t) - b_n}{a_n} \right\}_{t \in [T_1, T_2]}, \{\mathbb{M}(t)\}_{t \in [T_1, T_2]} \right) \\
& \leq \sup_{t \in [T_1, T_2]} |e_n(t) - \text{id}(t)| \vee \sup_{t \in [T_1, T_2]} \left| \frac{\mathbb{M}_n(t) - b_n}{a_n} - \mathbb{M}(e_n(t)) \right| \\
& \leq \frac{\delta}{2} \vee \left(\bigvee_{i=1}^2 \left| \frac{\mathbb{M}_n([0, T_i]) - b_n}{a_n} - \mathbb{M}(T_i) \right| \right) \vee \left(\bigvee_{s=1}^{\ell} \max_{t \in I_{k_s, i_s; n}} \left| \frac{X_t - b_n}{a_n} - \Lambda_{k_s, i_s} \right| \right) \\
& \leq \delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, while

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left\{ \Omega^{(1)} \cap \Omega^{(2)} \cap \Omega_n^{(3)} \cap \Omega_n^{(4)} \cap \Omega_n^{(5)} \right\} \geq 1 - 5\epsilon,$$

and $\epsilon > 0$ is also arbitrary, (3.64) follows. \blacksquare

APPENDIX A. RANGES AND LOCAL TIMES

This appendix is largely devoted to proving Theorem 2.3. Unless otherwise stated, F denotes the distribution in Assumption 2.1.

Let us begin by looking closely at the set $I_{0;n}$ in (2.29). By the definition of the probability measure μ_n , the random function $Y^{(k;n)}$ has its first zero coordinate between 0 and n , and the subsequent zero coordinates appear whenever the Markov chain returns back to zero. Therefore, $I_{0;n}$ is, in distribution, the restriction to $\{0, \dots, n\}$ of the range of a random walk with the step distribution F , starting at a random position in $\{0, \dots, n\}$. The following definition formalizes this type of random walks, which we will study in this appendix.

Assumption A.1. For $n \in \mathbb{N}_0$, $\mathbf{S}^{(n)} = \{S_t^{(n)}\}_{t \in \mathbb{N}_0}$ is a random walk such that

- (a) the initial position $S_0^{(n)}$ has the law of $\min\{0 \leq t \leq n : Y_t^{(n)} = 0\}$, where $\{Y_t^{(n)}\}_{t \in \mathbb{Z}}$ has the law μ_n in (2.28);
- (b) the steps $\{\xi_t\}_{t \in \mathbb{N}}$ are i.i.d. with distribution F , and are independent of $S_0^{(n)}$.

That is, the different random walks in Assumption A.1 only differ in the law of the initial position. Hereafter, $\{\{S_t^{(k;n)}\}_{t \in \mathbb{N}_0}\}_{k \in \mathbb{N}_0}$ denotes a collection of i.i.d. copies of $\mathbf{S}^{(n)}$. For the duration of this section we work with the sets $I_{k;n}$ in (2.29) written as

$$I_{k;n} = \{S_t^{(k;n)} : t \in \mathbb{N}_0\} \cap \{0, \dots, n\}, \quad n, k \in \mathbb{N}_0. \quad (\text{A.1})$$

By the definition these are nonempty random sets.

We will need several facts about these random walks, and we list these facts in the proposition below. Most of them are well known. In the sequel we use the notation $\min A$ ($\max A$) to denote the smallest (largest) point in a discrete set A .

Proposition A.2. (i) For every $k \in \mathbb{N}_0$,

$$\left(\frac{\min I_{k;n}}{n}, \frac{I_{k;n}}{n} \right) \Rightarrow (Z^*(0), \overline{R^*}) \quad \text{weakly in } [0, 1] \times \mathcal{F}([0, 1]) \quad (\text{A.2})$$

as $n \rightarrow \infty$, where $Z^*(0)$ and $\overline{R^*}$ are connected by (2.11), (2.12) and (2.16).

(ii) If A_0 denotes the full range $\{S_0^{(0)}, S_1^{(0)}, \dots\}$ of $\mathbf{S}^{(0)}$, then

$$\mathbb{P}\{n \in A_0\} \sim \frac{n^{\beta-1}\overline{F}(0)}{\Gamma(\beta)\Gamma(1-\beta)L(n)} \quad \text{as } n \rightarrow \infty \quad (\text{A.3})$$

and

$$\limsup_{n_0 \rightarrow \infty} \sup_{n > n_0} \max_{0 \leq k \leq n-1} \max_{m \in \mathbb{Z}} \frac{\#(A_0 \cap [m, m+2^k] \cap [2^{n_0}, 2^n])}{n/\overline{F}(2^k)} < \infty \quad \text{a.s.} \quad (\text{A.4})$$

(iii) For any $\gamma, \eta > 0$

$$\#\{k : S_k^{(0)} \leq \eta n, \xi_k \geq (\log n)^\gamma\} \xrightarrow{P} \infty, \quad n \rightarrow \infty. \quad (\text{A.5})$$

(iv) Let $\{Z(t)\}_{t \in \mathbb{R}_+}$ and $\{Z^*(t)\}_{t \in \mathbb{R}_+}$ be given by (2.8) and (2.11), respectively. Then with ϑ_n given by (2.35),

$$\left\{ \frac{1}{\vartheta_n} S_{[nt]}^{(0)\leftarrow} \right\}_{t \in \mathbb{R}_+} \Rightarrow \{Z^\leftarrow(t)\}_{t \in \mathbb{R}_+} \quad \text{in } (D(\mathbb{R}_+), J_1), \quad (\text{A.6})$$

$$\left\{ \frac{1}{\vartheta_n} S_{[nt]}^{(n)\leftarrow} \right\}_{t \in \mathbb{R}_+} \Rightarrow \{Z^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+} \quad \text{in } (D(\mathbb{R}_+), J_1). \quad (\text{A.7})$$

Proof. (i): The claim follows from Theorem 5.4 in Samorodnitsky and Wang (2019).

(ii): See (A.2) and Lemma A.1 in Appendix A in Chen and Samorodnitsky (2020).

(iii): See Lemma A.2 in Appendix A in Chen and Samorodnitsky (2020).

(iv): We first show (A.6). For a sequence $\{c_n\}$ satisfying $n\overline{F}(c_n) \sim 1/\Gamma(1-\beta)$ we have

$$\left\{ \frac{1}{c_n} S_{[nt]}^{(0)} \right\}_{t \in \mathbb{R}_+} \Rightarrow \{Z(t)\}_{t \in \mathbb{R}_+} \quad \text{in } (D(\mathbb{R}_+), J_1); \quad (\text{A.8})$$

see *e.g.* Theorem 4.5.3 in Whitt (2002), and it follows that (A.6) holds in the M_1 -topology (see Whitt (1971)). Because the process $\{Z^\leftarrow(t)\}_{t \in \mathbb{R}_+}$ is *a.s.* continuous, the convergence also holds in the J_1 -topology, see Section 12.4 in Whitt (2002). We combine this argument with part (i) of the proposition to get (A.7). \blacksquare

We will occasionally drop the superscript on our random walk whenever the discussion depends only on the step distribution of the walk. Denoting for $A \subset \mathbb{Z}$ and $a \in A$

$$\mathbb{P}_a\{\mathbf{S} \text{ escapes } A\} = \mathbb{P}\{\mathbf{S} \text{ does not hit } A \setminus \{a\} \mid S_0 = a\},$$

we define the capacity of A by

$$\text{cap}(A) = \sum_{a \in A} \mathbb{P}_a\{\mathbf{S} \text{ escapes } A\}. \quad (\text{A.9})$$

It is well known that $\text{cap}(A_1) \leq \text{cap}(A_2)$ if $A_1 \subset A_2$, and $\text{cap}(A_1 \cup A_2) \leq \text{cap}(A_1) + \text{cap}(A_2)$ for any A_1, A_2 ; see Spitzer (1964).

For $0 \leq m_1 < m_2 \leq \infty$ we consider the range

$$A_0(m_1, m_2) = \{S_{m_1}^{(0)}, \dots, S_{m_2-1}^{(0)}\},$$

so that $A_0 = A_0(0, \infty)$ is the full range.

Proposition A.3. (i) Fix $\gamma > (1-2\beta)^{-1}$, and let $\mathbf{S} = \{S_t\}_{t \in \mathbb{N}_0}$ be independent of A_0 . Then

$$\max_{0 \leq j \leq n - (\log n)^\gamma} \mathbb{P}\left(\mathbf{S} \text{ hits } A_0 \cap \{j + \lceil (\log n)^\gamma \rceil, \dots, n\} \mid S_0 = j, A_0\right) \rightarrow 0 \quad (\text{A.10})$$

a.s. as $n \rightarrow \infty$. In particular, if for $n = 1, 2, \dots$, $V_{1;n}$ and $V_{2;n}$ are measurable nonempty subsets of $A_0 \cap \{0, \dots, n\}$ with $\min V_{2;n} - \max V_{1;n} \geq (\log n)^\gamma$, then

$$\frac{1}{\#V_{1;n}} \left(\text{cap}(V_{1;n} \cup V_{2;n}) - \text{cap}(V_{1;n}) - \text{cap}(V_{2;n}) \right) \rightarrow 0 \text{ a.s.} \quad (\text{A.11})$$

(ii) Let $\tilde{\mathbf{S}}^{(0)}$ be an independent copy of $\mathbf{S}^{(0)}$, with ranges denoted by $\tilde{A}_0(\cdot, \cdot)$. Then

$$c_\infty := \mathbb{P} \left\{ A_0 \cap \tilde{A}_0 = \{0\} \right\} \in (0, 1). \quad (\text{A.12})$$

Furthermore,

$$\frac{\text{cap}(A_0(0, n))}{n} \rightarrow c_\infty \text{ a.s. as } n \rightarrow \infty. \quad (\text{A.13})$$

Proof. (i): For (A.10) we write for $0 \leq j \leq n - (\log n)^\gamma$,

$$\begin{aligned} & P(\mathbf{S} \text{ hits } A_0 \cap \{j + \lceil (\log n)^\gamma \rceil, \dots, n\} | S_0 = j, A_0) \\ & \leq \sum_{k=\lfloor \log_2 \lceil (\log n)^\gamma \rceil \rfloor}^{\lceil \log_2 n \rceil} \#(A_0 \cap [j + 2^k, j + 2^{k+1})) \cdot \max_{m \geq 2^k} \mathbb{P}_0 \{\mathbf{S} \text{ hits } m\}. \end{aligned}$$

By (A.4), there is an a.s. finite constant $B_1 = B_1(A_0)$ such that the first term in the sum does not exceed $B_1 \log n / \bar{F}(2^k)$, while by (A.3), the second term does not exceed $B_2 2^{(\beta-1)k} / L(2^k)$ for some finite constant B_2 . Therefore, for any $\epsilon > 0$, the sum above can be bounded by

$$B_1 B_2 \sum_{k=\lfloor \log_2(\log n)^\gamma \rfloor}^{\infty} \frac{\log n}{\bar{F}(2^k)} \cdot \frac{2^{(\beta-1)k}}{L(2^k)} \lesssim (\log n)^{1+(2\beta-1+\epsilon)\gamma}.$$

Choosing $0 < \epsilon < 1 - 2\beta - \gamma^{-1}$ proves (A.10).

For (A.11) we enumerate, for a fixed n and $i = 1, 2$, $V_{i;n}$ from left to right as

$$\{v_{i,1}, \dots, v_{i,n_i}\}.$$

Then

$$\begin{aligned} \text{cap}(V_{1;n} \cup V_{2;n}) &= \sum_{j=1}^{n_1} \mathbb{P}_{v_{1,j}} \{\mathbf{S} \text{ escapes } V_{1;n} \cup V_{2;n} | A_0\} + \text{cap}(V_{2;n}) \\ &= \text{cap}(V_{1;n}) - \sum_{j=1}^{n_1} q_j + \text{cap}(V_{2;n}), \end{aligned}$$

where in the obvious notation

$$q_j := \mathbb{P} \{\mathbf{S} \text{ escapes } V_{1;n} \text{ but hits } V_{2;n} | A_0, S_0 = v_{1,j}\}.$$

Now (A.11) follows from (A.10).

(ii): Note that by (A.3),

$$\mathbb{P}\{k \in A_0 \cap \tilde{A}_0\} = \mathbb{P}\{k \in A_0\} \cdot \mathbb{P}\{k \in \tilde{A}_0\} \in \mathbf{RV}_{2\beta-2},$$

so it is a summable sequence. This implies (A.12). To prove (A.13), we observe that the array $\{\text{cap}(A_0(m_1, m_2))\}_{m_1 < m_2}$ forms a stationary and subadditive family, so by the subadditive ergodic theorem (see Theorem 5 in Kingman (1968)) we have

$$\frac{\text{cap}(A_0(0, n))}{n} \rightarrow \Upsilon \text{ a.s.}$$

for some random variable $0 \leq \Upsilon \leq 1$. Since the invariant σ -field associated with the array is clearly trivial, it follows from Theorem 3 *ibid.* that Υ is a constant. It remains to show that the constant is equal to c_∞ . We have

$$\Upsilon = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \mathbb{P} \left\{ \tilde{A}_0 + S_i^{(0)} \cap A_0(i, n) = \{i\} \right\}$$

a.s., hence

$$\Upsilon = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \mathbb{P} \left\{ \tilde{A}_0 \cap A_0(0, n-i) = \{0\} \right\} \quad \textit{a.s.}$$

This is the arithmetic mean of a sequence that converges to c_∞ , so $\Upsilon = c_\infty$ *a.s.* \blacksquare

Proposition A.4. For $n \in \mathbb{N}$, let

$$\overline{A}_n = \{S_t^{(n)} : t = 0, 1, 2, \dots\} \cap \{0, \dots, n\}.$$

Let $\{Z^*(t)\}_{t \in \mathbb{R}_+}$, \overline{R}^* and ϑ_n be as in (2.11), (2.16) and (2.35) respectively.

(i) As $n \rightarrow \infty$

$$\left(\left\{ \frac{1}{\vartheta_n} S_{[nt]}^{(n)\leftarrow} \right\}_{t \in \mathbb{R}_+}, \frac{\overline{A}_n}{n} \right) \Rightarrow (\{Z^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+}, \overline{R}^*) \quad \textit{in } (D(\mathbb{R}_+), J_1) \times \mathcal{F}([0, 1]).$$

(ii) As $n \rightarrow \infty$

$$\begin{aligned} & \left\{ \frac{\text{cap}(\overline{A}_n \cap \{[nx], \dots, [ny]\})}{\vartheta_n}, 0 \leq x < y \leq 1 \right\} \\ & \Rightarrow \{c_\infty (Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x)), 0 \leq x < y \leq 1\} \end{aligned}$$

in finite-dimensional distributions.

Proof. (i): Since the marginal convergence has been established in Proposition A.2 (i) and (iv), the tightness is automatic. It suffices, therefore, to show uniqueness of subsequential weak limits. Suppose that for some subsequence $\{n_k\}$,

$$\left(\frac{1}{\vartheta_{n_k}} \left\{ S_{[n_k t]}^{(n_k)\leftarrow} \right\}_{t \in \mathbb{R}_+}, \frac{1}{n_k} \overline{A}_{n_k} \right) \Rightarrow (\{Z^\kappa(t)\}_{t \in \mathbb{R}_+}, R^\kappa)$$

for some Z^κ and R^κ . We will prove that, necessarily,

$$(\{Z^\kappa(t)\}_{t \in \mathbb{R}_+}, R^\kappa) \stackrel{d}{=} (\{Z^{*\leftarrow}(t)\}_{t \in \mathbb{R}}, \overline{R}^*). \quad (\text{A.14})$$

To this end, recall that the π -systems

$$\mathcal{C}_D = \left\{ \{x(t_i) > a_i : t_i \geq 0, a_i \in \mathbb{R}, 1 \leq i \leq \ell\} : \ell \in \mathbb{N} \right\}$$

and

$$\mathcal{C}_{\mathcal{F}} = \left\{ \mathcal{F}^T : T \text{ a finite union of open intervals} \right\}$$

generate the respective σ -fields in $D(\mathbb{R}_+)$ and \mathcal{F} , so it is enough to check that the laws of $(\{Z^\kappa(t)\}_{t \in \mathbb{R}_+}, R^\kappa)$ and $(\{Z^*(t)\}_{t \in \mathbb{R}_+}, \overline{R}^*)$ agree on $\mathcal{C}_D \times \mathcal{C}_{\mathcal{F}}$. That is, for any $\ell_1, \ell_2 \in \mathbb{N}_0$, $t_i > 0, a_i \in \mathbb{R}, i = 1, \dots, \ell_1$ and disjoint open intervals T_1, \dots, T_{ℓ_2} , we have

$$\mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \{R^\kappa \cap T_j \neq \emptyset, Z^\kappa(t_i) > a_i\} \right\} = \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \{\overline{R}^* \cap T_j \neq \emptyset, Z^{*\leftarrow}(t_i) > a_i\} \right\}.$$

Denote $T_1 = (c_1, d_1), \dots, T_{\ell_2} = (c_{\ell_2}, d_{\ell_2})$. We have

$$\begin{aligned}
& \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \{Z^\kappa(t_i) > a_i, R^\kappa \cap T_j \neq \emptyset\} \right\} \\
&= \lim_{k \rightarrow \infty} \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \left\{ \frac{S_{[n_k t_i]}^{(n_k)^\leftarrow}}{\vartheta_{n_k}} > a_i, \overline{A_{n_k}} \cap n_k T_j \neq \emptyset \right\} \right\} \\
&= \lim_{k \rightarrow \infty} \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \left\{ \frac{S_{[n_k t_i]}^{(n_k)^\leftarrow}}{\vartheta_{n_k}} > a_i, \frac{S_{[n_k d_j]}^{(n_k)^\leftarrow} - S_{[n_k c_j]}^{(n_k)^\leftarrow}}{\vartheta_{n_k}} > 0 \right\} \right\} \\
&= \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \{Z^{*\leftarrow}(t_i) > a_i, Z^{*\leftarrow}(d_j) - Z^{*\leftarrow}(c_j) > 0\} \right\} \\
&= \mathbb{P} \left\{ \bigcap_{\substack{1 \leq i \leq \ell_1 \\ 1 \leq j \leq \ell_2}} \{Z^{*\leftarrow}(t_i) > a_i, \overline{R^*} \cap T_j \neq \emptyset\} \right\},
\end{aligned}$$

as long as we can justify the penultimate equality.

Since each $Z^{*\leftarrow}(t_i)$ is a continuous random variable, by the Portmanteau Theorem we only need to check that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ 0 < \frac{S_{[nd]}^{(n)^\leftarrow} - S_{[nc]}^{(n)^\leftarrow}}{\vartheta_n} < \varepsilon \right\} = 0$$

for any $0 \leq c < d \leq 1$. This follows from the marginal convergence given in Proposition A.2 (i).

(ii): By the Skorokhod embedding theorem, the convergence in part (i) of the proposition holds as the *a.s.* convergence on some probability space, which will again be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$ for typographical convenience. Consider the partition $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 := \{\omega \in \Omega : \overline{R^*}(\omega) \cap (x, y) = \emptyset\}, \quad \Omega_2 := \{\omega \in \Omega : \overline{R^*}(\omega) \cap (x, y) \neq \emptyset\}.$$

We will show that the required convergence holds in probability on both Ω_1 and Ω_2 .

Since $\overline{R^*}$ does not hit fixed points, we have $Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x) = 0$ *a.s.* on Ω_1 . Furthermore, we can write for some null event $\Omega_{0,1}$,

$$\Omega_1 = \Omega_{0,1} \cup \bigcup_{k \geq 1} \Omega_1^k, \quad \Omega_1^k := \{\rho(\overline{R^*}(\omega), [x, y]) \geq 1/k\},$$

where ρ is defined in (2.5). Since $\frac{1}{n} \overline{A_n} \rightarrow \overline{R^s}$ *a.s.* in the Fell topology, the convergence holds in the Hausdorff metric ρ_H as well, so on each Ω_1^k ,

$$\overline{A_n} \cap \{[nx], \dots, [ny]\} = \emptyset$$

for all n large enough. Hence the required convergence holds *a.s.* on Ω_1 .

We now consider the event Ω_2 . Let

$$\tau_1 = \inf \{\overline{R^*} \cap [x, y]\}, \quad \tau_2 = \sup \{\overline{R^*} \cap [\tau_1, y]\} \quad (\text{both} = y \text{ if } \overline{R^*} \cap [x, y] = \emptyset).$$

For some null event $\Omega_{0,2}$ we can write

$$\Omega_2 = \Omega_{0,2} \cup \bigcup_{k \geq 1} \Omega_2^k, \quad \Omega_2^k := \{\tau_2(\omega) - \tau_1(\omega) \geq 1/k\}.$$

On Ω_2 , by the strong Markov property,

$$\text{cap}\left(\overline{A_n} \cap \{[nx], \dots, [ny]\}\right) \stackrel{d}{=} \text{cap}\left(A_0\left(0, S_{[ny]}^{(n)\leftarrow} - S_{[nx]}^{(n)\leftarrow}\right)\right).$$

Once again, since $\frac{1}{n}\overline{A_n} \rightarrow \overline{R^s}$ a.s. in the Hausdorff metric. So on each Ω_2^k , we have $S_{[ny]}^{(n)\leftarrow} - S_{[nx]}^{(n)\leftarrow} \rightarrow \infty$ a.s.. It follows by Proposition A.3 (ii) that

$$\frac{\text{cap}\left(\overline{A_n} \cap \{[nx], \dots, [ny]\}\right)}{S_{[ny]}^{(n)\leftarrow} - S_{[nx]}^{(n)\leftarrow}} \rightarrow c_\infty$$

in probability on each Ω_2^k , hence also on the entire Ω_2 . Finally,

$$\begin{aligned} & \frac{\text{cap}\left(\overline{A_n} \cap \{[nx], \dots, [ny]\}\right)}{\vartheta_n} \\ &= \frac{\text{cap}\left(\overline{A_n} \cap \{[nx], \dots, [ny]\}\right)}{S_{[ny]}^{(n)\leftarrow} - S_{[nx]}^{(n)\leftarrow}} \frac{S_{[ny]}^{(n)\leftarrow} - S_{[nx]}^{(n)\leftarrow}}{\vartheta_n} \rightarrow c_\infty (Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x)) \end{aligned}$$

in probability on Ω_2 . ■

We proceed with an important lemma. Switching back to the terminology of Subsection 2.2, we suppose that the random elements $\{Y^{(k;n)}\}_{k \in \mathbb{N}}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while $\{Y^{(0;n)}\}$ is defined on a different probability space, and the entire system is defined on the product probability space. We will use the notation \mathbb{P}_ω for the quenched (conditional) probability (computed with respect to $\{Y^{(0;n)}\}$). When needed in the sequel, the notion of quenched probability may change, and we will always specify its precise meaning.

Lemma A.5. *For any $K \in \mathbb{N}$ and $\epsilon > 0$ there exists a sequence of events $\{\Omega_{[K];n}^\epsilon\}_{n \geq 1}$ in Ω satisfying*

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left\{\Omega_{[K];n}^\epsilon\right\} > 1 - \epsilon, \quad (\text{A.15})$$

with the following properties.

(i) *There exists $C = C(\epsilon) > 0$ such that for all $1 \leq k \leq K$ and all large n ,*

$$C^{-1} \frac{\vartheta_n}{w_n} \leq \bar{p}_{k;n} \leq C \frac{\vartheta_n}{w_n} \quad \text{on } \Omega_{[K];n}^\epsilon. \quad (\text{A.16})$$

(ii) *For all $1 \leq k_1 \neq k_2 \leq K$, $I_{k_1;n} \cap I_{k_2;n} = \emptyset$ on $\Omega_{[K];n}^\epsilon$.*

(iii) *For $1 \leq k_1 \leq k_2 \leq K$ and $n \geq 1$, set*

$$\bar{p}_{[k_1:k_2];n} = \mathbb{P}\left\{I_{0;n} \cap (I_{k_1;n} \cup \dots \cup I_{k_2;n}) \neq \emptyset \mid I_{k_1;n} \dots I_{k_2;n}\right\}.$$

Then

$$\bar{p}_{[k_1:k_2];n} = \sum_{k=k_1}^{k_2} \bar{p}_{k;n} - o\left(\frac{\vartheta_n}{w_n}\right) \quad \text{on } \Omega_{[K];n}^\epsilon. \quad (\text{A.17})$$

Proof. The Skorohod embedding argument we have just used shows that there is a probability space (once again denoted by $(\Omega, \mathcal{F}, \mathbb{P})$) on which, for each $1 \leq k \leq K$,

$$\left(\left\{\frac{1}{\vartheta_n} S_{[nt]}^{(k;n)\leftarrow}\right\}_{t \in \mathbb{R}_+}, \frac{1}{n} I_{k;n}\right) \rightarrow (\{Z_k^{*\leftarrow}(t)\}_{t \in \mathbb{R}_+}, \overline{R_k^*}) \quad (\text{A.18})$$

a.s. in $(D(\mathbb{R}_+), J_1) \times \mathcal{F}([0, 1])$ and

$$\frac{\text{cap}(I_{k;n} \cap \{\lceil nx \rceil, \dots, \lfloor ny \rfloor\})}{\vartheta_n} \longrightarrow c_\infty (Z_k^{*\leftarrow}(y) - Z_k^{*\leftarrow}(x)) \quad (\text{A.19})$$

in probability for all $0 \leq x \leq y \leq 1$. In the remainder of the proof we work on this probability space. We spell out the argument in the case $K = 2$; the general case can be treated similarly.

(i): The last visit decomposition shows that

$$\bar{p}_{k;n} = \text{cap}(I_{k;n})/w_n, \quad k = 1, 2.$$

For $\varepsilon > 0$ choose $C_1 > 0$ so large that $\mathbb{P}(C_1^{-1} \leq Z^{*\leftarrow}(1) \leq C_1) > 1 - \varepsilon/2$. Letting

$$\Omega_{[2];n}^\varepsilon = \{c_\infty C_1^{-1} \leq \text{cap}(I_{k;n})/\vartheta_n \leq c_\infty C_1, k = 1, 2\},$$

we see by (A.19) with $x = 0, y = 1$ that (A.15) holds. Then (A.16) holds with $C = C_1/c_\infty$.

(ii): Since $\mathbb{P}(I_{1;n} \cap I_{2;n} \neq \emptyset) \rightarrow 0$, we can make the events $\Omega_{[2];n}^\varepsilon$ slightly smaller so that (A.15) still holds and the condition of (ii) also holds.

(iii): Recall that \bar{R}_1^* and \bar{R}_2^* intersect only on a null set. It follows from (A.18) that $\liminf \rho(I_{1;n}, I_{2;n})/n > 0$ a.s. on $\Omega_{[2];n}^\varepsilon$. After removing from $\Omega_{[2];n}^\varepsilon$ the null set, The claim (A.17) now follows from (A.10), where the initial point j is the leftmost point in $I_{1;n} \cup I_{2;n}$ that is in $I_{0;n}$, and A_0 the extension to the left of that among $I_{1;n}, I_{2;n}$ which does not contain j . \blacksquare

Lemma A.6. *For any $K, m \in \mathbb{N}$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \text{the numbers } \{j_{k,i;n}\}_{\substack{1 \leq k \leq K \\ 1 \leq i \leq m}} \text{ are all different} \right\} = 1. \quad (\text{A.20})$$

Proof. Again, we only spell out the argument in the case $K = 2$ and $m = 1$. For $0 < \varepsilon < 1$ let $\Omega_{[K];n}^\varepsilon$ and $C = C(\varepsilon) > 0$ be as in Lemma A.5. We have for a large $a > 0$

$$\begin{aligned} & \mathbb{P} \{j_{1,1;n} = j_{2,1;n}\} \\ & \leq \mathbb{P} \left\{ j_{1,1;n} = j_{2,1;n} \leq aC \frac{w_n}{\vartheta_n} \right\} + \mathbb{P} \left\{ j_{1,1;n} > aC \frac{w_n}{\vartheta_n} \right\}. \end{aligned}$$

Letting now \mathbb{P}_ω be the quenched probability given $\{Y^{(1;n)}\}$, we recall that, with respect to \mathbb{P}_ω , $j_{1,1;n}$ is geometrically distributed with success probability $\bar{p}_{1;n}$. Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ j_{1,1;n} > aC \frac{w_n}{\vartheta_n} \right\} = \int_\Omega \mathbb{P}_\omega \left\{ j_{1,1;n} > aC \frac{w_n}{\vartheta_n} \right\} \mathbb{P}(d\omega) \\ & \leq \varepsilon + \int_{\Omega_{[K];n}^\varepsilon} \mathbb{P}_\omega \left\{ j_{1,1;n} > aC \frac{w_n}{\vartheta_n} \right\} \mathbb{P}(d\omega) \\ & \leq \varepsilon + \left(1 - C^{-1} \frac{\vartheta_n}{w_n}\right)^{aC w_n / \vartheta_n - 1} \rightarrow \varepsilon + e^{-a}. \end{aligned}$$

On the other hand, by the inclusion-exclusion formula and Lemma A.5 (iii),

$$\begin{aligned} & \mathbb{P} \left\{ j_{1,1;n} = j_{2,1;n} \leq aC \frac{w_n}{\vartheta_n} \right\} \\ & \leq \epsilon + \frac{aCw_n}{\vartheta_n} \int_{\Omega_{[2];n}^\epsilon} \mathbb{P}_\omega \{ I_{0;n} \cap I_{1;n} \neq \emptyset, I_{0;n} \cap I_{2;n} \neq \emptyset \} \mathbb{P}(d\omega) \\ & \leq \epsilon + \frac{aCw_n}{\vartheta_n} \sup_{\omega \in \Omega_{[2];n}^\epsilon} \mathbb{P}_\omega \{ I_{0;n} \cap I_{1;n} \neq \emptyset, I_{0;n} \cap I_{2;n} \neq \emptyset \} \rightarrow \epsilon. \end{aligned}$$

Letting first $\epsilon \rightarrow 0$ and then $a \rightarrow \infty$ concludes the argument. \blacksquare

Proof of Theorem 2.3. For notational simplicity we consider the case $K = m = 2$. Our method easily carries over to arbitrary K and m . We will once again use the Skorohod embedding and assume that (A.18) and (A.19) hold. Then $I_{k;n}/n \rightarrow \bar{R}_k^*$ a.s. and by Proposition A.4 (ii), $w_n \bar{p}_{k;n}/\vartheta_n \rightarrow c_\infty Z_k^{* \leftarrow}(1)$ a.s. as well. We consider now the remaining components in (2.34).

Let \mathbb{P}_ω be the quenched probability given $\{Y^{(k;n)}\}$, $k = 1, 2$. To handle the second component in (2.34), it is enough to show that, for a.s. $\omega \in \Omega$,

$$(j_{1,1;n} \bar{p}_{1;n}, j_{2,1;n} \bar{p}_{2;n}) \Rightarrow (\Gamma_{1,1}, \Gamma_{2,1}) \quad (\text{A.21})$$

under \mathbb{P}_ω . For $0 < \epsilon < 1$, let $\Omega_{[2];n}^\epsilon$ be the event in Lemma A.5. Let $x_1, x_2 > 0$. On $\Omega_{[2];n}^\epsilon$, for any subsequence (n_m) over which $(\bar{p}_{2;n_m})^{-1} x_2 \geq (\bar{p}_{1;n_m})^{-1} x_1$ we have

$$\begin{aligned} & \mathbb{P}_\omega \{ j_{1,1;n_m} \bar{p}_{1;n_m} \geq x_1, j_{2,1;n_m} \bar{p}_{2;n_m} \geq x_2 \} \\ & = \left(1 - \bar{p}_{[1;2];n_m}\right)^{\lfloor (\bar{p}_{1;n_m})^{-1} x_1 \rfloor - 1} \cdot \left(1 - \bar{p}_{2;n_m}\right)^{\lfloor (\bar{p}_{2;n_m})^{-1} x_2 - (\bar{p}_{1;n_m})^{-1} x_1 \rfloor} \\ & = \left(\prod_{k=1}^2 \left(1 - \bar{p}_{k;n_m}\right) + o\left(\frac{\vartheta_{n_m}}{w_{n_m}}\right) \right)^{\lfloor (\bar{p}_{1;n_m})^{-1} x_1 \rfloor - 1} \\ & \quad \cdot \left(1 - \bar{p}_{2;n_m}\right)^{\lfloor (\bar{p}_{2;n_m})^{-1} x_2 - (\bar{p}_{1;n_m})^{-1} x_1 \rfloor} \\ & = (1 + o(1)) \prod_{k=1}^2 \left(1 - \bar{p}_{k;n_m} + o\left(\frac{\vartheta_{n_m}}{w_{n_m}}\right)\right)^{(\bar{p}_{k;n_m})^{-1} x_k} \rightarrow e^{-(x_1+x_2)} \end{aligned}$$

as $m \rightarrow \infty$. The same is true for any subsequence (n_m) over which $(\bar{p}_{2;n_m})^{-1} x_2 < (\bar{p}_{1;n_m})^{-1} x_1$. We thus see that

$$\mathbb{P}_\omega \{ j_{1,1;n_m} \bar{p}_{1;n_m} \geq x_1, j_{2,1;n_m} \bar{p}_{2;n_m} \geq x_2 \} \rightarrow e^{-(x_1+x_2)}$$

over $\liminf \Omega_{[2];n}^\epsilon$ for every $0 < \epsilon < 1$ and, hence, also on an event of probability 1. Since this is true for all $x_1, x_2 > 0$, (A.21) follows.

We now consider the last component in (2.34). By Lemma A.6 we only need to prove the following statement. Consider ℓ disjoint open intervals in $(0, 1)$, $\{B_i = (x_i, y_i) : i = 1, \dots, \ell\}$. Then for any $\epsilon, \delta > 0$, there exists a sequence of events $\{\Omega'_n\}_{n \in \mathbb{N}}$ in Ω such that $\liminf_{n \rightarrow \infty} \mathbb{P}\{\Omega'_n\} \geq 1 - \epsilon$ and

$$\sup_{\omega \in \Omega'_n} \left| \mathbb{P}_\omega \left\{ \bigcap_{r=1}^{\ell} \left\{ \frac{1}{n} I_{1,1;n} \cap B_r \neq \emptyset \right\} \right\} - \mathbb{P}_1 \left\{ \bigcap_{r=1}^{\ell} \{J_{1,1} \in B_r\} \right\} \right| \leq \delta \quad (\text{A.22})$$

where \mathbb{P}_ω is the quenched probability given $Y^{(1;n)}$ and \mathbb{P}_1 is the probability associated with an independent standard uniform random variable. We treat the cases $\ell = 1$ and $\ell \geq 2$ separately.

Suppose first that $\ell = 1$. By (2.15),

$$\mathbb{P}_1 \{J_{1,1} \in B_1\} = \frac{Z_1^{*\leftarrow}(y_1) - Z_1^{*\leftarrow}(x_1)}{Z_1^{*\leftarrow}(1)},$$

while by the last exit decomposition,

$$\mathbb{P}_\omega \left\{ \frac{I_{1,1;n}}{n} \cap B_1 \neq \emptyset \right\} = \frac{\text{cap}(I_{1;n}(\omega) \cap (nx_1, ny_1))}{\text{cap}(I_{1;n}(\omega))}.$$

Therefore, we can take $\Omega'_n = \Omega$ and (A.22) follows by (A.19).

If $\ell \geq 2$, then the second probability in (A.22) vanishes. Furthermore,

$$\begin{aligned} & \mathbb{P}_\omega \{I_{1,1;n} \cap nB_1 \neq \emptyset, \dots, I_{1,1;n} \cap B_\ell \neq \emptyset\} \\ & \leq \mathbb{P}_\omega \{I_{1,1;n} \cap nB_1 \neq \emptyset, I_{1,1;n} \cap nB_2 \neq \emptyset\}. \end{aligned}$$

Letting $\delta = x_2 - y_1 > 0$ we have by the strong Markov property,

$$\begin{aligned} & \mathbb{P} \{I_{1,1;n} \cap nB_1 \neq \emptyset, I_{1,1;n} \cap nB_2 \neq \emptyset\} \\ & \leq \mathbb{P} \left\{ A_0, \tilde{A}_0 \text{ have a common point } > n\delta \right\} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by (A.3), we immediately obtain (A.22). \blacksquare

Proof of Proposition 2.4. (i): The claim (2.36) follows from the obvious fact that

$$\#I_{1,1;n} \leq \#(A_0 \cap \tilde{A}_0)$$

and the latter cardinality has the geometric distribution with success probability $1 - c_\infty$ (with c_∞ defined in (A.12)).

(ii): The claim follows from (i) by the Markov inequality. \blacksquare

APPENDIX B. ADDITIONAL AUXILIARY RESULTS

This section contains several auxiliary results that are essential in the main proofs. We start with describing certain useful properties of the functions V and h in (3.4) and some related functions. Let

$$G(x) = \left(1/\gamma\bar{H}\right)^{\leftarrow}(x), \quad x \geq 1/\gamma; \quad (\text{B.1})$$

notice that by the inverse function theorem,

$$xG'(x) = h \circ G(x). \quad (\text{B.2})$$

Furthermore, by the Karamata theorem the function

$$x \mapsto \int_1^x \frac{du}{u^{1-\alpha}L_\alpha(u)}$$

is regularly varying at infinity with exponent α , so its inverse is regularly varying with exponent $1/\alpha$. Therefore, the function

$$\mathcal{L}(x) = x^{-1/\alpha} \left(\int_1^x \frac{du}{u^{1-\alpha}L_\alpha(u)} \right)^{\leftarrow} \quad (\text{B.3})$$

is slowly varying.

Proposition B.1. *The functions V , h , G and \mathcal{L} have the following properties at infinity:*

$$G(x) - V(x) = o(h \circ G(x)). \quad (\text{B.4})$$

$$V(x) \sim G(x) \sim (\log x)^{1/\alpha} \mathcal{L}(\log x), \quad (\text{B.5})$$

$$h \circ V(x) \sim h \circ G(x) \sim \alpha^{-1} (\log x)^{1/\alpha-1} \mathcal{L}(\log x). \quad (\text{B.6})$$

Furthermore, for every $t > 0$ the function V satisfies

$$\lim_{x \rightarrow \infty} \frac{V(tx) - V(x)}{h \circ V(x)} = \log t. \quad (\text{B.7})$$

Proof. The statement (B.4) follows from the properties of the tails in the Gumbel domain of attraction; see e.g. (2.4) in Chen and Samorodnitsky (2020). This now implies the first asymptotic equivalencies in (B.5) and (B.6). Since

$$G(x) = (\log(x\gamma))^{1/\alpha} \mathcal{L}(\log(x\gamma)),$$

the second asymptotic equivalence in (B.5) follows from the regular variation. Further, by Karamata's theorem,

$$L_\alpha(G(x)) \sim (\mathcal{L}(\log x))^\alpha / \alpha,$$

and the second asymptotic equivalence in (B.6) follows as well.

The version of the statement (B.7) with V replaced by G follows easily from the definition of G , and by (B.4) we may replace G by V . \blacksquare

We proceed with two lemmas used in the proof of Proposition 3.9. The first lemma is purely analytical, and we omit a straightforward proof.

Lemma B.2. (i) *The function*

$$\psi(r) = \frac{(1-\beta)^{1/\alpha} + \beta^{1/\alpha}}{(1-\beta-r)^{1/\alpha}} - 1, \quad 0 \leq r < 1-\beta$$

is increasing to infinity. Furthermore, the numbers r_m defined by $\psi(r_m) = m$ satisfy $r_m < m/(m+1) - \beta$ for $m \geq 1$.

(ii) *The function*

$$\tilde{\psi}(r) = (1-\beta-r)[\lfloor \psi(r) \rfloor + (\psi(r) - \lfloor \psi(r) \rfloor)^\alpha], \quad 0 \leq r < 1-\beta$$

is increasing and satisfies $\tilde{\psi}(r) > r + \beta$ on $(0, 1-\beta)$. Finally, $\tilde{\psi}(r) \rightarrow \infty$ as $r \rightarrow (1-\beta)^-$.

The next lemma is essential for Proposition 3.9.

Lemma B.3. *Fix any $m \in \mathbb{N}_0$.*

(i) *For $1 < b < \infty$ such that $\nu(1, b) > 0$ let $\{\xi_i\}_{i \in \mathbb{N}}$ be i.i.d. random variables whose law is the restriction of ν to $(1, b)$ normalized to a probability measure. Then for any $m \in \mathbb{N}_0$ there are $c_m, \gamma_m \geq 0$ depending on m only such that for any $y \in (mb, (m+1)b]$ and $d \geq m+1$,*

$$\mathbb{P} \left\{ \sum_{i=1}^d \xi_i \geq y \right\} \leq c_m^d b^{\gamma_m} (\bar{H}(b))^m \bar{H}(y - mb). \quad (\text{B.8})$$

(ii) For $0 < r < 1 - \beta$ consider sequences $z_n = V(w_n) + V(\vartheta_n) - V(w_n/n^r) - o(V(w_n))$ and $\bar{z}_n = V(w_n/n^r)$, $n \geq 1$. Let (r_m) be as in Lemma B.2. Then for any $m \geq 1$ there is $\gamma_m > 0$ such that for any $r \in (r_m, r_{m+1}]$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P} \left\{ \sum_{\Gamma_j > n^r} V(w_n/\Gamma_j) 1_{\{0 \in I_{j;n}\}} \geq z_n \right\}}{\bar{z}_n^{\gamma_m} (\overline{H}(\bar{z}_n))^m \overline{H}(z_n - m\bar{z}_n)} = 0. \quad (\text{B.9})$$

Proof. (i): Recall that $\overline{H}(x) = \exp\{-q(x)\}$ for $q \geq x_0$ with q increasing and concave, and $xq'(x) \leq q(x)$ for $x \geq x_1$, for some $x_1 \geq x_0$. Therefore, we can extend q in the obvious way from the range $[x_1, \infty)$ to an increasing and concave function on $[0, \infty)$, that vanishes at the origin. We work with this redefined H , while keeping the original notation H . There clearly is $C \geq 1$ so that $\mathbb{P}\{\xi_1 > x\} \leq C\overline{H}(x)$ for all $x > 0$. Since \overline{H} is the tail of a subexponential distribution, there is $c_0 > 0$ such that, in the usual notation for the convolution power, for all $y > 0$

$$\mathbb{P} \left\{ \sum_{i=1}^d \xi_i \geq y \right\} \leq C^d \overline{H}^{*d}(y) \leq c_0^d \overline{H}(y) \quad \text{for all } d \in \mathbb{N}, \quad (\text{B.10})$$

see Proposition 4.1.10 in Samorodnitsky (2016). This gives (B.8) in the case of $m = 0$ and all $d \geq 1$ (with $\gamma_0 = 0$).

We proceed in the inductive manner. Assume that (B.8) holds for all $0 \leq m \leq m_0$ and all $d \geq m + 1$. We first consider the case $m = m_0 + 1$ and $d = m + 1$. Let H_b be the restriction of H to $(1, b)$. We still have

$$\mathbb{P}\{\xi_1 > x\} \leq (C/\|H_b\|)\overline{H}_b(x) \quad \text{for all } x > 0.$$

Therefore, for $y > 0$

$$\begin{aligned} & \mathbb{P} \left\{ \sum_{i=1}^d \xi_i \geq y \right\} \leq (C/\|H_b\|)^d \overline{H}_b^{*d}(y) \\ &= (C/\|H_b\|)^d \int_{(0,b)^d} 1_{\{\sum_{i=1}^d z_i > y\}} \prod_{i=1}^d \exp\{-q(z_i)\} q'(z_i) dz_i \\ &\leq (C/\|H_b\|)^d (q(b))^{m+1} \\ &\quad \cdot \exp \left\{ - \inf \left\{ \sum_{i=1}^{m+1} q(z_i) : \sum_{i=1}^{m+1} z_i > y, 0 < z_1, \dots, z_{m+1} \leq b \right\} \right\}. \end{aligned}$$

Since $q(\cdot)$ is increasing and concave, for $y \in (mb, (m+1)b]$, the infimum is achieved at, say, $z_1 = \dots = z_m = b$, $z_{m+1} = y - mb$. Since for $b > 1$, $q(b) \leq C_1 b^{2\alpha}$ for some $C_1 > 0$, this establishes (B.8) in the case $d = m + 1$ with γ_m and c_m that must be at least 2α and CC_1 , correspondingly. Their final values will be set in the sequel.

We continue to induct on d while keeping the same m . Assume, therefore, that (B.8) is valid for $d = m + 1, \dots, m + \ell$, some $\ell \geq 1$. In the case $d = m + \ell + 1$ write for $y \in (mb, (m+1)b]$, in the obvious notation

$$\mathbb{P} \left\{ \sum_{i=1}^d \xi_i \geq y \right\} = \int_1^b F_\xi(dz) \mathbb{P} \left\{ \sum_{i=1}^{d-1} \xi_i \geq y - z \right\} =: T_1 + T_2,$$

where T_1 and T_2 are the integrals over $(1, y - mb]$ and $(y - mb, b)$, correspondingly. To estimate T_1 , note that in this range $y - z > mb$, so we may use the inductive assumption

over d to obtain

$$\mathbb{P} \left\{ \sum_{i=1}^{d-1} \xi_i \geq y - z \right\} \leq c_m^{d-1} b^{\gamma_m} (\overline{H}(b))^m \overline{H}(y - z - mb).$$

By (B.10)

$$\begin{aligned} \int_1^{y-mb} F_\xi(dz) \overline{H}(y - z - mb) &\leq C \int_1^\infty H(dz) \overline{H}(y - z - mb) \\ &\leq (c_0^2 C) \overline{H}(y - mb). \end{aligned}$$

It follows that

$$T_1 \leq (c_0^2 C) c_m^{d-1} b^{\gamma_m} (\overline{H}(b))^m \overline{H}(y - mb). \quad (\text{B.11})$$

To estimate T_2 , note that in this range $(m-1)b < y - z \leq mb$, and we use the inductive assumption over m to write

$$T_2 \leq c_{m-1}^{d-1} b^{\gamma_{m-1}} (\overline{H}(b))^{m-1} \int_{y-mb}^b F_\xi(dz) \overline{H}(y - (m-1)b - z).$$

Using the same optimization under concavity argument as above shows that

$$\begin{aligned} \int_{y-mb}^b F_\xi(dz) \overline{H}(y - (m-1)b - z) &\leq C \overline{H}^{*2}(y - (m-1)b) \\ &\leq C (q(b))^2 \overline{H}(b) \overline{H}(y - mb). \end{aligned}$$

Therefore,

$$T_2 \leq C c_{m-1}^{d-1} b^{\gamma_{m-1}} (q(b))^2 (\overline{H}(b))^m \overline{H}(y - mb). \quad (\text{B.12})$$

It follows from (B.11) and (B.12) that to complete the inductive argument we only need to make the final selection of γ_m and c_m to be so large as to satisfy

$$c_m^d \geq (c_0^2 C) c_m^{d-1} + C c_{m-1}^{d-1}, \quad b^{\gamma_m} \geq b^{\gamma_{m-1}} (q(b))^2.$$

Since this can clearly be done, this completes the proof of (B.8).

(ii): Note that

$$\sum_{\Gamma_j \geq n^r} V(w_n / \Gamma_j) 1_{\{0 \in I_{j;n}\}} \stackrel{d}{=} \sum_{i=1}^{N_n} \xi_i, \quad (\text{B.13})$$

where N_n is a Poisson random variable with mean $\bar{\nu}(x_0) - n^r / w_n$, and $\{\xi_i\}_{i \geq 1}$ is a family of *i.i.d.* random variables independent of N_n , whose law is the measure ν restricted to the interval (x_0, \bar{z}_n) and normalized to a probability measure there. Because of the range of r and (B.5) we see that for large n the event $\{\sum_{i=1}^d \xi_i > z_n\}$ requires d to be at least $m+1$. Therefore, in the notation of the first part of the lemma, by (B.8),

$$\begin{aligned} \mathbb{P} \left\{ \sum_{i=1}^{N_n} \xi_i > z_n \right\} &= \sum_{d=m+1}^{\infty} \mathbb{P} \left\{ \sum_{i=1}^d \xi_i > z_n \right\} \mathbb{P} \{N_n = d\} \\ &\leq \sum_{d=m+1}^{\infty} \frac{c_m^d \bar{z}_n^{\gamma_m} (\overline{H}(\bar{z}_n))^m \overline{H}(z_n - m\bar{z}_n)}{d!} \\ &\leq e^{c_m \bar{z}_n^{\gamma_m}} (\overline{H}(\bar{z}_n))^m \overline{H}(z_n - m\bar{z}_n), \end{aligned}$$

Since $\bar{z}_n \rightarrow \infty$, using $\gamma_m + 1$ from the first part of the lemma as γ_m in (B.9) gives us (B.9). \blacksquare

REFERENCES

- J. AARONSON (1997): *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence.
- Z. CHEN and G. SAMORODNITSKY (2020): Extremal clustering under moderate long range dependence and moderately heavy tails. Preprint.
- R. DAVIS (1982): Limit laws for the maximum and minimum of stationary sequences. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 61:31–42.
- R. DONEY (1997): One-sided local large deviation and renewal theorems in the case of infinite mean. *Probability Theory and Related Fields* 107:451–465.
- P. EMBRECHTS, C. GOLDIE and N. VERVAERBEKE (1979): Subexponentiality and infinite divisibility. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 49:335–347.
- S. FOSS, T. KONSTANTOPOULOS and S. ZACHARY (2007): The principle of a single big jump: discrete and continuous time modulated random walks with heavy-tailed increments. *Journal of Theoretical Probability* 20:581–612.
- C. GOLDIE and S. RESNICK (1988): Distributions that are both subexponential and in the domain of attraction of an extreme-value distribution. *Advances in Applied Probability* 20:706–718.
- T. HARRIS and H. ROBBINS (1953): Ergodic theory of Markov chains admitting an infinite invariant measure. *Proceedings of the National Academy of Sciences* 39:860–864.
- J. KINGMAN (1968): The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* 30:499–510.
- C. LACAUX and G. SAMORODNITSKY (2016): Time-changed extremal process as a random sup measure. *Bernoulli* 22:1979–2000.
- M. LEADBETTER (1983): Extremes and local dependence of stationary sequences. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 65:291–306.
- I. MOLCHANOV (2017): *Theory of Random Sets*. Springer, London, 2nd edition.
- G. O'BRIEN, P. TORFS and W. VERVAAT (1990): Stationary self-similar extremal processes. *Probability Theory and Related Fields* 87:97–119.
- T. OWADA and G. SAMORODNITSKY (2015): Maxima of long memory stationary symmetric α -stable processes, and self-similar processes with stationary max-increments. *Bernoulli* 21:1575–1599.
- E. PITMAN (1980): Subexponential distribution functions. *Journal of Australian Mathematical Society, Series A* 29:337–347.
- S. RESNICK (1987): *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York.
- S. RESNICK (2007): *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, New York.
- S. RESNICK and M. RUBINOVITCH (1973): The structure of extremal processes. *Advances in Applied Probability* 5:287–307.
- S. RESNICK, G. SAMORODNITSKY and F. XUE (2000): Growth rates of sample covariances of stationary symmetric α -stable processes associated with null recurrent Markov chains. *Stochastic Processes and Their Applications* 85:321–339.
- G. SAMORODNITSKY (2016): *Stochastic Processes and Long Range Dependence*. Springer, Cham, Switzerland.
- G. SAMORODNITSKY and Y. WANG (2019): Extremal theory for long range dependent infinitely divisible processes. *Annals of Probability* 47:2529–2562.

- K. SATO (2013): *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 2nd edition.
- F. SPITZER (1964): *Principle of Random walk*. D. van Nostrand Company, Princeton.
- S. TAYLOR and J. G. WENDEL (1966): The Exact Hausdorff Measure of the Zero Set of a Stable Process. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 6:170–180.
- W. WHITT (1971): Weak convergence of first passage time processes. *Journal of Applied Probability* 8:417–422.
- W. WHITT (2002): *Stochastic-Process Limits. An Introduction to Stochastic-Process Limits and Their Applications to Queues*. Springer, New York.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14850, USA
Email address: zc288cornell.edu

SCHOOL OF OPERATION RESEARCH AND INFORMATION ENGINEERING, CORNELL UNIVERSITY, ITHACA,
NEW YORK 14850, USA
Email address: gs18@cornell.edu