CALCULATION OF RUIN PROBABILITIES FOR A DENSE CLASS OF HEAVY TAILED DISTRIBUTIONS

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Abstract. In this paper we propose a class of infinite–dimensional phase–type distributions with finitely many parameters as models for heavy tailed distributions. The class of finite–dimensional distributions is dense in the class of distributions on the positive reals and may hence approximate any such distribution. We prove that formulas from renewal theory, and with a particular attention to ruin probabilities, which are true for common phase–type distributions also hold true for the infinite–dimensional case. We provide algorithms for calculating functionals of interest such as the renewal density and the ruin probability. It might be of interest to approximate a given heavy–tailed distribution of some other type by a distribution from the class of infinite–dimensional phase–type distributions and to this end we provide a calibration procedure which works for the approximation of distributions with a slowly varying tail. An example from risk theory, comparing ruin probabilities for a classical risk process with Pareto distributed claim sizes, is presented and exact known ruin probabilities for the Pareto case are compared to the ones obtained by approximating by an infinite–dimensional hyper–exponential distribution.

1. Introduction

The purpose of this paper is to propose a large class of (potentially) heavy tailed distributions which allows for explicit or exact solutions to a variety of problems in applied probability, such as e.g. ruin probabilities in risk theory or waiting time distributions in queueing theory. The class proposed is based on infinite–dimensional phase–type distributions. [13], [3], [14], [6]

A phase–type distribution of finite dimension is the distribution of the absorption time in a finite state–space Markov jump process with one absorbing state and the rest being transient. The class of phase–type distributions is dense in the class of distributions on the positive reals, see e.g. [5]. The

1991 *Mathematics Subject Classification.* Primary .
Key words and phrases.
Samorodnitsky's research was partially supported by the ARO grants W911NF-07-1-0078 and W911NF-12-10385, NSF grant DMS-1005903 and NSA grant H98230-11-1-0154 at Cornell University.

quality of the resulting approximation can be improved if a Markov process was allowed to have an infinite number of states (stages); we are allowing an infinite number of stages in this paper. In general, this would involve an infinite number of parameters but here we shall restrict our attention to models with only a finite number of freely varying parameters (but, generally, infinite state space).

Phase–type distributions of finite dimension, i.e. where the underlying Markov jump process lives on a finite state–space, have been employed in a variety of contexts in applied probability since they often provide exact, or even explicit, solutions to important problems in complex stochastic models. This is for example the case in renewal theory where the renewal density is known explicitly, in queueing theory where waiting time distributions are of phase–type if the service times are such or in risk theory where the deficit at ruin is phase–type distributed if the claims sizes are such.

In this paper, we shall prove that many formulas which are known explicitly as being exact for the case of finite–dimensional phase–type distributions, holds true also for a class of infinite–dimensional phase–type distributions, and we shall provide methods for their numerical evaluation since, due to their infinite dimension, straightforward calculations are not feasible.

Our method could be illustrated by applications to any of the above mentioned areas of interest. In this paper, however, we are mainly concerned with an application of our method to the ruin probabilities in risk theory, when the claim sizes have heavy tails. This is due to the fact that the ruin probabilities are very tail sensitive.

Specifically, we are concerned with a problem which in a general formulation, may be described as follows. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with a common law F, concentrated on $(0,\infty)$. Let N be a homogeneous Poisson process on $(0,\infty)$ independent of the sequence $X_1, X_2, \ldots$, with rate β. Assume that $EX_1 < \infty$ and, moreover, $\beta EX_1 < 1$. Then the stochastic process

$$S(t) = \sum_{j=1}^{N(t)} X_j - t,\ t \geq 0\,, \tag{1.1}$$

drifts to $-\infty$ as the time grows, and the probability

$$\Psi(u) = P\big(\sup_{t\geq 0} S(t) > u\big), u > 0\,, \tag{1.2}$$

is the object we are interested in calculating. In actuarial applications, assuming a deterministic unit rate stream of premiums paid to an insurance company, and if the claims arrive according to the Poisson process N, and the size of the jth claim is X_j, then $\Psi(u)$ is the probability that the excess of submitted claims over the premiums ever exceeds the initial capital u of the company. This is, therefore, the (infinite horizon) ruin probability. See, for instance, [11]. However, quantities of the same type as $\Psi(u)$ are known to represent the tail of the stationary workload in a stable queue (see e.g. [7]), and appear in many other applications.

Calculating the ruin probability is, therefore, of the utmost importance, and studying this problem has been a source of much research. We refer the reader to [4] and [11] for some detailed accounts. Many of the known results are asymptotic in nature, that is, they describe the behavior of the ruin probability as the level u grows to infinity. In this paper we describe a general procedure that, in principle, allows to calculate the ruin probability for any level u. We devote our attention to the case of *heavy tailed claim sizes*, however, we expect the procedure to be applicable when the tails are lighter as well.

We provide a method for calibrating the approximating infinite phase–type distribution to a claim size distribution F which has a regularly varying tail. That is,

$$\overline{F}(x) := 1 - F(x) = x^{-\phi} L(x),\ x > 0 \tag{1.3}$$

for some $\phi > 1$. Here L is a slowly varying at infinity function. That means that L never vanishes and $L(cx)/L(x) \to 1$ as $x \to \infty$ for any $c > 0$. This is a very common class of models for heavy tailed phenomena; we refer the reader to [19] for an introduction to regular variation and to [9] for an encyclopedic treatment. Calculating the ruin probability for regularly varying claims has attracted much attention. An asymptotic result for the subexponential case, including the regularly varying case as a special situation, has been established by [12]. The effect on the asymptotics of dependence between claims is studied e.g. in [15] and [2].

For practical applications it is, of course, important to have a way to estimate the ruin probability for moderate levels u as well. This has turned out to be difficult. Practical computation algorithms have been developed for claim sizes with different versions of the pure Pareto distribution; see [17, 18] and [1]. Our approach for calculating the ruin probability will apply to a wide range of levels of u and to a wide variety of claim size distributions. The main idea is to understand what happens when the claim size distribution is an infinite dimensional phase–type distribution. We provide an algorithm for calibrating a distribution from this class to any other distribution with a regular varying tail. By comparing the results we obtain in the case of Pareto distributed claims with the previously published numbers we see that our approach works well up to reasonable high levels of u.

A number of papers have addressed the issue of approximating heavy tailed distributions using mixtures. The paper [13] proposes the use of a finite mixture of exponential distributions with a logarithmic scaling of their means. A similar approach was applied in [3] to approximate arrival processes with long range dependence. In [14] an infinite dimensional mixture of exponential distributions is used to model heavy tailed data and an example of application from queueing theory is provided. The paper [6] applies the model of [14] in the context of the GI/M/1 queue. One major difference from the previous papers is our systematic use of phase–type methodology and carrying through calculations in infinite dimensions without previous truncation.

The paper is organized as follows. In the Section 2 we describe the general construction of the class of infinite dimensional phase–type distributions and prove some of the basic results concerning renewal theory and with applications to risk theory in Section 3. We describe the calibration algorithm in Section 4. In Section 4.3 we test the algorithm on Pareto distributed claims and compare our results with the previously published results.

2. Infinite dimensional phase–type distributions

We start with presenting some background on phase-type distributions. For further details the reader can consult [16].

Consider a Markov jump process $\mathbf{X} = \{X_t\}_{t\geq 0}$ on the finite state space $E = \{\Delta, 1, 2, ..., p\}$, where states $1, 2, ..., p$ are transient and state Δ absorbing. Then $\mathbf{X}$ has an intensity matrix $\mathbf{Q}$ of the form

$$\mathbf{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix}, \tag{2.1}$$

where $\mathbf{T}$ is a $p \times p$ sub–intensity matrix of jump intensities between the transient states, $\mathbf{t}$ is a p–dimensional column vector of intensities for jumping to the absorbing state and $\mathbf{0}$ is a p–dimensional zero row vector. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_p)$ denote a p–dimensional row vector of initial probabilities, $\alpha_i = \mathbb{P}(X_0 = i)$, $i = 1, ..., p$. Let $\mathbf{e} = (1, 1, 1, \cdots, 1)'$ be the p dimensional column vector of ones. We assume that $\boldsymbol{\alpha}\mathbf{e} = 1$, i.e. with probability 1 the process initiates in one of the transient states. Since the rows of an intensity matrix sum to 0, $\mathbf{t} = -\mathbf{T}\mathbf{e}$. Let $\tau = \inf\{t > 0 | X_t = p + 1\}$ denote the time until absorption. Then we say that τ has a phase–type distribution with parameters $(\boldsymbol{\alpha}, \mathbf{T})$ and we write $\tau \sim \mathrm{PH}(\boldsymbol{\alpha}, \mathbf{T})$.

Phase–type distributions generalize mixtures and convolutions of exponential distributions, and are widely used in applied probability due to their tractability. If $\tau \sim \mathrm{PH}(\boldsymbol{\alpha}, \mathbf{T})$ then the density of τ is given by

$$f_\tau(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}, \; x > 0\,, \tag{2.2}$$

where $\exp(\mathbf{T}x) = \sum_{n=0}^{\infty} \mathbf{T}^n x^n/n!$ denotes the usual matrix–exponential. The survival function of τ is given by

$$\overline{F}_\tau(x) = 1 - F_\tau(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{e}, \; x \geq 0\,,$$

where F_τ denotes the distribution function of τ, and the Laplace transform of τ is given by

$$Ee^{-s\tau} = \boldsymbol{\alpha}\,(s\mathbf{I} - \mathbf{T})^{-1}\,\mathbf{t}, \; s \geq 0\,,$$

where $\mathbf{I}$ denotes the $p \times p$ identity matrix. The nth order moment is given by

$$\mathbb{E}(\tau^n) = (-1)^n n!\, \boldsymbol{\alpha}\mathbf{T}^{-n}\mathbf{e}.$$

Similarly, we can consider an infinite dimensional Markov jump process on the state space $E = \{\Delta\} \cup \{1, 2, ...\}$, where states $1, 2...$ are transient and the state Δ is absorbing. It is assumed that absorption occurs with probability one. We define an infinite dimensional phase–type distribution in the same way as for a finite state space, namely the time until absorption. We let $\boldsymbol{\pi} = (\pi_1, \pi_2, ...)$ be the initial probabilities; these are concentrated on the transient states (i.e. $\pi_\Delta = 0$). We still write the intensity matrix $\mathbf{\Lambda}$ of the Markov jump process in the same way as for the finite dimensional case, namely

$$\mathbf{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{\lambda} & \mathbf{\Lambda} \end{pmatrix},$$

but now $\mathbf{\Lambda}$ is a doubly infinite matrix.

The question which we may raise is to which extent all of the above formulas generalize to the infinite dimensional case. A sufficient condition for matrix–exponentials of infinite matrices to be well–defined is requiring the matrices to be bounded, which shall indeed be the case for our applications.

Below is an example of infinite–dimensional phase–type distributions we will consider. Let $Y \sim \mathrm{PH}(\boldsymbol{\alpha}, \mathbf{T})$ be a random variable with a (finite dimensional) phase-type distribution; by (2.2) it has a density given by $g(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}$, $x > 0$. Here and in the sequel we still use the notation (2.1).

Let $\mathbf{q} = (q_1, q_2, \ldots)$ be a vector of probabilities on $\mathbb{N}$. We define a new probability law on $(0, \infty)$ by

$$\tilde{F}(x) = \sum_{i=1}^{\infty} q_i P(Y \le x/i), \ x > 0 \, .$$

It is easy to see that this is an infinite dimensional phase-type distribution with a representation of the form

$$\mathbf{\Lambda} = \mathrm{diag}(\mathbf{T}_1, \mathbf{T}_2, ...) = \begin{pmatrix} \mathbf{T}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & ... \\ \mathbf{0} & \mathbf{T}_2 & \mathbf{0} & \mathbf{0} & ... \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_3 & \mathbf{0} & ... \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_4 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.3}$$

where $\mathbf{T}_i = \mathbf{T}/i$ and $\mathbf{t}_i = -\mathbf{T}_i\mathbf{e} = \mathbf{t}/i$, $i = 1, 2, \ldots$. For convenience we denote the infinite vector of the intensities of transition to the absorbing state by $\boldsymbol{\lambda} = \big(\mathbf{t}_1, \mathbf{t}_2, \ldots\big)'$. The density of F is then given by

$$f(x) = \sum_{i=1}^{\infty} q_i \boldsymbol{\alpha} e^{\mathbf{T}_i x} \mathbf{t}_i,\ x > 0\,. \tag{2.4}$$

This infinite dimensional phase–type distribution can be thought of as having a representation $(\boldsymbol{\pi}, \boldsymbol{\Lambda})$, with $\boldsymbol{\pi} = (\mathbf{q} \otimes \boldsymbol{\alpha})$, the notation $\mathbf{q} \otimes \boldsymbol{\alpha}$ meaning that the initial probability of the jth position in the ith block is equal to $q_i \alpha_j$. Correspondingly, the density f in (2.4) can be written compactly as

$$f(x) = (\mathbf{q} \otimes \boldsymbol{\alpha}) e^{\boldsymbol{\Lambda} x} \boldsymbol{\lambda},\ x > 0\,,$$

and the distribution function $\tilde{F}$ as

$$F(x) = 1 - (\mathbf{q} \otimes \boldsymbol{\alpha}) e^{\boldsymbol{\Lambda} x} \mathbf{e}_\infty,\ x \geq 0\,. \tag{2.5}$$

Here $\mathbf{e}_\infty$ is the infinite dimensional column vector of ones. The Laplace transform of this infinite dimensional phase-type distribution is given by

$$(\mathbf{q} \otimes \boldsymbol{\alpha}) \left(s\mathbf{I}_\infty - \boldsymbol{\Lambda}\right)^{-1} \boldsymbol{\lambda},\ s \geq 0\,,$$

where $\mathbf{I}_\infty$ is the infinite dimensional identity matrix. The nth moment, if it exists, is given by $n!(\mathbf{q} \otimes \boldsymbol{\alpha})(-\boldsymbol{\Lambda})^{-n} \mathbf{e}_\infty$.

Example 2.1. Consider $g(x) = x^{29}/29!\ e^{-x}$, an Erlang distribution with 30 stages. Then $\mathbf{T}$ is a 30–dimensional matrix with $t_{ii} = -1$ and $t_{i,i+1} = 1$ and all other entrances zero. $\boldsymbol{\alpha} = (1, 0, 0, ..., 0)$ (dimension 30) and $\mathbf{t}' = (0, 0, 0, ..., 0, 1)$ (dimension 30). We choose $q_i = i^{-1.5}/\zeta(1.5)$, the Zeta distribution, and construct f as in (2.4). The following figure shows the original Erlang density versus f.

Example 2.2. Though the basic construction uses phase–type distributions, we can easily generalize the construction to distributions having a rational Laplace transform in general. Most applications will go through under this larger class, as is certainly the case for the ruin formulas in the next section. Let $g(x) = (1 + \frac{1}{4\pi^2})(1 - \cos(2\pi x))e^{-x}$. Then g is not a phase type distribution but a distribution with a rational Laplace transform which may be represented as a matrix–exponential distribution in the following way

$$\boldsymbol{\alpha} = (1, 0, 0) \quad \mathbf{T} = \begin{pmatrix} 0 & -1 - 4\pi^2 & 1 + 4\pi^2 \\ 3 & 2 & -6 \\ 2 & 2 & -5 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

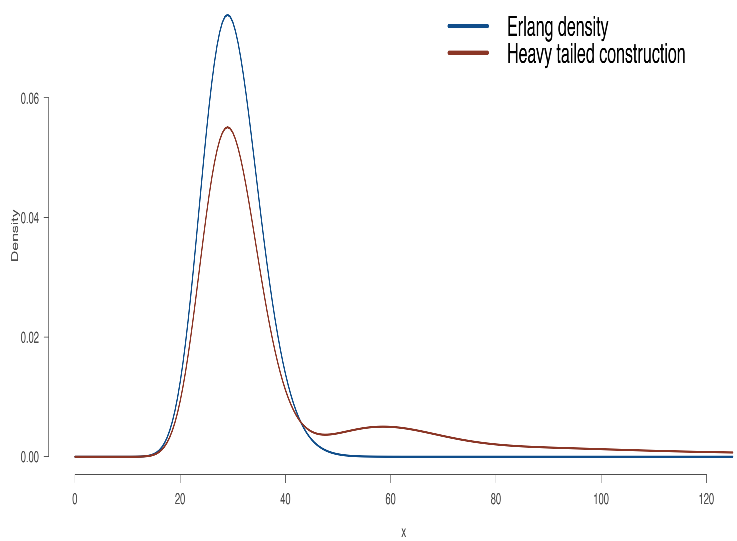

Figure 1. Erlang density versus a heavy tailed modification based on the same Erlang distribution and the ζ–distribution

We choose $q_i = i^{-1.5}/\zeta(1.5)$, the Zeta distribution, and construct f as in (2.4) . The following figure shows the original matrix–exponential density versus its heavy–tailed version f.

3. Renewal theory, ladder heights and ruin probabilities

If $\boldsymbol{\Lambda}$ is a bounded operator then so is $\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha})$. The matrix $\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha})$ is the intensity matrix (defective or not) of the Markov jump process obtained by concatenating the underlying Markov jump processes of the corresponding (infinite dimensional) phase–type renewal process. Hence

$$(\mathbf{q}\otimes\boldsymbol{\alpha})e^{(\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha}))x}$$

is the distribution at time x of the concatenated Markov jump process. This distribution is of crucial interest for calculating various functionals of interest. Let $\mathbf{v}$ be an infinite–dimensional vector and consider

$$f_{\mathbf{v}}(x) = (\mathbf{q}\otimes\boldsymbol{\alpha})e^{(\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha}))x}\mathbf{v}.$$

If $\mathbf{v}=\boldsymbol{\lambda}$, then $f_{\mathbf{v}}(x)$ equals the renewal density. If the renewal process is terminating ($\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha})$ being defective), then for $\mathbf{v}=\mathbf{e}$, $f_{\mathbf{v}}(x)$ equals the probability that the renewal process will ever reach time x. If the renewal

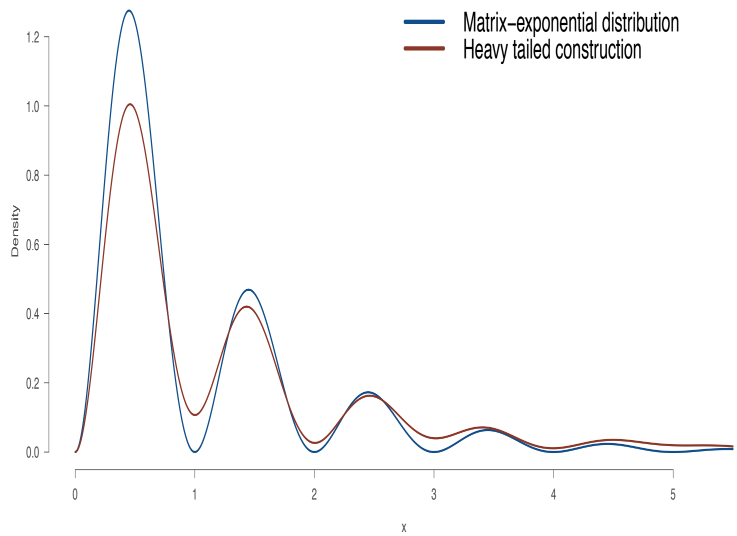

FIGURE 2. Matrix–exponential distribution versus a heavy–tailed modification of the same. The broken line in the heavy tailed version is caused by numerical difficulties.

process is not defective, this probability is of course equal to one. If $\mathbf{v} = e^{\mathbf{\Lambda} y}\boldsymbol{\lambda}$, then $y \to f_{\mathbf{v}}(x) = f_{\mathbf{v}}(x,y)$ equals the over–shoot density in the renewal process, i.e. the density of the time until the next arrival after time x. The following theorem provides an algorithm for the calculation of the functional $f_{\mathbf{v}}(x)$.

Theorem 3.1.

$$f_{\mathbf{v}}(x) = e^{-\theta x}\sum_{n=0}^{\infty}\kappa_n\frac{(\theta x)^n}{n!},$$

where

$$\kappa_n = (\mathbf{q}\otimes\boldsymbol{\alpha})\left(\mathbf{I}+\theta^{-1}(\mathbf{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha}))\right)^n\mathbf{v}.$$

The κ_n's can be calculated by the following recursion scheme:

$$\kappa_{n+1} = \sum_{i=0}^{n}\theta^{-1}(\mathbf{q}\otimes\boldsymbol{\alpha})(\mathbf{I}+\theta^{-1}\mathbf{\Lambda})^i\boldsymbol{\lambda}\kappa_{n-i} + (\mathbf{q}\otimes\boldsymbol{\alpha})(\mathbf{I}+\theta^{-1}\mathbf{\Lambda})^{n+1}\mathbf{v},$$

with initial condition $\kappa_0 = (\mathbf{q}\otimes\boldsymbol{\alpha})\mathbf{v}$

Proof. Since the diagonal elements of $\mathbf{\Lambda}$ are bounded, we can choose a $\theta > 0$ such that $\mathbf{I}+\theta^{-1}(\mathbf{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha}))$ is a sub–transition matrix. With such θ we write

$$\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha}) = -\theta \mathbf{I} + \theta \left(\mathbf{I} + \theta^{-1}(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right) ,$$

so that

$$e^{(\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha}))u} = e^{-\theta u} e^{(\mathbf{I}+\theta^{-1}(\boldsymbol{\Lambda}+\boldsymbol{\lambda}(\mathbf{q}\otimes\boldsymbol{\alpha})))\theta u} .$$

Therefore

$$\begin{aligned} f_{\mathbf{v}}(x) &= (\mathbf{q} \otimes \boldsymbol{\alpha}) e^{-\theta u} \sum_{n=0}^{\infty} \left(\mathbf{I} + \theta^{-1}(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha}))\right)^n \frac{(\theta u)^n}{n!} \mathbf{v} \\ &= e^{-\theta u} \sum_{n=0}^{\infty} \kappa_n \frac{(\theta u)^n}{n!}. \end{aligned}$$

Concerning the recursion scheme, we have that

$$\begin{aligned} \kappa_{n+1} &= (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \left(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right)\right)^{n+1} \mathbf{v} \\ &= (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \left(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right)\right) \left(\mathbf{I} + \theta^{-1} \left(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right)\right)^{n} \mathbf{v} \\ &= \theta^{-1} (\mathbf{q} \otimes \boldsymbol{\alpha}) \boldsymbol{\lambda} \kappa_n + (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \boldsymbol{\Lambda}\right) \left(\mathbf{I} + \theta^{-1} \left(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right)\right)^{n} \mathbf{v} \\ &= \theta^{-1} (\mathbf{q} \otimes \boldsymbol{\alpha}) \boldsymbol{\lambda} \kappa_n + \theta^{-1} (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \boldsymbol{\Lambda}\right) \boldsymbol{\lambda} \kappa_{n-1} \\ &\quad + (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \boldsymbol{\Lambda}\right)^2 \left(\mathbf{I} + \theta^{-1} \left(\boldsymbol{\Lambda} + \boldsymbol{\lambda}(\mathbf{q} \otimes \boldsymbol{\alpha})\right)\right)^{n-1} \mathbf{v} \\ &= \sum_{i=0}^{n} \theta^{-1} (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \boldsymbol{\Lambda}\right)^i \boldsymbol{\lambda} \kappa_{n-i} + (\mathbf{q} \otimes \boldsymbol{\alpha}) \left(\mathbf{I} + \theta^{-1} \boldsymbol{\Lambda}\right)^{n+1} \mathbf{v} . \end{aligned}$$

□

Consider the claim surplus process $\{S(t)\}_{t\geq 0}$ in (1.1) with i.i.d. claims with the infinite–dimensional phase–type distribution whose density is given by (2.4). Let $\tau_+ = \tau_+(1) = \inf\{t > 0 | S(t) > 0\}$ and $\tau_+(n+1) = \inf\{t > \tau_+(n) | S(t) > S(\tau_+(n))\}$, $n \geq 1$ denote the successive ladder epochs for the ascending ladder process $\{S(\tau_+(n))\}_{n\geq 1}$. This is a terminating renewal process since the ladder height distribution is defective. The density of this distribution is $g_+(x) = \beta(1 - F(x))$ (see e.g. [5]), $x > 0$. Thus by (2.5)

$$g_+(x) = \beta(\mathbf{q} \otimes \boldsymbol{\alpha}) e^{\boldsymbol{\Lambda} x} \mathbf{e} = \beta \sum_{i=1}^{\infty} q_i \boldsymbol{\alpha} e^{\mathbf{T}_i x} \mathbf{e},\ x > 0.$$

Rewriting for $x > 0$

$$\begin{aligned} g_+(x) &= \beta(\mathbf{q} \otimes \boldsymbol{\alpha}) \boldsymbol{\Lambda}^{-1} e^{\boldsymbol{\Lambda} x} \boldsymbol{\Lambda} \mathbf{e} \\ &= \left(-\beta(\mathbf{q} \otimes \boldsymbol{\alpha}) \boldsymbol{\Lambda}^{-1}\right) e^{\boldsymbol{\Lambda} x} \boldsymbol{\lambda}, \end{aligned}$$

we see that the ascending ladder height distribution, though defective, is itself of the infinite dimensional phase–type, with a representation $(\boldsymbol{\pi}_+, \boldsymbol{\Lambda})$, where $\boldsymbol{\pi}_+ = -\beta(\mathbf{q}\otimes\boldsymbol{\alpha})\boldsymbol{\Lambda}^{-1}$. It is defective because these initial probabilities do not sum to one.

Since $\boldsymbol{\Lambda}$ is block diagonal, the ascending ladder height density can also be written as

$$g_+(x) = -\beta \sum_{i=1}^{\infty} q_i \boldsymbol{\alpha}\mathbf{T}_i^{-1} e^{\mathbf{T}_i x}\mathbf{t}_i = -\beta \sum_{i=1}^{\infty} q_i \boldsymbol{\alpha}\mathbf{T}^{-1} e^{\mathbf{T} i x}\mathbf{t}.$$

Thus, if $\hat{g}$ is the density of the finite dimensional defective phase–type distribution $\mathrm{PH}(\boldsymbol{\alpha}_+, \mathbf{T})$, where $\boldsymbol{\alpha}_+ = -\beta\boldsymbol{\alpha}\mathbf{T}^{-1}$, then

$$g_+(x) = \sum_{i=1}^{\infty} q_i \hat{g}(x/i), \; x > 0\,.$$

In fact, $\hat{g}$ is the ascending ladder height density corresponding to the finite dimensional phase–type distribution with the representation $(\boldsymbol{\alpha}, \mathbf{T})$.

Let $\hat{G}$ denote the distribution function corresponding to $\hat{g}$. Then the total mass of the defective distribution G_+ corresponding to g_+ is

$$||G_+|| = \int_0^{\infty} g_+(x)dx = \sum_{i=1}^{\infty} i q_i \|\hat{G}\| = \mu_{\mathbf{q}}\, \boldsymbol{\alpha}_+ \mathbf{e},$$

where $\mu_{\mathbf{q}}$ is the mean of the discrete distribution $\mathbf{q}$. Of course we also have $||G_+|| = \boldsymbol{\pi}_+\mathbf{e}$.

Since the ascending ladder process is terminating, there is a finite (geometrically distributed) number of ascending ladder epochs. Let M denote the sum of all the ladder heights (i.e. the life of the terminating renewal process). Then M also has an infinite dimensional phase–type distribution with a representation $(\boldsymbol{\pi}_+, \boldsymbol{\Lambda} + \boldsymbol{\lambda}\boldsymbol{\pi}_+)$. The ruin occurs if and only if $M > u$, so the corresponding ruin probability is given by

$$\Psi(u) = \mathbb{P}(M > u) = \boldsymbol{\pi}_+ e^{(\boldsymbol{\Lambda} + \boldsymbol{\lambda}\boldsymbol{\pi}_+)u}\mathbf{e}. \tag{3.1}$$

Note, further, that the random variable M has a compound geometric distribution, because it can be represented as $M = G_1 + ... + G_K$, where K is a geometric variable with parameter $\delta = ||G_+||$, independent of an i.i.d. sequence $G_1, G_2, ...$ with the common law G_+/δ. We will find this interpretation useful in the sequel.

We may use Theorem 3.1 with $\mathbf{v} = \mathbf{e}$ to compute (3.1). However, using the special structure of $\mathbf{\Lambda}$ we can further refine the algorithm to the following

$$\begin{aligned}
\kappa_0 &= \beta\boldsymbol{\alpha}(-\mathbf{T})^{-1}\sum_{j=1}^{\infty} jq_j\mathbf{e}\,,\\
\kappa_n &= \theta^{-1}\beta\boldsymbol{\alpha}\sum_{i=0}^{n-1}\sum_{j=1}^{\infty} q_j\left(\mathbf{I}+(j\theta)^{-1}\mathbf{T}\right)^i\mathbf{e}\kappa_{n-1-i}\\
&\quad+\beta\boldsymbol{\alpha}(-\mathbf{T})^{-1}\sum_{j=1}^{\infty} jq_j\left(\mathbf{I}+(j\theta)^{-1}\mathbf{T}\right)^n\mathbf{e},\ n\geq 1\,.
\end{aligned}$$

The computations for the ruin probability $\Psi(u)$ developed above work well up to reasonably high levels u. Estimates of ruin probabilities for reserves u larger than those that could be handled by the proposed recursion scheme, might be carried out by the asymptotic expansions developed by [8]. We reproduce this expansion here. Suppose that the claim sizes have a distribution with regular varying tails with index $\phi > 1$, as in (1.3).

Let $m < \phi - 1$ be a nonnegative integer. If $m \geq 2$, assume that the claim size distribution tail $\overline{F}$ has, for large values of its argument, $m-1$ derivatives, with the $m-1$th derivative regularly varying at infinity with exponent $-(\phi+m-1)$. Note that about any distribution F used in practice with $\phi > 3$ satisfies this assumption with $m = \lceil\phi\rceil - 2$. Let $\rho = \beta EX$; recall that we are assuming that $\rho < 1$. Denote

$$H(x) = \frac{1}{EX}\int_0^x \overline{F}(y)\,dy,\ x\geq 0\,,$$

the stationary renewal distribution function corresponding to F. Note that by the Karamata theorem, $\overline{H}(u) = 1 - H(u)$ is regularly varying with exponent $-\phi+1$; see [19]. Then by Proposition 4.5.1 in [8],

$$\Psi(u) = \rho(1-\rho)\sum_{j=0}^{m}\gamma_j(\rho)\frac{d^j\overline{H}(u)}{du^j} + o\big(u^{1-m}\overline{F}(u)\big) \tag{3.2}$$

as $u\to\infty$. Here

$$\gamma_j(\rho) = \frac{1}{j!}\frac{d^j}{d\theta^j}\Big(1-\rho L_H(\theta)\Big)^{-2}\Big|_{\theta=0},\ j=0,\ldots,m\,,$$

with

$$L_H(\theta) = \int_0^\infty e^{-\theta x}\,H(dx),\ \theta\geq 0$$

being the Laplace transform of H. The first few values of $\gamma_j(\rho)$ are

$$\gamma_0(\rho) = \frac{1}{(1-\rho)^2}, \quad \gamma_1(\rho) = -\frac{2\rho}{(1-\rho)^3} m_{1,H}\,,$$

$$\gamma_2(\rho) = \frac{\rho}{(1-\rho)^3} m_{2,H} + \frac{3\rho^2}{(1-\rho)^4}(m_{1,H})^2,$$

$$\gamma_3(\rho) = -\left(\frac{\rho}{3(1-\rho)^3} m_{3,H} + \frac{3\rho^2}{(1-\rho)^4} m_{1,H} m_{2,H} + \frac{4\rho^3}{(1-\rho)^5}(m_{1,H})^3\right),$$

$$\gamma_4(\rho) = \frac{\rho}{12(1-\rho)^3} m_{4,H} + \frac{\rho^2}{(1-\rho)^4} m_{1,H} m_{3,H} + \frac{3\rho^2}{4(1-\rho)^4}(m_{2,H})^2 + \frac{6\rho^3}{(1-\rho)^5}(m_{1,H})^2 m_{2,H} + \frac{5\rho^4}{(1-\rho)^6}(m_{1,H})^4\,.$$

Here

$$m_{j,H} = \frac{EX^{j+1}}{(j+1)EX}, \; j = 0, \ldots, m\,.$$

4. Calibration to a known distribution with a regularly varying tail

Let X be a generic claim size random variable whose law is the claim size distribution F, which is assumed to have a regularly varying tail with exponent $-\phi < -1$. We assume that $EX > 1$; any other case can be reduced to this case by scaling the problem.

Let Y be another (positive) random variable with $EY^\phi = 1$. In the algorithm Y will have a phase–type distribution, of the type described in Section 2. For now it is enough to assume that Y has a significantly lighter tail than X does; we will be more specific in a moment. We start by approximating the law F of X by a suitable mixture of the scaled versions of Y.

For $i = 1, 2, \ldots$ let

$$q_i = P(i-1 < X \le i) = F(i) - F(i-1)\,.$$

Next, let

$$I = \max\left\{ \min\{j : jF(j) + \sum_{i=j+1}^{\infty} i q_i \ge EX/EY\}, \; \min\{j : F(j) + \sum_{i=j+1}^{\infty} i q_i \le EX/EY\} \right\},$$

and for $s \in [0,\infty]$ denote

$$p_i(s) = \frac{q_i s^i}{\sum_{j=1}^I q_j s^j} F(I),\ i = 1,\dots,I\,.$$

Then

$$\tilde F_s(x) = \sum_{i=1}^{I} p_i(s) P(Y \le x/i) + \sum_{i=I+1}^{\infty} q_i P(Y \le x/i),\ \ x>0\,,$$

defines a probability distribution on $(0,\infty)$. If X_s is a generic random variable with this law, we see, by the definition of I, that

$$\lim_{s\to 0} EX_s = \Big(F(I) + \textstyle\sum_{i=I+1}^{\infty} iq_i\Big) EY \le EX,$$
$$\lim_{s\to \infty} EX_s = \Big(IF(I) + \textstyle\sum_{i=I+1}^{\infty} iq_i\Big) EY \ge EX\,.$$

Therefore, there is a unique $s_* \in [0,\infty]$ such that $EX_{s_*} = EX$. The approximating claim size law is defined by

$$\tilde F(x) = F_{s_*}(x),\ \ x > 0\,, \tag{4.1}$$

and we denote by $\tilde X$ a generic random variable with this distribution. By construction, $E\tilde X = EX$. The following proposition shows that the asymptotic tail of $\tilde X$ matches the tail of X as well.

Proposition 4.1. (i) *Assume that $EY^{\phi+\varepsilon} < \infty$ for some $\varepsilon > 0$, and that $EY^\phi = 1$. Then*

$$\lim_{x\to\infty} \frac{P(\tilde X > x)}{P(X > x)} = 1\,.$$

(ii) *Assume, additionally, that Y has a bounded density g, satisfying for some $p > \phi + 1$,*

$$g(x) \le Ax^{-p} \ \ \text{for some } A > 0, \text{ all } x > 0. \tag{4.2}$$

Then $\tilde X$ has a density $\tilde f$ satisfying

$$\lim_{x\to\infty} \frac{\tilde f(x)}{\phi x^{-1} P(X > x)} = 1\,.$$

Proof. Denote by D a discrete random variable with

$$P(D = i) = \begin{cases} p_i(s_*) & \text{for } i = 1,\dots,I, \\ q_i & \text{for } i = I+1,\dots. \end{cases}$$

Note that for $x > I$,

$$P(D > x) = P(X > \lceil x\rceil) \sim P(X > x)$$

as $x \to \infty$. Since $\tilde{X} \stackrel{d}{=} DY$, with D and Y independent in the right hand side, we obtain by the Breiman lemma (see [10]), that

$$P(\tilde{X} > x) \sim EY^{\phi} P(D > x) \sim P(X > x)$$

as $x \to \infty$. This establishes the claim of part (i).

For part (ii), existence of a density $\tilde{f}$ follows from existence of the density of Y. We will use the results in Section 4.4 of [9]. To this end we write the density $\tilde{f}$ as a Mellin–Stieltjes convolution

$$\tilde{f}(x) = \frac{1}{x} \int_0^{\infty} k(x/y)\, U(dy) = \frac{1}{x} \left(k \overset{S}{*} U \right)(x), \ \ x > 0\,;$$

see (4.0.3) in [9]. Here $k(x) = xg(x)$, $x > 0$, while U is a non-decreasing right continuous function on $(0, \infty)$ defined by $U(x) = -P(D > x)$, $x > 0$. We check the conditions of Theorem 4.4.2 in [9]. Choose two real numbers, σ and τ, such that

$$-(p-1) < \sigma < -\phi < 0 < \tau < 1\,,$$

and consider the amalgam norm

$$\|k\|_{\sigma,\tau} = \sum_{n=-\infty}^{\infty} \max\left(e^{-\sigma n}, e^{-\tau n}\right) \sup_{e^n \le x \le e^{n+1}} |k(x)|\,.$$

Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \max\left(e^{-\sigma n}, e^{-\tau n}\right) \sup_{e^n \le x \le e^{n+1}} |k(x)| &= \sum_{n=0}^{\infty} e^{-\sigma n} \sup_{e^n \le x \le e^{n+1}} |k(x)| \\
&\le A \sum_{n=0}^{\infty} e^{-\sigma n} e^{n+1} e^{-pn} < \infty
\end{aligned}$$

by (4.2) and the choice of σ. Next, denoting by M an upper bound on the density g, we have

$$\begin{aligned}
\sum_{n=-\infty}^{0} \max\left(e^{-\sigma n}, e^{-\tau n}\right) \sup_{e^n \le x \le e^{n+1}} |k(x)| &= \sum_{n=-\infty}^{0} e^{-\tau n} \sup_{e^n \le x \le e^{n+1}} |k(x)| \\
&\le M \sum_{n=-\infty}^{0} e^{-\tau n} e^{n+1} < \infty
\end{aligned}$$

by the choice of τ. We conclude that $\|k\|_{\sigma,\tau} < \infty$. Finally, $U(0+) = -1 = O(x^{\sigma})$ as $x \to 0+$. Therefore, we may apply Theorem 4.4.2 in [9] to conclude that, as $x \to \infty$,

$$\tilde{f}(x) \sim (-\alpha)\left(\frac{1}{x}U(x)\right)\int_0^\infty u^{-\alpha-1}k(1/u)\,du$$
$$= \phi x^{-1} E Y^\phi P(D > x) \sim \phi x^{-1} P(X > x)\,,$$

as required. □

In our algorithm we choose Y above to be a phase–type distribution, discussed in Section 2. Note that, in particular, all the conditions of Proposition 4.1 hold.

A useful conclusion from Proposition 4.1 is the fact that a subclass of infinite–dimensional phase–type distributions is dense within the class of all distrbutions in $(0,\infty)$ with regularly varying tails, both in the weak topology, and in a tail–related topology, that we presently explain. Let F be a probability law on $(0,\infty)$, and (t_i) a dense countable subset of $(0,\infty)$ consisting of continuity points of F. Let (w_i) be an arbitrary countable family of positive numbers adding up to 1. Then a sequence (G_n) of probability laws on $(0,\infty)$ converges weakly to F if and only if

$$d_0(G_n, F) := \sum_i w_i \big|G_n(t_i) - F(t_i)\big| \to 0$$

as $n \to \infty$. In order to show that the approximation works well in the tails in the distribution, we would like to to show convergence using the distance

$$d_1(G_n, F) := \sum_i w_i \left(\left|1 - \frac{1 - F(t_i)}{1 - G_n(t_i)}\right| + \left|1 - \frac{1 - G_n(t_i)}{1 - F(t_i)}\right| \right)\,. \tag{4.3}$$

Clearly, $d_1(G_n, F) \to 0$ still implies weak convergence, but it also implies that the convergence to F is good "in the tails" as well.

Consider the class $\mathcal{I}$ of infinite–dimensional phase-type distributions with a representation of the form, generalizing that in (2.3):

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{T}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_3 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.4}$$

where $\mathbf{T}_i$, $i \geq 1$ are as in in (2.3), and $\mathbf{T}_0$ is a finite sub–intensity matrix. We claim that for any probability law F on $(0,\infty)$ with a regularly varying tail with exponent $-\phi < -1$, any dense countable subset (t_i) of $(0, \infty)$ consisting of continuity points of F, we can find a sequence (G_n) of laws in $\mathcal{I}$ converging to F in the distance d_1 in (4.3).

To see this, let X be a random variable with distribution F. Let $\theta > 0$ be large enough so that $P(X \leq \theta) > 0$. By Proposition 4.1 there is a random variable $\tilde{X}_\theta$ with an infinite–dimensional phase-type distributions with a representation of the form (2.3) such that

$$\lim_{x\to\infty} \frac{P(\tilde{X}_\theta > x)}{P(X > x)} = P(X > \theta)\,.$$

Let Y_θ have the conditional law of X given $X \leq \theta$. By Theorem 4.2 in [5], for any (small) $\varepsilon > 0$ we can find a random variable $\tilde{Y}_{\theta,\varepsilon}$ with a finite phase–type distribution such that both

$$d_0(Y_\theta, \tilde{Y}_{\theta,\varepsilon}) \leq \varepsilon$$

and

$$E(Y_{\theta,\varepsilon})^{2\phi} \leq M_\theta$$

for some finite M_θ independent of ε. If we define (the law of) a random variable $Z_{\theta,\varepsilon}$ by

$$\text{the law of } Z_{\theta,\varepsilon} = \begin{cases} \text{the law of } \tilde{Y}_{\theta,\varepsilon} & \text{with probability } P(X \leq \theta) \\ \text{the law of } \tilde{X}_\theta & \text{with probability } P(X > \theta), \end{cases}$$

then the law of $Z_{\theta,\varepsilon}$ is in $\mathcal{I}$, and it is straightforward to check that, by taking θ large enough, and $\varepsilon > 0$ small enough, we can make the law of $Z_{\theta,\varepsilon}$ arbitrarily close to the law F of X in the distance d_1 in (4.3).

Our next result shows how to calculate the non–integer moments of phase–type distributions.

Theorem 4.2. *Let $X \sim PH(\boldsymbol{\alpha}, \mathbf{S})$. For any real $\phi > 1$, the non-integer moments $\mathbb{E}\left(X^\phi\right)$ can be calculated using the formula*

$$\mathbb{E}\left(X^\phi\right) = \Gamma(\phi+1)\theta^{-\phi}\boldsymbol{\alpha}\sum_{i=0}^{\infty}\binom{-\phi-1}{i}(-\mathbf{K})^i\mathbf{k}$$

where $\mathbf{S} = \theta\mathbf{K} - \theta\mathbf{I}$, $\mathbf{k} = \theta^{-1}\mathbf{s}$ and θ, $\mathbf{K}$ are chosen such that spectral radius of $\mathbf{K}$ is less than 1. The formula can be restated as

$$\mathbb{E}\left(Y^\phi\right) = \Gamma(\phi+1)\boldsymbol{\alpha}(-\mathbf{S})^{-\phi}\mathbf{e}.$$

Proof. First we choose a positive θ with a value that is at least as large as the absolute values of all eigenvalues of $\mathbf{S}$. The matrix $\mathbf{I} + \theta^{-1}\mathbf{S}$ will then have all eigenvalues within the unit circle, as all eigenvalues of $\mathbf{S}$ have negative real part. Additionally we define $\mathbf{k} = \theta^{-1}\mathbf{s}$. We then write

$$e^{\mathbf{S}y} = e^{(\theta\mathbf{K}-\theta\mathbf{I})y} = e^{-\theta y}\sum_{i=0}^{\infty}\frac{(\mathbf{K}\theta y)^i}{i!}$$

and use this expression when evaluating the moments.

$$\begin{aligned}
\mathbb{E}\left(Y^{\phi}\right) &= \boldsymbol{\alpha}\int_0^{\infty} y^{\phi}\sum_{i=0}^{\infty}\frac{(\mathbf{K}\theta y)^i}{i!}e^{-\theta y}dy\mathbf{s} = \boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{\mathbf{K}^i}{i!}\theta^{-\phi}\int_0^{\infty}\theta(\theta y)^{\phi+i}e^{-\theta y}dy\frac{\mathbf{s}}{\theta} \\
&= \boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{\mathbf{K}^i}{i!}\theta^{-\phi}\Gamma(\phi+1+i)\frac{\mathbf{s}}{\theta} = \theta^{-\phi}\boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{\mathbf{K}^i}{i!}\Gamma(\phi+1+i)\mathbf{k} \\
&= \Gamma(\phi+1)\theta^{-\phi}\boldsymbol{\alpha}\sum_{i=0}^{\infty}\frac{\prod_{\ell=1}^{i}(-\phi-\ell)}{i!}(-\mathbf{K})^i\mathbf{k} \\
&= \Gamma(\phi+1)\theta^{-\phi}\boldsymbol{\alpha}\sum_{i=0}^{\infty}\binom{-\phi-1}{i}(-\mathbf{K})^i\mathbf{k},
\end{aligned}$$

which converges for $\mathrm{SP}(\mathbf{K}) < 1$, where SP - the spectral value - is defined as the largest absolute value of the eigenvalues of $\mathbf{K}$. For the second part of the theorem we apply a well-known result. For $\mathrm{SP}(\mathbf{K}) < 1$ it is well-known that

$$(\mathbf{I}+\mathbf{K})^r = \sum_{i=0}^{\infty}\binom{r}{i}\mathbf{K}^i = \mathbf{I} + r\mathbf{K} + \frac{r(r-1)}{2!}\mathbf{K}^2\ldots$$

so

$$\mathbb{E}\left(Y^{\phi}\right) = \Gamma(\phi+1)\theta^{-\phi}(\mathbf{I}-\mathbf{K})^{-\phi-1}\mathbf{k}.$$

Now

$$(-\mathbf{S})^r = (\theta(\mathbf{I}-\mathbf{K}))^r = \theta^r(\mathbf{I}-\mathbf{K})^r$$

and we get

$$\mathbb{E}\left(Y^{\phi}\right) = \Gamma(\phi+1)\boldsymbol{\alpha}(-\mathbf{S})^{-\phi}\mathbf{e}.$$

□

In the next example we examine how our calibration and ruin probability evaluation procedures for certain infinite dimensional phase–type distributions combine in the case of Pareto distributed claims. We have chosen this

u	approximation	Ramsay	Deviation
0	0.2001	0.2000	0.1%
1	0.2561	0.2551	0.4%
5	0.3593	0.3523	2.0%
10	0.4226	0.4148	1.9%
30	0.5405	0.5349	1.0%
50	0.5997	0.5954	0.7%
100	0.6793	0.6765	0.4%
500	0.8321	0.8313	0.1%
1000	0.8779	0.8774	0.1%

TABLE 1. Pareto claims, $\rho = 0.80$ $\phi = 1.5$

example because exact calculations of the ruin probabilities are available in this case, due to [17, 18].

Example 4.3. Consider a generic claim size X having a Pareto distribution with tail index $\phi > 1$, whose survival function is given by

$$\mathbb{P}(X > x) = \left(1 + \frac{x}{(\phi - 1)\mu_X}\right)^{-\phi}$$

for $x > 0$ and where μ_X is the mean of X. Let, further, Y have the exponential distribution with an intensity such that

$$\mathbb{E}(Y^{\phi}) = 1.$$

We apply the calibration procedure descrubed in Section 4 and approximate the distribution of X as a suitable mixture of scaled versions of Y.

Having calibrated the approximate claim size distribution suitable for an application of Theorem 3.1, we use (3.1) to calculate the correponding ruin probability (see the discussion following (3.1)). In Tables 1 – 4 we compare the obtained approximate ruin probability with the exact (for Pareto claims) numbers of [17, 18]. The comparison is performed for a risk reserve process with unit premium rate in terms of the expected net claim amount per unit time, ρ, and the parameter ϕ of the Pareto distribution. The results show that the approximate results are reasonably close to the true value. Interstingly, the largest deviation is found for moderately large values of the initial reserve u.

u	approximation	Ramsay	Deviation
0	0.0501	0.0500	0.2 %
1	0.0674	0.0669	0.7 %
5	0.1031	0.1003	2.8 %
10	0.1287	0.1251	2.9 %
30	0.1872	0.1833	2.1 %
50	0.2238	0.2200	1.7 %
100	0.2845	0.2809	1.3 %
500	0.4713	0.4685	0.6 %

TABLE 2. Pareto claims, $\rho = 0.95$ $\phi = 1.5$

u	approximation	Ramsay	Deviation
0	0.2000	0.2000	0.0 %
1	0.3097	0.3090	0.2 %
5	0.5201	0.5050	3.0 %
10	0.6446	0.6273	2.8 %
30	0.8314	0.8217	1.2 %
50	0.8949	0.8895	0.6 %
100	0.9495	0.9448	0.5 %
500	0.9913	0.9913	0.0 %
1000	0.9958	0.9958	0.0 %

TABLE 3. Pareto claims, $\rho = 0.80$ $\phi = 2.0$

u	approximation	Ramsay	Deviation
0	0.0500	0.0500	0.0 %
1	0.0847	0.0845	0.2 %
5	0.1693	0.1628	4.0 %
10	0.2399	0.2294	4.6 %
30	0.4160	0.4010	3.7 %
50	0.5255	0.5104	3.0 %
100	0.6872	0.6747	1.9 %
500	0.9425	0.9409	0.2 %
1000	0.9758	0.9755	0.0 %

TABLE 4. Pareto claims, $\rho = 0.95$ $\phi = 2.0$

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