MAXIMA OF LONG MEMORY STATIONARY SYMMETRIC α-STABLE PROCESSES, AND SELF-SIMILAR PROCESSES WITH STATIONARY MAX-INCREMENTS

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Abstract. We derive a functional limit theorem for the partial maxima process based on a long memory stationary α-stable process. The length of memory in the stable process is parameterized by a certain ergodic theoretical parameter in an integral representation of the process. The limiting process is no longer a classical extremal Fréchet process. It is a self-similar process with α-Fréchet marginals, and it has the property of stationary max-increments, which we introduce in this paper. The functional limit theorem is established in the space $D[0,\infty)$ equipped with the Skorohod M_1-topology; in certain special cases the topology can be strengthened to the Skorohod J_1-topology.

1. Introduction

The asymptotic behavior of the partial maxima sequence $M_n = \max_{1\le k\le n} X_k$, $n = 1, 2, \ldots$ for an i.i.d. sequence $(X_1, X_2, \ldots)$ of random variables is the subject of the classical extreme value theory, dating back to Fisher and Tippett (1928). The basic result of this theory says that only three one-dimensional distributions, the Fréchet distribution, the Weibull distribution and the Gumbel distribution, have a max-domain of attraction. If Y has one of these three distributions, then for a distribution in its domain of attraction, and a sequence of i.i.d. random variables with that distribution,

$$\frac{M_n - b_n}{a_n} \Rightarrow Y \tag{1.1}$$

for properly chosen sequences (a_n), (b_n); see e.g. Chapter 1 in Resnick (1987) or Section 1.2 in de Haan and Ferreira (2006). Under the same max-domain of attraction assumption, a functional version of (1.1) was established in Lamperti (1964): with the same sequences (a_n), (b_n) as in (1.1),

$$\left(\frac{M_{\lfloor nt\rfloor} - b_n}{a_n},\ t \ge 0\right) \Rightarrow (Y(t),\, t \ge 0) \tag{1.2}$$

for a nondecreasing right continuous process $(Y(t),\, t \ge 0)$, and the convergence is weak convergence in the Skorohod J_1 topology on $D[0,\infty)$. The limiting process is often called *the extremal process*; its properties were established in Dwass (1964, 1966) and Resnick and Rubinovitch (1973).

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Much of the more recent research in extreme value theory concentrated on the case when the underlying sequence $(X_1, X_2, \ldots)$ is stationary, but may be dependent. In this case the extrema of the sequence may cluster, and it is natural to expect that the limiting results (1.1) and (1.2) will, in general, have to be different. The extremes of moving average processes have received special attention; see e.g. Rootzén (1978), Davis and Resnick (1985) and Fasen (2005). The extremes of the GARCH(1,1) process were investigated in Mikosch and Stărică (2000). The classical work on this subject of the extremes of dependent sequences is Leadbetter et al. (1983); in some cases this clustering of the extremes can be characterized through the *extremal index* (introduced, originally, in Leadbetter (1983)). The latter is a number $0 \leq \theta \leq 1$. Suppose that a stationary sequence $(X_1, X_2, \ldots)$ has this index, and let $(\tilde{X}_1, \tilde{X}_2, \ldots)$ be an i.i.d. sequence with the same one-dimensional marginal distributions as $(X_1, X_2, \ldots)$. If (1.1) and (1.2) hold for the i.i.d. sequence, then the corresponding limits will satisfy $\tilde{Y} \overset{d}{=} \tilde{Y}(1)$, but the limit in (1.1) for the dependent sequence $(X_1, X_2, \ldots)$ will satisfy $Y \overset{d}{=} \tilde{Y}(\theta)$. In particular, the limit will be equal to zero if the extremal index is equal to zero. This case can be viewed as that of long range dependence in the extremes, and it has been mostly neglected by the extreme value community. Long range dependence is, however, an important phenomenon in its own right, and in this paper we take a step towards understanding how long range dependence affects extremes.

A random variable X is said to have a regularly varying tail with index $-\alpha$ for $\alpha > 0$ if

$$P(X > x) = x^{-\alpha} L(x), \ x > 0 \,,$$

where L is a slowly varying at infinity function, and the distribution of any such random variable is in the max-domain of attraction of the Fréchet distribution with the same parameter α; see e.g. Resnick (1987). Recall that the Fréchet law $F_{\alpha,\sigma}$ on $(0, \infty)$ with the tail index α and scale $\sigma > 0$ satisfies

$$F_{\alpha,\sigma}(x) = \exp\bigl\{-\sigma^\alpha x^{-\alpha}\bigr\}, \ x > 0 \,. \tag{1.3}$$

Sometimes the term α-*Fréchet* is used. In this paper we discuss the case of regularly varying tails and the resulting limits in (1.2). The limits obtained in this paper belong to the family of the so-called *Fréchet processes*, defined below. We would like to emphasize that, even for stationary sequences with regularly varying tails, non-Fréchet limits may appear in (1.2). We are postponing a detailed discussion of this point to a future publication.

A stochastic process $(Y(t), \, t \in T)$ (on an arbitrary parameter space T) is called a Fréchet process if for all $n \geq 1$, $a_1, \ldots, a_n > 0$ and $t_1, \ldots, t_n \in T$, the weighted maximum $\max_{1 \leq j \leq n} a_j Y(t_j)$ follows a Fréchet law as in (1.3). The best known Fréchet process is the extremal Fréchet process obtained in the scheme (1.2) starting with an i.i.d. sequence with regularly varying tails. The extremal

Fréchet process $\big(Y(t),\, t \geq 0\big)$ has finite dimensional distributions defined by

$$\big(Y(t_1), Y(t_2), \ldots, Y(t_n)\big) \stackrel{d}{=} \Bigg(X^{(1)}_{\alpha, t_1^{1/\alpha}},\ \max\Big(X^{(1)}_{\alpha, t_1^{1/\alpha}},\, X^{(2)}_{\alpha, (t_2-t_1)^{1/\alpha}}\Big), \ldots \\ \max\Big(X^{(1)}_{\alpha, t_1^{1/\alpha}},\, X^{(2)}_{\alpha, (t_2-t_1)^{1/\alpha}}, \ldots, X^{(n)}_{\alpha, (t_n-t_{n-1})^{1/\alpha}}\Big)\Bigg) \tag{1.4}$$

for all n and $0 \leq t_1 < t_2 : \ldots < t_n$. The different random variables in the right hand side of (1.4) are independent, with $X^{(k)}_{\alpha,\sigma}$ having the Fréchet law $F_{\alpha,\sigma}$ in (1.3), for any $k = 1, \ldots, n$. This independence makes the extremal Fréchet processes the extremal Fréchet versions of the better known Lévy processes. The structure of general Fréchet processes has been extensively studied in the last several years. These processes were introduced in Stoev and Taqqu (2005), and their representations (as a part of a much more general context) were studied in Kabluchko and Stoev (2012). Stationary Fréchet processes (in particular, their ergodicity and mixing) were discussed in Stoev (2008), Kabluchko et al. (2009) and Wang and Stoev (2010).

In this paper we concentrate on the maxima of stationary α-stable processes with $0 < \alpha < 2$. Recall that a random vector $\mathbf{X}$ in $\mathbb{R}^d$ is called α-stable if for any A and $B > 0$ we have

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{d}{=} (A^\alpha + B^\alpha)^{1/\alpha}\mathbf{X} + \mathbf{y}\,,$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are i.i.d. copies of $\mathbf{X}$, and $\mathbf{y}$ is a deterministic vector (unless $\mathbf{X}$ is deterministic, necessarily, $0 < \alpha \leq 2$). A stochastic process $(X(t),\, t \in T)$ is called α-stable if all of its finite-dimensional distributions are α-stable. We refer the reader to Samorodnitsky and Taqqu (1994) for information on α-stable processes. When $\alpha = 2$, an α-stable process is Gaussian, while in the case $0 < \alpha < 2$, both the left and the right tails of a (nondegenerate) α-stable random variable X are (generally) regularly varying with exponent α. That is,

$$P(X > x) \sim c_+ x^{-\alpha}, \quad P(X < -x) \sim c_- x^{-\alpha} \ \text{ as } x \to \infty$$

for some $c_+,\, c_- \geq 0$, $c_+ + c_- > 0$. That is, if $(X_1, X_2, \ldots)$ is an i.i.d. sequence of α-stable random variables, then the i.i.d. sequence $(|X_1|, |X_2|, \ldots)$ satisfies (1.1) and (1.2) with $a_n = n^{1/\alpha}$ (and $b_n = 0$), $n \geq 1$. Of course, we are not planning to study the extrema of an i.i.d. α-stable sequence. Instead, we will study the maxima of (the absolute values of) a stationary α-stable process. The reason we have chosen to work with stationary α-stable processes is that their structure is very rich, and is also relatively well understood. This will allow us to study the effect of that structure on the limit theorems (1.1) and (1.2). We are specifically interested in the long range dependent case, corresponding to the zero value of the extremal index.

The structure of stationary symmetric α-stable (SαS) processes has been clarified in the last several years in the works of Jan Rosiński; see e.g. Rosiński (1995, 2006). The integral representation of such a process can be chosen to have a very special form. The class of stationary SαS processes we will investigate requires a representation slightly more restrictive than the one generally allowed. Specifically, we will consider processes of the form

$$X_n = \int_E f \circ T^n(x)\, dM(x), \quad n = 1, 2, \dots , \tag{1.5}$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite *infinite* control measure μ. The map $T : E \to E$ is a measurable map that preserves the measure μ. Further, $f \in L^\alpha(\mu)$. See Samorodnitsky and Taqqu (1994) for details on α-stable random measures and integrals with respect to these measures. It is elementary to check that a process with a representation (1.5) is, automatically, stationary. Recall that any stationary SαS process has a representation of the form:

$$X_n = \int_E f_n(x) dM(x), \quad n = 1, 2, \dots , \tag{1.6}$$

with

$$f_n(x) = a_n(x) \left(\frac{d\mu \circ T^n}{d\mu}(x) \right)^{1/\alpha} f \circ T^n(x), \quad x \in E \tag{1.7}$$

for $n = 1, 2, \dots$, where $T : E \to E$ is a one-to-one map with both T and T^{-1} measurable, mapping the control measure μ into an equivalent measure, and

$$a_n(x) = \prod_{j=0}^{n-1} u \circ T^j(x), \quad x \in E$$

for $n = 1, 2, \dots$, with $u : E \to \{-1, 1\}$ a measurable function. Here M is SαS (and $f \in L^\alpha(\mu)$). See Rosiński (1995). The main restriction of (1.5) is, therefore, to a measure preserving map T.

It has been observed that the ergodic-theoretical properties of the map T, either in (1.5) or in (1.7), have a major impact on the memory of a stationary α-stable process. See e.g. Surgailis et al. (1993), Samorodnitsky (2004, 2005), Roy (2008), Resnick and Samorodnitsky (2004), Owada and Samorodnitsky (2012), Owada (2013). The most relevant for this work is the result of Samorodnitsky (2004), who proved that, if the map T in (1.5) or in (1.7) is conservative, then using the normalization $a_n = n^{1/\alpha}$ ($b_n = 0$) in (1.1), as indicated by the marginal tails, produces the zero limit, so the partial maxima grow, in this case, strictly slower than at the rate of $n^{1/\alpha}$. On the other hand, if the map T is not conservative, then the normalization $a_n = n^{1/\alpha}$ in (1.1) is the correct one, and it leads to a Fréchet limit (we will survey the ergodic-theoretical notions in the next section). Therefore, the extrema of SαS processes corresponding to conservative flows cluster so much that the sequence of the partial maxima grows at a slower rate than that indicated by the marginal tails. This case can be thought of as indicating long range dependence. It is, clearly, inconsistent with a positive extremal index.

The Fréchet limit obtained in (1.1) by Samorodnitsky (2004) remains valid when the map T is conservative (but with the normalization of a smaller order than $n^{1/\alpha}$), as long as the map T satisfies a certain additional assumption. If one views the stationary α-stable process as a natural function of the Poisson points forming the random measure M in (1.6) then, informally, this assumption guarantees that only the largest Poisson point contributes, distributionally, to the asymptotic behavior of the partial maxima of the process. In this paper we restrict ourselves to this situation as well. However, we will look at the limits obtained in the much more informative functional scheme (1.2). In this paper the assumption on the map T will be expressed in terms of the rate of growth of the so-called return sequence, which we define in the sequel. We would like to emphasize that, when this return sequence grows at a rate slower than the one assumed in this paper, new phenomena seem to arise. Multiple Poisson points may contribute to the asymptotic distribution of the partial maxima, and non-Fréchet limit may appear in (1.2). We leave a detailed study of this to a subsequent work.

In the next section we provide the elements of the infinite ergodic theory needed for the rest of the paper. In Section 3 we introduce a new notion, that of a process with stationary max-increments. It turns out that the possible limits in the functional maxima scheme (1.2) (with $b_n = 0$) are self-similar with stationary max-increments. We discuss the general properties of such processes and then specialize to the concrete limiting process we obtain in the main result of the paper, stated and proved in Section 4.

2. Ergodic Theoretical Notions

In this section we present some basic notation and notions of, mostly infinite, ergodic theory used in the sequel. The main references are Krengel (1985), Aaronson (1997), and Zweimüller (2009).

Let $(E, \mathcal{E}, \mu)$ be a σ-finite, infinite measure space. We will say that $A = B \bmod \mu$ if $A, B \in \mathcal{E}$ and $\mu(A \triangle B) = 0$. For $f \in L^1(\mu)$ we will often write $\mu(f)$ for the integral $\int f \, d\mu$.

Let $T : E \to E$ be a measurable map preserving the measure μ. The sequence (T^n) of iterates of T is called a *flow*, and the ergodic-theoretical properties of the map and the flow are identified. A map T is called *ergodic* if any T-invariant set A (i.e., a set such that $T^{-1}A = A \bmod \mu$) satisfies $\mu(A) = 0$ or $\mu(A^c) = 0$. A map T is said to be *conservative* if

$$\sum_{n=1}^{\infty} \mathbf{1}_A \circ T^n = \infty \quad \text{a.e. on } A$$

for any $A \in \mathcal{E}$, $0 < \mu(A) < \infty$; if T is also ergodic, then the restriction "*on* A" is not needed.

The *conservative part* of a measure-preserving T is the largest T-invariant subset C of E such that the restriction of T to C is conservative. The set $D = E \setminus C$ is the *dissipative part* of T (and the decomposition $E = C \cup D$ is called *the Hopf decomposition* of T).

The *dual operator* $\widehat{T} : L^1(\mu) \to L^1(\mu)$ is defined by

$$\widehat{T} f = \frac{d\big(\nu_f \circ T^{-1}\big)}{d\mu}, \ f \in L^1(\mu) \,, \tag{2.1}$$

where ν_f is the signed measure $\nu_f(A) = \int_A f d\mu$, $A \in \mathcal{E}$. The dual operator satisfies the duality relation

$$\int_E \widehat{T} f \cdot g \, d\mu = \int_E f \cdot g \circ T \, d\mu \tag{2.2}$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$. Note that (2.1) makes sense for any nonnegative measurable function f on E, and the resulting $\widehat{T} f$ is again a nonnegative measurable function. Furthermore, (2.2) holds for arbitrary nonnegative measurable functions f and g.

A conservative, ergodic and measure preserving map T is said to be *pointwise dual ergodic*, if there exists a normalizing sequence $a_n \nearrow \infty$ such that

$$\frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k f \to \mu(f) \quad \text{a.e. for every } f \in L^1(\mu) \,. \tag{2.3}$$

The property of pointwise dual ergodicity rules out invertibility of the map T.

Sometimes we require that for some functions the above convergence takes place uniformly on a certain set. A set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$ is said to be a *uniform set* for a conservative, ergodic and measure preserving map T, if there exist a normalizing sequence $a_n \nearrow \infty$ and a nontrivial nonnegative measurable function $f \in L^1(\mu)$ (nontriviality means that f is different from zero on a set of positive measure) such that

$$\frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k f \to \mu(f) \quad \text{uniformly, a.e. on } A \,. \tag{2.4}$$

If (2.4) holds for $f = \mathbf{1}_A$, the set A is called *a Darling-Kac set*. A conservative, ergodic and measure preserving map T is pointwise dual ergodic if and only if T admits a uniform set; see Proposition 3.7.5 in Aaronson (1997). In particular, it is legitimate to use the same normalizing sequence (a_n) both in (2.3) and (2.4).

Let $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. The frequency of visits to the set A along the trajectory $(T^n x)$, $x \in E$, is naturally related to the *wandering rate* sequence

$$w_n = \mu\Big(\bigcup_{k=0}^{n-1} T^{-k} A \Big) \,. \tag{2.5}$$

If we define the first entrance time to A by

$$\varphi_A(x) = \min\{n \geq 1 : T^n x \in A\}$$

(notice that $\varphi_A < \infty$ a.e. on E since T is conservative and ergodic), then $w_n \sim \mu(\varphi_A < n)$ as $n \to \infty$. Since T is also measure preserving, we have $\mu(A \cap \{\varphi_A > k\}) = \mu(A^c \cap \{\varphi_A = k\})$ for $k \geq 1$. Therefore, alternative expressions for the wandering rate sequence are

$$w_n = \mu(A) + \sum_{k=1}^{n-1} \mu(A^c \cap \{\varphi_A = k\}) = \sum_{k=0}^{n-1} \mu(A \cap \{\varphi_A > k\}) .$$

Suppose now that T is a pointwise dual ergodic map, and let A be a uniform set for T. It turns out that, under an assumption of regular variation, there is a precise connection between the wandering rate sequence (w_n) and the normalizing sequence (a_n) in (2.3) and (2.4). Specifically, let RV_γ represent the class of regularly varying at infinity sequences (or functions, depending on the context) of index γ. If either $(w_n) \in RV_\beta$ or $(a_n) \in RV_{1-\beta}$ for some $\beta \in [0, 1]$, then

$$a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{w_n} \quad \text{as } n \to \infty . \tag{2.6}$$

Proposition 3.8.7 in Aaronson (1997) gives one direction of this statement, but the argument is easily reversed.

We finish this section with a statement on distributional convergence of the partial maxima for pointwise dual ergodic flows. It will be used repeatedly in the proof of the main theorem. For a measurable function f on E define

$$M_n(f)(x) = \max_{1 \leq k \leq n} |f \circ T^k(x)| , \quad x \in E, \ n \geq 1 .$$

In the sequel we will use the convention $\max_{k \in K} b_k = 0$ for a nonnegative sequence (b_n), if $K = \emptyset$.

Proposition 2.1. *Let T be a pointwise dual ergodic map on a σ-finite, infinite, measure space $(E, \mathcal{E}, \mu)$. We assume that the normalizing sequence (a_n) is regularly varying with exponent $1-\beta$ for some $0 < \beta \leq 1$. Let $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, be a uniform set for T. Define a probability measure on E by $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi_A \leq n\})/\mu(\{\varphi_A \leq n\})$. Let $f : E \to \mathbb{R}$ be a measurable bounded function supported by the set A. Let $\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e. on } A\}$. Then*

$$\left(M_{\lfloor nt \rfloor}(f), \ 0 \leq t \leq 1\right) \Rightarrow \|f\|_\infty \left(\mathbf{1}_{\{V_\beta \leq t\}}, \ 0 \leq t \leq 1\right) \quad \textit{in the } M_1 \textit{ topology on } D[0,1],$$

where the law of the left hand side is computed with respect to μ_n, and V_β is a random variable defined on a probability space $(\Omega', \mathcal{F}', P')$ with $P'(V_\beta \leq x) = x^\beta$, $0 < x \leq 1$.

Proof. For $0<\varepsilon<1$ let $A_\varepsilon=\{x\in A:\, |f(x)|\geq (1-\varepsilon)\|f\|_\infty\}$. Note that each A_ε is uniform since A is uniform. Clearly,

$$(1-\varepsilon)\|f\|_\infty \mathbf{1}_{\{\varphi_{A_\varepsilon}(x)\leq nt\}} \leq M_{\lfloor nt\rfloor}(f)(x) \leq \|f\|_\infty \mathbf{1}_{\{\varphi_A(x)\leq nt\}} \quad \mu\text{-a.e.}$$

for all $n\geq 1$ and $0\leq t\leq 1$. Since for monotone functions weak convergence in the M_1 topology is implied by convergence in finite dimensional distributions, we can use Theorem 3.2 in Billingsley (1999) in a finite-dimensional situation. The statement of the proposition will follow once we show that, for a uniform set B (which could be either A or A_ε) the law of φ_B/n under μ_n converges to the law of V_β. Let $(w_n^{(B)})$ be the corresponding wandering rate sequence. Since (2.6) holds for $(w_n^{(B)})$ with the same normalizing constants a_n,

$$\mu_n\left(\frac{\varphi_B}{n}\leq x\right)=\frac{\mu(\varphi_B\leq \lfloor nx\rfloor)}{\mu(\varphi_A\leq n)}\sim \frac{w^{(B)}_{\lfloor nx\rfloor}}{w_n}\to x^\beta$$

for all $0<x\leq 1$, because the wandering rate sequence is regularly varying with index β by (2.6). □

3. Self-similar processes with stationary max-increments

The limiting process obtained in the next section shares with any possible limits in the functional maxima scheme (1.2) (with $b_n=0$) two very specific properties, one of which is classical, and the other is less so. Recall that a stochastic process $\big(Y(t),\, t\geq 0\big)$ is called self-similar with exponent H of self-similarity if for any $c>0$

$$\big(Y(ct),\, t\geq 0\big)\overset{d}{=}\big(c^H Y(t),\, t\geq 0\big)$$

in the sense of equality of all finite-dimensional distributions. The best known classes of self-similar processes arise in various versions of a functional central limit theorem for stationary processes, and they have an additional property of stationary increments. Recall that a stochastic process $\big(Y(t),\, t\geq 0\big)$ is said to have stationary increments if for any $r\geq 0$

$$\big(Y(t+r)-Y(r),\, t\geq 0\big)\overset{d}{=}\big(Y(t)-Y(0),\, t\geq 0\big)\,; \tag{3.1}$$

see e.g. Embrechts and Maejima (2002) and Samorodnitsky (2006). In the context of the functional limit theorem for the maxima (1.2), a different property appears.

Definition 3.1. A stochastic process $(Y(t),\, t\geq 0)$ is said to have stationary max-increments if for every $r\geq 0$, there exists, perhaps on an enlarged probability space, a stochastic process $\big(Y^{(r)}(t),\, t\geq 0\big)$ such that

$$\begin{aligned}\big(Y^{(r)}(t),\, t\geq 0\big) &\overset{d}{=} (Y(t),\, t\geq 0)\,,\\ (Y(t+r),\, t\geq 0) &\overset{d}{=} \big(Y(r)\vee Y^{(r)}(t),\, t\geq 0\big)\,.\end{aligned} \tag{3.2}$$

Notice the analogy between the definition (3.1) of stationary increments (when $Y(0)=0$) and Definition 3.1. Since the operations of taking the maximum is not invertible (unlike summation), the latter definition, by necessity, is stated in terms of existence of the max-increment process $\big(Y^{(r)}(t),\, t\geq 0\big)$.

Theorem 3.2. *Let $(X_1, X_2, \ldots)$ be a stationary sequence. Assume that for some sequence $a_n \to \infty$, and a stochastic process $(Y(t),\, t\geq 0)$ such that, for every $t>0$, $Y(t)$ is not a constant,*

$$\left(\frac{1}{a_n} M_{\lfloor nt\rfloor},\ t\geq 0\right) \Rightarrow (Y(t),\, t\geq 0)$$

in terms of convergence of finite-dimensional distributions. Then $(Y(t),\, t\geq 0)$ is self-similar with exponent $H>0$ of self-similarity, and has stationary max-increments. Furthermore, the sequence (a_n) is regularly varying with index H.

Proof. The facts that the limiting process $(Y(t),\, t\geq 0)$ is self-similar with exponent $H>0$ of self-similarity, and that the sequence (a_n) is regularly varying with index H, follow from the Lamperti theorem; see Lamperti (1962), or Theorem 2.1.1 in Embrechts and Maejima (2002). Note that the distinction between convergence along a discrete sequence $n\to\infty$, or along a continuous sequence $\lambda\to\infty$, disappears for maxima of stationary processes.

We check now the stationarity of the max-increments of the limiting process. Let $r>0$, and $t_i>0$, $i=1,\ldots,k$, some $k\geq 1$. Write

$$\frac{1}{a_n} M_{\lfloor n(t_i+r)\rfloor} = \frac{1}{a_n} M_{\lfloor nr\rfloor} \bigvee \frac{1}{a_n} \max_{nr<j\leq n(t_i+r)} X_j,\ i=1,\ldots,k\,. \tag{3.3}$$

By the assumption of the theorem and stationarity of the process $(X_1, X_2, \ldots)$,

$$\frac{1}{a_n} M_{\lfloor nr\rfloor} \Rightarrow Y(r),\quad \left(\frac{1}{a_n} \max_{nr<j\leq n(t_i+r)} X_j,\ i=1,\ldots,k\right) \Rightarrow \big(Y(t_1),\ldots,Y(t_k)\big)$$

as $n\to\infty$. Using the standard tightness argument, we conclude that for every sequence $n_m\to\infty$ there is a subsequence $n_{m(l)}\to\infty$ such that

$$\left(\frac{1}{a_{n_{m(l)}}} M_{\lfloor n_{m(l)} r\rfloor},\ \left(\frac{1}{a_{n_{m(l)}}} \max_{n_{m(l)} r<j\leq n_{m(l)}(t_i+r)} X_j,\ i=1,\ldots,k\right)\right) \Rightarrow \Big(Y(r),\ \big(Y^{(r)}(t_1),\ldots,Y^{(r)}(t_k)\big)\Big)$$

as $l\to\infty$, where $\big(Y^{(r)}(t_1),\ldots,Y^{(r)}(t_k)\big) \stackrel{d}{=} \big(Y(t_1),\ldots,Y(t_k)\big)$.

Let now τ_i, $i=1,2,\ldots$ be an enumeration of the rational numbers in $[0,\infty)$. A diagonalization argument shows that there is a sequence $n_m\to\infty$ and a stochastic process $\big(Y^{(r)}(\tau_i),\, i=1,2,\ldots\big)$ with $\big(Y^{(r)}(\tau_i),\, i=1,2,\ldots\big) \stackrel{d}{=} \big(Y(\tau_i),\, i=1,2,\ldots\big)$ such that

$$\left(\frac{1}{a_{n_m}} M_{\lfloor n_m r\rfloor},\ \left(\frac{1}{a_{n_m}} \max_{n_m r<j\leq n_m(\tau_i+r)} X_j,\ i=1,2,\ldots\right)\right) \Rightarrow \Big(Y(r),\ \big(Y^{(r)}(\tau_i),\ i=1,2,\ldots\big)\Big) \tag{3.4}$$

in finite-dimensional distributions, as $m \to \infty$. We extend the process $Y^{(r)}$ to the entire positive half-line by setting

$$Y^{(r)}(t) = \frac{1}{2}\Bigl(\lim_{\tau \uparrow t, \text{ rational}} Y^{(r)}(\tau) + \lim_{\tau \downarrow t, \text{ rational}} Y^{(r)}(\tau)\Bigr), \quad t \geq 0\,.$$

Then (3.4) and monotonicity imply that as $m \to \infty$,

$$\Bigl(\frac{1}{a_{n_m}} M_{\lfloor n_m r \rfloor}, \; \Bigl(\frac{1}{a_{n_m}} \max_{n_m r < j \leq n_m (t+r)} X_j, \; t \geq 0\Bigr)\Bigr) \Rightarrow \Bigl(Y(r), \; \bigl(Y^{(r)}(t), \; t \geq 0\bigr)\Bigr) \tag{3.5}$$

in finite-dimensional distributions. Now the stationarity of max-increments follows from (3.3), (3.5) and continuous mapping theorem. □

Remark 3.3. Self-similar processes with stationary max-increments arising in a functional maxima scheme (1.2) are close in spirit to the stationary self-similar extremal processes of O'Brien et al. (1990), while extremal processes themselves are defined as random sup measures. A random sup measure is, as its name implies, indexed by sets. They also arise in a limiting maxima scheme similar to (1.2), but with a stronger notion of convergence. Every stationary self-similar extremal processes trivially produces a self-similar process with stationary max-increments via restriction to sets of the type $[0, t]$ for $t \geq 0$, but the connection between the two objects remains unclear. Our limiting process in Theorem 4.1 below can be extended to a stationary self-similar extremal processes, but the extension is highly nontrivial, and will not be pursued here.

It is not our goal in this paper to study in details the properties of self-similar processes with stationary max-increments, so we restrict ourselves to the following basic result,

Proposition 3.4. *Let $\bigl(Y(t), \, t \geq 0\bigr)$ be a nonnegative self-similar process with stationary max-increments, and exponent H of self-similarity. Suppose $\bigl(Y(t), \, t \geq 0\bigr)$ is not identically zero. Then $H \geq 0$, and the following statements hold.*

(a) *If $H = 0$, then $Y(t) = Y(1)$ a.s. for every $t > 0$.*
(b) *If $0 < EY(1)^p < \infty$ for some $p > 0$, then $H \leq 1/p$.*
(c) *If $H > 0$, $\bigl(Y(t), \, t \geq 0\bigr)$ is continuous in probability.*

Proof. By the stationarity of max-increments, $Y(t)$ is stochastically increasing with t. This implies that $H \geq 0$.

If $H = 0$, then $Y(n) \stackrel{d}{=} Y(1)$ for each $n = 1, 2 \ldots$. We use (3.2) with $r = 1$. Using $t = 1$ we see that, in the right hand side of (3.2), $Y(1) = Y^{(1)}(1)$ a.s. Since $Y^{(1)}(n) \geq Y^{(1)}(1)$ a.s., we conclude, using $t = n$ in the right hand side of (3.2), that $Y(1) = Y^{(1)}(n)$ a.s for each $n = 1, 2, \ldots$. By monotonicity, we conclude that the process $\bigl(Y^{(1)}(t), \, t \geq 0\bigr)$, hence also the process $\bigl(Y(t), \, t \geq 0\bigr)$, is a.s. constant on $[1, \infty)$ and then, by self-similarity, also on $(0, \infty)$.

Next, let $p>0$ be such that $0<EY(1)^p<\infty$. It follows from (3.2) with $r=1$ that

$$2^H Y(1) \stackrel{d}{=} Y(2) \stackrel{d}{=} \max\bigl(Y(1),\, Y^{(1)}(1)\bigr)\,.$$

Therefore,

$$2^{pH} EY(1)^p = EY(2)^p = E\bigl[Y(1)^p \vee Y^{(1)}(1)^p\bigr] \le 2EY(1)^p$$

This means that $pH \le 1$.

Finally, we take arbitrary $0<s<t$. We use (3.2) with $r=s$. For every $\eta>0$,

$$P\bigl(Y(t)-Y(s)>\eta\bigr) = P\bigl(Y(s)\vee Y^{(s)}(t-s)-Y(s)>\eta\bigr)$$
$$\le P\bigl(Y^{(s)}(t-s)>\eta\bigr) = P\bigl((t-s)^H Y(1)>\eta\bigr).$$

Hence continuity in probability. $\square$

We now define the limiting process obtained in the main limit theorem of Section 4, and place it in the general framework introduced earlier in this section. Let $\alpha>0$, and consider the extremal Fréchet process $Z_\alpha(t)$, $t\ge 0$, defined in (1.4), with the scale $\sigma=1$. For $0<\beta<1$ we define a new stochastic process by

$$Z_{\alpha,\beta}(t) = Z_\alpha(t^\beta),\ t\ge 0\,.$$

Since the extremal Fréchet process is self-similar with $H=1/\alpha$, it is immediately seen that the process $Z_{\alpha,\beta}$ is self-similar with $H=\beta/\alpha$.

We claim that the process $Z_{\alpha,\beta}$ has stationary max-increments as well. To show this, we start with a useful representation of the extremal Fréchet process $Z_\alpha(t)$, $t\ge 0$ in terms of the points of a Poisson random measure. Let $\bigl((j_k,s_k)\bigr)$ be the points of a Poisson random measure on $\mathbb{R}_+^2$ with mean measure $\rho_\alpha\times\lambda$, where $\rho_\alpha(x,\infty)=x^{-\alpha}$, $x>0$ and λ is the Lebesgue measure on $\mathbb{R}_+$. Then an elementary calculation shows that

$$\bigl(Z_\alpha(t),\, t\ge 0\bigr) \stackrel{d}{=} \Bigl(\sup\{j_k:\, s_k\le t\},\, t\ge 0\Bigr)\,.$$

Therefore, $\bigl(Z_{\alpha,\beta}(t),\, t\ge 0\bigr) \stackrel{d}{=} \bigl(U_{\alpha,\beta}(t),\, t\ge 0\bigr)$, where

$$U_{\alpha,\beta}(t) = \sup\{\, j_k : s_k \le t^\beta\}\,,\quad t\ge 0\,. \tag{3.6}$$

Given $r>0$, we define

$$U_{\alpha,\beta}^{(r)}(t) = \sup\{\, j_k : (t+r)^\beta - t^\beta \le s_k \le (t+r)^\beta\}\,.$$

Since

$$\bigl((t_1+r)^\beta - t_1^\beta,\, (t_1+r)^\beta\bigr) \subset \bigl((t_2+r)^\beta - t_2^\beta,\, (t_2+r)^\beta\bigr)$$

for $0\le t_1<t_2$, it follows that

$$\bigl(U_{\alpha,\beta}^{(r)}(t),\, t\ge 0\bigr) \stackrel{d}{=} \bigl(U_{\alpha,\beta}(t),\, t\ge 0\bigr)\,.$$

Furthermore, since $(t+r)^\beta - t^\beta \le r^\beta$, we see that

$$U_{\alpha,\beta}(t+r) = U_{\alpha,\beta}(r) \vee U^{(r)}_{\alpha,\beta}(t) \ \text{ for all } t \ge 0\,.$$

This means that the process $U_{\alpha,\beta}$ has stationary max-increments and, hence, so does the process $Z_{\alpha,\beta}$.

It is interesting to note that, by part (b) of Proposition 3.4, any H-self-similar process with stationary max-increments and α-Fréchet marginals, must satisfy $H \le 1/\alpha$. The exponent $H = \beta/\alpha$ with $0 < \beta \le 1$ of the process $Z_{\alpha,\beta}$ (with $\beta = 1$ corresponding to the extremal Fréchet process Z_α) covers the entire interval $(0, 1/\alpha]$. Therefore, the upper bound of part (b) of Proposition 3.4 is, in general, the best possible.

As a general phenomenon, the stationary increments of a self-similar stochastic process arising in a functional central limit theorem (for partial sums) exhibit dependence reflecting, at least partially the dependence in the original stationary process. This can be seen, for example, in Owada and Samorodnitsky (2012) that starts with a stationary process similar in its nature to the one considered in this paper. Do the stationary max-increments of a process arising in the functional maxima scheme (1.2) reflect similarly the dependence structure in the original process?

The fact the operation of taking partial maxima tends to reduce the memory in the original process has been noticed before; for example, for stationary Gaussian processes partial maxima converge, with the same normalization, to the extremal Gumbel process, as long as the correlations decay more rapidly than $(\log n)^{-1}$; see Berman (1964). Within that range of correlation decay, on the other hand, one would observe that the partial sums stop converging to the Brownian motion, and start converging to the Fractional Brownian motion, and the normalization will change as well. A related example with power tails is given by the infinite moving average models of the form $X_n = \sum_{j=0}^\infty c_j Z_{n-j}$, where (Z_n) are i.i.d. random variables, belonging to the max-domain of attraction of an α-Fréchet law, and (c_j) is a deterministic sequence satisfying certain conditions for the process to be well-defined. In this case it was shown by Davis and Resnick (1985) that the normalized partial maxima of (X_n) converge in the functional sense to an extremal α-Fréchet process; this paper assumes absolute summability of (c_j), but it is only needed for the tail estimate (2.7), which is valid without the absolute summability, see Mikosch and Samorodnitsky (2000). In contrast, partial sums of the infinite moving average processes can converge, if $0 < \alpha < 2$, in certain cases (some of which prevent lack of summability of the coefficients), to the linear fractional α-stable motions, and not to the α-stable Lévy motions; see e.g. Section 4.7 in Whitt (2002).

With the discussion in view, we investigate the dependence properties of the stationary max-increments of the process $Z_{\alpha,\beta}$ defined above. This process will appear, in the case $0 < \alpha < 2$ and $1/2 < \beta < 1$, in the main limit theorem, Theorem 4.1 below. Equivalently, we will look at the max-increments of the process $U_{\alpha,\beta}$ in (3.6). Let

$$V_n^{(\alpha,\beta)} = \sup\{\, j_k : n^\beta - 1 < s_k \le n^\beta \,\}\,, \quad n = 1, 2, \ldots\,. \tag{3.7}$$

An important observation is that this sequence is independent only when $\beta = 1$, that is, when $Z_{\alpha,\beta}$ and $U_{\alpha,\beta}$ coincide with the extremal α-Fréchet process. In general, we have the following statement.

Proposition 3.5. *The stationary process* $(V_n^{(\alpha,\beta)},\, n \ge 1)$ *is mixing and is generated by a dissipative flow.*

Proof. Notice that

$$(V_n^{(\alpha,\beta)},\, n \ge 1) \stackrel{d}{=} \big(M_\alpha((n^\beta - 1, n^\beta]),\, n \ge 1\big)\,,$$

where M_α is an independently scattered α-Fréchet sup-measure on $\mathbb{R}_+$ with the Lebesgue control measure. See Stoev and Taqqu (2005). It follows by Theorem 3.4 in Stoev (2008) that $(V_n^{(\alpha,\beta)},\, n \ge 1)$ is mixing. Further, $\sum_{n=1}^\infty \mathbf{1}_{(n^\beta-1,n^\beta]}(x) < \infty$ a.e., and Theorem 5.2 of Wang and Stoev (2010) shows that $V_n^{(\alpha,\beta)}$ is generated by a dissipative flow. □

The main result in Theorem 4.1 below deals with the partial maxima of stationary SαS processes for which the underlying flows are conservative; these are associated with long memory in the process. The resulting limiting process is not an extremal process, hence its max-increments are dependent. The memory in the original SαS is long enough to guarantee that. On the other hand, the max-increments of the latter are generated by a dissipative flow, which is associated with shorter memory. We observe, therefore, another example of "memory reduction" by the operation of taking partial maxima.

We finish this section by mentioning that an immediate conclusion from (3.6) is the following representation of the process $Z_{\alpha,\beta}$ on the interval $[0,1]$:

$$(Z_{\alpha,\beta}(t),\, 0 \le t \le 1) \stackrel{d}{=} \left(\bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{V_j \le t\}},\, 0 \le t \le 1 \right)\,, \tag{3.8}$$

where Γ_j, $j = 1, 2, \ldots$, are arrival times of a unit rate Poisson process on $(0, \infty)$, and (V_j) are i.i.d. random variables with $P(V_1 \le x) = x^\beta$, $0 < x \le 1$, independent of (Γ_j).

4. A Functional Limit Theorem for Partial Maxima

In this section we state and prove our main result, a functional limit theorem for the partial maxima of the process $\mathbf{X} = (X_1, X_2, \ldots)$ given in (1.5). Recall that T is a conservative, ergodic and measure preserving map on a σ-finite, infinite, measure space $(E, \mathcal{E}, \mu)$. We will assume that T is a pointwise dual ergodic map with normalizing sequence (a_n) that is regularly varying with exponent $1 - \beta$; equivalently, the wandering sequence (w_n) in (2.5) is assumed to be regularly varying with exponent β. Crucially, we will assume that $1/2 < \beta < 1$. See Remark 4.2 after the statement of Theorem 4.1 below.

Define

$$b_n = \left(\int_E \max_{1 \le k \le n} \left| f \circ T^n(x) \right|^\alpha \mu(dx) \right)^{1/\alpha}, \quad n = 1, 2, \dots . \tag{4.1}$$

The sequence (b_n) is known to play an important role in the rate of growth of partial maxima of an α-stable process of the type (1.5). It also turns out to be a proper normalizing sequence for our functional limit theorem. In Samorodnitsky (2004) it was shown that, for a canonical kernel (1.7), if the map T is conservative, then the sequence (b_n) grows at a rate strictly slower than $n^{1/\alpha}$. The extra assumptions imposed in this paper will guarantee a more precise statement: $(b_n) \in RV_{\beta/\alpha}$. Specifically,

$$\lim_{n\to\infty} \frac{b_n^\alpha}{w_n} = \|f\|_\infty \tag{4.2}$$

(where (w_n) is the wandering sequence). This fact has an interesting message, because it explicitly shows that the rate of growth of the partial maxima is determined both by the heaviness of the marginal tails (through α) and by the length of memory (through β).

In contrast, if the map T has a non-trivial dissipative component, then the sequence (b_n) grows at the rate $n^{1/\alpha}$, and so do the partial maxima of a stationary SαS process; see Samorodnitsky (2004). This is the limiting case of the setup in the present paper, as β gets closer to 1. Intuitively, the smaller is β, the longer is the memory in the process.

The functional limit theorem in Theorem 4.1 below involves weak convergence in the space $D[0, \infty)$ equipped with two different topologies, the Skorohod J_1-topology and the Skorohod M_1-topology, introduced in Skorohod (1956). The details could be found, for instance, in Billingsley (1999) (for the J_1-topology), and in Whitt (2002) (for the M_1-topology). See Remark 4.3.

We recall the tail constant of an α-stable random variable given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \begin{cases} (1-\alpha)/\big(\Gamma(2-\alpha)\cos(\pi\alpha/2)\big) & \text{if } \alpha \ne 1, \\ 2/\pi & \text{if } \alpha = 1; \end{cases}$$

see Samorodnitsky and Taqqu (1994).

Theorem 4.1. *Let T be a conservative, ergodic and measure preserving map on a σ-finite infinite measure space $(E, \mathcal{E}, \mu)$. Assume that T is a pointwise dual ergodic map with normalizing sequence $(a_n) \in RV_{1-\beta}$, $0 \le \beta \le 1$. Let $f \in L^\alpha(\mu) \cap L^\infty(\mu)$, and assume that f is supported by a uniform set A for T. Let $\alpha > 0$. Then the sequence (b_n) in (4.1) satisfies (4.2).*

Assume now that $0 < \alpha < 2$ and $1/2 < \beta < 1$. If M is a $S\alpha S$ random measure on $(E, \mathcal{E})$ with control measure μ, then the stationary $S\alpha S$ process $\mathbf{X}$ given in (1.5) satisfies

$$\Big(\frac{1}{b_n} \max_{1\le k\le \lfloor nt\rfloor} |X_k|,\, t \ge 0\Big) \Rightarrow \big(C_\alpha^{1/\alpha} Z_{\alpha,\beta}(t),\, t\ge 0\big) \quad \textit{in } D[0,\infty) \tag{4.3}$$

in the Skorohod M_1-topology. Moreover, if $f = \mathbf{1}_A$, then the above convergence occurs in the Skorohod J_1-topology as well.

Remark 4.2. Convergence to a Fréchet limit in (4.3) no longer holds in the range $0 < \beta < 1/2$. We will discuss elsewhere what non-Fréchet limiting processes may appear in that case. In the boundary case $\beta = 1/2$, however, the statement (4.3) may still hold. This will be the case when the "one jump" property (4.6) below is satisfied. This is the case, for example, for the Markov shift operator T presented at the end of the paper. See also Samorodnitsky (2004).

Remark 4.3. It is not difficult to see why the weak convergence in (4.3) holds in the J_1-topology for indicator functions, but only in the M_1-topology in general. Indeed, for a general function f the values of the process $M_{\lfloor n\cdot\rfloor}(f)$ in Proposition 2.1 may have multiple jumps on the time scale $o(n)$ before reaching the limiting value $\|f\|_\infty$. Since the limiting process has a single jump, one does not expect the J_1-convergence. On the other hand, if $f = \mathbf{1}_A$, then the value $\|f\|_\infty = 1$ is reached in a single jump, matching the single jump in the limiting process.

Proof of Theorem 4.1. We start with verifying (4.2). Obviously,

$$b_n^\alpha \le \|f\|_\infty \mu(\varphi_A \le n)\,,$$

and the definition of the wandering rate sequence gives us the upper bound

$$\limsup_{n\to\infty} \frac{b_n^\alpha}{w_n} \le \|f\|_\infty\,.$$

On the other hand, take an arbitrary $\epsilon \in \big(0, \|f\|_\infty\big)$. The set

$$B_\epsilon = \big\{x \in A : |f(x)| \ge \|f\|_\infty - \epsilon\big\}\,.$$

is a uniform set for T. Let $\big(w_n^{(\varepsilon)}\big)$ be the corresponding wandering rate sequence. Then a lower bound for b_n^α is obtained by

$$b_n^\alpha \ge \big(\|f\|_\infty - \epsilon\big)\, \mu\Big(\bigcup_{j=1}^n T^{-j} B_\epsilon\Big)\,,$$

so

$$\liminf_{n\to\infty} \frac{b_n^\alpha}{w_n} = \liminf_{n\to\infty} \frac{b_n^\alpha}{w_n^{(\varepsilon)}} \ge \|f\|_\infty - \epsilon\,.$$

Letting $\varepsilon \to 0$, we obtain (4.2).

Suppose now that $0 < \alpha < 2$ and $1/2 < \beta < 1$. We continue with proving convergence in the finite dimensional distributions in (4.3). Since for random elements in $D[0,\infty)$ with nondecreasing sample paths, weak convergence in the M_1-topology is implied by the finite-dimensional weak convergence, this will also establish (4.3) in the sense of weak convergence in the M_1-topology.

Fix $0 = t_0 < t_1 < \cdots < t_d$, $d \geq 1$. We may and will assume that $t_d \leq 1$. We use a series representation of the random vector $(X_1, \ldots, X_n)$: with $f_k = f \circ T^k$, $k = 1, 2, \ldots$,

$$(X_k,\, k = 1, \ldots, n) \overset{d}{=} \left(b_n\, C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1 \leq i \leq n} |f_i(U_j^{(n)})|},\, k = 1, \ldots, n \right). \tag{4.4}$$

Here (ϵ_j) are i.i.d. Rademacher random variables (symmetric random variables with values ± 1), (Γ_j) are the arrival times of a unit rate Poisson process on $(0, \infty)$, and $(U_j^{(n)})$ are i.i.d. E-valued random variables with the common law η_n defined by

$$\frac{d\eta_n}{d\mu}(x) = \frac{1}{b_n^\alpha} \max_{1 \leq k \leq n} |f_k(x)|^\alpha, \quad x \in E. \tag{4.5}$$

The sequences (ϵ_j), (Γ_j), and $(U_j^{(n)})$ are taken to be independent. We refer to Section 3.10 of Samorodnitsky and Taqqu (1994) for series representations of α-stable random vectors. The representation (4.4) was also used in Samorodnitsky (2004), and the argument below is structured similarly to the corresponding argument *ibid.*

The crucial consequence of the assumption $1/2 < \beta < 1$ is that, in the series representation (4.4), only the largest Poisson jump will play an important role. It is shown in Samorodnitsky (2004) that, under the assumptions of Theorem 4.1, for every $\eta > 0$,

$$\varphi_n(\eta) \equiv P\left(\bigcup_{k=1}^{n} \left\{ \Gamma_j^{-1/\alpha} \frac{|f_k(U_j^{(n)})|}{\max_{1 \leq i \leq n} |f_i(U_j^{(n)})|} > \eta \ \text{ for at least 2 different } j = 1, 2, \ldots \right\} \right) \to 0 \tag{4.6}$$

as $n \to \infty$.

We will proceed in two steps. First, we will prove that

$$\left(\bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1 \leq k \leq \lfloor nt_i \rfloor} |f_k(U_j^{(n)})|}{\max_{1 \leq k \leq n} |f_k(U_j^{(n)})|},\, i = 1, \ldots, d \right) \Rightarrow (Z_{\alpha,\beta}(t_i),\, i = 1, \ldots, d) \quad \text{in } \mathbb{R}_+^d. \tag{4.7}$$

Next, we will prove that, for fixed $\lambda_1, \ldots, \lambda_d > 0$, for every $0 < \delta < 1$,

$$P\big(b_n^{-1} \max_{1 \leq k \leq \lfloor nt_i \rfloor} |X_k| > \lambda_i, \quad i = 1, \ldots, d\big)$$

$$\leq P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1 \leq k \leq \lfloor nt_i \rfloor} |f_k(U_j^{(n)})|}{\max_{1 \leq k \leq n} |f_k(U_j^{(n)})|} > \lambda_i(1 - \delta), \quad i = 1, \ldots, d \right) + o(1) \tag{4.8}$$

and that

$$P\big(b_n^{-1} \max_{1 \leq k \leq \lfloor nt_i \rfloor} |X_k| > \lambda_i, \quad i = 1, \ldots, d\big)$$

$$\geq P\left(C_\alpha^{1/\alpha}\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\frac{\max_{1\leq k\leq \lfloor nt_i\rfloor}|f_k(U_j^{(n)})|}{\max_{1\leq k\leq n}|f_k(U_j^{(n)})|}>\lambda_i(1+\delta),\quad i=1,\ldots,d\right)+o(1)\,. \tag{4.9}$$

Since the Fréchet distribution is continuous, the weak convergence

$$\left(b_n^{-1}\max_{1\leq k\leq \lfloor nt_i\rfloor}|X_k|,\quad i=1,\ldots,d\right)\Rightarrow \left(Z_{\alpha,\beta}(t_i),\, i=1,\ldots,d\right)\quad \text{in } \mathbb{R}_+^d$$

will follow by taking δ arbitrarily small.

We start with proving (4.7). For $n=1,2,\ldots$, $N_n=\sum_{j=1}^\infty \delta_{(\Gamma_j,U_j^{(n)})}$ is a Poisson random measure on $(0,\infty)\times\cup_{k=1}^n T^{-k}A$ with mean measure $\lambda\times\eta_n$. Define a map $S_n:\mathbb{R}_+\times\cup_{k=1}^n T^{-k}A\to\mathbb{R}_+^d$ by

$$S_n(r,x)=r^{-1/\alpha}\big(M_n(f)(x)\big)^{-1}\big(M_{\lfloor nt_1\rfloor}(f)(x),\ldots,M_{\lfloor nt_d\rfloor}(f)(x)\big)\,,\quad r>0,\ x\in\cup_{k=1}^n T^{-k}A\,.$$

Then, for $\lambda_1,\ldots,\lambda_d>0$,

$$\begin{aligned}
&P\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\frac{\max_{1\leq k\leq \lfloor nt_i\rfloor}|f_k(U_j^{(n)})|}{\max_{1\leq k\leq n}|f_k(U_j^{(n)})|}\leq\lambda_i\,,\quad i=1,\ldots,d\right)\\
&=P\left[N_n\left(S_n^{-1}\Big((0,\lambda_1]\times\cdots\times(0,\lambda_d]\Big)^c\right)=0\right]\\
&=\exp\left\{-(\lambda\times\eta_n)\left(S_n^{-1}\Big((0,\lambda_1]\times\cdots\times(0,\lambda_d]\Big)^c\right)\right\}\\
&=\exp\left\{-(\lambda\times\eta_n)\left\{(r,x):\bigvee_{j=1}^d\lambda_j^{-\alpha}\frac{\big(M_{\lfloor nt_j\rfloor}(f)(x)\big)^\alpha}{\big(M_n(f)(x)\big)^\alpha}>r\right\}\right\}\\
&=\exp\left\{-b_n^{-\alpha}\int_E\bigvee_{j=1}^d\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^\alpha d\mu\right\}\,.
\end{aligned}$$

We use (4.2) and the weak convergence in Proposition 2.1 to obtain

$$\begin{aligned}
&b_n^{-\alpha}\int_E\bigvee_{j=1}^d\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^\alpha d\mu\sim\|f\|_\infty^{-1}\int_E\bigvee_{j=1}^d\lambda_j^{-\alpha}M_{\lfloor nt_j\rfloor}(f)^\alpha d\mu_n\\
&\to\int_{\Omega'}\bigvee_{j=1}^d\lambda_j^{-\alpha}\mathbf{1}_{\{V_\beta\leq t_j\}}dP'=\sum_{i=1}^d(t_i^\beta-t_{i-1}^\beta)\left(\bigwedge_{j=i}^d\lambda_j\right)^{-\alpha}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\frac{\max_{1\leq k\leq \lfloor nt_i\rfloor}|f_k(U_j^{(n)})|}{\max_{1\leq k\leq n}|f_k(U_j^{(n)})|}\leq\lambda_i\,,\quad i=1,\ldots,d\right)\\
&\to\exp\left\{-\sum_{i=1}^d(t_i^\beta-t_{i-1}^\beta)\left(\bigwedge_{j=i}^d\lambda_j\right)^{-\alpha}\right\}=P\big(Z_{\alpha,\beta}(t_i)\leq\lambda_i,\quad i=1,\ldots,d\big)\,.
\end{aligned}$$

The claim (4.7) has, consequently, been proved.

We continue with the statements (4.8) and (4.9). Since the arguments are very similar, we only prove (4.8). Let $K \in \mathbb{N}$ and $0 < \epsilon < 1$ be constants so that

$$K + 1 > \frac{4}{\alpha} \quad \text{and} \quad \delta - \epsilon K > 0\,.$$

Then

$$P\big(b_n^{-1} \max_{1\le k\le \lfloor nt_i\rfloor} |X_k| > \lambda_i, \quad i = 1, \dots, d\big)$$

$$\le P\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt_i\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} > \lambda_i(1-\delta), \quad i = 1, \dots, d \right) + \varphi_n\big(C_\alpha^{-1/\alpha} \epsilon \min_{1\le i\le d} \lambda_i\big)$$

$$+ \sum_{i=1}^{d} P\Bigg(C_\alpha^{1/\alpha} \max_{1\le k\le \lfloor nt_i\rfloor} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^n)}{\max_{1\le l\le n} |f_l(U_j^{(n)})|} \right| > \lambda_i\,,$$

$$C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt_i\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} \le \lambda_i(1-\delta)\,, \quad \text{and for each } m = 1, \dots, n,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_m(U_j^{(n)})|}{\max_{1\le l\le n} |f_l(U_j^{(n)})|} > \epsilon \min_{1\le l\le d} \lambda_l \ \text{ for at most one } j = 1, 2, \dots \Bigg)\,.$$

By (4.6), it is enough to show that for all $\lambda > 0$ and $0 \le t \le 1$,

$$P\Bigg(C_\alpha^{1/\alpha} \max_{1\le k\le \lfloor nt\rfloor} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \right| > \lambda\,, \tag{4.10}$$

$$C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{\max_{1\le k\le \lfloor nt\rfloor} |f_k(U_j^{(n)})|}{\max_{1\le k\le n} |f_k(U_j^{(n)})|} \le \lambda(1-\delta)\,, \quad \text{and for each } m = 1, \dots, n,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_m(U_j^{(n)})|}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} > \epsilon\lambda \ \text{ for at most one } j = 1, 2, \dots \Bigg) \to 0\,.$$

For every $k = 1, 2, \dots, n$, the Poisson random measure represented by the points

$$\big(\epsilon_j \Gamma_j^{-1/\alpha} f_k(U_j^{(n)}) \big(\max_{1\le i\le n} |f_i(U_j^{(n)})|\big)^{-1},\ j = 1, 2, \dots\big)$$

has the same mean measure as that represented by the points

$$\big(\epsilon_j \Gamma_j^{-1/\alpha} \|f\|_\alpha b_n^{-1},\ j = 1, 2, \dots\big)\,.$$

Hence, these two Poisson random measures coincide distributionally. We conclude that the probability in (4.10) is bounded by

$$\sum_{k=1}^{\lfloor nt\rfloor} P\Bigg(C_\alpha^{1/\alpha} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \right| > \lambda\,,\ C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \frac{f_k(U_j^{(n)})}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} \le \lambda(1-\delta)\,,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} \frac{|f_k(U_j^{(n)})|}{\max_{1\le i\le n} |f_i(U_j^{(n)})|} > \epsilon\lambda \ \text{ for at most one } j = 1, 2, \dots \Bigg)$$

$$= \lfloor nt \rfloor P\Bigg(C_\alpha^{1/\alpha} \Bigg| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \Bigg| > \lambda \|f\|_\alpha^{-1} b_n \,, \quad C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \le \lambda(1-\delta) \|f\|_\alpha^{-1} b_n \,,$$

$$C_\alpha^{1/\alpha} \Gamma_j^{-1/\alpha} > \epsilon\lambda \|f\|_\alpha^{-1} b_n \ \text{ for at most one } j=1,2,\ldots \Bigg)$$

$$\le nP\left(C_\alpha^{1/\alpha} \left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right| > (\delta - \epsilon K)\lambda \|f\|_\alpha^{-1} b_n \right)$$

$$\le \frac{n\|f\|_\alpha^4 C_\alpha^{4/\alpha}}{(\delta-\epsilon K)^4 \lambda^4 b_n^4} \, E \left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right|^4 .$$

Due to the choice $K+1 > 4/\alpha$,

$$E \left| \sum_{j=K+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right|^4 < \infty;$$

see Samorodnitsky (2004) for a detailed proof. Since $n/b_n^4 \to 0$ as $n \to \infty$, (4.10) follows.

Suppose now that $f = \mathbf{1}_A$. In that case the probability measure η_n defined in (4.5) coincides with the probability measure μ_n of Proposition 2.1. In order to prove weak convergence in the J_1-topology, we will use a truncation argument. First of all, we may and will restrict ourselves to the space $D[0,1]$. Let $K = 1, 2, \ldots$. First of all, we show, in the notation of (3.8), the convergence

(4.11)

$$\left(C_\alpha^{1/\alpha} \max_{1\le k\le \lfloor nt \rfloor} \left| \sum_{j=1}^{K} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \right| ,\ 0 \le t \le 1 \right) \Rightarrow \left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{V_j \le t\}},\ 0 \le t \le 1 \right)$$

in the J_1-topology on $D[0,1]$. Indeed, by (4.6), outside of an event of asymptotically vanishing probability, the process in the left hand side of (4.11) is

$$\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \max_{1\le k\le \lfloor nt \rfloor} \mathbf{1}_A \circ T^k(U_j^{(n)}),\ 0 \le t \le 1 \right) . \tag{4.12}$$

By Proposition 2.1, we can put all the random variables involved on the same probability space so that the time of the single step in the jth term in (4.12) converges a.s. for each $j = 1, \ldots, K$ to V_j. Then, trivially, the process in (4.12) converges a.s. in the J_1-topology on $D[0,1]$ to the process in the right hand side of (4.11). Therefore, the weak convergence in (4.11) follows.

Next, we note that in the J_1-topology on the space $D[0,1]$,

$$\left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{K} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{V_j \le t\}},\ 0 \le t \le 1 \right) \to \left(C_\alpha^{1/\alpha} \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{V_j \le t_i\}} \ 0 \le t \le 1 \right) \quad \text{as } K \to \infty \quad \text{a.s.}$$

This is so because, as $K \to \infty$,

$$\sup_{0\le t\le 1}\left(\bigvee_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}} - \bigvee_{j=1}^{K}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{V_j\le t\}}\right) \le \Gamma_{K+1}^{-1/\alpha} \to 0 \quad \text{a.s..}$$

According to Theorem 3.2 in Billingsley (1999), the J_1-convergence in (4.3) will follow once we show that

$$\lim_{K\to\infty}\limsup_{n\to\infty} P\left(\max_{1\le k\le n}\left|\sum_{j=K+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) = 0$$

for every $\epsilon > 0$. Write

$$P\left(\max_{1\le k\le n}\left|\sum_{j=K+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right)$$

$$\le \int_0^{(\epsilon/2)^{-\alpha}} e^{-x}\frac{x^{K-1}}{(K-1)!}dx + \int_{(\epsilon/2)^{-\alpha}}^{\infty} e^{-x}\frac{x^{K-1}}{(K-1)!}\, P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) dx\,.$$

Clearly, the first term vanishes when $K \to \infty$. Therefore, it is sufficient to show that for every $x \ge (\epsilon/2)^{-\alpha}$,

$$P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) \to 0 \tag{4.13}$$

as $n \to \infty$.

To this end, choose $L \in \mathbb{N}$ and $0 < \xi < 1/2$ so that

$$L+1 > \frac{4}{\alpha} \quad \text{and} \quad \frac{1}{2} - \xi L > 0\,. \tag{4.14}$$

By (4.6) we can write

$$\begin{aligned} &P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\right) \\ &\le P\Bigg(\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)})\right| > \epsilon\,, \quad \text{and for each } m=1,\dots,n, \\ &\qquad \big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^m(U_j^{(n)}) > \xi\epsilon \ \text{ for at most one } j=1,2,\dots\Bigg) + o(1)\,. \end{aligned} \tag{4.15}$$

Notice that for every $k = 1,\dots,n$, the Poisson random measure represented by the points

$$\big(\epsilon_j\big(\Gamma_j+x\big)^{-1/\alpha}\mathbf{1}_A\circ T^k(U_j^{(n)}),\, j=1,2,\dots\big)$$

is distributionally equal to the Poisson random measure represented by the points

$$\big(\epsilon_j\big(b_n^{\alpha}\mu(A)^{-1}\Gamma_j+x\big)^{-1/\alpha},\, j=1,2,\dots\big)\,.$$

Therefore, the first term on the right hand side of (4.15) can be bounded by

$$\sum_{k=1}^{n} P\Bigg(\Bigg| \sum_{j=1}^{\infty} \epsilon_j \big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) \Bigg| > \epsilon \,,$$

$$\big(\Gamma_j + x\big)^{-1/\alpha} \mathbf{1}_A \circ T^k(U_j^{(n)}) > \xi\epsilon \ \text{ for at most one } j = 1, 2, \ldots \Bigg)$$

$$= nP\Bigg(\Bigg| \sum_{j=1}^{\infty} \epsilon_j \big(b_n^{\alpha}\mu(A)^{-1}\Gamma_j + x\big)^{-1/\alpha} \Bigg| > \epsilon \,,$$

$$\big(b_n^{\alpha}\mu(A)^{-1}\Gamma_j + x\big)^{-1/\alpha} > \xi\epsilon \ \text{ for at most one } j = 1, 2, \ldots \Bigg)$$

$$\leq nP\left(\left| \sum_{j=L+1}^{\infty} \epsilon_j \big(b_n^{\alpha}\mu(A)^{-1}\Gamma_j + x\big)^{-1/\alpha} \right| > \left(\frac{1}{2} - \xi L\right)\epsilon \right) .$$

In the last step we used the fact that, for $x \geq (\epsilon/2)^{-\alpha}$, the magnitude of each term in the infinite sum does not exceed $\epsilon/2$. By the contraction inequality for Rademacher series (see e.g. Proposition 1.2.1 of Kwapień and Woyczyński (1992)),

$$nP\left(\left| \sum_{j=L+1}^{\infty} \epsilon_j \big(b_n^{\alpha}\mu(A)^{-1}\Gamma_j + x\big)^{-1/\alpha} \right| > \left(\frac{1}{2} - \xi L\right)\epsilon \right)$$

$$\leq 2nP\left(\left| \sum_{j=L+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right| > \left(\frac{1}{2} - \xi L\right)\epsilon\mu(A)^{-1/\alpha} b_n \right) .$$

As before, by Markov's inequality and using the constraints of the constants $L \in \mathbb{N}$ and $0 < \xi < 1/2$ given in (4.14),

$$2nP\left(\left| \sum_{j=L+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right| > \left(\frac{1}{2} - \xi L\right)\epsilon\mu(A)^{-1/\alpha} b_n \right) \leq \frac{2n\mu(A)^{4/\alpha}}{(2^{-1} - \xi L)^4 \epsilon^4 b_n^4} \ E\left| \sum_{j=L+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \right|^4 \to 0$$

as $n \to \infty$ and, hence, (4.13) follows. □

Remark 4.4. There is not doubt that the convergence result in Theorem 4.1 can be extended to more general infinitely divisible random measures M in (1.5), under appropriate assumptions of regular variation of the Lévy measure of M and integrability of the function f. In particular, processes $Z_{\alpha,\beta}$ with any $\alpha > 0$ are likely to appear in the limit in (4.3) (and not only with $0 < \alpha < 2$ allowed by the assumption of stability). Furthermore, the symmetry of the process $\mathbf{X}$ has very little to do with the limiting distribution of the partial maxima. For example, a straightforward symmetrization argument allows one to extend (4.3) to skewed α-stable processes, at least in the sense of convergence of finite-dimensional distributions. The reason we decided to restrict the presentation to the symmetric stable case had to do with a particularly simple form of the series representation (4.4) available in this case. This has allowed us to avoid certain technicalities that might have otherwise blurred the main message, which is the effect of memory on the functional limit theorem for the partial maxima.

We conclude by mentioning that the result of Theorem 4.1 applies, among others, to the two examples in Owada and Samorodnitsky (2012), that of a flow generated by a null recurrent Markov chain, and that of an AFN-system. We only remind the reader of the setup of the former example, since it also appears in Samorodnitsky (2004).

Consider an irreducible null recurrent Markov chain $(x_n,\, n \geq 0)$ defined on an infinite countable state space $\mathbb{S}$ with the transition matrix (p_{ij}). Let $(\pi_i,\, i \in \mathbb{S})$ be its unique (up to constant multiplication) invariant measure with $\pi_0 = 1$. Note that (π_i) is necessarily an infinite measure. Define a σ-finite and infinite measure on $(E, \mathcal{E}) = (\mathbb{S}^{\mathbb{N}}, \mathcal{B}(\mathbb{S}^{\mathbb{N}}))$ by

$$\mu(B) = \sum_{i \in \mathbb{S}} \pi_i P_i(B), \quad B \subseteq \mathbb{S}^{\mathbb{N}},$$

where $P_i(\cdot)$ denotes the probability law of (x_n) starting in state $i \in \mathbb{S}$. Let

$$T(x_0, x_1, \dots) = (x_1, x_2, \dots)$$

be the usual left shift operator on $\mathbb{S}^{\mathbb{N}}$. Then T preserves μ. Since the Markov chain is irreducible and null recurrent, T is conservative and ergodic (see Harris and Robbins (1953)).

Let $A = \{x \in \mathbb{S}^{\mathbb{N}} : x_0 = i_0\}$ for a fixed state $i_0 \in \mathbb{S}$, and

$$\varphi_A(x) = \min\{n \geq 1 : x_n \in A\}, \quad x \in \mathbb{S}^{\mathbb{N}}$$

be the first entrance time. Assume that

$$\sum_{k=1}^{n} P_0(\varphi_A \geq k) \in RV_\beta$$

for some $\beta \in (1/2, 1)$. Then all of the assumptions of Theorem 4.1 are satisfied for any $f \in L^\alpha(\mu) \cap L^\infty(\mu)$, supported by A; see Aaronson (1997).

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