

# LONG RANGE DEPENDENCE

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ABSTRACT. The notion of long range dependence is discussed from a variety of points of view, and a new approach is suggested. A number of related topics is also discussed, including connections with non-stationary processes, with ergodic theory, self-similar processes and fractionally differenced processes, heavy tails and light tails, limit theorems and large deviations.

## 1. INTRODUCTION

*Long range dependence* and *long memory* are synonymous notions, that are arguably very important. This importance can be judged, for example, by the very large number of publications having one of these notions in the title, in areas such as finance (e.g. Lo (2001)), econometrics (e.g. Robinson (2003)), internet modelling (e.g. Karagiannis et al. (2004)), hydrology (e.g. Painter (1998)), climate studies (e.g. Varotsos and Kirk-Davidoff (2006)), linguistics (e.g. Alvarez-Lacalle et al. (2006)) or DNA sequencing (e.g. Karmeshu and Krishnamachari (2004)). These publications address a great variety of issues: detection of long memory in the data, statistical estimation of parameters of long range dependence, limit theorems under long range dependence, simulation of long memory processes, and many others. Surprisingly, very few of these publications address what long range dependence is. When definitions are given, they vary from author to author (the econometric survey Guégan (2005) mentions 11 different definitions). The notion of long range dependence can also be applied to different aspects of a given stochastic process (e.g. Heyde (2002)). More diverse definitions become possible if, instead of looking at the “usual” stationary processes, one studies *stationary point processes*, as in Daley and Vesilo (1997). It is

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the purpose of this survey to discuss what is meant (often implicitly) by long range dependence, clarify why this notion is important, mention different points of view on the topic and, hopefully, remove some of the mystery that surrounds it.

The notion of long range dependence has, clearly, something to do with memory in a stochastic process. Memory is, by definition, something that lasts. It is the requirement that the memory has to be “long” that is special. Why is it important that in one model the memory is “a bit longer” than in another model? The first serious argument that this can be important is in a series of papers of B. Mandelbrot and his co-authors, e.g. Mandelbrot (1965) and Mandelbrot and Wallis (1968). It is also due to the influence of these early papers and subsequent publications of Mandelbrot (especially Mandelbrot (1983)) that long range dependence has also become associated with scaling and fractal behaviour. We survey some of the early history in Section 2.

The “specialness” of long memory indicates that most stationary stochastic processes do not have it. This also makes it intuitive that non-stationary processes can provide an alternative explanation to the empirical phenomena that the notion of long range dependence is designed to address. This connection between long memory and lack of stationarity is very important. It is related to such well known phenomena as unit root problem (see Phillips (1987)) and regime switching (Diebold and Inoue (2001)). We discuss the connections with non-stationary processes in Section 3.

A very attractive point of view on long range dependence is based on ergodic-theoretical properties of the dynamical system on which a stationary stochastic process is constructed. Many features that are intuitively associated with long memory are automatically found in such an approach. For several reasons this approach has not become widely accepted. We discuss this in Section 4.

Most of the definitions of long range dependence appearing in literature are based on the second order properties of stochastic process. Such properties include asymptotic behaviour of covariances, spectral density, and

variances of partial sums. The reasons for popularity of the second order properties in this context are both historical and practical: second order properties are relatively simple conceptually and easy to estimate from the data. This approach to the notion of long memory is discussed in Section 5.

The term “fractional” appears very frequently in the context of long range dependence. This usually refers to a model constructed using a generalized operation of a non-integer order, whereas the “usual” order of the operation has to be integer. The examples include differencing or differentiation “non-integral number of times”. Certain features often associated with long memory can sometimes be obtained by doing so. Models obtained in this way are discussed in Section 6.

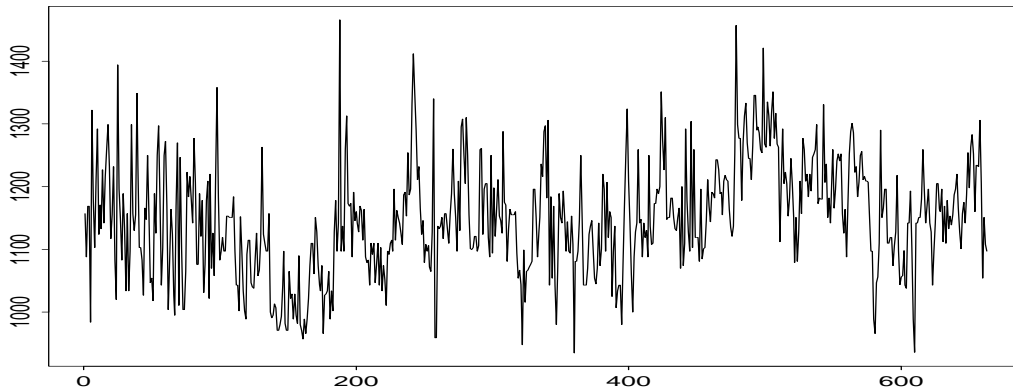
It is, once again, largely due to the early history that the notion of long range dependence has become closely associated with self-similar processes. Self-similar processes are stochastic models with the property that a scaling in time is equivalent to an appropriate scaling in space. The connection between the two types of scaling is determined by a constant often called *the Hurst exponent*, and it has been argued that the value of this exponent determines whether or not the increments of a self-similar process with stationary increments possess long range dependence. We discuss self-similar processes in Section 7.

The final part of this survey, Section 8, introduces a different approach to understanding long memory, a one that is related to the notion of phase transitions. We argue that this approach makes the notion of long range dependence both intuitive and practical. One should hope for major future research effort in this direction.

## 2. SOME HISTORY. THE HURST PHENOMENON

The history of long range dependence as a concrete phenomenon believed to be important in its own right should be regarded as beginning in the 1960s with a series of papers of Benoit Mandelbrot and his co-workers, even though even earlier empirical findings had occurred. The cause was a need to explain an empirical phenomenon observed by Hurst (1951, 1955) who studied the flow of water in the Nile river. A particular data set Hurst

looked at appears on Figure 2.1. There are many things that are interesting



**Figure 2.1.** Annual minima of the water level in the Nile river for the years 622 to 1281, measured at the Roda gauge near Cairo.

about about this data set (one of which is how far back in time the data go). Harold Hurst, who was interested in dam design, looked at this data through a particular statistic. Given a sequence of  $n$  observations  $X_1, X_2, \dots, X_n$ , define the partial sum sequence  $S_m = X_1 + \dots + X_m$  for  $m = 0, 1, \dots$  (with  $S_0 = 0$ ). The statistics Hurst looked at is

$$(2.1) \quad \frac{R}{S}(X_1, \dots, X_n) = \frac{\max_{0 \leq i \leq n}(S_i - \frac{i}{n}S_n) - \min_{0 \leq i \leq n}(S_i - \frac{i}{n}S_n)}{(\frac{1}{n} \sum_{i=1}^n (X_i - \frac{1}{n}S_n)^2)^{1/2}}.$$

Note that  $S_n/n$  is the sample mean of the data. Therefore,  $\max_{0 \leq i \leq n}(S_i - \frac{i}{n}S_n)$ , for example, measures how far the partial sums get above the straight line they would follow if all observations were equal (to the sample mean), and the difference between the maximum and the minimum of the numerator in (2.1) is the difference between the highest and lowest positions of the partial sums with respect to the straight line of uniform growth. It is referred to as the range of the observations. The denominator of (2.1) is, of course, the sample standard deviation. The entire statistic in (2.1) has, then, been called *the rescaled range* or *the  $R/S$  statistic*.

Suppose now that  $X_1, X_2, \dots$  is a sequence of random variables. One can apply the  $R/S$  statistic to the first  $n$  observations  $X_1, X_2, \dots, X_n$  for

increasing values of  $n$ . What would one expect the resulting sequence of values of the  $R/S$  statistic to be like? Let us do some simple calculations.

Consider the space  $D[0, 1]$  of right continuous and having left limits functions on  $[0, 1]$  equipped with the Skorohod  $J_1$  topology (see Billingsley (1999)). The function  $f : D[0, 1] \rightarrow \mathbb{R}$  defined by

$$f(\mathbf{x}) = \sup_{0 \leq t \leq 1} (x(t) - tx(1)) - \inf_{0 \leq t \leq 1} (x(t) - tx(1)),$$

$\mathbf{x} = (x(t), 0 \leq t \leq 1) \in D[0, 1]$ , is easily seen to be continuous. We would like to apply this function to the  $D[0, 1]$ -valued version of the partial sum sequence, the so-called partial sum process.

Suppose that  $X_1, X_2, \dots$  is, in fact, a stationary sequence of random variables with a finite variance, and a common mean  $\mu$ . The partial sum process is defined by

$$(2.2) \quad S^{(n)}(t) = S_{[nt]} - [nt]\mu, \quad 0 \leq t \leq 1.$$

The classical Functional Central Limit Theorem (Donsker's Theorem, invariance principle) says that, if  $X_1, X_2, \dots$  are i.i.d., then

$$(2.3) \quad \frac{1}{\sqrt{n}} S^{(n)} \Rightarrow \sigma_* B \quad \text{weakly in } D[0, 1],$$

where  $\sigma_*^2$  is equal to the common variance  $\sigma^2$  of the observations, and  $B$  is the standard Brownian motion on  $[0, 1]$  (Theorem 14.1 in Billingsley (1999)). In fact, the Functional Central Limit Theorem is known to hold for stationary processes with a finite variance that are much more general than an i.i.d. sequence (with the limiting standard deviation  $\sigma_*$  not equal, in general, to the standard deviation of the  $X_i$ 's); see a recent survey in Merlevéde et al. (2006).

It is straightforward to check that the range of the first  $n$  observations (the numerator in the  $R/S$  statistic) is equal to  $f(S^{(n)})$ . Therefore, if the invariance principle (2.3) holds, then by the continuous mapping theorem,

$$\begin{aligned} \frac{1}{\sqrt{n}}(\text{ the range of the first } n \text{ observations } ) &= f\left(\frac{1}{\sqrt{n}} S^{(n)}\right) \\ \Rightarrow f(\sigma_* B) &= \sigma_* \left[ \sup_{0 \leq t \leq 1} (B(t) - tB(1)) - \inf_{0 \leq t \leq 1} (B(t) - tB(1)) \right] \\ &:= \sigma_* \left[ \sup_{0 \leq t \leq 1} B_0(t) - \inf_{0 \leq t \leq 1} B_0(t) \right], \end{aligned}$$

where  $B_0$  is a Brownian bridge on  $[0, 1]$ . Furthermore, if the stationary sequence  $X_1, X_2, \dots$  is ergodic, then the sample standard deviation is a consistent estimator of the population standard deviation, and so

$$\left(\frac{1}{n} \sum_{i=1}^n (X_i - \frac{1}{n} S_n)^2\right)^{1/2} \rightarrow \sigma \quad \text{with probability 1.}$$

Under these circumstances we see that

$$(2.4) \quad \frac{1}{\sqrt{n}} \frac{R}{S}(X_1, \dots, X_n) \Rightarrow \frac{\sigma^*}{\sigma} \left[ \sup_{0 \leq t \leq 1} B_0(t) - \inf_{0 \leq t \leq 1} B_0(t) \right].$$

That is, the  $R/S$  statistic grows as the square root of the sample size.

When Harold Hurst calculated the  $R/S$  statistic on the Nile river data on Figure 2.1 he found, however, the empirical rate of growth closer to  $n^{.74}$  (with  $n$  being the number of observations). This phenomenon became known as *the Hurst phenomenon*, and finding a stochastic model that would explain it proved to be tricky. The assumptions made above to guarantee the convergence in (2.4) are reasonably mild, and one would expect that even if exact convergence in (2.4) was difficult to establish, the square root of the sample of the sample size was still the order of magnitude of the  $R/S$  statistic. A drastic departure from the assumptions was needed.

One such departure was suggested in Moran (1964), and it consisted of dropping the assumption of a finite variance of the observations  $X_1, X_2, \dots$  and assuming, instead, that the observations are in the domain of attraction of an infinite variance  $\alpha$ -stable distribution with  $0 < \alpha < 2$  (Moran, actually, assumed the observations to have a symmetric  $\alpha$ -stable law). It was pointed out in Mandelbrot and Taqqu (1979), however, that the self-normalizing feature of the  $R/S$  statistic prevents infinite variance alone from explaining the Hurst phenomenon. Let us sketch why.

We will assume, for simplicity, that the observations (in addition to being i.i.d.) have *balanced power tails*:

$$(2.5) \quad P(|X_1| > x) \sim cx^{-\alpha} \quad \text{as } x \rightarrow \infty, \text{ and}$$

$$\lim_{x \rightarrow \infty} \frac{P(X_1 > x)}{P(|X_1| > x)} = p, \quad \lim_{x \rightarrow \infty} \frac{P(X_1 < -x)}{P(|X_1| > x)} = q$$

for some  $0 < \alpha < 2$ ,  $c > 0$  and  $0 \leq p, q \leq 1$ ,  $p + q = 1$ . The general domain of attraction assumption allows a slowly varying function in the tail of  $|X_1|$  (see

e.g. Feller (1971)). The greater generality may introduce a slowly varying (in the sample size) function in the order of magnitude of the  $R/S$  statistic, but it cannot change the crucial exponent of  $n$  in that order of magnitude.

Assuming (2.5), one can use a very important point process convergence result (from which a large number of other *heavy tailed* limit theorems follow). For  $n \geq 1$  we define

$$N_n = \sum_{j=1}^n \delta_{(j/n, X_j/n^{1/\alpha})}$$

and view  $N_n$  as a point process on  $[0, 1] \times ([-\infty, \infty] \setminus \{0\})$ . Here  $\delta_{(t,x)}$  is the point mass at a point  $(t, x)$ . Then

$$(2.6) \quad N_n \Rightarrow N := \sum_{j=1}^{\infty} \delta_{(U_j, J_j)}$$

weakly in the space of Radon discrete random measures on the space  $[0, 1] \times ([-\infty, \infty] \setminus \{0\})$ , where the Radon property means that a measure assigns finite values to sets bounded away from the origin. In the right hand side of (2.6),  $(J_j)$  are the points of a Poisson process on  $\mathbb{R}$  with mean measure  $\mu$  given by

$$m((x, \infty)) = c p x^{-\alpha}, \quad m((-\infty, -x)) = c q x^{-\alpha}$$

for  $x > 0$ , while  $(U_j)$  are i.i.d. standard uniform random variables independent of the Poisson process. The space of Radon discrete random measures is endowed with the topology of vague convergence. The result is (4.70) in Resnick (1986), which can also be consulted for technical details.

It is possible to apply (2.6) to understand the “size” of the  $R/S$  statistic, starting with a typical “truncation” step, needed because various sums of points are not continuous functionals of point processes in the topology of vague convergence (but see Davydov and Egorov (2005) for recent progress towards topologies that may make certain sums of the points continuous functionals). Using Theorem 3.2 in Billingsley (1999) and verifying certain technical conditions, one obtains

$$(2.7) \quad \frac{1}{\sqrt{n}} \frac{R}{S}(X_1, \dots, X_n) \Rightarrow g(N),$$

where

$$g(N) = \frac{\sup_{0 \leq t \leq 1} \sum_{j=1}^{\infty} (\mathbf{1}(U_j \leq t) - t)J_j - \inf_{0 \leq t \leq 1} \sum_{j=1}^{\infty} (\mathbf{1}(U_j \leq t) - t)J_j}{(\sum_{j=1}^{\infty} J_j^2)^{1/2}};$$

we omit the details. Note that (2.7) means that even in the heavy tailed case the  $R/S$  statistic grows as the square root of the sample size.

We conclude, therefore, as did Mandelbrot and Taqqu (1979), that infinite variance alone cannot explain the Hurst phenomenon. A different drastic departure from the assumptions leading to the square root of the sample size rate of growth of the  $R/S$  statistic was suggested in Mandelbrot (1965) (see also Mandelbrot and Wallis (1968)), and it had nothing to do with heavy tails. The idea was, instead, to take as a model a stationary process with a finite variance, but with correlations decaying so slowly as to invalidate the Functional Central Limit Theorem (2.3). The simplest model of that sort is the *Fractional Gaussian Noise*.

Let us start with a zero mean Gaussian process  $(B_H(t), t \geq 0)$  satisfying  $B_H(0) = 0$  and  $E(B_H(t) - B_H(s))^2 = \sigma^2|t - s|^{2H}$  for some  $\sigma > 0$  and  $0 < H \leq 1$ . We will see below that such a process does exist, and it has stationary increments (that is, the law of  $(B_H(t+h) - B_H(h), t \geq 0)$  does not depend on  $h \geq 0$ ). It is called a *Fractional Brownian motion*, or *FBM*, and it becomes the usual Brownian motion when  $H = 1/2$ . Clearly, this process has the self-similarity property  $(B_H(ct), t \geq 0) \stackrel{d}{=} (c^H B_H(t), t \geq 0)$  for any  $c > 0$ .

The power-like behaviour of the incremental variance immediately allows one to check the metric entropy condition (see Dudley (1967)) or the Kolmogorov criterion (see e.g. Durrett (1996)) to conclude that a Fractional Brownian motion has a continuous version, and we always assume that we are working with such a version. Furthermore, an easy computation of the covariance function shows that for  $H = 1$ ,  $B_1(t) = tB_1(1)$  with probability 1 for each  $t \geq 0$ , and so to avoid trivialities we always take  $0 < H < 1$ .

A Fractional Gaussian Noise, or *FGN*, is a discrete step increment process of a Fractional Brownian motion defined by  $X_j = B_H(j) - B_H(j-1)$  for



$j = 1, 2, \dots$ . The stationarity of the increments of the FBM implies that this is a stationary Gaussian process. Using the fact  $ab = (a^2 + b^2 - (a - b)^2)/2$  and the incremental variance of the FBM, we easily see that

$$(2.8) \quad \text{Cov}(X_{j+n}, X_j) = \frac{\sigma^2}{2} \left[ (n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right]$$

for  $j \geq 1, n \geq 0$ . That is,

$$(2.9) \quad \rho_n := \text{Corr}(X_{j+n}, X_j) \sim H(2H-1)n^{-2(1-H)} \quad \text{as } n \rightarrow \infty.$$

In particular,  $\rho_n \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that the FGN is a mixing, hence ergodic, process; see Cornfeld et al. (1982). Furthermore, by the self-similarity of the FBM, for every  $n$

$$(2.10) \quad \text{Var}(X_1 + \dots + X_n) = \text{Var}B_H(n) = \sigma^2 n^{2H}.$$

Suppose now that a set of observations  $X_1, X_2, \dots$  forms a Fractional Gaussian Noise as defined above, and let us consider the behaviour of the  $R/S$  statistic on these observations. The ergodicity of the FGN implies that the denominator of the statistic converges a.s. to the standard deviation of the observations,  $\sigma$ . For the numerator of the  $R/S$  statistic we recall that  $S_i = B_H(i)$  for every  $i$ , and the self-similarity of the FBM gives us

$$\begin{aligned} & \max_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) - \min_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) \\ &= \max_{0 \leq i \leq n} (B_H(i) - \frac{i}{n} B_H(n)) - \min_{0 \leq i \leq n} (B_H(i) - \frac{i}{n} B_H(n)) \\ &\stackrel{d}{=} n^H \left[ \max_{0 \leq i \leq n} (B_H(\frac{i}{n}) - \frac{i}{n} B_H(1)) - \min_{0 \leq i \leq n} (B_H(\frac{i}{n}) - \frac{i}{n} B_H(1)) \right]. \end{aligned}$$

By the continuity of the sample paths of the FBM we have

$$\begin{aligned} & \max_{0 \leq i \leq n} (B_H(\frac{i}{n}) - \frac{i}{n} B_H(1)) - \min_{0 \leq i \leq n} (B_H(\frac{i}{n}) - \frac{i}{n} B_H(1)) \\ &\rightarrow \sup_{0 \leq t \leq 1} (B_H(t) - t B_H(1)) - \inf_{0 \leq t \leq 1} (B_H(t) - t B_H(1)) \end{aligned}$$

with probability 1. That is, for the FGN,

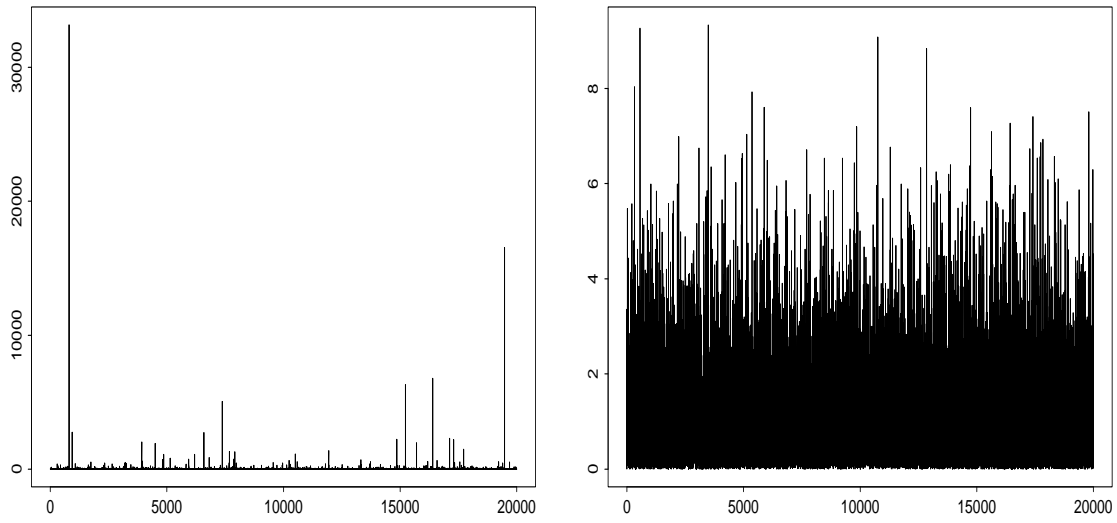
$$n^{-H} \frac{R}{S}(X_1, \dots, X_n) \Rightarrow \sup_{0 \leq t \leq 1} (B_H(t) - t B_H(1)) - \inf_{0 \leq t \leq 1} (B_H(t) - t B_H(1)),$$

and so the  $R/S$  statistic grows at the rate  $n^H$  as a function of the sample size. Therefore, selecting an appropriate  $H$  in the model will, finally, explain the Hurst phenomenon. In particular, the parameter  $H$  of Fractional Brownian motion is often referred to as *Hurst parameter*.

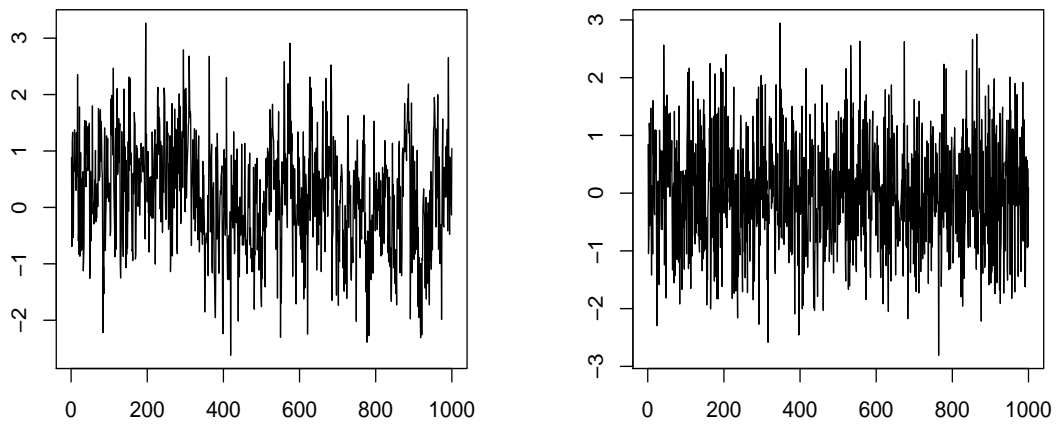
This success of the Fractional Gaussian Noise model was, and still is, striking. Of course, self-similarity of the FBM was used in the above computation, but it was quickly realized that the really important fact was the unusually slow decay of correlations in (2.9), especially for high values of  $H$  (i.e. close to 1). For these values of  $H$  the variance of the partial sums in (2.10) also increases unusually fast. Unlike the previous unsuccessful attempt to explain the Hurst phenomenon by introducing in the model unusually heavy tails (infinite variance in this case), the FGN model succeeds here by introducing unusually long memory. Particularly vivid terminology was introduced in Mandelbrot and Wallis (1968), in the context of weather and precipitation: unusually heavy tails were designated as *Noah effect*, referring to the biblical story of Noah and extreme incidents of precipitation, while unusually long memory was designated as *Joseph effect*, referring to the biblical story of Joseph and long stretches (seven years) of time higher than average and lower than average precipitation. One can visually see the domination of extreme observations in the left plot of Figure 2.2, where the observations are Pareto random variables with parameter 1 (and so even fail to have a finite mean), as opposed to a much less pronounced domination of extreme observations in the right plot of Figure 2.2, where the observations are standard exponential random variables.

Joseph effect, on the other hand, is clearly visible on Figure 2.3: in the left plot, where the observations form FGN with Hurst parameter  $H = 0.8$ , there are long stretches of time (hundreds of observations) where the observations tend to be on one side of the true mean 0. This is, clearly, not the case on the right plot of i.i.d. normal observations. Returning momentarily to the Nile river data on Figure 2.1 we see evidence of Joseph effect there as well.

This brought the fact that memory of a certain length can make a big difference to the attention of many. The terms “long range dependent process” and “long memory” came into being; they can already be found in the early papers by Mandelbrot and co-authors. A number of surveys throughout the years helped to maintain clarity in this otherwise mysterious subject;



**Figure 2.2.** i.i.d. Pareto random variables with parameter 1 (left plot) and i.i.d. exponential random variables with parameter 1 (right plot)



**Figure 2.3.** Fractional Gaussian noise with  $H = 0.8$  (left plot) and i.i.d. standard Gaussian random variables (right plot)

we mention Cox (1984), Rosenblatt (1984), a bibliographic guide of Taqqu (1986), and the monograph of Beran (1994).

### 3. LONG MEMORY AND NON-STATIONARITY

It is standard in theory of stochastic processes to apply the notion of long range dependence only to stationary processes. This is not necessarily the case in certain areas of application (such as, for example, physics, or self-organizing criticality), where closely related terms (long-term correlations,  $1/f$  noise, often in the context of power laws) are sometimes applied to non-stationary models, such as Brownian motion. See e.g. Berg-Sørensen and Flyvberg (2005). Because of that, it has also been suggested, e.g. in Heyde and Yang (1997), to modify certain (second-order) definitions of long range dependence to apply to non-stationary processes as well.

In general, the relationship between long range dependence and non-stationarity is delicate in a number of ways. We have seen that the *Joseph effect* involves long stretches of time when the process tends to be above the mean, and long stretches of time when the process tends to be below the mean. This and related phenomena can, of course, be taken to indicate non-stationarity. Quoting a description in Mandelbrot (1983), page 251, of a Fractional Gaussian noise with  $H = 1/2$ : “Nearly every sample looks like a “random noise” superimposed upon a background that performs several cycles, whichever the sample’s duration. However, these cycles are *not* periodic, that is, *cannot* be extrapolated as the sample lengthens.” In application to real data, either stationary long memory models or appropriate non-stationary models can and have been used. There is, obviously, no “right” or “wrong” way to go here, beyond the principle of parsimony.

Among the first to demonstrate the difficulty of distinguishing between stationary long memory models and certain non-stationary models was the paper Bhattacharya et al. (1983) who suggested that, instead of Fractional Gaussian noise or another model with long memory, the Hurst phenomenon can be explained by a simple non-stationary model as follows. Let  $Y_1, Y_2, \dots$  be a sequence of independent identically distributed random variables with

a finite variance  $\sigma^2$ . Let  $0 < \beta < 1/2$ , choose  $a \geq 0$  and consider the model

$$(3.1) \quad X_i = Y_i + (a + i)^{-\beta}, \quad i = 1, 2, \dots$$

Clearly, the stochastic process  $X_1, X_2, \dots$  is non-stationary, for it contains a non-trivial drift. However, it is asymptotically stationary (as the time increases), and the drift can be taken to be very small to start with (by taking  $a$  to be large). This process has no memory at all, as the sequence  $Y_1, Y_2, \dots$  is i.i.d. It does, however, cause the  $R/S$  statistic to behave in the same way as if the sequence  $X_1, X_2, \dots$  were a FGN, or another long range dependent process. To see why, assume for simplicity that  $a = 0$  above, and note that for this model, the numerator of the  $R/S$  statistic is bounded between

$$r_n - R_n^Y \leq \max_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) - \min_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) \leq r_n + R_n^Y,$$

where

$$r_n = \max_{0 \leq i \leq n} (s_i - \frac{i}{n} s_n) - \min_{0 \leq i \leq n} (s_i - \frac{i}{n} s_n),$$

$$R_n^Y = \max_{0 \leq i \leq n} (S_i^Y - \frac{i}{n} S_n^Y) - \min_{0 \leq i \leq n} (S_i^Y - \frac{i}{n} S_n^Y),$$

and  $S_m^Y = Y_1 + \dots + Y_m$ ,  $s_m = \sum_{j=1}^m j^{-\beta}$  for  $m = 0, 1, 2, \dots$ . Since  $s_m$  is a sum of a decreasing sequence of numbers, we see that  $\min_{0 \leq i \leq n} (s_i - \frac{i}{n} s_n) = 0$ . On the other hand, the extremum in the max part of  $r_n$  is achieved at  $i = \lfloor (\frac{1}{n} \sum_{j=1}^n j^{-\beta})^{-1/\beta} \rfloor$ , and elementary computations show that  $\max_{0 \leq i \leq n} (s_i - \frac{i}{n} s_n) \sim C_\beta n^{1-\beta}$  with  $C_\beta = \beta(1-\beta)^{1/\beta-2}$ . Since  $R_n^Y$  grows as  $n^{1/2}$ , we immediately conclude that

$$\frac{1}{n^{1-\beta}} \left[ \max_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) - \min_{0 \leq i \leq n} (S_i - \frac{i}{n} S_n) \right] \rightarrow C_\beta$$

in probability as  $n \rightarrow \infty$ . Similarly, in the denominator of the  $R/S$  statistic we have a bound

$$D_n^Y - d_n \leq \left( \sum_{i=1}^n (X_i - \frac{1}{n} S_n)^2 \right)^{1/2} \leq D_n^Y + d_n,$$

where

$$D_n^Y = \left( \sum_{i=1}^n (Y_i - \frac{1}{n} S_n^Y)^2 \right)^{1/2}, \quad d_n = \left( \sum_{i=1}^n (i^{-\beta} - \frac{1}{n} s_n)^2 \right)^{1/2}.$$

We know that  $D_n^Y/n^{1/2} \rightarrow \sigma$  a.s. as  $n \rightarrow \infty$ , while an elementary computation leads to  $d_n/n^{1/2-1/\beta} \rightarrow C'_\beta$  with  $C'_\beta = \beta^2(1-\beta)^{-2}(1-2\beta)$ . Therefore,

$$n^{-1/2} \left( \sum_{i=1}^n (X_i - \frac{1}{n} S_n)^2 \right)^{1/2} \rightarrow \sigma$$

a.s. and we conclude that

$$\frac{1}{n^{1-\beta}} \frac{R}{S}(X_1, \dots, X_n) \rightarrow \frac{C_\beta}{\sigma}$$

in probability as  $n \rightarrow \infty$ . Therefore, for the model (3.1) the  $R/S$  statistic grows as  $n^{1-\beta}$ , same rate as for the FGN with  $H = 1 - \beta$ , and so the  $R/S$  statistic cannot distinguish between these two models. Apart from fooling the  $R/S$  statistic, however, the model (3.1) is not difficult to tell apart from a stationary process with correlations decaying as in (2.9). That this can be done using *the periodogram* was quickly shown in Künsch (1986).

A very important class of non-stationary models that empirically resemble long memory stationary models is that of *regime switching models*. The name is descriptive, and makes it clear where the lack of stationarity comes from. The Fractional Gaussian noise also appear to exhibit different “regimes” (the Joseph effect), but the non-stationary regime switching models are usually those with break points, whose location changes with the sample size, in either random or non-random manner.

One class of regime switching models obtains by taking a parametric model that would be stationary if its parameters were kept constant and then changing the parameters along a sequence of non-random time points, again chosen relatively to the sample size. In Mikosch and Stărică (1999) and Mikosch and Stărică (2000) such a procedure was applied to the GARCH( $p, q$ ) model. Such a change can affect the mean and the variance (among many other things) of the process after break points, and to many sample statistics this will look like long memory.

To see what might happen here consider a sample  $X_1, \dots, X_n$ , where the observations come from  $r$  subsamples of lengths proportional to the overall sample size. That is, given fixed proportions  $0 < p_i < 1$ ,  $i = 1, \dots, r$  with

$p_1 + \dots + p_r = 1$ , the sample has the form

$$(3.2) \quad X_1^{(1)}, \dots, X_{[np_1]}^{(1)}, X_{[np_1]+1}^{(2)}, \dots, X_{[n(p_1+p_2)]}^{(2)}, \dots, X_{[n(1-p_r)]}^{(r)}, \dots, X_n^{(r)},$$

where the  $i$ th subsample forms a stationary ergodic process with a finite variance,  $i = 1, \dots, r$ . Since one of the common ways to try to detect long range dependence is by looking for a slow decay of covariances and correlations, let us check the behaviour of the sample covariance on the sample (3.2). Note that for a fixed time lag  $m$

$$\hat{R}_m(n) = \frac{1}{n} \sum_{j=1}^{n-m} (X_j - \bar{X})(X_{j+m} - \bar{X}) = A_m(n) + B_m(n),$$

where  $\bar{X}$  is the overall sample mean,

$$A_m(n) = \frac{1}{n} \sum_{j=1}^{n-m} X_j X_{j+m} - (\bar{X})^2,$$

and

$$B_m(n) = \frac{1}{n} \bar{X} \left( \sum_{j=1}^m X_j + \sum_{j=n-m+1}^n X_j \right) - \frac{m}{n} (\bar{X})^2.$$

Obviously  $B_m(n) \rightarrow 0$  in probability as  $n \rightarrow \infty$ . By ergodicity, also  $\bar{X} \rightarrow \sum_{i=1}^r p_i \mu_i$ , where  $\mu_i$  is the mean of the  $i$ th subsample. Finally, if  $I_i$  denotes the set of indices corresponding to the  $i$ th subsample,  $i = 1, \dots, r$ , then by the same ergodicity,

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^{n-m} X_j X_{j+m} \\ &= \sum_{i=1}^r \frac{\text{Card}(I_i \cap (I_i - m))}{n} \frac{1}{\text{Card}(I_i \cap (I_i - m))} \sum_{j \in I_i \cap (I_i - m)} X_j^{(i)} X_{j+m}^{(i)} \\ & \quad + \frac{1}{n} \sum_{i=1}^r \sum_{\substack{j \in \{1, \dots, n-m\} \\ j \in I_i, j+m \in I_{i+1}}} X_j X_{j+m} \rightarrow \sum_{i=1}^r p_i (R_m^{(i)} + \mu_i^2), \end{aligned}$$

where  $R_m^{(i)}$  is the covariance at lag  $m$  of the  $i$ th subsample. We conclude that

$$\begin{aligned} (3.3) \quad \hat{R}_m(n) & \rightarrow \sum_{i=1}^r p_i (R_m^{(i)} + \mu_i^2) - \left( \sum_{i=1}^r p_i \mu_i \right)^2 \\ & = \sum_{i=1}^r p_i R_m^{(i)} + \sum_{i_1=1}^r \sum_{i_2=i_1+1}^r p_{i_1} p_{i_2} (\mu_{i_1} - \mu_{i_2})^2 \end{aligned}$$

in probability as  $n \rightarrow \infty$ . What (3.3) indicates is that, if there is regime switching as we have described, and (some of) the mean values in different regimes are different, then the estimated from the sample covariance function will tend to stabilize, at large lags, at a positive value. This is what often observed in practice and long memory is suspected. Of course, this regime switching model is simply a deterministic way of mimicking Joseph effect (recall Figure 2.3). Various other regime switching models mimicking long range dependence are suggested in Diebold and Inoue (2001).

Overall, stationary long memory models have become more popular than regime switching models. An important reason for this is, undoubtedly, parsimony. Statistical goodness of fit should be and has been taken into account as well. For example, the stationary Fractional Gaussian noise fits the Nile river data very well (see Beran (1994), Chapter 10). On the other hand, certain workload data in computer networks, often modeled as long memory processes, can be well fit by nonstationary ARMA( $p, 1, q$ ) models (Stegeman (2001)).

More generally, ARMA models (otherwise known as linear models in time series) provide further connections between stationary long memory processes and non-stationary models. A linear model is described by two functions applied to the backshift operator, the autoregressive function and moving average functions (both often polynomials); we refer the reader to Brockwell and Davis (1987) for details. Stationarity of the model is easier to achieve if the autoregressive function does not vanish on the unit circle in the complex plane; if the order of a root there is at least 1, stationarity is impossible - this is the so called unit root situation. On the other hand, roots of certain fractional orders allow for stationary, long memory models. These fractional models will be considered in Section 6 below. Distinguishing between non-stationary unit root models and stationary fractional models is an important problem in econometrics; see e.g. Barkoulas et al. (1999).

It is possible to summarize the discussion of long memory and non-stationarity by saying that the stationary long memory processes form a layer among the stationary processes that is “near the boundary” with



non-stationary processes, or, alternatively, as the layer separating the non-stationary processes from the “usual” stationary processes (Samorodnitsky (2006)). The processes in the “layer” resemble non-stationary models (the Joseph effect), and they are unusual stationary processes to such an extent that one can talk about a phase transition. This is discussed in details in Section 8.

#### 4. LONG MEMORY, ERGODIC THEORY AND STRONG MIXING

The notion of memory in a stationary stochastic process is by definition, related to the connections between certain observations and those occurring after an amount of time has passed. If  $X_1, X_2, \dots$  is the process then the passage of time corresponds to a *shifted process*:  $X_{k+1}, X_{k+2}, \dots$ , for a time shift  $k$ . In other words, the notion of memory is related to the connections between the process and its shifts. Since the process is stationary, the shifts do not change the distribution of the process. This makes various notions from ergodic theory (of measure preserving transformations on measure spaces) a very attractive language in describing memory of a stationary process. We refer the reader to Krengel (1985) and Aaronson (1997) for the ergodic theoretical notions used in this survey.

It is convenient (but not necessary) to assume that the sample space associated with a stationary process is a space of sequences, on which shifts are naturally defined. It is even more convenient (even though, once again, not necessary) to take the sample space to be the space of two-sided sequences  $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, x_2, \dots)$  since the shifts are invertible on such spaces. Let us, therefore, assume in this section that a stationary process  $\mathbf{X} = (\dots, X_{-1}, X_0, X_1, X_2, \dots)$  is defined as the identity map on a probability space  $(\Omega, \mathcal{F}, P)$  corresponding to such a sequence sample space (and equipped with the usual cylindrical  $\sigma$ -field). Let  $T$  be the left shift on  $\Omega$  defined by

$$T(\dots, x_{-1}, x_0, x_1, x_2, \dots) = (\dots, x_0, x_1, x_2, x_3, \dots).$$

The basic notion in ergodic theory is that of *ergodicity* of a transformation. A transformation  $T$  is ergodic if there is no shift invariant measurable set  $A$

(i.e. a set satisfying  $P(A \Delta (T^{-1}A)) = 0$ ) with  $0 < P(A) < 1$ . Equivalently, the shift transformation  $T$  is ergodic if for every measurable function  $f \in L_1(\Omega, \mathcal{F}, P)$

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \mathbf{X}) = Ef(\mathbf{X}) \quad \text{a.s.}$$

Here  $T^j \mathbf{X}$  is the  $j$ th shift of the process  $\mathbf{X}$ :  $T^j(\dots, X_{-1}, X_0, X_1, X_2, \dots) = (\dots, X_{j-1}, X_j, X_{j+1}, X_{j+2}, \dots)$ .

The usual terminology is to call a stationary stochastic process ergodic if the corresponding shift transformation  $T$  defined on some (equivalently, any) sequence sample space supporting the process is ergodic. It is easy to see that a stationary stochastic process is not ergodic if and only if it can be represented a non-trivial mixture of two different stationary stochastic processes:

$$(4.2) \quad \mathbf{X} = \begin{cases} \mathbf{Y} & \text{with probability } p \\ \mathbf{Z} & \text{with probability } 1 - p \end{cases},$$

where  $0 < p < 1$  and  $\mathbf{Y}$  and  $\mathbf{Z}$  are stationary stochastic processes with different finite dimensional distributions. Indeed, suppose that  $\mathbf{X}$  is not ergodic, and take a shift invariant measurable subset  $A$  of the sequence space with  $p = P(A) \in (0, 1)$ . Then (4.2) holds with both  $\mathbf{Y}$  and  $\mathbf{Z}$  being the canonical processes on the sequence space equipped with probability measures  $P_1 = p^{-1}P|_A$  and  $P_2 = (1-p)^{-1}P|_{A^c}$ , accordingly. Since  $A$  is shift invariant,  $P_1$  and  $P_2$  are not affected by the shift, and so the two processes are stationary. Since  $A$  and  $A^c$  are disjoint, the two probability measures are different, and so  $\mathbf{Y}$  and  $\mathbf{Z}$  have different finite dimensional distributions. Conversely, suppose that (4.2) holds. Then there is a bounded measurable function  $f$  such that  $Ef(\mathbf{Y}) \neq Ef(\mathbf{Z})$ , and then by the ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \mathbf{X}) = \begin{cases} L_1 & \text{with probability } p \\ L_2 & \text{with probability } 1 - p \end{cases},$$

where  $L_1$  and  $L_2$  are random variables satisfying  $EL_1 = Ef(\mathbf{Y})$  and  $EL_2 = Ef(\mathbf{Z})$ , implying that (4.1) fails, and so  $\mathbf{X}$  is not ergodic.

Note that it is very natural to say that a non-ergodic stationary process  $\mathbf{X}$  has infinite memory. Indeed, a non-ergodic process has the structure

given in (4.2), and so the result of a single “coin toss” (with probabilities  $p$  and  $1 - p$ ) will be “remembered forever”. Therefore, it certainly makes sense to call stationary ergodic processes “processes with finite memory”, and stationary non-ergodic processes “processes with infinite memory”.

It is, then, very tempting to try to find another ergodic theoretical notion, stronger than ergodicity, corresponding to stationary processes with finite and short memory. Then ergodic stationary processes that lack this stronger property will be naturally called processes with long memory. A very natural notion to try is *mixing*. Recall that a transformation  $T$  of a probability space is mixing if for every two measurable sets  $A$  and  $B$  we have  $P(A \cap T^{-n}B) \rightarrow P(A)P(B)$  as  $n \rightarrow \infty$ , and a stationary process is called mixing if the corresponding shift transformation  $T$  is mixing. The shift transformation can, once again, be defined on any sequence sample space supporting the process.

It is obvious that a mixing stationary process is ergodic, and it is easy to construct examples of ergodic but non-mixing stationary processes. A candidate definition of a long range dependent process would then refer to an ergodic but non-mixing process.

Such a definition has not become standard, for the reasons that will be discussed below. Note, however, that the approaches to memory of a stationary process via the ergodic theoretical properties of the corresponding shift transformation are very attractive from the following point of view. Let  $\mathbf{X}$  be a stationary process, and let the process  $\mathbf{Y}$  be derived from the process  $\mathbf{X}$  by means of a point transformation  $Y_n = g(X_n)$  for all  $n$ , where  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function. Clearly,  $\mathbf{Y}$  is also a stationary process. It is intuitively clear that the process  $\mathbf{X}$  “remembers at least as much” as the process  $\mathbf{Y}$  does. If, in particular,  $g$  is a one-to-one map, and  $g^{-1}$  is also measurable, then this intuition says that the processes  $\mathbf{X}$  and  $\mathbf{Y}$  should have “the same length of memory”: if one of them has long memory, then so should do the other one.

This, apparently very natural, requirement has proved to be difficult to satisfy by many of the proposed definitions of long range dependence. It

is, however, automatic with ergodic theoretical-based definitions. Indeed, it follows from the definition of the ergodicity and mixing that  $\mathbf{X}$  is ergodic (mixing) if and only if  $\mathbf{Y}$  has this property.

It is instructive to record what the ergodic theoretical-based notions of memory discussed above mean for stationary Gaussian processes. Let  $\mathbf{X}$  be a (real-valued) stationary Gaussian process with covariance function  $R_k$ ,  $k \geq 0$  and spectral measure  $F$  on  $(-\pi, \pi]$ . That is,  $R_k = \int_{(-\pi, \pi]} \cos(kx) F(dx)$  for  $k \geq 0$ . Then

- the process  $\mathbf{X}$  is ergodic if and only if the spectral measure  $F$  is atomless;
- the process  $\mathbf{X}$  is mixing if and only if  $R_k \rightarrow 0$  as  $k \rightarrow \infty$ ;

see Cornfeld et al. (1982). The requirement that the covariance function vanishes at the high lag limit has, however, proved to be insufficient when dealing with long memory for Gaussian processes. Indeed, many “unusual” phenomena have been observed for Gaussian processes whose covariance function does vanish in the limit, but sufficiently slowly, as we have already seen on the example of the Fractional Gaussian noise. Therefore, the mixing property is not believed to be sufficiently strong to say that a stationary process with this property has short memory. A stronger requirement is needed.

Several such stronger requirements have been introduced; they are collectively known under the name “strong mixing conditions” (even though one of them carries that same name separately). We refer the reader to the recent survey Bradley (2005) for a detailed discussion of the notions we introduce here and their relation to the other possible conditions.

Let, once again,  $\mathbf{X} = (\dots, X_{-1}, X_0, X_1, X_2, \dots)$  be a stationary process. Define for  $n \geq 1$

$$(4.3) \quad \alpha_{\mathbf{X}}(n) = \sup \left\{ |P(A \cap B) - P(A)P(B)|, \right. \\ \left. A \in \sigma(X_k, k \leq 0), B \in \sigma(X_k, k \geq n) \right\}.$$

The process  $\mathbf{X}$  is called *strongly mixing* if  $\alpha_{\mathbf{X}}(n) \rightarrow 0$  as  $n \rightarrow \infty$ . A possible connection between the strong mixing property and lack of long memory

(i.e. short memory) has been observed beginning with Rosenblatt (1956). Specifically, it turns out that strong mixing is related to the fact whether or not the partial sums of the process  $\mathbf{X}$  satisfy the Functional Central Limit Theorem (2.3).

Let, as before,  $S_m = X_1 + \dots + X_m$ ,  $m = 0, 1, \dots$ , and define the partial sum process by (2.2).

**Theorem 4.1.** *Assume that  $\mathbf{X}$  is a zero mean strongly mixing process such that for some  $\delta > 0$ ,  $|X_0|^{2+\delta} < \infty$ . Assume that  $\text{Var}(S_m) \rightarrow \infty$  as  $m \rightarrow \infty$ , and for some  $K < \infty$*

$$E|S_m|^{2+\delta} \leq K (\text{Var}(S_m))^{1+\delta/2} \text{ for all } m.$$

*Then the properly normalized partial sum process converges weakly to the Brownian motion, i.e.*

$$\frac{1}{(\text{Var}(S_n))^{1/2}} S^{(n)} \Rightarrow B \text{ weakly in } D[0, 1],$$

*where  $B$  is the standard Brownian motion. Moreover, assume, in addition, that the covariances  $R_n = \text{Cov}(X_0, X_n)$ ,  $n = 0, 1, \dots$  are summable:*

$$\sum_{n=0}^{\infty} |R_n| < \infty,$$

*Then the limit  $\sigma_*^2 = \lim_{n \rightarrow \infty} \text{Var}(S_n)/n$  exists, is finite and positive, and (2.3) holds.*

See Rosenblatt (1956), Merlevéde et al. (2006), Proposition 34, and Bradley (1999) for the last comment.

Theorem 4.1 indicates that a strongly mixing process behaves, as far as the central limit theorem is concerned, similarly to an i.i.d. sequence and, hence, can be viewed as having short memory. Extra moment conditions involved are somewhat disappointing, and turns out that imposing “a stronger strong mixing condition” allows one to get rid of these extra conditions. For a stationary process  $\mathbf{X}$  define

$$(4.4) \quad \alpha_{\mathbf{X}}^*(n) = \sup_{S,T} \sup \left\{ |P(A \cap B) - P(A)P(B)|, \right. \\ \left. A \in \sigma(X_k, k \in S), B \in \sigma(X_k, k \in T) \right\},$$

where the first supremum in (4.4) is taken over all subsets  $S$  and  $T$  of integers satisfying

$$\text{dist}(S, T) := \min_{k_1 \in S, k_2 \in T} |k_1 - k_2| \geq n.$$

Clearly,  $\alpha_{\mathbf{X}}(n) \leq \alpha_{\mathbf{X}}^*(n)$ . The process  $\mathbf{X}$  is *interlaced strongly mixing* if  $\alpha_{\mathbf{X}}^*(n) \rightarrow 0$  as  $n \rightarrow \infty$ . The following is a central limit theorem under the this stronger mixing assumption.

**Theorem 4.2.** *Assume that  $\mathbf{X}$  is a zero mean finite variance interlaced strongly mixing process, and  $\text{Var}(S_m) \rightarrow \infty$  as  $m \rightarrow \infty$ . Then the properly normalized partial sum process converges weakly to the Brownian motion, i.e.*

$$\frac{1}{(\text{Var}(S_n))^{1/2}} S^{(n)} \Rightarrow B \quad \text{weakly in } D[0, 1],$$

where  $B$  is the standard Brownian motion. Furthermore, the limit  $\sigma_*^2 = \lim_{n \rightarrow \infty} \text{Var}(S_n)/n$  exists, is finite and positive, and (2.3) holds.

See Peligrad (1998) for a proof under the so-called “interlaced  $\rho$ -mixing condition, and Bradley (1993) for the equivalence of the two “interlaced conditions”. In fact, Theorem 4.2 holds for a strongly mixing stationary process that satisfies  $\alpha_{\mathbf{X}}^*(n) < 1$  for some  $n \geq 1$ .

For a stationary Gaussian process to be strongly mixing, it is necessary that its spectral measure be absolutely continuous with respect to the Lebesgue measure on  $(-\pi, \pi]$ . If the spectral density (i.e. the derivative of the spectral measure with respect to the Lebesgue measure on  $(-\pi, \pi]$ ) is continuous and positive, then the process is interlaced strong mixing (see Kolmogorov and Rozanov (1960) and Rosenblatt (1985)). Necessary and sufficient conditions for strong mixing of a stationary Gaussian process were later established in Helson and Sarason (1967). Explicit necessary and sufficient conditions for interlaced strong mixing of stationary Gaussian process do not appear to have been stated.

The above results explain why absence of one or another strong mixing condition (as opposed to the ergodic-theoretical mixing) is sometimes taken as the definition of long range dependence. The strong mixing properties share with the ergodic-theoretical notions of ergodicity and mixing the very

desirable feature discussed above: if a process  $\mathbf{Y}$  is derived from a process  $\mathbf{X}$  by means of a one-to-one point transformation  $Y_n = g(X_n)$  for all  $n$ , then the process  $\mathbf{X}$  has long memory in the sense of lacking of one of the strong mixing properties if and only if the process  $\mathbf{Y}$  does.

In spite of these attractive features of the strong mixing conditions have not become standard. To some extent this is due to the fact that if one is interested not in the partial sums of a stationary process but in, say, *partial maxima*, then strong mixing conditions, while relevant, allow clustering and, hence, limits different from the ones seen for i.i.d. sequences; see e.g. Leadbetter et al. (1983). More importantly, the strong mixing conditions are not easily related to the natural building blocks of many stochastic models, and are difficult to verify, with the possible exception of Gaussian processes and Markov chains. Even in the latter cases necessary and sufficient conditions are not always available, and the picture is not completely clear.

## 5. SECOND-ORDER THEORY

By far the most popular point of view on long range dependence is through a slow decay of correlations. This is related to the original explanation of the Hurst phenomenon by Mandelbrot, discusses in Section 2, and to the simple fact that correlations are one of the easiest to understand and estimate features of a stochastic model. Clearly, such approaches to the notion of long memory are restricted to second-order stationary processes, and this is the assumption that will be made throughout this section. A related, if not entirely equivalent, second-order approach is through the behaviour of the spectral density of the process (assuming its existence) at the origin. These issues are discussed in this section.

Let, then,  $\mathbf{X} = (X_1, X_2, \dots)$  be a zero mean stationary stochastic process with a finite variance,  $EX_1^2 = \sigma^2 \in (0, \infty)$ , covariances  $R_n = \text{Cov}(X_1, X_{n+1})$  and correlations  $\rho_n = R_n/\sigma^2$ ,  $n = 0, 1, \dots$ . We start with an obvious computation of the variance of the partial sum  $S_n = X_1 + \dots + X_n$ . We have

$$(5.1) \quad \text{Var}S_n = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j)$$

$$= \sigma^2 \sum_{i=1}^n \sum_{j=1}^n \rho_{|i-j|} = \sigma^2 \left( n + 2 \sum_{i=1}^{n-1} (n-i) \rho_i \right),$$

and the behaviour of the last sum is closely related to how fast the correlations of the process decay. Assume that they are summable:

$$(5.2) \quad \sum_{n=0}^{\infty} |\rho_n| < \infty.$$

Then by the dominated convergence theorem the limit

$$(5.3) \quad \lim_{n \rightarrow \infty} \frac{\text{Var} S_n}{n} = \sigma^2 \left( 1 + 2 \sum_{i=1}^{\infty} \rho_i \right) := \sigma_*^2$$

exists, and is finite. However, it is possible that the limit  $\sigma_*^2$  is equal to zero. Assuming that it is not equal to zero, we conclude that the variance of the partial sums of the process  $\mathbf{X}$  grows linearly fast with the number of terms. On the other hand, suppose that the correlations of the process are, in fact, regularly varying at infinity:

$$(5.4) \quad \rho_n = n^{-d} L(n)$$

for  $n \geq 1$ , where  $0 \leq d < 1$ , and  $L$  is a slowly varying at infinity function, i.e.  $L$  is eventually non-zero, and for all  $t > 0$ ,  $L(tx)/L(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

Then by Karamata's theorem (see e.g. Theorem 0.6 in Resnick (1986)),

$$(5.5) \quad \text{Var} S_n \sim \frac{2\sigma^2}{(1-d)(2-d)} L(n) n^{2-d} \quad \text{as } n \rightarrow \infty.$$

That is, regular variation of correlations as in (5.4) implies that the variance of the partial sums grows much faster than in the case of summable correlations. This is, of course, the case for the Fractional Gaussian Noise (with  $H > 1/2$ ) of Section 2, whose asymptotic behaviour is given by (2.9).

When the variance of the partial sums of a stationary process grows linearly fast with the number of terms, at least from that point of view, the process “is not far” from an i.i.d. sequence. Furthermore, we have seen in Section 4 that this, under certain strong mixing and moment assumptions, means that the classical invariance principle holds (modulo a different variance of the limiting Brownian motion). This is also true, for example, under the assumption of association - see Newman and Wright (1981). On the other hand, when the variance of the partial sums grows as a regularly



varying function with the exponent larger than 1 - as in (5.5) - it follows immediately from Lamperti's theorem (see e.g. Theorem 2.1.1 in Embrechts and Maejima (2002)) that convergence to the Brownian motion is impossible, no matter what normalization one uses, and so the invariance principle does not hold.

This is the main reason why the summability of correlations in (5.2) is often taken as the indication of short memory, and its opposite, the divergence of the series in the left hand side of (5.2), as the definition of the long range dependence. On the other hand, it is also possible to take the rate of increase of the variance of the partial sums itself do draw the bounadry. From this point of view, one could say that a second order stationary process has short memory if

$$(5.6) \quad \lim_{n \rightarrow \infty} \frac{\text{Var} S_n}{n} < \infty,$$

and the infinite limit in (5.6) would then be taken as an indication of long range dependence. This is sometimes referred to as *Allen variance* short and long memory; see e.g. Heyde and Yang (1997).

Of course, the summability of correlations (5.2) is not necessary for an at most linear rate of increase of the variance in (5.6). In fact, rewrite (5.1) as

$$(5.7) \quad \frac{\text{Var} S_n}{n} = \sigma^2 \left( 1 + 2 \frac{1}{n} \sum_{j=1}^{n-1} \sum_{i=1}^j \rho_i \right).$$

In particular, if

$$(5.8) \quad \text{the sum } \sum_{n=0}^K \rho_n \text{ converges as } K \rightarrow \infty,$$

then, since the usual convergence implies the Cesaro convergence, we will still obtain (5.3), regardless of the summability of the correlations. Such situations are, clearly, possible. A simple example is  $\rho_n = \sin na/na$ ,  $n = 1, 2, \dots$  for  $0 < a < \pi$ . However, even the convergence in (5.8) is not necessary for (5.6), as another simple example  $\rho_n = (-1)^n/2$ ,  $n = 1, 2, \dots$  shows.

To get better understanding of the condition (5.6) we need to concentrate on the spectrum of the covariance function of the process. Recall

that the spectral measure  $F$  is a measure on  $(-\pi, \pi]$ , satisfying  $R_k = \int_{(-\pi, \pi]} \cos(kx) F(dx)$  for  $k \geq 0$ . Recall, further, that if the correlations are absolutely summable as in (5.2), then the spectral measure has a continuous density with respect to the Lebesgue measure on  $(-\pi, \pi]$ , *the spectral density*, given by

$$(5.9) \quad f(x) = \frac{\sigma^2}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} \rho_n \cos nx \right), \quad -\pi < x < \pi.$$

A simple computation will allow us to relate the right hand side of (5.7) to the spectral measure. Assuming that the spectral measure does not have atoms at zero and at  $\pi$ , we have for every  $j \geq 1$

$$\begin{aligned} \sum_{i=1}^j \rho_i &= \frac{1}{\sigma^2} \int_{(-\pi, \pi]} \sum_{i=1}^j \rho_i \cos(ix) F(dx) \\ &= \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{1}{\sin x} (\sin(j+1)x + \sin jx - \sin x) F(dx) \\ &= \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{\sin(j+1)x}{\sin x} F(dx) + \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{\sin jx}{\sin x} F(dx) - \frac{1}{2}. \end{aligned}$$

Furthermore, for  $n \geq 1$ ,

$$\begin{aligned} \sum_{j=1}^{n-1} \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{\sin(j+1)x}{\sin x} F(dx) &= \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{1}{\sin x} \sum_{j=1}^{n-1} \sin(j+1)x F(dx) \\ &= \frac{1}{4\sigma^2} \int_{(-\pi, \pi]} \frac{1}{\sin^2 x} (\cos x + \cos 2x - \cos nx - \cos(n+1)x) F(dx). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{j=1}^{n-1} \frac{1}{2\sigma^2} \int_{(-\pi, \pi]} \frac{\sin jx}{\sin x} F(dx) \\ = \frac{1}{4\sigma^2} \int_{(-\pi, \pi]} \frac{1}{\sin^2 x} (1 + \cos x - \cos(n-1)x - \cos nx) F(dx), \end{aligned}$$

and so

$$(5.10) \quad \frac{\text{Var} S_n}{n} = a_{n-1} + 2a_n + a_{n+1} + O(1)$$

where

$$(5.11) \quad a_n = \frac{1}{2n} \int_{(-\pi, \pi]} \frac{1 - \cos nx}{x^2} F(dx), \quad n = 1, 2, \dots$$

One immediate conclusion is as follows.

**Proposition 5.1.** *Suppose that for some  $\epsilon > 0$  the spectral measure  $F$  has a density in the interval  $(-\epsilon, \epsilon)$ , and*

$$(5.12) \quad \text{the density has a continuous at the origin version } f.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\text{Var} S_n}{n} = 2\pi f(0).$$

*Proof.* Suppose first that the spectral measure has no atom at  $\pi$ . Then, clearly,

$$\begin{aligned} a_n &= \frac{1}{2n} \int_{-\epsilon}^{\epsilon} \frac{1 - \cos nx}{x^2} f(x) dx + O(1) \\ &= \frac{f(0)}{n} \int_0^{\epsilon} \frac{1 - \cos nx}{x^2} dx + O(1) \\ &= f(0) \int_0^{n\epsilon} \frac{1 - \cos y}{y^2} dy + O(1) \\ &\rightarrow f(0) \int_0^{\infty} \frac{1 - \cos y}{y^2} dy = \frac{\pi}{2} f(0), \end{aligned}$$

and our statement follows from (5.10). Observing that adding an atom at the point  $\pi$  does not change the rate of growth of the variance of the partial sums, we see that the proof is complete.  $\square$

In particular, the condition

$$(5.13) \quad \text{the process has a continuous at the origin spectral density}$$

is sometimes taken as another definition of a process with short memory. This condition is also known to be sufficient for the Central Limit Theorem for linear processes; see e.g. Corollary 5.2 in Hall and Heyde (1980). Of course, Proposition 5.1 shows that this condition is not necessary for an at most linear rate of increase of the variance as in (5.6), since it allows arbitrarily “bad” spectral measure outside of a neighbourhood of zero. In fact, (5.6) can happen even if there is no neighbourhood of zero where the process has a continuous at the origin spectral density as the example of the process with the spectral density  $f(x) = 1 + \cos(1/x)$ ,  $-\pi < x < \pi$ , shows.

Summarizing, the assumptions (5.2), (5.6), (5.8) and (5.13) have all been used to define a short memory process in the sense of the variance of the

partial sums of the process increasing at most linearly fast. These assumptions are, of course, not equivalent. Moreover, it is also possible to have a process with a bounded from zero and infinity spectral density for which

$$0 < \liminf_{n \rightarrow \infty} \frac{\text{Var} S_n}{n} < \limsup_{n \rightarrow \infty} \frac{\text{Var} S_n}{n} < \infty;$$

an example is constructed in Bradley (1999).

Let us now see what can cause the variance of the partial sums grow faster than linearly fast. Here is a counterpart to Proposition 5.1.

**Proposition 5.2.** *Suppose that for some  $\epsilon > 0$  the spectral measure  $F$  has a density in the interval  $(-\epsilon, \epsilon)$ , with a version  $f$  such that*

$$f(x) = x^{-(1-d)} L_1(x), \quad 0 < x < \epsilon,$$

for some  $0 < d < 1$ , where  $L_1$  is a slowly varying at zero function (i.e.  $L_1(1/\cdot)$  is slowly varying at infinity). Then

$$\text{Var} S_n \sim \frac{4\Gamma(d) \cos(\pi d/2)}{(1-d)(2-d)} L_1(1/n) n^{2-d} \quad \text{as } n \rightarrow \infty.$$

*Proof.* As in the proof of Proposition 5.1 we may assume that the spectral measure of the process has no atom at the point  $\pi$ . For  $a_n$  defined in (5.11) we have

$$\begin{aligned} a_n &\sim \frac{1}{n} \int_0^\epsilon \frac{1 - \cos nx}{x^2} f(x) dx = \frac{1}{n} \int_0^\epsilon \frac{1 - \cos nx}{x^{3-d}} L_1(x) dx \\ &= n^{1-d} \int_0^{n\epsilon} \frac{1 - \cos y}{y^{3-d}} L_1(y/n) dy. \end{aligned}$$

By Potter's bounds (see e.g. Proposition 0.8 in Resnick (1987)) there is a finite positive constant  $C$  such that  $L_1(y/n)/L_1(1/n) \leq C y^{-d/2}$  for all  $n \geq 1$  and  $0 < y < n\epsilon$ . By the dominated convergence theorem we obtain

$$\begin{aligned} a_n &\sim n^{1-d} L_1(1/n) \int_0^\infty \frac{1 - \cos y}{y^{3-d}} dy \\ &= \frac{\Gamma(d) \cos(\pi d/2)}{(1-d)(2-d)} L_1(1/n) n^{1-d} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Now an appeal to (5.10) completes the proof.  $\square$

Comparing the statement of Proposition 5.2 with the consequence (5.5) of the regular variation of the correlations in (5.4) we see that, for  $0 < d < 1$ ,

the assumptions

$$(5.14) \quad \rho_n \sim n^{-d} L(n) \quad \text{as } n \rightarrow \infty$$

and existence in the neighbourhood of the origin of a spectral density satisfying

$$(5.15) \quad f(x) \sim x^{-(1-d)} L(1/x) \frac{\sigma^2}{2\Gamma(d) \cos(\pi d/2)} \quad \text{as } x \downarrow 0$$

lead to the same asymptotic behaviour of the variance of the partial sums.

**Example 5.3.** The Fractional Gaussian Noise with covariance function given by (2.8) has a spectral density given by the formula

$$(5.16) \quad f(x) = \frac{\sigma^2}{2} C(H) (1 - \cos x) \sum_{j=-\infty}^{\infty} |2\pi j + x|^{-(1+2H)},$$

where

$$C(H) = \frac{2H(1-2H)}{\Gamma(2-2H)} \frac{1}{\cos \pi H}, \quad H \neq 1/2.$$

Indeed, with  $f$  as above,

$$\begin{aligned} \int_{(-\pi, \pi]} \cos(nx) f(x) dx &= \frac{\sigma^2}{2} C(H) \int_{-\infty}^{\infty} |x|^{-(1+2H)} (1 - \cos x) \cos(nx) dx \\ &= \frac{\sigma^2}{2} C(H) \left[ \int_0^{\infty} x^{-(1+2H)} (1 - \cos(n+1)x) dx \right. \\ &\quad \left. + \int_0^{\infty} x^{-(1+2H)} (1 - \cos(n-1)x) dx - 2 \int_0^{\infty} x^{-(1+2H)} (1 - \cos nx) dx \right] \\ &= \frac{\sigma^2}{2} C(H) \int_0^{\infty} x^{-(1+2H)} (1 - \cos x) dx \left[ (n+1)^{2H} + |n-1|^{2H} - 2n^{2H} \right] \\ &= \frac{\sigma^2}{2} \left[ (n+1)^{2H} + |n-1|^{2H} - 2n^{2H} \right], \end{aligned}$$

as in (2.8). Clearly, for  $1/2 < H < 1$  the spectral density in (5.16) satisfies

$$f(x) \sim \frac{\sigma^2}{4} C(H) x^{-(2H-1)} \quad \text{as } x \downarrow 0,$$

and it is easily verified that for FGN the asymptotic behaviour of correlations in (2.9) and of the spectral density at zero are related as in (5.14) and (5.15).

In fact, all three statements (5.5), (5.14) and (5.15) have been taken as definitions of long range dependence. The behaviour of the partial sum variance in (5.5) is, clearly, the least demanding of these 3 statements. For

example, a process with a spectral density equal in a neighbourhood of the origin to

$$f(x) = (1 + \cos(1/x))x^{-(1-d)}L(1/x) \frac{\sigma^2}{2\Gamma(d) \cos(\pi d/2)},$$

which is not regularly varying, will still satisfy (5.5). However, the relationship between (5.14) and (5.15) has been somewhat of a mystery, and in the literature one can sometimes read that these latter statements are equivalent. In fact, these statements are not equivalent; we will present examples shortly. The claims of equivalence appear to stem from several sources, one being a casual treatment of similar conditions in an influential paper of Cox (1984), and the second a definition of a slowly varying function in Zygmund (1968) that is different from, and more restrictive than, what is understood by this notion today.

We start with a positive result, that gives sufficient conditions under which the statements (5.14) and (5.15) imply each other. We will call an eventually positive function  $L$  *nice* if

(5.17) for every  $\delta > 0$  the function  $g_1 = x^\delta L(x)$  is eventually increasing

and the function  $g_2 = x^{-\delta} L(x)$  is eventually decreasing.

Note that a nice function is automatically slowly varying, but there are slowly varying functions that are not nice.

**Theorem 5.4.** (i) *Assume that the correlations are regularly varying in the sense of (5.14), and the function  $L$  is nice. Then the process has a spectral density that is regularly varying at zero, in the sense that (5.15) holds.*

(ii) *Conversely, assume that the process has a spectral density that is regularly varying at zero, in the sense that (5.15) holds, and the function  $L$  is nice. Then the correlations are regularly varying in the sense of (5.14).*

Part (i) of Theorem 5.4 is in Theorem (2-6) in Chapter V of Zygmund (1968). The proof of part (ii) will appear separately.

The following is an example of a situation where a spectral density is regularly varying at the origin as in (5.15), but correlations are not regularly varying as in (5.14).

**Example 5.5.** Let  $0 < \epsilon < \pi/2$ , and  $g$  a positive integrable function on  $(0, \epsilon)$  satisfying (5.15). Let

$$f(x) = g(|x|)\mathbf{1}(0 < |x| < \epsilon) + g(|\pi - x|)\mathbf{1}(\pi - \epsilon < |x| < \pi)$$

for  $-\pi < x < \pi$ . Then  $f$  is a spectral density satisfying (5.15). Notice that

$$\begin{aligned} R_n &= \int_{-\pi}^{\pi} \cos nx f(x) dx = \int_{-\epsilon}^{\epsilon} \cos nx g(x) dx + \int_{-\epsilon}^{\epsilon} \cos n(\pi - x) g(x) dx \\ &= \left(1 + (-1)^n\right) \hat{R}_n, \end{aligned}$$

where  $\hat{R}_n = \int \cos nx g(x) dx$ . Since  $R_n$  vanishes for all odd lags  $n$ , the correlations are not regularly varying, and (5.14) fails. Examples of this sort can also be constructed by letting the spectral density “blow up” around points other than  $x = \pi$ .

Next is an example of a situation where the correlations are regularly varying as in (5.14), but there is no version of a spectral density that is regularly varying at the origin as in (5.15).

**Example 5.6.** Let us start with a spectral density  $g$  that does satisfy (5.15), and such that the correlations are also regularly varying as in (5.14) (for example, one can take the spectral density of a FGN as in Example 5.3, or any spectral density satisfying part (ii) of Theorem 5.4.) We will construct a continuous nonnegative integrable function  $g_1$  on  $(0, \pi)$  such that

$$(5.18) \quad \liminf_{x \downarrow 0} x^2 g_1(x) > 0$$

and

$$(5.19) \quad \int_{-\pi}^{\pi} \cos nx g_1(x) dx = o\left(\int_{-\pi}^{\pi} \cos nx g(x) dx\right)$$

as  $n \rightarrow \infty$ . Then we will set  $f(x) = g(x) + g_1(|x|)$  for  $-\pi < x < \pi$ . It will follow from (5.18) that some pieces of  $f$  are too large near the origin to permit  $f$  to be regularly varying as in (5.15), but (5.19) and regular variation of the correlations corresponding to the density  $g$  mean that the correlations corresponding to the density  $f$  will also satisfy (5.14).

We proceed with a construction of a function  $g_1$ . Define

$$(5.20) \quad g_1(x) = 2^{2j} \quad \text{if } 2^{-j} \leq x \leq 2^{-j} + 2^{-2j} \quad \text{for } j = 0, 1, \dots$$

Clearly for  $x = 2^{-j}$ ,  $g_1(x) = x^{-2}$ , so (5.18) holds. Further,

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx g_1(x) dx &= \sum_{j=0}^{\infty} 2^{2j} \int_{2^{-j}}^{2^{-j}+2^{-2j}} \cos nx dx \\ &= \frac{2}{n} \sum_{j=0}^{\infty} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right) \cos\left(n(2^{-j} + 2^{-2j-1})\right) \leq \frac{2}{n} \sum_{j=0}^{\infty} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right) \\ &= \frac{2}{n} \sum_{j \leq \log_2 \log_2 n} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right) + \frac{2}{n} \sum_{j > \log_2 \log_2 n} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right). \end{aligned}$$

Clearly,

$$\begin{aligned} \frac{2}{n} \sum_{j \leq \log_2 \log_2 n} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right) &\leq \frac{2}{n} \sum_{j \leq \log_2 \log_2 n} 2^{2j} \\ &\leq cn^{-1} (\log_2 n)^2 \quad \text{for some } 0 < c < \infty \end{aligned}$$

and

$$\begin{aligned} \frac{2}{n} \sum_{j > \log_2 \log_2 n} 2^{2j} \sin\left(\frac{n}{2}2^{-2j}\right) &\leq \frac{2}{n} \sum_{j > \log_2 \log_2 n} 2^{2j} \left(\frac{n}{2}2^{-2j}\right) \\ &= \sum_{j > \log_2 \log_2 n} 2^{2j} 2^{-2j} \leq cn^{-1} (\log_2 n)^2 \quad \text{for some } 0 < c < \infty \end{aligned}$$

as well. This clearly implies (5.19).

Of course, the function  $g_1$  in (5.20) is not continuous, but it can be easily made such by appropriately ‘‘connecting the dots’’ at the jump points of the function in (5.20).

If one uses the second order approach to long range dependence in one of related, but not equivalent, ways discussed above, it would be nice to know that there is stability under point transformations discusses in Section 4. Namely, if  $\mathbf{X}$  is a stationary process with a finite variance with, say, regularly varying correlations as in (5.14), and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a one-to-one measurable function such that  $Eg(X_i)^2 < \infty$ , then the process  $Y_n = g(X_n)$  for  $n = 0, 1, \dots$  will also have a similar second-order behaviour. Unfortunately, this turns out not to be the case, and the correlations of the process  $\mathbf{Y}$  may turn out to decay much slower or much faster than those of the process  $\mathbf{X}$ . To construct examples of this type we need the notion of Hermite polynomials.

For  $n \geq 0$  define a function of a real variable  $x$  by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$



(some authors include an extra factor of  $1/n!$  in the definition of  $H_n$ ; see e.g. Nualart (1995), which can also be consulted for more details). Then  $H_0(x) = 1$ ,  $H_1(x) = x$ ,  $H_2(x) = x^2 - 1$  and, in general,

$$H_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n!}{m!(n-2m)!} (-2)^{-m} x^{n-2m}$$

for real  $x$ . That is, each  $H_n$  is a polynomial of degree  $n$ , and is called *the  $n$ th Hermite polynomial*. If  $X$  is standard normal, then

$$EH_n(X) = 0 \text{ for all } n \geq 1, \text{ and } \text{Var}H_n(X) = n!.$$

More generally, if  $X$  and  $Y$  are jointly normal, with zero mean, unit variance, and correlation  $\rho$ , then

$$(5.21) \quad EH_n(X)H_m(Y) = \begin{cases} 0 & \text{if } n \neq m \\ \rho^n n! & \text{if } n = m \end{cases}.$$

Furthermore, the family  $(H_n)_{n \geq 0}$  forms an orthogonal basis in the space  $L^2(\mathbb{R}, \mu_G)$ , where  $\mu_G$  is the law of the standard normal random variable. That is, if  $X$  is standard normal, and  $Eg(X)^2 < \infty$ , then

$$(5.22) \quad g(X) = \sum_{n=0}^{\infty} \frac{a_n}{n!} H_n(X),$$

where for  $n \geq 0$ ,  $a_n = E(H_n(X)g(X))$ , and the sum converges in  $L^2$ . This is the so-called *Hermite expansion* of the function  $g$ , and the smallest  $n \geq 1$  such that  $a_n \neq 0$  is called *the Hermite rank* of the function  $g$ .

Suppose now that  $\mathbf{X}$  is a stationary process Gaussian process, with zero mean and unit variance, and correlations satisfying (5.14). Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that  $Eg(X_i)^2 < \infty$ , and define a new stationary process  $\mathbf{Y}$  by  $Y_n = g(X_n)$  for  $n = 0, 1, \dots$ . As above, we tend to think that the process  $\mathbf{X}$  "remembers at least as much" as the process  $\mathbf{Y}$  does and, if the function  $g$  is, additionally, one-to-one, then we expect the two processes to have the same "length of memory". On the other hand, let  $k$  be the Hermite rank of the function  $g$ . It turns out that it is possible to have a one-to-one measurable function  $g$  such that  $kd > 1$  (where  $0 < d < 1$  is the exponent in (5.14)), as the following example indicates.

**Example 5.7.** Take  $a > 0$  such that

$$ae^{-a} \int_a^\infty xe^x e^{-x^2/2} dx = \int_0^a x^2 e^{-x^2/2} dx,$$

and define

$$g(x) = \begin{cases} -\frac{1}{a}x & \text{if } 0 \leq x < a \\ e^{x-a} & \text{if } x \geq a \end{cases}.$$

Set  $g(x) = -g(-x)$  for  $x < 0$ . Clearly, the function  $g$  is odd, measurable, one-to-one, and  $Eg(X)^2 < \infty$ . Furthermore, by the choice of the number  $a$ ,  $E(H_n(X)g(X)) = 0$  for  $n = 1$ , and, by the fact that  $g$  is odd, the same is true for  $n = 2$ . Therefore, the Hermite rank of the function  $g$  is at least 3 (in fact, it is not difficult to check that in this case the rank  $k$  of  $g$  is exactly equal to 3, but one can modify the construction and obtain one-to-one functions of arbitrarily high rank). Therefore, if the exponent  $d$  in (5.14) satisfies  $d > 1/3$ , then we have  $kd > 1$ .

For a process  $Y_n = g(X_n)$  for  $n = 0, 1, 2, \dots$ , it follows from (5.22), (5.21) and  $L^2$  convergence that the covariance function  $R^{(Y)}$  of the process  $\mathbf{Y}$  satisfies

$$R_j^{(Y)} = \sum_{n=k}^{\infty} a_n^2 \rho_j^n \sim a_k^2 \rho_j^k$$

as  $j \rightarrow \infty$ , where  $k$  is the Hermite rank of  $g$ . If  $g$  is a one-to-one function  $g$  satisfying  $kd > 1$ , then the two processes,  $\mathbf{X}$  and  $\mathbf{Y}$ , have correlations functions decaying at vastly different rates. In particular, the process  $\mathbf{Y}$  will have summable correlations, while the process  $\mathbf{X}$  does not; recall that *we expected  $\mathbf{X}$  and  $\mathbf{Y}$  to have the same lengthy of memory as the function  $g$  is one-to-one!* In fact, depending on the Hermite rank of the function  $g$ , the process  $\mathbf{Y}$  satisfies the Central Limit Theorem only if  $kd \geq 1$ , whereas in the case  $kd < 1$  it satisfies a so-called *Non-Central Limit Theorem*, see Dobrushin and Major (1979) and Breuer and Major (1983).

Therefore, using the behaviour of the correlations as a definition of long memory has the weakness that such behaviour can drastically change when applying a one-to-one point map. This, unfortunately, is a problem with many alternative definitions, other than ergodic and strong mixing notions of Section 4. Still, this is a warning sign against relying too much on correlations. Incidentally, in the example we have just considered, it definitely

makes sense to view the process  $\mathbf{X}$  as long range dependent, since it is a centered Gaussian process, and the covariances carry full information about such processes. The same cannot be said about the transformed process  $\mathbf{Y}$ . In general, the covariances may carry very little information about a process unless it is similar to a Gaussian one.

## 6. FRACTIONAL PROCESSES AND RELATED MODELS WITH LONG MEMORY

One often encounters the adjective "fractional" in the names of processes purportedly having long range dependence (the Fractional Gaussian Noise we encountered early on is an example). Partly this is due to the connotation "unusual" the adjective "fractional" carries. A deeper connection exists, however, and it goes back to the issues of stationarity and non-stationarity.

If  $\mathbf{X} = (\dots, X_{-1}, X_0, X_1, \dots)$  is a stationary process (note that we have switched, once again, to two-sided processes, as in Section 4), then the differenced process  $\mathbf{Y}$  with  $Y_n = X_n - X_{n-1}$  for  $n \in \mathbb{Z}$  is, clearly, also stationary. The typical notation is  $\mathbf{Y} = (I - B)\mathbf{X}$ , where  $I$  is the identity operator on the space of sequences  $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, x_2, \dots)$ , and  $B$  is the *backward shift operator* on that space:

$$B(\dots, x_{-1}, x_0, x_1, x_2, \dots) = (\dots, x_{-2}, x_{-1}, x_0, x_1, \dots)$$

( $B$  is the inverse of the shift operator  $T$  of Section 4). On the other hand, not every stationary process  $\mathbf{Y}$  is of the form  $\mathbf{Y} = (I - B)\mathbf{X}$  for some stationary process  $\mathbf{X}$ ; e.g. a sequence of i.i.d. not identically zero random variables is not of this form. If, however,  $\mathbf{Y}$  is of this form, one can write  $\mathbf{X} = (I - B)^{-1}\mathbf{Y}$  and call the process  $\mathbf{X}$  an integrated process (specifically, an integrated process  $\mathbf{Y}$ ). Obviously, if an integrated process exists, it is not uniquely determined: one can add the same random variable to each  $X_n$ , as long as doing so preserves stationarity.

It is intuitive that the differencing operator on stationary processes,  $\Delta = I - B$ , makes the memory in the process "less positive, more negative"; this is simply a consequence of alternating plus and minus signs attached to the same random variables. A simple illustration is obtained by considering what happens to a sequence of i.i.d. random variables under differencing.

Similarly, if it is possible "to integrate" a stationary process (i.e. to apply the inverse operator  $\Delta^{-1} = (I - B)^{-1}$ ) and obtain a stationary process, the integrated process will tend to have "more positive" memory than the original process. Long memory, when present, is usually "of the positive kind", so one can try to obtain a process with long range dependence by integrating some stationary process, and as many times as possible.

The problem is that, as we know, many "natural" stationary processes cannot be integrated even once, while preserving stationarity. It turns out, however, that sometimes one can integrate a process a fractional number of times, while preserving stationarity. This leads to a class of models known as *fractionally integrated processes*. The success of the construction depends on a definition of a fractional power of the differencing operator  $\Delta$ , and the starting point is the generalized Binomial formula (or the Taylor expansion): for all real  $d$

$$(6.1) \quad (1 - x)^d = \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} x^j,$$

where

$$\binom{d}{j} = \frac{d(d-1)\dots(d-j+1)}{j!}.$$

If  $d$  is a nonnegative integer, (6.1) is just the classical Binomial formula, and a sum with finitely many terms; otherwise it is an infinite sum, and then it can be rewritten in the form

$$(6.2) \quad (1 - x)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} x^j.$$

Using the Stirling formula for the Gamma function it is easy to check that

$$(6.3) \quad \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} \sim \frac{j^{-(d+1)}}{\Gamma(-d)}$$

as  $j \rightarrow \infty$ , and so the infinite series in (6.2) converges absolutely for all complex  $x$  with  $|x| < 1$  (inside the unit circle), and also on the boundary of that circle if  $d > 0$ .

Given a stationary process  $\mathbf{Y}$  we can formally define the process  $\mathbf{X} = \Delta^{-d}\mathbf{Y}$  by expanding  $\Delta^{-d} = (I - B)^{-d}$  for  $d$  that are not nonpositive integers into powers of the backward shift operator  $B$  as in (6.2) by formally

identifying the identity operator with the unity and the backshift operator  $B$  with  $x$  to obtain

$$(6.4) \quad X_n = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} Y_{n-j},$$

$n = \dots, -1, 0, 1, 2, \dots$ . If  $d > 0$ , we view the process  $\mathbf{X}$  as an integrated process, while if  $d < 0$ , we view it as a differenced process  $\mathbf{Y}$ . We are interested in the "integrated" case, with  $0 < d < 1$ ; if one needs to get beyond this range, one can first perform the usual "non-fractional" integration.

It is clear that, if the series in (6.4) converges in probability, then the resulting process  $\mathbf{X}$  is automatically stationary. Therefore, first of all we need to make sure that the infinite series in (6.4) converges. This requires imposing restrictions on the initial process  $\mathbf{Y}$ .

We start with assuming that the process  $\mathbf{Y}$  is a stationary zero mean finite variance process with variance  $\sigma^2$  and correlation function  $\rho$ , satisfying (5.2), that is, a process with absolutely summable correlations. Denoting the  $j$ th coefficient in (6.4) by  $a_j$  we note that for  $m, k \geq 1$

$$E \left( \sum_{j=m+1}^{m+k} a_j Y_{n-j} \right)^2 = \sigma^2 \sum_{j=m+1}^{m+k} a_j^2 + 2\sigma^2 \sum_{j=m+1}^{m+k} a_j \sum_{i=j+1}^{m+k} a_i \rho_{i-j}.$$

Since the sequence  $(a_j)$  (with  $0 < d < 1$ ) is easily seen to be decreasing, we conclude that

$$E \left( \sum_{j=m+1}^{m+k} a_j Y_{n-j} \right)^2 \leq \left( 1 + 2 \sum_{n=1}^{\infty} |\rho_n| \right) \sigma^2 \sum_{j=m+1}^{m+k} a_j^2.$$

If  $0 < d < 1/2$ , then the sum  $\sum_j a_j^2$  converges by (6.3), and so the series (6.4) converges in  $L^2$  to a stationary process.

Under somewhat stronger assumptions than the absolute summability of the correlations of the initial process  $\mathbf{Y}$ , the rate of decay of the correlation function of the partially integrated process is determined by the order of partial integration, as the following proposition shows.

**Proposition 6.1.** *Let  $\mathbf{Y}$  be a stationary zero mean finite variance process with variance  $\sigma^2$  and absolutely summable correlation function  $\rho$ . Let  $0 <$*

$d < 1/2$  and assume that

$$(6.5) \quad \Psi_n := \sum_{m=n}^{\infty} \rho_m = o\left(n^{-(1-2d)}\right) \quad \text{as } n \rightarrow \infty.$$

Then the process  $\mathbf{X}$  defined by (6.4) is a well defined zero mean stationary process whose covariance function  $R^*$  satisfies

$$(6.6) \quad R_n^* \sim \left( \sigma^2 \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \sum_{m=-\infty}^{\infty} \rho_m \right) n^{-(1-2d)}$$

as  $n \rightarrow \infty$ .

*Proof.* We have already established that  $\mathbf{X}$  is a well defined zero mean stationary process with finite variance. Its covariance function is given by

$$R_n^* = \sigma^2 \lim_{M \rightarrow \infty} \sum_{i=0}^M \sum_{j=0}^M a_i a_j \rho_{n+i-j} = \sigma^2 \lim_{M \rightarrow \infty} \sum_{m=n-M}^M b_{n-m}^{(M)} \rho_m,$$

where

$$b_k^{(M)} = \sum_{i=-k \vee 0}^{(M-k) \wedge M} a_i a_{i+k}.$$

Since the numbers  $b_k^{(M)}$  are uniformly bounded (by  $\sum_{-\infty}^{\infty} a_i^2$ ) and the correlations of the process  $\mathbf{X}$  are absolutely summable, we can use the dominated convergence theorem to obtain

$$R_n^* = \sigma^2 \sum_{m=-\infty}^{\infty} b_{n-m} \rho_m,$$

with

$$b_k = \sum_{i=-k \vee 0}^{\infty} a_i a_{i+k} = b_{-k}.$$

It follows from (6.3) that

$$(6.7) \quad b_k \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} k^{-(1-2d)} \quad \text{as } k \rightarrow \infty.$$

Similarly, since

$$(6.8) \quad a_k - a_{k+1} = \frac{1-d}{k+1} a_k \sim \frac{1-d}{\Gamma(d)} k^{-(2-d)},$$

we obtain also that

$$(6.9) \quad g_k := b_k - b_{k+1} \sim c k^{-2(1-d)} \quad \text{as } k \rightarrow \infty$$

for some  $c > 0$ . Clearly, the statement (6.6) will follow from (6.7) once we check that

$$(6.10) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1-2d} \left| \sum_{m=-\infty}^{-M} b_{n-m} \rho_m \right| = 0$$

and

$$(6.11) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1-2d} \left| \sum_{m=M}^{\infty} b_{n-m} \rho_m \right| = 0.$$

First of all, observe that by monotonicity, for  $M > 0$

$$\left| \sum_{m=-\infty}^{-M} b_{n-m} \rho_m \right| \leq b_n \sum_{m=-\infty}^{-M} |\rho_m|,$$

and so (6.10) follows from (6.7) and summability of the correlations of the process  $\mathbf{Y}$ . Next, using summation by parts we see that

$$\sum_{m=M}^{\infty} b_{n-m} \rho_m = b_{n-M+1} \Psi_M + \sum_{m=M}^{\infty} g_{n-m} \Psi_m.$$

By (6.7), for a  $c > 0$ ,

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1-2d} b_{n-M+1} \Psi_M = \lim_{M \rightarrow \infty} c \Psi_M = 0.$$

Furthermore, write

$$\sum_{m=M}^{\infty} g_{n-m} \Psi_m = \sum_{m \leq n/2} + \sum_{m > n/2} := S_n^{(1)}(M) + S_n^{(2)}(M).$$

By (6.9) and the assumption (6.5) we see that for some constant  $c$  and large  $n$

$$\begin{aligned} \left| S_n^{(1)}(M) \right| &\leq cn^{-2(1-d)} \sum_{m=M}^{[n/2]} |\Psi_m| \\ &\leq cn^{-2(1-d)} \sum_{m=M}^{[n/2]} m^{-(1-2d)} \leq cn^{-2(1-2d)}, \end{aligned}$$

and so for all  $M > 0$

$$\lim_{n \rightarrow \infty} n^{1-2d} S_n^{(1)}(M) = 0.$$

Finally, by the assumption (6.5) we have

$$\left| S_n^{(2)}(M) \right| \leq o(1) n^{-(1-2d)} \sum_{m=-\infty}^{\infty} |g_m|.$$

The sum in the right hand side above is finite by (6.9) and the fact that  $g_{-k} = -g_{k-1}$ . Therefore, for all  $M > 0$

$$\lim_{n \rightarrow \infty} n^{1-2d} S_n^{(2)}(M) = 0,$$

and we have checked (6.11).  $\square$

In particular, if

$$(6.12) \quad \sum_{m=-\infty}^{\infty} \rho_m \neq 0,$$

then the correlations  $\rho_n^*$  of the fractionally integrated process  $\mathbf{X}$  satisfy  $\rho_n^* \sim cn^{-(1-2d)}$  as  $n \rightarrow \infty$  for some  $0 < c < \infty$ .

Note that it is not surprising that we could only perform above a construction of a fractionally integrated process of the order  $0 < d < 1/2$ . Indeed, our intuition tells us that the higher the degree of "negative dependence" in a stationary process, the easier it is to integrate it while preserving stationarity. The assumptions in the discussion preceding Proposition 6.1 (in this case, absolute summability of correlations), while preventing the process from having "too much of positive dependence", do not imply any "negative dependence" either. Therefore, the dependence in initial process  $\mathbf{Y}$  can be viewed as only assumed to be "midway", between a very negatively dependent process, that can be integrated completely (of order  $d = 1$ ), and a very positively dependent process, that cannot be integrated at all. Hence, intuitively at least, the boundary  $d < 1/2$  is understandable. Processes with certain negative dependence can be integrated to a higher order, as we will see in the sequel. Such negative dependence will, in particular, imply that (6.12) breaks down.

In practice one often starts with  $\mathbf{Y}$  being an i.i.d. sequence, or a stationary ARMA model (see Brockwell and Davis (1987)). In this case the process  $\mathbf{Y}$  has exponentially fast decaying correlations, and Proposition 6.1 applies. The resulting models are typically called ARIMA models or, more explicitly, fractional ARIMA (or FARIMA, alternatively ARFIMA) models, and were originally introduced by Granger and Joyeux (1980) and Hosking (1981).

In the spectral domain things are even more transparent. Suppose that the original process  $\mathbf{Y}$  has absolutely summable correlations, and so it has a



continuous spectral density  $f$  given by (5.9). Since the series (6.4) converges in  $L^2$ , the fractionally integrated process  $\mathbf{X}$  has also a spectral density,  $f^*$ , given by  $f^*(x) = |\sum_{m=0}^{\infty} a_m e^{imx}|^2 f(x)$ , (where  $(a_j)$  are the coefficients in (6.4)), and the infinite sum in the expression for the density converges in  $L^2((-\pi, \pi], f(x)dx)$ ; see e.g. Theorem 4.10.1 in Brockwell and Davis (1987). Note that  $|\sum_{m=0}^{\infty} a_m z^m|^2 = |1 - z|^{-2d}$  for complex numbers  $z$  with  $|z| < 1$ , and the right hand side of this relation has a continuous extension to the part of the unit circle that is bounded away from the point  $z = 1$ . It follows that  $|\sum_{m=-\infty}^{\infty} a_m e^{imx}|^2 = |1 - e^{ix}|^{-2d}$  for almost every  $x \in (-\pi, \pi]$  (with respect to the Lebesgue measure), and so the integrated process  $\mathbf{X}$  has a spectral density given by

$$(6.13) \quad f^*(x) = |1 - e^{ix}|^{-2d} f(x), \quad x \in (-\pi, \pi].$$

In particular,

$$(6.14) \quad f^*(x) \sim x^{-2d} f(0) = \left( \frac{\sigma^2}{2\pi} \sum_{m=-\infty}^{\infty} \rho_m \right) x^{-2d} \quad \text{as } x \downarrow 0.$$

If the correlations of the original process  $\mathbf{Y}$  do not add up to zero (i.e. (6.12) holds), then the asymptotic behaviour at infinity of the correlations of the fractionally integrated process and asymptotic behaviour of its spectral density at the origin correspond, once again, to each other as (5.14) corresponds to (5.15).

To what extent can one integrate a stationary process that does not have a finite second moment, and what is the effect of existing negative dependence in the original process  $\mathbf{Y}$ ? Here is one simple situation. Let  $S_n^{(Y)} = Y_1 + \dots + Y_n$ ,  $n = 0, 1, \dots$  be the partial sum sequence of the process  $\mathbf{Y}$ . The rate of growth of the partial sum sequence depends both on the memory in the stationary process  $\mathbf{Y}$  and on the marginal tails of the process. Assume that for some  $\theta \in (0, 1)$  there is  $c > 0$  such that

$$(6.15) \quad E|S_n^{(Y)}| \leq cn^\theta, \quad n = 1, 2, \dots$$

Recall that, if  $\mathbf{Y}$  is a zero mean finite variance stationary process with absolutely summable correlations (or such that the series in (5.8) converges), then (6.15) holds with  $\theta = 1/2$ , while certain slow decay of correlations (or

a pole of the spectral density at the origin) can guarantee (6.15), but with  $\theta$  larger than  $1/2$ ; see (5.5) and Proposition 5.2. On the other hand, for the Fractional Gaussian Noise with  $0 < H < 1/2$ , the relation (6.15) holds with  $\theta$  smaller than  $1/2$ ; see (2.10).

**Proposition 6.2.** *Let a stationary process  $\mathbf{Y}$  be such that (6.15) holds for some  $0 < \theta < 1$ . Then for any  $0 < d < 1 - \theta$  the series (6.4) converges in  $L^1$  and the resulting process  $\mathbf{X}$  is a well defined stationary process.*

*Proof.* We may consider the sum (6.4) for  $n = 0$ , and we may also reverse the time in the process  $\mathbf{Y}$  noting that, marginally, the partial sums of the time reversed process have the same law as those of the original process. Using summation by parts we see that for  $m, k \geq 1$ ,

$$\sum_{j=m+1}^{m+k} a_j Y_j = \sum_{j=m+1}^{m+k} (a_j - a_{j+1}) S_j^{(Y)} + a_{m+k} S_{m+k}^{(Y)} - a_{m+1} S_m^{(Y)}.$$

The assumption  $d < 1 - \theta$  together with (6.15) and (6.3) shows that  $a_n S_n^{(Y)} \rightarrow 0$  in  $L^1$  as  $n \rightarrow \infty$ . Therefore, the last two terms in the above relation converge to zero in  $L^1$  as  $m \rightarrow \infty$  uniformly in  $k$ . Similarly, for some  $c > 0$  we have by (6.15) and (6.8),

$$E \left| \sum_{j=m+1}^{m+k} (a_j - a_{j+1}) S_j^{(Y)} \right| \leq c \sum_{j=m+1}^{\infty} j^{-(2-d)} j^\theta \rightarrow 0$$

as  $m \rightarrow \infty$  because  $d < 1 - \theta$ . This shows the  $L^1$  convergence, and stationarity is obvious.  $\square$

It is interesting to note that, if  $\mathbf{Y}$  is a sequence of i.i.d. zero mean random variables in the domain of attraction of an  $\alpha$ -stable law with  $1 < \alpha < 2$ , then (6.15) holds with any  $\theta < 1/\alpha$  (see e.g. Feller (1971)), and so by Proposition 6.2 such sequences can be integrated up to the order  $1 - 1/\alpha$ . I.i.d. sequences with even fatter tails (e.g. in the domain of attraction of an  $\alpha$ -stable law with  $0 < \alpha \leq 1$ ) cannot be fractionally integrated at all! However, assuming appropriate negative dependence, even “very fat tailed” stationary processes can be integrated up to some order (with the series (6.4) also converging in an appropriately weaker sense).

It is clear that much of the previous discussion depends on little more than the asymptotic order of the magnitude of the coefficients in the infinite series (6.4) and their differences. The specific choice arising from fractional differencing is attractive both because of its intuitive meaning and because of parsimony arising from dependence on a single parameter  $0 < d < 1$ .

Fractionally integrated models, especially FARIMA models, have found numerous applications in economics and econometrics; two examples are Crato and Rothman (1994) and Gil-Alana (2004). In this area of application one would like to combine fractional integration with the so-called *clustering of volatility*, or *conditional heteroscedasticity*. The standard (but non-fractional) model with clustering of volatility is the Generalized Autoregressive Conditionally Heteroscedastic (or GARCH) process, introduced in Engle (1982) in its original (non-generalized) form and generalized by Bollerslev (1986). A possible way of introducing clustering of volatility into a fractionally integrated model is to start with a process  $\mathbf{Y}$  in (6.4) that has the clustering of volatility property, for example with a GARCH process. This approach is suggested in Hauser and Kunst (1998). Even earlier on, an alternative model was suggested by Baillie et al. (1996). This model directly combines fractional differencing/integration with the recursion for computation of the conditional variance of each subsequent observation, and has become known as a Fractionally Integrated GARCH (or FIGARCH) model. This model has proved difficult to analyze; even existence of a stationary version of the model that has desired properties is an issue. Recent progress has been made in Douc et al. (2006); see also Zaffaroni (2004).

## 7. SELF-SIMILAR PROCESSES

Recall that a stochastic process  $\mathbf{Y} = (Y(t), t \geq 0)$  is called self-similar if there is  $H$  such that for all  $c > 0$  one has

$$(Y(ct), t \geq 0) \stackrel{d}{=} (c^H Y(t), t \geq 0).$$

The number  $H$  is alternatively referred to as the exponent of self-similarity, the scaling exponent, or the Hurst exponent. If  $X_i = Y(i) - Y(i-1)$ ,  $i = 1, 2, \dots$  is the increment process of  $\mathbf{Y}$ , then the partial sum process  $S_n =$

$X_1 + \dots + X_n$ ,  $n = 1, 2, \dots$ , clearly satisfies for  $n \geq 1$ ,

$$(7.1) \quad S_n = Y(n) - Y(0) \stackrel{d}{=} n^H(Y(1) - Y(0)) = n^H S_1.$$

If the process  $\mathbf{Y}$  also has stationary increments, then the process  $\mathbf{X} = (X_1, X_2, \dots)$  is stationary, and then (7.1) shows that the scaling exponent  $H$  determines the *distributional* order of magnitude of the partial sum process of the stationary model  $\mathbf{X}$ .

We have seen in Section 2 that the success of the Fractional Gaussian Noise in explaining the Hurst phenomenon is, at least in part, related to the fact that this stationary process is the increment process of the Fractional Brownian motion, a self-similar process, with the scaling exponent in the range  $0 < H < 1$ . Recall also that the correlations of the Fractional Gaussian Noise are summable when  $0 < H \leq 1/2$ , and not summable when  $1/2 < H < 1$  (cf. (2.9)), while the spectral density of the Fractional Gaussian Noise, given by (5.16), is continuous at the origin when  $0 < H \leq 1/2$ , while diverging at the origin when  $1/2 < H < 1$ . It is, therefore, attractive to consider the class of stationary models given as increments of general self-similar processes with stationary increments, and to call these stationary processes long range dependent if the scaling exponent is large enough; see e.g. Taquq (1987) and Beran (1994).

This program has the advantage of being applicable to stationary processes with or without finite second moment. The boundary between short and long memory is, further, given by a single number - a certain critical value of the scaling exponent. This last feature is also a drawback of the approach: a single number does not usually represent well the dependence structure of a stochastic process, despite the example of certain Gaussian models. Another drawback of this approach is that a reasonably limited family of the models is thus considered - the increments of self-similar stationary increments processes. To overcome this one can distinguish between *exactly self-similar* models as above, and those that are only self-similar in a certain asymptotic sense; see e.g. Lopez-Ardao et al. (2000). This class of models has become subject of intense research since it was pointed out in Leland et al. (1993) that Ethernet traffic data have features strikingly in

common with certain models of this type and a logically attractive explanation of the connection of the network traffic to self-similarity was offered in Willinger et al. (1997). Models arising from self-similar processes have also been used in risk theory (see e.g. Michna (1998)) and finance (see e.g. Cont (2005)).

Attractiveness of using the increment processes of self-similar processes with stationary increments as “canonical” models with shorter or longer types of memory is particularly obvious because such processes turn out to be *the only possible weak limit* in a common class of limiting procedures. Specifically, let  $(U(t), t \geq 0)$  be a stochastic process, and  $a_n \uparrow \infty$  be a sequence of positive numbers. If

$$(7.2) \quad \left( \frac{1}{a_n} U(nt), t \geq 0 \right) \Rightarrow (Y(t), t \geq 0)$$

in terms of finite-dimensional distributions, and the limiting process  $\mathbf{Y}$  is non-degenerate in the sense that  $P(Y(t) \neq 0) > 0$  for all  $t > 0$ , then  $\mathbf{Y}$  is  $H$ -self-similar for some  $H > 0$  (and the sequence  $(a_n)$  is automatically regularly varying with exponent  $H$ ). This was proved (in a slightly different form) by Lamperti (1962), and is often referred to as the Lamperti theorem. Since great many of the limiting results in probability theory and its applications can be formulated in the form (7.2), it is not surprising the self-similar models are ubiquitous. If, in addition, the process  $\mathbf{U}$  in (7.2) has stationary increments (as often happens in applications), the limiting process  $\mathbf{Y}$  will have stationary increments as well. Furthermore, the “type” of memory the increments of the process  $\mathbf{U}$  have, often translates into the “type” of memory that the increments of the limiting self-similar process  $\mathbf{Y}$  possess. For example, the strong mixing properties in Theorem 4.1 and Theorem 4.2 above is the sort of short range dependence that guarantees that any memory completely disappears in the limit, which in both theorems is a Brownian motion, that has independent increments. On the other hand, there are examples of processes  $\mathbf{U}$  whose memory is so strong that it persists in the limit; see e.g. Taqqu (1975) and Dobrushin and Major (1979). Then the limiting self-similar process  $\mathbf{Y}$  is not a Brownian motion; some of the possible limiting processes are discussed below, and their increments can be

strongly dependent. The results of the type (7.2) where the limiting process  $\mathbf{Y}$  is different from the Brownian motion are often referred to as non-central limit theorems.

Many facts on self-similar processes can be found in Chapter 7 of Samorodnitsky and Taqqu (1994) and in a recent book of Embrechts and Maejima (2002).

Let  $\mathbf{Y} = (Y(t), t \geq 0)$  be a self-similar process with stationary increments (commonly abbreviated to an SSSI process). There are restrictions on the feasible values of the scaling exponent  $H$ . It is immediate that the only self-similar process with  $H < 0$  is the trivial zero process  $Y(t) = 0$  a.s. for each  $t \geq 0$ . The value  $H = 0$  of the scaling exponent does allow some non-trivial SSSI processes (the process for which  $Y(t_j)$ ,  $j = 1, \dots, k$  are i.i.d. for any  $t_1, \dots, t_k$  and  $k = 1, 2, \dots$  is an example), but, assuming that  $\mathbf{Y}$  has a measurable version, leaves only the constant process  $Y(t) = Y(1)$  a.s. for each  $t \geq 0$  as a possibility (Vervaat (1985)). In modeling one assumes a positive scaling exponent  $H$ , as we will do from now on. This assumption, clearly, means that  $Y(0) = 0$  a.s. We will assume in the sequel that we are not dealing with the trivial zero process.

Further restrictions on the value of the scaling exponent of an SSSI process  $\mathbf{Y}$  are related to finiteness of the marginal moments of the process. For example, suppose that for some  $0 < \gamma < 1$  we have  $E|Y(1)|^\gamma < \infty$ . The assumption that  $Y(1) \neq 0$  with positive probability implies that for  $n$  large enough, on a set of positive probability, at least 2 of the variables in a finite stationary sequence  $(Y(1), Y(2) - Y(1), \dots, Y(n) - Y(n-1))$  are different from zero at the same time. Then by the self-similarity and stationarity of the increments

$$n^{\gamma H} E|Y(1)|^\gamma = E|Y(n)|^\gamma = E \left| \sum_{j=1}^n (Y(j) - Y(j-1)) \right|^\gamma$$

$$< \sum_{j=1}^n |Y(j) - Y(j-1)|^\gamma = n E|Y(1)|^\gamma,$$

which implies that  $H < 1/\gamma$ . In particular, the finite mean assumption  $E|Y(1)|^\gamma < \infty$  implies that also  $E|Y(1)|^\gamma < \infty$  for all  $0 < \gamma < 1$ , and so we

must have  $H < 1/\gamma$  for all such  $\gamma$  which is, clearly, equivalent to  $H \leq 1$ . Summarizing,

$$(7.3) \quad \begin{cases} H < \frac{1}{\gamma} & \text{if } E|Y(1)|^\gamma < \infty \text{ for } 0 < \gamma < 1 \\ H \leq 1 & \text{if } E|Y(1)| < \infty. \end{cases}$$

In fact, the only SSSI process with a finite mean for which  $H = 1$  is *the straight line process* for which  $Y(t) = tY(1)$  a.s. for every  $t > 0$ , as the following argument (due to Vervaat (1985)) shows. By self-similarity,  $Y(n)/n \stackrel{d}{=} Y(1)$  for all  $n \geq 1$ . By the stationarity of the increments and ergodic theorem

$$\frac{Y(n)}{n} = \frac{1}{n} \sum_{j=1}^n (Y(j) - Y(j-1)) \rightarrow E(Y(1)|\mathcal{I})$$

with probability 1, where  $\mathcal{I}$  is the invariant  $\sigma$ -field for the increment process  $\mathbf{X}$ . Therefore,  $Y(1) \stackrel{d}{=} E(Y(1)|\mathcal{I})$ . As Smit (1983) showed, this implies that  $Y(1)$  is measurable with respect to the completion of  $\mathcal{I}$ , and so  $Y(1) = Y(n) - Y(n-1)$  a.s. for all  $n \geq 1$ , implying that  $Y(t) = tY(1)$  a.s. for every  $t = 1, 2, \dots$ . Now one can use self-similarity to extend this relation first to  $t = 1, 1/2, 1/3, \dots$ , then to all rational  $t > 0$  and, finally, by the continuity in probability, which all SSSI processes with  $H > 0$ , clearly, possess, to all  $t > 0$ .

Non-trivial finite mean SSSI models exist, therefore, only for  $0 < H < 1$ , and we will, correspondingly, restrict ourselves to that range when the mean is finite. Since self-similarity forces  $EY(n) = n^H EY(1)$ , while stationarity of the increments implies  $EY(n) = nEY(1)$ , the non-trivial finite mean SSSI models must have zero mean.

Suppose that  $\mathbf{Y}$  is a zero mean *finite variance* SSSI process with  $0 < H < 1$ . Denoting  $\sigma^2 = EY(1)^2$ , we immediately see that for all  $s, t \geq 0$

$$(7.4) \quad \text{Cov}(Y(s), Y(t)) = \frac{\sigma^2}{2} [t^{2H} + s^{2H} - |t - s|^{2H}] ,$$

so the self-similarity and stationarity of the increments uniquely determine the correlation function of any such process, which is then also the correlation function of the Fractional Brownian motion introduced in Section 2. It turns out that for any  $0 < H < 1$  the expression in the right hand side of (7.4) is, in fact, nonnegative definite and, hence, a legitimate covariance

function. This can be demonstrated by simply exhibiting a Gaussian process whose covariance function is given by the right hand side of (7.4).

Let  $(B(t), t \in \mathbb{R})$  be the standard Brownian motion. Choose a real number  $1/2 < \gamma < 1$ ,  $\gamma \neq 3/2 - H$ , and define a stochastic process by

$$(7.5) \quad B_H(t) = \frac{\sigma}{C(H, \gamma)} \int_{-\infty}^{\infty} \left( \int_x^{\infty} (v-x)^{-\gamma} (|v|^{H+\gamma-3/2} - |v-t|^{H+\gamma-3/2}) dv \right) B(dx),$$

where

$$C(H, \gamma) = \left( \int_{-\infty}^{\infty} \left( \int_x^{\infty} (v-x)^{-\gamma} (|v|^{H+\gamma-3/2} - |v-1|^{H+\gamma-3/2}) dv \right)^2 dx \right)^{1/2}.$$

This is a well defined centered Gaussian process, and its covariance function is easily checked to be given by the right hand side of (7.4) (independently of  $\gamma$ ). Since that characteristic function has the property  $\text{Cov}(Y(cs), Y(ct)) = c^{2H} \text{Cov}(Y(s), Y(t))$  for all  $c > 0$ , we conclude that a centered Gaussian process with that characteristic function is, in fact, self-similar with exponent  $H$ . Since (7.4) is equivalent to the incremental variance statement  $E(Y(t) - Y(s))^2 = \sigma^2 |t - s|^{2H}$ , it implies, for a Gaussian process, stationarity of the increments as well. Therefore, we have constructed in (7.5) a Fractional Brownian motion, which is then the only SSSI Gaussian process. For  $H = 1/2$  and  $0 < \gamma < 1$  (7.5) gives different representations of the standard Brownian motion.

Other important finite variance SSSI processes, different from the Fractional Brownian motion, can be represented as multiple Wiener-Itô integrals with respect to the Brownian motion; we refer the reader to Major (1981) or Section 1.1.2 in Nualart (1995) for basic information on the multiple integrals. For  $k = 1, 2, \dots$  and  $1/2 < \gamma < 1/2 + 1/(2k)$ ,  $H + k\gamma \neq 1 + k/2$ , define

$$(7.6) \quad Y^{(k)}(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_t^{(k)}(x_1, \dots, x_k) B(dx_1) \dots B(dx_k)$$

for  $t \geq 0$ , where  $(B(t), t \in \mathbb{R})$  is still the standard Brownian motion, and the kernel  $Q_t^{(k)}$  is defined by

$$(7.7) \quad Q_t^{(k)}(x_1, \dots, x_k) = \int_{\max\{x_1, \dots, x_k\}}^{\infty} \prod_{j=1}^k (v - x_j)^{-\gamma} (|v|^{H+k\gamma-1-k/2}$$



$$-|v - t|^{H+k\gamma-1-k/2} dv.$$

This process is mentioned in Mori and Oodaira (1986), following a similar process introduced in Rosenblatt (1979) (for  $k = 2$ ). It is, obviously, a generalization of the Fractional Brownian motion in (7.5). If the latter can be viewed as a linear functional of the sample paths of the Brownian motion, the process in (7.6) can be viewed as a polynomial functional of order  $k$  of these sample paths. The fact that the process  $\mathbf{Y}$  in (7.6) is well defined follows from the fact that

$$(7.8) \quad \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_t^{(k)}(x_1, \dots, x_k)^2 dx_1 \dots dx_k < \infty \text{ for all } t \geq 0;$$

the verification is standard, if somewhat tedious. It can be easily checked that, in addition, the kernel  $Q_t^{(k)}$  has also the following properties. For all  $0 \leq s < t$  and  $c > 0$

$$(7.9) \quad Q_t^{(k)}(x_1, \dots, x_k) - Q_s^{(k)}(x_1, \dots, x_k) = Q_{t-s}^{(k)}(x_1 - s, \dots, x_k - s),$$

and

$$(7.10) \quad Q_{ct}^{(k)}(cx_1, \dots, cx_k) = c^{H-k/2} Q_t^{(k)}(x_1, \dots, x_k)$$

for almost all  $(x_1, \dots, x_k)$ . Every stochastic process given in the form (7.6) with the functions  $(Q_t^{(k)}, t \geq 0)$  satisfying (7.9), has stationary increments, and every stochastic process given in the form (7.6) with the functions  $(Q_t^{(k)}, t \geq 0)$  satisfying (7.10) is self-similar with exponent of self-similarity  $H$ . Both statements are heuristically obvious when one makes the appropriate change of variable in the defining multiple integral in (7.6), and uses the stationary of the increments of the Brownian motion, and its self-similarity with exponent  $1/2$ . This argument can be made precise by approximating the kernel  $Q_t^{(k)}$  by simple symmetric kernels.

Therefore, the stochastic process defined in (7.6) with the kernel given by (7.7) is SSSI, with the scaling exponent  $H$ . As all other multiple Wiener-Itô integrals with respect to the Brownian motion, it has finite moments of all orders. It shares with the Fractional Brownian motion its correlation function, but is not a Gaussian process if  $k \geq 2$ .

It is not difficult to check that the properties (7.9) and (7.10) (with  $k = 1$ ) of the kernel  $Q_t^{(1)}(x)$  in the representation (7.5) of the Fractional Brownian

motion imply that for  $0 < H < 1$ ,  $H \neq 1/2$ ,

$$(7.11) \quad Q_t^{(1)}(x) = g_t(c_1, c_2; H; x) := c_1 \left[ ((t-x)_+)^{H-1/2} - ((-x)_+)^{H-1/2} \right] \\ + c_2 \left[ ((t-x)_-)^{H-1/2} - ((-x)_-)^{H-1/2} \right],$$

where  $a_+ := \max(a, 0)$  is the positive part of a real number  $a$ , and  $a_- := \max(-a, 0)$  is its negative part, and  $0^a$  is interpreted as 0 for all  $a \in \mathbb{R}$ . Here  $c_i = c_i(H, \gamma)$ ,  $i = 1, 2$ , are real numbers; in fact, one can start with choosing  $c_1$  and  $c_2$  in such a way that  $Q_1^{(1)}(-1) = g_1(c_1, c_2; H; -1)$  and  $Q_1^{(1)}(2) = g_1(c_1, c_2; H; 2)$ , and then show that the equality extends to all  $t \geq 0$  and  $x \neq 0, t$ . For  $H = 1/2$  a similar argument shows that

$$(7.12) \quad Q_t^{(1)}(x) = g_t(c_1, c_2; 1/2; x) := c_1 \mathbf{1}_{[0,t]}(x) + c_2 (\log |t-x| - \log |x|),$$

once again for some real  $c_i = c_i(\gamma)$ ,  $i = 1, 2$ .

In fact, alternative representations of the Fractional Brownian motion (up to a multiplicative constant) are obtained via

$$B_H(t) = \int_{-\infty}^{\infty} g_t(c_1, c_2; H; x) B(dx), \quad t \geq 0$$

for *arbitrary* real  $c_1, c_2$ . These are the so called *moving average representations* of the Fractional Brownian motion, originating with Mandelbrot and Van Ness (1968); see Section 7.2.1 in Samorodnitsky and Taqqu (1994).

Moving average representations of the Fractional Brownian motion different from the representation (7.5) can themselves be extended to SSSI processes represented by multiple Wiener-Itô integrals. For example, take  $c_2 = 0$  in (7.11). In the case  $1/2 < H < 1$  one can rewrite the resulting expression in an equivalent form, and extend it, leading to a family of processes

$$(7.13) \quad Y^{(k)}(t) = \int_{-\infty}^t \dots \int_{-\infty}^t \left( \int_0^t \prod_{j=1}^k ((v-x_j)_+)^{-(1/2+(1-H)/k)} dv \right) \\ B(dx_1) \dots B(dx_k), \quad t \geq 0,$$

introduced in Taqqu (1978); it appeared as a limit in a “non-central limit theorem” in Taqqu (1979) and, in a more general situation, in Surgailis

(1981a) (see also Avram and Taqqu (1987)). In the case  $0 < H < 1/2$  a similar procedure leads to a family of processes

$$(7.14) \quad Y^{(k)}(t) = \int_{-\infty}^t \dots \int_{-\infty}^t \left( \int_t^{\infty} \prod_{j=1}^k (v - x_j)^{-(1/2+(1-H)/k)} dv \right. \\ \left. - \mathbf{1}(\max(x_1, \dots, x_k) < 0) \int_0^{\infty} \prod_{j=1}^k (v - x_j)^{-(1/2+(1-H)/k)} dv \right) \\ B(dx_1) \dots B(dx_k), \quad t \geq 0.$$

One can check that in both cases the kernels in the multiple integrals satisfy (7.8), (7.9) and (7.10) and, hence, the processes defined in (7.13) and (7.14) are SSSI processes, with the corresponding scaling exponent  $H$ . In fact, for  $k \geq 2$  the process given in (7.14) is well defined for all  $0 < H < 1$ .

The SSSI processes with representations as in (7.6), (7.13) and (7.14) are examples of such processes in *the  $k$ th Gaussian chaos*, in the terminology of Wiener (1938). If, for  $k \geq 1$ ,  $\mathbf{Y}^{(k)}$  has the representation (7.6), with the kernels  $Q_t^{(k)}$  satisfying, for each  $k$ , (7.8), (7.9) and (7.10), then for any sequence of constants  $(a_k)$  such that

$$\sum_{k=1}^{\infty} a_k^2 k! \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_t^{(k)}(x_1, \dots, x_k)^2 dx_1 \dots dx_k < \infty,$$

the new process

$$(7.15) \quad Y(t) = \sum_{k=1}^{\infty} a_k Y_k(t) \\ = \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_t^{(k)}(x_1, \dots, x_k) B(dx_1) \dots B(dx_k)$$

is a well defined second order stochastic process (see Nualart (1995)), and the previous argument using stationarity of the increments of the Brownian motion and its self-similarity implies that this process is SSSI, with exponent  $H$  of self-similarity. Of course, this process is no longer, in general, a polynomial-like functional of the Brownian motion.

Yet more representations of the Fractional Brownian motion exist; see e.g. Decreusefond and Üstünel (1999) and Norros et al. (1999). We mention only one more, as it has an impact on our discussion of long range dependence.

Let  $(B^{(j)}(t), t \geq 0)$  for  $j = 1, 2$  be independent standard Brownian motions, and extend  $\mathbf{B}^{(1)}$  to the entire real line as an even function, and  $\mathbf{B}^{(2)}$  as an odd function. For  $0 < H < 1$  the following is, up to a multiplicative constant, a representation of the Fractional Brownian motion:

$$(7.16) \quad B_H(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1}{ix} |x|^{-(H-1/2)} \tilde{B}(dx), \quad t \geq 0,$$

where  $\tilde{B}(t) = B^{(1)}(t) + iB^{(2)}(t)$ ,  $t \in \mathbb{R}$ . This is the so called *harmonizable representation* of the Fractional Brownian motion; its origins go back to Kolmogorov (1940) and Yaglom (1955). The harmonizable representation also has a natural extension to a SSSI process in the  $k$ th Gaussian chaos for  $k = 2, 3, \dots$ , appearing (as a limit) in Dobrushin and Major (1979), who used the techniques introduced in Dobrushin (1979). The obtained processes have been showed by Taqqu (1979) to coincide, in the case  $1/2 < H < 1$ , with those defined in (7.13) via a moving average representation.

The above discussion presented a large number of SSSI processes with a finite variance and exponent of self-similarity  $H \in (0, 1)$ ; even more can be obtained by replacing some of the repeated Brownian motion in the multiple integrals by independent copies of a Brownian motion, or by more generally correlated Brownian motions. The increments of each one of them form a stationary process, whose covariance function coincides with that of the Fractional Gaussian Noise given in (2.8). One often says that these stationary processes have long range dependence if  $1/2 < H < 1$  (and short memory if  $0 < H \leq 1/2$ ); see e.g. Beran (1994), Willinger et al. (1998) and Embrechts and Maejima (2002). This is, of course, entirely understandable from the point of view of the rate of decay of correlations, as in (5.14), or from the point of view of the behaviour of the spectral density at the origin, as in (5.15), or from the point of view of the rate of increase of the variance of the partial sums, as in (5.5). While no further justification seems to be necessary for the Fractional Gaussian noise, the increment process of the Fractional Brownian motion; for the other models the second order measures provides, of course, only partial information. A further important point is the distributional rate of growth of the partial sums of the increment

processes: for  $1/2 < H < 1$  this rate of growth is above what is allowed for a central limit theorem and convergence to a Brownian motion.

Notice that, if  $\mathbf{Y}$  is an SSSI process with an exponent  $H$  of self-similarity, and  $A$  is a random variable independent of  $\mathbf{Y}$ , then the process

$$(7.17) \quad Z(t) = AY(t), \quad t \geq 0$$

is also an SSSI process with the same scaling exponent  $H$  as  $\mathbf{Y}$ . If  $\mathbf{Y}$  is a finite variance process, and  $A$  also has a finite variance, then the resulting SSSI process in (7.17) will have a finite variance as well. In particular, its increment process will have exactly the same second order properties as the Fractional Gaussian noise, and it will have the same distributional rate of growth of the partial sums as the latter. However, apart from a small number of degenerate situations, the increments of  $\mathbf{Z}$  will be non-ergodic and, hence, will arguably have infinite memory, *regardless of the value of the scaling exponent  $H$* .

This is, of course, not the situation with the SSSI processes with a chaos representation as in (7.15). The increment processes of the latter are always ergodic because of the property (7.9) and ergodicity of Bernoulli shifts applied to the increments of the Brownian motion (see Section 1.4 in Krengel (1985)). However the example of the process (7.17) emphasizes the limited amount information provided by the scaling exponent alone.

The most common *infinite variance* SSSI processes are  $\alpha$ -stable processes,  $0 < \alpha < 2$ ; we refer the reader to Samorodnitsky and Taqqu (1994) for information on such models. For an  $\alpha$ -stable random variable  $X$  one has power-like tails:  $P(|X| > x) \sim cx^{-\alpha}$  as  $x \rightarrow \infty$  for some  $c > 0$ ; this implies that the mean is finite in the case  $1 < \alpha < 2$ , and infinite if  $0 < \alpha \leq 1$ . We will consider here *symmetric*  $\alpha$ -stable (SaS) SSSI processes.

There are many similarities between finite variance Gaussian SSSI models and processes related to them, and SaS and related SSSI processes, the most important of which is the fact that both arise in a number of natural limit theorems. In the finite mean case  $1 < \alpha < 2$  the exponent of self-similarity of any SaS SSSI process is still restricted (to avoid trivialities) to the range  $0 < H < 1$ , while in the infinite mean case  $0 < \alpha \leq 1$ , the tail

behaviour of the marginal distributions restricts the scaling exponent of a S $\alpha$ S SSSI process to the range  $0 < H \leq 1/\alpha$ . In a significant departure from the Gaussian case, where the exponent of self-similarity determines the correlation function and, hence, the law of the SSSI process (up to a multiplicative constant), for every feasible pair  $(\alpha, H)$  of the index of stability and scaling exponent, there are generally *many different* S $\alpha$ S SSSI models; some of them will be discussed below. The only exception is the case  $0 < \alpha < 1$ ,  $H = 1/\alpha$ , which corresponds to a single process, the S $\alpha$ S Lévy motion; see Samorodnitsky and Taqqu (1990). It is common to use the increments of certain S $\alpha$ S SSSI processes as canonical heavy tailed models with long range dependence.

A S $\alpha$ S Lévy process (motion)  $(Y(t), t \geq 0)$  is the heavy tailed equivalent of the Brownian motion, a process with stationary and independent increments. It is self-similar, and its scaling exponent is  $H = 1/\alpha$ . No other (symmetric) Lévy processes are self-similar; see Sato (1999). When deciding which S $\alpha$ S SSSI processes should be said to have long range dependent increments, the Gaussian case has been often taken as guidance. This means using the S $\alpha$ S Lévy motion as the benchmark, and viewing the processes with  $H > 1/\alpha$  as long range dependent, see e.g. Abry et al. (2000), Stoev and Taqqu (2004) and Kotulska (2007). Unfortunately, this range of the exponent of self-similarity is only possible when  $\alpha > 1$ . A number of limit theorems in which  $\alpha$ -stable SSSI processes appear have been established, most of them apply to partial sums of linear infinite order moving average processes; in situations where long memory is believed to be present the exponent of self-similarity of the resulting SSSI process turned out to be in the range  $H > 1/\alpha$ . See Maejima (1983b), Astrauskas (1983b), Kasahara et al. (1988) or Levy and Taqqu (2000). A continuous-time version for a shot noise model is in Giraitis and Surgailis (1991).

Let  $\mathbf{Y}$  be a S $\alpha$ S SSSI process, and  $X_n = Y(n) - Y(n-1)$  for  $n = 1, 2, \dots$  be its stationary increment process. Since  $\alpha$ -stable processes with  $0 < \alpha < 2$  have infinite variance, it is impossible to relate the case  $H > 1/\alpha$  to a slow decay of correlations of  $\mathbf{X}$ . However, when  $H > 1/\alpha$ , the partial sums

$S_n = X_1 + \dots + X_n = Y(n)$ ,  $n = 1, 2, \dots$ , grow distributionally at the rate  $n^H$ , larger than the "usual" rate of  $n^{1/\alpha}$  (times a slowly varying function) associated with the heavy tailed version of the Functional Central Limit Theorem, where the limit is the  $\alpha$ -stable Lévy motion; see e.g. Durrett and Resnick (1978) (for interesting topological difficulties that may arise see Avram and Taqqu (1992)). While for  $1 < \alpha < 2$  the benchmark  $H = 1/\alpha$  is in the middle of the feasible range  $0 < H < 1$  of the exponent of self-similarity, in the case  $0 < \alpha \leq 1$  it is its *right endpoint*. This, of course, means that the rate of  $n^{1/\alpha}$  is the fastest possible rate at which the partial sums of the increment process of a SaS SSSI process with such an index of stability  $\alpha$  can grow; in fact, the partial sums of *any* stationary SaS process with  $0 < \alpha \leq 1$  can grow at most at the rate of  $n^{1/\alpha}$ . However, it means that, according to the rule  $H > 1/\alpha$ , no SaS SSSI process can have long range dependent increments. This is, clearly, unfortunate.

Most of SaS SSSI processes discussed in the extensive literature on the subject are constructed with, once again, guidance from the Gaussian case. One starts, typically, with a representation of the Fractional Brownian motion and modifies it appropriately. The best known SaS self-similar process with stationary increments originates with the moving average representation (see (7.11)) of the latter. Let  $(L(t), t \in \mathbb{R})$  be a SaS Lévy motion. Choose  $0 < H < 1$  (with  $H \neq 1/\alpha$  if  $1 < \alpha < 2$ ) and define for real  $c_1, c_2$

$$(7.18) \quad Y(t) = \int_{-\infty}^{\infty} Q_t(x) L(dx), \quad t \geq 0,$$

where

$$(7.19) \quad Q_t(x) = c_1 \left[ ((t-x)_+)^{H-1/\alpha} - ((-x)_+)^{H-1/\alpha} \right] \\ + c_2 \left[ ((t-x)_-)^{H-1/\alpha} - ((-x)_-)^{H-1/\alpha} \right].$$

This is a well defined SaS process. The kernel defined by (7.19) satisfies, for  $0 \leq s < t$  and  $c > 0$

$$(7.20) \quad Q_t(x) - Q_s(x) = Q_{t-s}(x-s)$$

and

$$(7.21) \quad Q_{ct}(cx) = c^{H-1/\alpha} Q_t(x)$$

for almost all  $x \in \mathbb{R}$ . It is clear that the intuition we used in the Gaussian case still works here: the properties (7.20) and (7.21) of the kernel imply that the process in (7.18) has stationary increments and is self-similar (and, once again, the argument can be made precise). The SSSI process defined by (7.18) with the kernel given by (7.19) is called *Linear Fractional Stable Motion*. This process originates in Taqqu and Wolpert (1983) and Maejima (1983a). It is a general phenomenon that the stable integrals are much more “rigid” than similarly looking Gaussian integrals. Whereas any choice of the constants  $c_1$  and  $c_2$  in (7.11) produces, up to a multiplicative constant, a representation of the same Fractional Brownian motion, different pairs  $(c_1, c_2)$  in (7.19) will produce *different* Linear Fractional Stable Motions, unless these parameters are proportional (see Cambanis and Maejima (1989) and Samorodnitsky and Taqqu (1989)).

If  $1 < \alpha < 2$  and  $H = 1/\alpha$ , the process corresponding to the Linear Fractional Stable Motion is given in the form (7.18) with the kernel

$$(7.22) \quad Q_t^{(1)}(x) = c_1 \mathbf{1}_{[0,t]}(x) + c_2 (\log |t-x| - \log |x|),$$

for real  $c_1, c_2$ , which is, of course, identical to (7.12). When  $c_2 = 0$  this gives back the SaS Lévy motion, for  $c_1 = 0$  we obtain the so called *Log-fractional Stable Motion* introduced by Kasahara et al. (1988); it is easily seen not to have independent increments.

In the case  $1 < \alpha < 2$  it is also possible to start with the kernel in a representation of the Fractional Brownian motion as given in (7.5). The corresponding kernel in the  $\alpha$ -stable case is

$$Q_t(x) = \int_x^\infty (v-x)^{-\gamma} (|v|^{H+\gamma-1-1/\alpha} - |v-t|^{H+\gamma-1-1/\alpha}) dv$$

for  $0 < H < 1$  and  $1/\alpha < \gamma < 1$ ,  $\gamma \neq 1 + 1/\alpha - H$ . This kernel satisfies (7.20) and (7.21) and, hence, can be shown to coincide with (7.19) when  $H \neq 1/\alpha$ , and with (7.22) when  $H = 1/\alpha$ , in both cases for some  $c_1$  and  $c_2$  depending on  $\gamma$ .

One can also start with the harmonizable representation (7.16) of the Fractional Brownian motion and extend it to the  $\alpha$ -stable case,  $0 < \alpha < 2$ . This is usually done by starting with a complex-valued *isotropic* SaS Lévy



motion  $\tilde{M}$  (see Samorodnitsky and Taqqu (1994) for details) and defining

$$(7.23) \quad Y(t) = \mathcal{R}e \int_{-\infty}^{\infty} \frac{e^{itx} - 1}{ix} |x|^{-(H-1+1/\alpha)} \tilde{M}(dx), \quad t \geq 0.$$

This process is often referred to as *Harmonizable Fractional Stable Motion*, and it was introduced in Cambanis and Maejima (1989). The Harmonizable Fractional Stable Motion is a *different process* from the Linear Fractional Stable Motion; in certain cases this follows from Cambanis and Soltani (1984), more generally for  $1 < \alpha < 2$  this is in Cambanis and Maejima (1989), and in full generality with  $0 < \alpha < 2$  it is in Chapter 7 of Samorodnitsky and Taqqu (1994). In fact, the stationary increment process of the latter is a mixing stationary process (this is implicit in Cambanis et al. (1987) and explicit in Surgailis et al. (1993)), while the increment process of the former is not even ergodic (this statement is in Cambanis et al. (1987), and it also follows from the fact that real *stationary* harmonizable processes have a representation as mixtures of stationary Gaussian processes, discovered by Marcus and Pisier (1984)).

The above classes of S $\alpha$ S SSSI processes can be viewed as linear functionals of S $\alpha$ S Lévy motions in their integral representations. Analogously to the Gaussian case, new SSSI models can be constructed as polynomial-type functions of S $\alpha$ S Lévy motions, as multiple stochastic integrals, i.e. as stochastic processes of the form

$$(7.24) \quad Y^{(k)}(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Q_t^{(k)}(x_1, \dots, x_k) L(dx_1) \dots L(dx_k)$$

for  $t \geq 0$ , where  $(L(t), t \in \mathbb{R})$  is a S $\alpha$ S Lévy motion. The conditions on the kernel  $Q_t^{(k)}$  for the integral in (7.24) to exist are “more complicated” in the stable case than in the Gaussian case. A sufficient condition for integrability is

$$(7.25) \quad \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |Q_t^{(k)}(x_1, \dots, x_k)|^\alpha \left( \log_+ \frac{|Q_t^{(k)}(x_1, \dots, x_k)|^\alpha}{\psi(x_1) \dots \psi(x_k)} \right)^{k-1} dx_1 \dots dx_k < \infty, \quad t \geq 0,$$

for a strictly positive probability density  $\psi$  on  $\mathbb{R}$ , where  $\log_+ x = \log x$  for  $x > 1$ , and  $= 0$  otherwise. See Rosiński et al. (1991). Once the process

is well defined, if the kernel  $Q_t^{(k)}$  also satisfies the condition (7.9) and the following analog of (7.10): for all  $c > 0$

$$(7.26) \quad Q_{ct}^{(k)}(cx_1, \dots, cx_k) = c^{H-k/\alpha} Q_t^{(k)}(x_1, \dots, x_k)$$

for almost all  $(x_1, \dots, x_k)$ , then the process defined in (7.24) is SSSI, with exponent  $H$  of self-similarity.

A model with these properties was introduced in Surgailis (1981b), and it is a SaS version of the process in (7.13). Assume that  $1 < \alpha < 2$  and  $H \in (1/\alpha, 1)$ . Then the choice

$$(7.27) \quad Q_t^{(k)}(x_1, \dots, x_k) = \int_0^t \prod_{j=1}^k ((v - x_j)_+)^{-(1/\alpha + (1-H)/k)} dv$$

leads to a well defined SSSI process, which is, of course, a direct generalization of the multiple Wiener-Itô integral process in (7.13). This process appears as a limit in the “non-central limit theorem” setting (in the case  $k = 2$ ), as shown in Astrauskas (1983a). Similarly, the multiple Wiener-Itô integral process in (7.14) can be generalized to the  $\alpha$ -stable multiple integral situation as well, resulting in another SSSI model with

$$(7.28) \quad \begin{aligned} Q_t^{(k)}(x_1, \dots, x_k) &= \mathbf{1}(\max(x_1, \dots, x_k) < t) \int_t^\infty \prod_{j=1}^k (v - x_j)^{-(1/\alpha + (1-H)/k)} dv \\ &- \mathbf{1}(\max(x_1, \dots, x_k) < 0) \int_0^\infty \prod_{j=1}^k (v - x_j)^{-(1/\alpha + (1-H)/k)} dv. \end{aligned}$$

For single integrals this is well defined only if  $H < 1/\alpha$ , but for  $k \geq 2$  this process is well defined for all  $0 < H < 1$  (for both (7.27) and (7.28) the condition (7.25) can be verified with, for example,  $\psi$  being the standard Cauchy density). We are not aware of limit theorems in which the process with the kernel as in (7.28) appears as the limit.

There is no doubt that other SSSI processes in the form of a finite order symmetric  $\alpha$ -stable chaos can be defined, for example by extending yet other SaS processes, or by a direct analogy with the Gaussian case. Using the recipe (7.15) one can construct even more models (even though necessary and sufficient conditions on the sequence  $(a_k)$  for the series in (7.15) to converge when each SSSI process  $\mathbf{Y}^{(k)}$  is in the  $k$ th SaS chaos do not seem

to be known, it is obvious that the series will converge if the coefficients are “small enough”). Similarly to the Gaussian case, one can also replace some of the repeated S $\alpha$ S Lévy motions in a  $k$ -tuple integrals by the components of a  $k$ -dimensional S $\alpha$ S Lévy motion (see Sato (1999)). Unlike the Gaussian case, this last exercise can even be performed on certain S $\alpha$ S SSSI processes by, for example, integrating each one of the two parts in (7.19) or (7.22) with respect to different components of a bivariate S $\alpha$ S Lévy motion.

This provides for an even greater variety of SSSI processes with infinite variance, based on stochastic integrals with respect to S $\alpha$ S Lévy processes with  $0 < \alpha < 2$ , than of the finite variance models we considered above. In the case  $1 < \alpha < 2$  the value  $H = 1/\alpha$  is considered to be “the critical value” for exponent of self-similarity, with the range  $(1/\alpha, 1)$  of  $H$  corresponding to long memory of the increment process. This, of course, cannot be justified any longer by looking at the change in the behaviour of the covariance function of the increments, which is not defined now. Certain substitutes have been used, mostly for SSSI processes that are themselves symmetric  $\alpha$ -stable. Two such substitutes have appeared in literature, the *covariation* and the *codifference*.

Let  $X_1$  and  $X_2$  be jointly S $\alpha$ S random variables. Their joint characteristic function can be written in the form

$$(7.29) \quad E^{i(\theta_1 X_1 + \theta_2 X_2)} = \exp \left\{ - \int_{S_2} |\theta_1 s_1 + \theta_2 s_2|^\alpha \Gamma(ds) \right\}$$

for real  $\theta_1, \theta_2$ , where  $S_2$  is the unit circle, and  $\Gamma$  is a uniquely determined finite symmetric measure on the unit circle, the so called *spectral measure* of  $(X_1, X_2)$ . See Samorodnitsky and Taqqu (1994). If  $1 < \alpha < 2$ , one defines the covariation of  $X_1$  and  $X_2$  by

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 s_2^{\langle \alpha-1 \rangle} \Gamma(ds),$$

where for real  $a, b$ , the notation  $a^{\langle b \rangle}$  stands for the signed power  $|a|^b \text{sign}(a)$ . The covariation can also be defined for  $\alpha = 1$ , but it appears to be less useful in that case; an extension to the case  $0 < \alpha < 1$  is only partially possible, and requires restrictions on the spectral measure of  $(X_1, X_2)$ . The covariation is not, generally, symmetric in its arguments. It reduces to

half the covariance in the Gaussian case  $\alpha = 2$ , if one chooses to write the characteristic function as (7.29) in that case (the spectral measure is not uniquely defined if  $\alpha = 2$ .) The notion of covariation was introduced in Miller (1978). If  $(X_n, n = 1, 2, \dots)$  is a stationary  $S\alpha S$  process, its covariation function can be defined via  $\gamma(k) = [X_{n+k}, X_n]_\alpha$  for  $k = 0, 1, 2, \dots$  (but changing the order of the arguments will lead, in general, to a different function).

The codifference can be defined for any random variables and stochastic processes. For a random vector  $(X_1, X_2)$  we define

$$\tau(X_1, X_2) = \log Ee^{i(X_1 - X_2)} - \log Ee^{iX_1} - \log Ee^{iX_2}$$

(where we take the continuous branch of the logarithm equal to zero at point 1.) The codifference is symmetric in its arguments, and is equal to the covariance for a Gaussian vector  $(X_1, X_2)$ . The term ‘‘codifference’’ appeared first in Kokoszka and Taqqu (1994), but related notions had been used many times before. For a stationary process  $(X_n, n = 1, 2, \dots)$  its codifference function is defined by  $\tau(k) = \tau(X_{n+k}, X_n)$  for  $k = 0, 1, 2, \dots$

Both covariation and codifference are equal to zero in the case of independence, but the converse is not true. On the other hand, zero covariation does imply a kind of Banach space orthogonality, the so called *James orthogonality*; see Cambanis et al. (1988). Furthermore, it is possible to characterize independence of infinitely divisible random vectors based on codifference (see Rosiński and Żak (1997)), and for a certain class of stationary infinitely processes (including stable processes) mixing is equivalent to convergence the codifference function to zero at large lags (see Rosiński and Żak (1996)).

Suppose  $\mathbf{Y}$  is a SSSI process, that is  $S\alpha S$  with  $1 < \alpha < 2$ . In order to understand if there is a significant change in the properties of the increment process  $X_n = Y(n) - Y(n - 1)$ ,  $n = 1, 2, \dots$  at the value  $H = 1/\alpha$  of the exponent of self-similarity, one can try to see if the behaviour of either the covariation function or codifference function changes significantly at that point.

The behaviour of the codifference function of the increment process of the Linear Fractional Stable Motions as in (7.18) with (7.19), was considered in

Astrauskas et al. (1991). They discovered that

$$\tau(k) \sim \begin{cases} Ct^{-(1+\frac{1}{\alpha}-H)} & \text{if } 0 < H < 1 - \frac{1}{\alpha(\alpha-1)} \\ Ct^{-\alpha(1-H)} & \text{if } 1 - \frac{1}{\alpha(\alpha-1)} < H < 1, \quad H \neq \frac{1}{\alpha} \end{cases}$$

as  $t \rightarrow \infty$ , where  $C$  is a constant depending on the parameters in (7.19). Here a change in the rate of decay of the codifference function does occur, but at the point  $1 - 1/(\alpha(\alpha - 1))$ , and not at the point  $1/\alpha$ . A similar computation was performed by Kokoszka and Taqqu (1994) for the FARIMA model with S $\alpha$ S noise, and the results for both the covariation function and the codifference function were similar (in the case  $1 < \alpha < 2$ ) to the above.

It is not easy to evaluate the evidence provided by the behaviour of the covariation and codifference functions. One expects it to be smaller than that provided by the covariances for the second order stationary processes. However, even this available evidence does not necessarily point to a particular importance of the point  $H = 1/\alpha$  when deciding whether or not the increments of a S $\alpha$ S SSSI processes are long range dependent or not.

In fact, the ergodic theory seems to provide a better guidance to the memory of the increment process than the rate of decay the covariation and codifference functions. Recall that the increments of the Harmonizable Fractional Stable Motions are not ergodic, while the increments of the Linear Fractional Stable Motions are mixing. This already says that the memory of the latter is shorter than that of the former, and this is regardless of the value of the scaling exponent  $H$ . In fact, in studying the memory of any stationary S $\alpha$ S process the ergodic theory enters the picture in yet another, even more informative way, as will be seen in the next section.

## 8. LONG RANGE DEPENDENCE AS A PHASE TRANSITION

A different point of view on long range dependence was suggested by Samorodnitsky (2004). Suppose that we are given a family of shift-invariant probability measures  $(P_\theta, \theta \in \Theta)$  on  $\mathbb{R}^{\mathbb{Z}}$ ; that is, each  $P_\theta$  describes the finite-dimensional distributions of a two-sided (for convenience) stationary stochastic process  $\mathbf{X} = (\dots, X_{-1}, X_0, X_1, X_2, \dots)$ . Assume that, as  $\theta$  varies over the parameter space  $\Theta$ , the one-dimensional marginal distributions of

the process do not change significantly. This requirements allows, for a example, a change in scale, or other changes not relevant for the application of interest. We do not usually want to allow a serious change in the marginal tails, for instance loss/gain of a finite variance. For example,  $(P_\theta, \theta \in \Theta)$  might describe a family of correlations functions of unit variance stationary Gaussian processes, or a a family of coefficients of an infinite moving average model. A subset  $\Theta_0$  of the parameter space corresponds to the choices of the parameters under which the process  $\mathbf{X}$  is a sequence of i.i.d. random variables; sometimes  $\Theta_0$  is a singleton.

Let  $\phi = \phi(\mathbf{X})$  be a measurable functional of the process; typical examples are the sequence of the partial sums, for which  $\phi : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^\infty$ ,  $\phi_n(\mathbf{x}) = \sum_{j=1}^n x_j$ , and the sequence of the partial maxima, for which also  $\phi : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^\infty$ , but this time  $\phi_n(\mathbf{x}) = \max_{j=1}^n x_j$ . The behaviour of this functional will, in general, be different under different probability measures  $P_\theta, \theta \in \Theta$ .

Suppose that there is a partition  $\Theta = \Theta_1 \cup \Theta_2$  of the parameter space into disjoint parts, with  $\Theta_0 \subset \Theta_1$ , such that the behaviour of the functional  $\phi$  changes significantly as the parameter crosses the boundary between  $\Theta_1$  and  $\Theta_2$ . This means that, as long as the parameter stays within  $\Theta_1$ , the behaviour of the functional  $\phi$  does not change much, and it remains the same as in the i.i.d. case  $\theta \in \Theta_0$ , perhaps “up to a multiplicative constant”. Once the parameter  $\theta$  crosses the boundary into  $\Theta_2$ , there is a change “in the order of magnitude” in the behaviour of the functional  $\phi$ . Moreover, there continue to be significant changes as the parameter  $\theta$  moves within  $\Theta_2$ .

Under these conditions we view the part  $\Theta_1$  of the parameter space as corresponding to short memory models, and the part  $\Theta_2$  of the parameter space as corresponding to long memory models. From this point of view, the boundary between  $\Theta_1$  and  $\Theta_2$  is the boundary between short and long memory, and it is natural to regard the appearance of long range dependence as a phase transition.

This approach has drawbacks. It is “an approach”, and not a rigorous definition. It ties the notion of long range dependence to a particular functional and, perhaps, to a particular aspect of the behaviour of that functional.

This, however, appears to be inevitable. One of the reasons it has been so difficult to define long range dependence is that one has tried to give a single definition to what is, really, a series of phenomena. Moreover, studying the change in behaviour of a functional relevant in applications is, arguably, more important than trying to find a single critical parameter. If one adopts this point of view on long range dependence, the problem reduces to that of finding critical boundaries. It will undoubtedly turn out that for many models there will be more than one such boundary.

A functional of major interest, and the one that historically generated most interest, is the sequence of the partial sums of the process. When considering stationary processes with a finite second moment, the second-order approach to long range dependence concentrates on the behaviour of the variances of the partial sums.

If  $(P_\theta, \theta \in \Theta)$  is the family of laws of all stationary stochastic processes with marginal variance equal to (say) 1, then for every  $\theta \in \Theta_0$  (corresponding to the law of a sequence of i.i.d. random variables) the variance of the partial sums grows linearly with the sample size. It is natural then to define  $\Theta_1$  to be that subset of  $\Theta$  such that, under the law  $P_\theta$  with  $\theta \in \Theta_1$ , the variance of the partial sums grows at most linearly fast with the sample size. Then  $\Theta_2$ , the complement of  $\Theta_1$  in  $\Theta$ , is the collection of the laws of finite second order stationary processes that will be considered as long range dependent, and the notion relies strictly on the variance of the partial sums growing, at least along a subsequence, faster than linearly fast. From this point of view, the various alternative notions of a short memory process discussed in Section 5: (5.2), (5.6), (5.8) or (5.13) are entirely reasonable when viewed as *sufficient conditions* for the law of the stochastic process to be one of  $P_\theta$  with  $\theta \in \Theta_1$ , hence of short memory, but it is quite a bit less reasonable to view the failure of one of these conditions as an indication of long memory.

Similarly, the conditions (5.4), (5.5) or those of Proposition 5.2 can very reasonably be viewed as *sufficient conditions* for long memory, but their absence should not be viewed as an indication of short memory.

It is not by any means obvious that the change from situation where the variance of the partial sums grows at most linearly fast with the sample size, to the situation where this is not the case, is, by itself, important enough to justify calling this change a passage from short memory to long memory. The exception is, of course, the Gaussian case. If  $(P_\theta, \theta \in \Theta)$  is the family of laws of all stationary *Gaussian* processes with marginal variance 1, then the analogous to the above partition of the parameter space  $\Theta$  into  $\Theta_1$  and  $\Theta_2$  is a natural basis for distinction between short and long memory.

Still concentrating on the behaviour of the partial sums of a stationary second order process, one can partition the parameter space  $\Theta$  into two parts, depending on whether or not the partial sums satisfy the invariance principle with convergence to the Brownian motion. From this point of view, the strong mixing conditions in Theorems 4.1 and 4.2 should be viewed as sufficient conditions for short memory, but their absence should not be regarded as an indication of long memory. The Fractional Gaussian noise with *any*  $H \neq 1/2$  (and not only  $H > 1/2$ ) will then be considered to be long range dependent.

In principle, basing the decision on whether or not a given stationary process has long memory on the order of magnitude of the partial sums of the process is possible also for infinite variance processes. Even more specifically, suppose  $(P_\theta, \theta \in \Theta)$  is the family of laws of all stationary stochastic processes whose one dimensional marginal distributions are in the domain of attraction of an  $\alpha$ -stable law with  $0 < \alpha < 2$ . Then for every  $\theta \in \Theta_0$  (corresponding to the law of a sequence of i.i.d. random variables in the  $\alpha$ -stable domain of attraction), the partial sums satisfy an invariance principle with convergence to an  $\alpha$ -stable Lévy motion, and one can define a partition of the parameter space  $\Theta$  into  $\Theta_1$  and  $\Theta_2$  depending on whether such an invariance principle still holds (an example is in Astrauskas (1983a) or Kasahara and Maejima (1988)). From this point of view, the stationary increment process



of any  $\alpha$ -stable SSSI process, other than the strictly stable Lévy motion, has long memory *regardless of the value of the Hurst exponent  $H$* , which can even be equal to  $1/\alpha$  if  $1 < \alpha < 2$ ; see the example of the Log-fractional Stable Motion in (7.22).

As the marginal tails of a stationary process become heavier, concentrating on the partial sums of the process, particularly on their rate of growth, to draw the boundary between short and long memory becomes less useful. The following proposition (whose proof will appear elsewhere) shows that, when the marginal tails are sufficiently heavy, the partial sums cannot grow faster than those of an i.i.d. sequence. For simplicity we state it in the symmetric case, but the statement holds in a much greater generality.

**Proposition 8.1.** *Let  $X$  be a symmetric random variable such that  $E|X|^\beta = \infty$  for some  $0 < \beta < 1$ . Let  $\mathbf{X} = (X_1, X_2, \dots)$  be a stochastic process with each  $X_i \stackrel{d}{=} X$ , and let  $\mathbf{Y} = (Y_1, Y_2, \dots)$  be a sequence of independent copies of  $X$ . Let  $a_n \uparrow \infty$  be a sequence of positive numbers such that*

$$\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < \infty.$$

If

$$(8.1) \quad \limsup_{n \rightarrow \infty} P(|X_1 + X_2 + \dots + X_n| > a_n) > 0$$

then also

$$(8.2) \quad \limsup_{n \rightarrow \infty} P(|Y_1 + Y_2 + \dots + Y_n| > a_n) > 0.$$

When the marginal tails of a stationary process are heavy, extreme values are, often, important. The partial maxima of the process are a natural functional to use in this case in order to draw the boundary between short and long memory. In the case of stationary S $\alpha$ S processes such boundary was found in Samorodnitsky (2004).

A stationary S $\alpha$ S process has an integral representation

$$(8.3) \quad X_n = \int_E f_n(x) M(dx), \quad n = 1, 2, \dots,$$

where  $M$  is a S $\alpha$ S random measure on a standard Borel space  $(E, \mathcal{E})$  with a  $\sigma$ -finite control measure  $m$ . The functions  $f_n$ ,  $n = 1, 2, \dots$  can be chosen

to be of the form

$$(8.4) \quad f_n(x) = a_n(x) \left( \frac{dm \circ \phi^{n-1}}{dm}(x) \right)^{1/\alpha} f \circ \phi^{n-1}(x) \quad x \in E,$$

for  $n = 1, 2, \dots$ , where  $\phi : E \rightarrow E$  is a measurable non-singular map (i.e. a one-to-one map with both  $\phi$  and  $\phi^{-1}$  measurable, mapping the control measure  $m$  into an equivalent measure),

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x), \quad x \in E,$$

for  $n = 0, 1, 2, \dots$ , with  $u : E \rightarrow \{-1, 1\}$  a measurable function and  $f \in L^\alpha(m)$ . See Samorodnitsky and Taqqu (1994) and Rosiński (1995) for the details.

A basic fact from ergodic theory is the existence of the Hopf decomposition of the set  $E$  with respect to the flow  $(\phi^n, n = 0, 1, 2, \dots)$ : a decomposition of  $E$  into a disjoint (modulo a null set with respect to  $m$ ) union  $E = C \cup D$ , such that  $C$  and  $D$  are measurable  $\phi$ -invariant sets, and *the flow is conservative on  $C$  and dissipative on  $D$* ; we refer the reader to Krengel (1985) for the details. This allows us to write

$$(8.5) \quad X_n = \int_C f_n(x) M(dx) + \int_D f_n(x) M(dx) := X_n^C + X_n^D,$$

$n = 1, 2, \dots$ , a unique in law decomposition of a stationary symmetric  $\alpha$ -stable process into a sum of two independent such processes, one of which is *generated by a conservative flow*, and the other is *generated by a dissipative flow*. The i.i.d. S $\alpha$ S sequence is generated by a dissipative flow (i.e. the component  $\mathbf{X}^C$  in the decomposition (8.5) vanishes). See Rosiński (1995).

Let

$$(8.6) \quad M_n = \max_{j=1,2,\dots,n} |X_j|, \quad n = 1, 2, \dots,$$

be the sequence of the partial maxima. The following result was proved in Samorodnitsky (2004): if the component  $\mathbf{X}^D$  generated by a dissipative flow in the decomposition (8.5) does not vanish, then

$$(8.7) \quad n^{-1/\alpha} M_n \Rightarrow C Z_\alpha,$$

where  $C$  is a finite positive constant, and  $Z_\alpha$  is the standard Fréchet extreme value random variable with the distribution

$$(8.8) \quad P(Z_\alpha \leq z) = e^{-z^{-\alpha}}, \quad z > 0.$$

If, on the other hand, the component  $\mathbf{X}^D$  generated by a dissipative flow vanishes, then

$$(8.9) \quad n^{-1/\alpha} M_n \rightarrow 0$$

in probability as  $n \rightarrow \infty$ .

This is a phase transition that qualifies as a change from short memory to long memory. Let us call stationary SaS processes with only one nondegenerate component in the decomposition (8.5) *single component processes*. We parametrize the family of laws of single component stationary SaS processes (with the scale fixed to, say, 1) by a space  $(E, \mathcal{E}, m)$ , a *flow*  $(\phi^n, n = 0, 1, 2, \dots)$ , a function  $f \in L^\alpha(m)$  and a *cocycle*  $(a_n, n = 0, 1, 2, \dots)$ . The collection of these parameters forms the parameter space  $\Theta$ . Then the set  $\Theta_1$  of parameters for which the flow is dissipative corresponds to short memory processes, while the set  $\Theta_2$  of parameters for which the flow is conservative corresponds to long memory processes, and the boundary between the two is the phase transition boundary.

It is interesting that *the partial maxima grow the fastest in the short memory case* (including the i.i.d. case). In particular, if a stationary SaS process has both a nondegenerate dissipative component and a nondegenerate conservative component in (8.5), then the long range dependent conservative component will be hidden by the faster growing maximum of the short memory dissipative component. Therefore, if we use the same parameters as above to parametrize the family of laws of *all* stationary SaS processes with the same scale, then the phase transition becomes less interesting, because in this case the short memory part  $\Theta_1$  of the parameter space becomes the set of the parameters in which the flow has a non-vanishing dissipative component. This will allow for short memory processes with a nondegenerate long memory component generated by a conservative flow. This is an

indication that for certain functionals of a stationary process it is important to choose the parametrization carefully.

In the light of the present discussion let us revisit the question of short or long memory in the increments of SSSI S $\alpha$ S processes considered in Section 7. If  $\mathbf{Y}$  is the Linear Fractional Stable Motion defined in (7.18) and (7.19), then its increment process is a single component process generated by a dissipative flow (see Rosiński (1995)), hence a short memory process, regardless of the value of index of stability  $\alpha$  and Hurst exponent  $H$ . Similarly, if  $\mathbf{Y}$  is the Harmonizable Fractional Stable Motion defined in (7.23), then its increment process is a single component process generated by a conservative flow (see once again Rosiński (1995)), hence a long memory process, once again regardless of the values of  $\alpha$  and  $H$ . In fact, since the increment process of the Harmonizable Fractional Stable Motion is not ergodic, we can view this process as having infinite memory. On the other hand, Cohen and Samorodnitsky (2006) constructed a family of SSSI S $\alpha$ S process for which the increment process is also a single component process generated by a conservative flow, but this time the process is ergodic (even mixing). Thus we can view it as having finite but long memory.

It is not difficult to see that this classification of the memory of the increments of SSSI S $\alpha$ S processes is more informative than making a distinction based on a single critical value of the Hurst exponent or on decay rates of covariance substitutes such as covariation and codifference.

It is interesting that for stationary *Gaussian* processes the change in the rate of increase of the partial maxima occurs “much later”, from the point of view of the rate of decay of correlations, than what is needed to change the rate of increase of the partial sums as discussed above. For example, if the condition

$$R_n \log n \rightarrow 0 \text{ as } n \rightarrow \infty$$

is satisfied by the covariance function of a stationary Gaussian process (i.e. if the correlations decay faster than *logarithmically* fast), then the partial maxima of the process increase at the same rate as in the i.i.d. case (and even the constants are preserved!); see Berman (1964).

More examples of a phase transition of the type discussed above are given by infinite moving average models. These are models of the form

$$(8.10) \quad X_n = \sum_{j=-\infty}^{\infty} \varphi_{n-j} \varepsilon_j, \quad n = 1, 2, \dots,$$

where  $(\varepsilon_n, n = \dots, -1, 0, 1, 2, \dots)$  are i.i.d. *noise variables*, and  $(\varphi_n)$  are deterministic coefficients; the latter, clearly, have to satisfy certain conditions for the series to converge and the process to be well defined. We will assume that the noise variables have a finite mean, in which case the absolute summability of the coefficients

$$(8.11) \quad \sum_{j=-\infty}^{\infty} |\varphi_j| < \infty$$

guarantees that the series in (8.10) converges (absolutely) a.s., but weaker assumptions (depending on the noise distribution) will suffice for a.s. convergence as well. Obviously, if the moving average process (8.10) is well defined, it is automatically a stationary process. It is also known as a *linear process* (as is the ARMA model discussed in Section 3, which is a special case of the infinite moving average model).

Fixing the distribution of the noise variables, the law of the stationary moving average process is determined by its sequence of coefficients, so in this case the parameter space  $\Theta$  is the collection of sequences  $(\varphi_n)$  for which the series (8.10) converges. Unless the noise variables are Gaussian or  $\alpha$ -stable, different choices of the parameter will affect the one-dimensional marginal distributions of the moving average process by more than a scale factor. Still, it makes sense to restrict the parameter space in an appropriate way to keep the marginal distributions from varying too much.

The part of the parameter space  $\Theta$  defined by (8.11) is sometimes viewed as leading to short memory models, often under the additional assumption

$$(8.12) \quad \sum_{j=-\infty}^{\infty} \varphi_j \neq 0,$$

see e.g. Section 13.2 of Brockwell and Davis (1987). (Intuitively, the case of the zero sum in (8.12) corresponds to negative memory, not dissimilar with the Fractional Gaussian Noise with  $0 < H < 1/2$ .) Obviously, the laws of

i.i.d. sequences belong to this part of the parameter space. Is there a phase transition that occurs when (8.11) breaks down?

Assume first that the noise variables have a finite variance  $\sigma^2 > 0$ . Then the necessary and sufficient condition for convergence of the series in (8.10) is

$$(8.13) \quad S_\varphi := \sum_{j=-\infty}^{\infty} \varphi_j^2 < \infty.$$

For the purpose of normalization of the marginal distributions one can define the parameter space  $\Theta$  to consist of the sequences of the coefficients satisfying  $S_\varphi = 1$ .

As before, a reasonable partition of  $\Theta$  into two parts, corresponding to short and long memory processes, is obtained depending on whether or not the partial sums of the moving average process satisfy the invariance principle with convergence to the Brownian motion. Moving average processes with absolutely summable coefficient have this property (see Hannah (1979)). However, also every parameter point satisfying, for example, condition (5.37) in Hall and Heyde (1980) guarantees such an invariance principle, hence a short memory process, and this condition may be satisfied even when (8.11) fails. From this point of view, the summability of the coefficients (8.11) is a sufficient condition for a short memory process, but its failure does not automatically imply a long memory moving average.

Still concentrating on the partial sums of a stationary process, important aspects of their behaviour are related to large deviations. Suppose that the noise random variables satisfy

$$(8.14) \quad Ee^{\lambda\varepsilon_0} < \infty$$

for  $\lambda \in (-\epsilon, \epsilon)$ , some  $\epsilon > 0$ , i.e. have exponentially fast decaying tails.

Let  $\mathbf{X} = (X_1, X_2, \dots)$  be a stationary process. We say that the large deviation principle holds for the sample averages of the process for some *speed sequence*  $b_n \uparrow \infty$  and upper and lower rate function  $I_u(\cdot)$  and  $I_l(\cdot)$ , respectively,

$$(8.15) \quad - \inf_{x \in A^\circ} I_l(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{b_n} \log P \left( \frac{X_1 + \dots + X_n}{n} \in A \right)$$

$$\leq \limsup_{n \rightarrow \infty} \frac{1}{b_n} \log P \left( \frac{X_1 + \dots + X_n}{n} \in A \right) \leq - \inf_{x \in A} I_u(x)$$

for every Borel set  $A$ , where  $A^\circ$  and  $\bar{A}$  denote the interior and closure of a set  $A$ , correspondingly. Detailed accounts of large deviations are in Dembo and Zeitouni (1993) and Deuschel and Stroock (1989). The rate sequence has the single most important role in the large deviation principle (8.15). For i.i.d. sequences satisfying (8.14), the classical Cramer theorem says that the large deviation principle holds with the speed  $b_n = n$ ,  $n = 1, 2, \dots$ .

Returning to infinite moving averages with the noise variables satisfying the exponential tail condition (8.14), define a partition of the parameter space  $\Theta$  (consisting of the sequences of the coefficients satisfying  $S_\varphi = 1$  in (8.13)) into parts  $\Theta_1$  and  $\Theta_2$  by declaring  $\Theta_1$  to be that set of parameters  $\theta$  for which the large deviation principle holds under  $P_\theta$  with the speed  $b_n \equiv n$ , and  $\Theta_2$ , corresponding to the long memory moving averages, to be its complement. Similar partitions can be created based on the functional version of the large deviation principle, known to hold in the i.i.d. case with the linear speed by the Mogulskii theorem (Mogulski (1976) or Theorem 5.1.2 in Dembo and Zeitouni (1993), assuming that that (8.14) holds for all  $\lambda \in \mathbb{R}$ ), or on other versions of the large deviation principle.

The fact that, under the assumptions (8.11) and (8.12), the moving average process satisfies the large deviation principle with the speed  $b_n \equiv n$  is well established, albeit under a variety of tail assumptions on the noise variables: see Burton and Dehling (1990) and Jiang et al. (1995). However, such large deviation principle also holds under the weaker assumption (5.13); see Djellout and Guillin (2001) or Wu (2004). (All these authors establish their large deviation principles at different levels of generality, but always covering the simplest one-dimensional version formulated in (8.15)). From this point of view, the assumptions (8.11) and (8.12) are sufficient for a short memory linear process, but not necessary.

Certain situations where the large deviation principle with a linear rate no longer holds were presented in Ghosh and Samorodnitsky (2007). Specifically, they assumed that the coefficients  $(\varphi_n)$  are *balanced regularly varying*: there is a regularly varying at infinity with exponent  $-\beta$ ,  $1/2 < \beta \leq 1$ ,

function  $\psi : [0, \infty) \rightarrow [0, \infty)$  and  $0 \leq p \leq 1$ , such that

$$(8.16) \quad \lim_{n \rightarrow \infty} \frac{\phi_n}{\psi(n)} = p \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\phi_{-n}}{\psi(n)} = 1 - p.$$

If  $\beta = 1$ , assume further that (8.11) fails.

Denote  $\Psi_n = \sum_{1 \leq i \leq n} \psi(i)$ ,  $n = 1, 2, \dots$ . It turns that, under the balanced regular variation assumption (8.16), the moving average process satisfies the large deviation principle with the speed  $b_n = n/\Psi_n^2$ ,  $n = 1, 2, \dots$ . Observe that (by Karamata's theorem) this speed sequence is regular varying with exponent  $2\beta - 1$ . Even in the case  $\beta = 1$  the speed sequence has the form  $b_n = nL_n$ , with a slowly varying function  $L$  converging to zero, and so it grows strictly slower than linearly fast. More general versions of the large deviation principle also exhibit similar behaviour; see Ghosh and Samorodnitsky (2007).

From this point of view, moving average processes with coefficients satisfying (8.16) are long range dependent.

Several other large deviation-related ways to look at short and long memory exist, the better known ones of which are related to *ruin probabilities* and *long strange segments*; a recent extensive account of the former is in Asmussen (2000), the latter have been analyzed since Erdős and Rényi (1970). We will look at the behaviour of the long strange segments, defined as follows. For a Borel set  $A$  and  $n = 1, 2, \dots$  let

$$(8.17) \quad R_n(A) = \sup \left\{ j - i : 0 \leq i < j \leq n, \frac{X_{i+1} + \dots + X_j}{j - i} \in A \right\},$$

(defined to be equal to zero if the supremum is taken over the empty set). If the closure of  $A$  does not contain the mean of the stationary process (which we have assumed to be equal to zero), then the long segments over which the sample mean belongs to  $A$  are “strange” because the law of large numbers seems to break down there. For the i.i.d. sequence under the finite exponential moment assumption (8.14),

$$(8.18) \quad \frac{1}{\inf_{x \in A^\circ} I_l(x)} \leq \liminf_{n \rightarrow \infty} \frac{R_n(A)}{\log n} \leq \limsup_{n \rightarrow \infty} \frac{R_n(A)}{\log n} \leq \frac{1}{\inf_{x \in \bar{A}} I_u(x)}$$

with probability 1, where  $I_l$  and  $I_u$  are the rate functions in the large deviation principle (8.15); see Theorem 3.2.1 in Dembo and Zeitouni (1993).



Let  $\Theta_1$  be that part of the parameter space  $\Theta$  (still consisting of the sequences of the coefficients satisfying  $S_\varphi = 1$  in (8.13)) where the length of the long strange segments grows at the logarithmic rate, as in (8.18), and  $\Theta_2$  to be its complement. Since the statement (8.18) is set-dependent, one can restrict the test set  $A$  to the form  $A = \{x : |x - EX| > \theta\}$  for small  $\theta > 0$ . It was shown in Ghosh and Samorodnitsky (2007) that, under the assumptions (8.11) and (8.12), the logarithmic rate of increase still holds for the long strange segments (but with a generally different rate functions in (8.18)), whereas under the assumption (8.16) of the balanced regular variation assumption, the long strange segments grow at a strictly faster rate: the rate is now  $h(\log n)$ , where the function  $h : (0, \infty) \rightarrow (0, \infty)$  satisfies

$$\frac{h(s)}{[\Psi(\lfloor h(s) \rfloor)]^2} \rightarrow 1 \text{ as } s \rightarrow \infty.$$

Note that the function  $h$  is regularly varying with exponent  $(2\beta - 1)^{-1}$ . Therefore, also from the point of view of the long strange segments, the assumptions (8.11) and (8.12) are sufficient for a short memory of the moving average process, while the balanced regular variation assumption (8.16) is sufficient for long memory.

In summary, the change from short to long memory in infinite moving average processes with finite exponential moments as in (8.14) is of a phase transition nature. Only sufficient conditions for being on either side of the boundary are known at the moment; future research will undoubtedly tell us more about the description of the boundary in terms of the coefficients in the model.

We conclude by briefly looking at related phase transitions for infinite moving averages, where the noise variables do not have finite exponential moments. Suppose that the noise variables have, in fact, balanced regularly varying tails; this is a notion slightly more general than the balanced power tails in (2.5). Specifically, assume that

$$(8.19) \quad \begin{cases} P(|\varepsilon_0| > \lambda) = L(\lambda) \lambda^{-\alpha}, \\ \lim_{\lambda \rightarrow \infty} \frac{P(\varepsilon_0 > \lambda)}{P(|\varepsilon_0| > \lambda)} = p_\varepsilon, \quad \lim_{\lambda \rightarrow \infty} \frac{P(\varepsilon_0 < -\lambda)}{P(|\varepsilon_0| > \lambda)} = q_\varepsilon, \end{cases}$$

as  $\lambda \rightarrow \infty$ , for some  $\alpha > 1$  and  $0 < p_\varepsilon = 1 - q_\varepsilon \leq 1$ . Here  $L$  is a slowly varying function at infinity. Note that, if  $\alpha \leq 2$ , then the assumption of the finite variance of the noise variables may fail, and it certainly does fail if  $\alpha < 2$ . If  $\alpha > 2$ , then the variance of the noise variables is still finite, and the square summability condition (8.13) is still the necessary and sufficient condition for the linear process (8.10) to be well defined; in the case  $1 < \alpha \leq 2$  a sufficient condition is

$$(8.20) \quad \sum_{j=-\infty}^{\infty} |\varphi_j|^{\alpha-\epsilon} < \infty \quad \text{for some } \epsilon > 0.$$

In both cases the resulting moving average process is a stationary process whose 1-dimensional marginal tails are proportional to the tails of the noise variables; see Mikosch and Samorodnitsky (2000).

Once again, let  $\Theta$  be the parameter space appropriate to the situation at hand: this will be the space of the coefficients satisfying (8.13) if  $\alpha > 2$ , or the space of the coefficients satisfying (8.20) if  $1 < \alpha \leq 2$ . Keeping the marginal distributions of the process from varying too much as the parameter changes is desirable; with “power-like” tails it is often a good idea to control the tails: the normalization  $\sum_j |\varphi_j|^\alpha = 1$  achieves that (see Mikosch and Samorodnitsky (2000)), and it can be taken as a part of the description of the parameter space.

Full large deviation principles have not been extended for stochastic processes with “power-like” tails far beyond the i.i.d. case (see Nagaev (1979) and Hult et al. (2005) for what happens in that case), so we will only consider the long strange segments. It turns out that the absolutely summable coefficients satisfying (8.11) still belong to that part  $\Theta_1$  of the parameter space such that, under the law  $P_\theta$  with  $\theta \in \Theta_1$  the long strange segments of the moving average process behave as in the i.i.d. case, while certain balanced regularly varying coefficients belong to that part  $\Theta_2$  of the parameter space such that, under the law  $P_\theta$  with  $\theta \in \Theta_2$  the long strange segments of the moving average process behave in a drastically different way. We need to modify the assumption of the balanced regularly varying coefficients in

(8.16) as follows: assume that (8.16) is satisfied with  $1/2 < \beta < 1$  if  $\alpha > 2$ , and with  $1/\alpha < \beta < 1$  if  $1 < \alpha \leq 2$ .

We consider long strange intervals defined in (8.17), with test sets of the form  $A = (\theta, \infty)$  for  $\theta > 0$  (these are “strange” because of the assumption of zero mean). Let  $F$  denote the distribution function of the noise random variable  $|\varepsilon_0|$ , and for  $n \geq 1$  define

$$(8.21) \quad a_n = \left( \frac{1}{1-F} \right)^{\leftarrow} (n),$$

where for a nondecreasing function  $U$ ,  $U^{\leftarrow}(y) = \inf\{s : U(s) \geq y\}$ ,  $y > 0$ , is the left continuous inverse of  $U$ . Clearly  $(a_n)$  is regularly varying at infinity with exponent  $1/\alpha$ . It was proved by Mansfield et al. (2001) that under the absolutely summability assumption (8.11),

$$a_n^{-1} R_n((\theta, \infty)) \Rightarrow \frac{M(\varphi)}{\theta} Z_\alpha,$$

where  $Z_\alpha$  is the standard Fréchet random variable defined in (8.8), and

$$M(\varphi) = (p_\varepsilon M_+(\varphi)^\alpha + q_\varepsilon M_-(\varphi)^\alpha)^{1/\alpha},$$

with

$$M_+(\varphi) = \max \left\{ \sup_{-\infty < k < \infty} \left( \sum_{j=-\infty}^k \varphi_j \right)_+, \sup_{-\infty < k < \infty} \left( \sum_{j=k}^{\infty} \varphi_j \right)_+ \right\}$$

and

$$M_-(\varphi) = \max \left\{ \sup_{-\infty < k < \infty} \left( \sum_{j=-\infty}^k \varphi_j \right)_-, \sup_{-\infty < k < \infty} \left( \sum_{j=k}^{\infty} \varphi_j \right)_- \right\}.$$

Therefore, under the assumption (8.11), the length of the long strange segments grows at the rate  $a_n$  (which is regularly varying at infinity with exponent  $1/\alpha$ ); this rate is the same as the rate of growth of the long strange segments for i.i.d. sequences with the same marginal tails. Note that the assumption (8.12) is not needed here.

On the other hand, under the assumption (8.16) of the balanced regularly variation of the coefficients (modified to the present case of the “power-like” tails), it was shown in Rachev and Samorodnitsky (2001) that

$$\tilde{a}_n^{-1} R_n((\theta, \infty)) \Rightarrow \frac{p_\varepsilon^{1/(\alpha\beta)} (p^{1/\beta} + q^{1/\beta})}{(1-\beta)^{1/\beta} \theta^{1/\beta}} Z_{\alpha\beta},$$

where  $Z_{\alpha\beta}$  is, once again, the standard Fréchet random variable, but this time with the exponent equal to  $\alpha\beta$ , and the sequence  $(\tilde{a}_n)$  satisfies

$$\frac{\psi(\tilde{a}_n)}{a_n} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Therefore, under the (modified) assumption of the balanced regularly variation of the coefficients, the length of the long strange segments grows at the rate  $\tilde{a}_n$ , which is regularly varying at infinity with exponent  $1/(\alpha\beta)$  and, hence, faster than in the case of absolutely summable coefficients.

We conclude that the behaviour of long strange segments in the infinite moving average processes with noise variables with balanced regularly varying tails exhibits a phase transition (similar to that in the case of linear processes with the “light-tailed” noise variables) that may qualify as a phase transition that separates between short memory and long memory processes. The assumption of absolute summability of the coefficients is sufficient for a short memory process, while the assumption of a certain regular variation of the coefficients is sufficient for a long memory process.

In summary, it appears that connecting the notion of long range dependence to certain types of phase transitions is promising. It fits well with our intuition of the term “long memory” describing a model that is out of the ordinary. Furthermore, it allows us to concentrate on the behaviour of really important functionals. Much remains to be done to clarify both possible types of such phase transitions and the relevant boundaries for concrete families of stochastic processes.

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