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CLUSTERING OF LARGE DEVIATIONS IN MOVING AVERAGE PROCESSES: THE LONG MEMORY REGIME

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We investigate how large deviations events cluster in the framework of an infinite moving average process with light-tailed noise and long memory. The long memory makes clusters larger, and the asymptotic behaviour of the size of the cluster turns out to be described by the first hitting time of a randomly shifted fractional Brownian motion with drift.

1. Introduction.

We consider an infinite moving average process of the form

$$X_n = \sum_{i=0}^{\infty} a_i Z_{n-i}\,,\ n \geq 0\,, \tag{1.1}$$

where the noise variables $(Z_n : n \in \mathbb{Z})$ are assumed to bef i.i.d. non-degenerate random variables. The noise distribution F_Z is assumed have finite exponential moments:

$$\int_{\mathbb{R}} e^{tz}\, F_Z(dz) < \infty \text{ for all } t \in \mathbb{R}\,. \tag{1.2}$$

Furthermore, assuming that the noise is centred:

$$\int_{\mathbb{R}} z\, F_Z(dz) = 0\,, \tag{1.3}$$

the series defining the process in (1.1) converges if and only if the coefficients $a_0, a_1, a_2 \ldots$ satisfy

$$\sum_{j=0}^{\infty} a_j^2 < \infty\,. \tag{1.4}$$

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In this case (X_n) is a zero mean stationary ergodic process. For $\varepsilon > 0$ we consider the sequence of large deviation events

$$E_j(n,\varepsilon) = \left\{ \frac{1}{n} \sum_{i=j}^{n+j-1} X_i \geq \varepsilon \right\}, \; j \geq 0. \tag{1.5}$$

By stationarity, each event $E_j(n,\varepsilon)$ is equally rare, and we are interested in the cluster of these events that occur given that the event $E_0(n,\varepsilon)$ occurs.

In Chakrabarty and Samorodnitsky (2022) the short memory case was considered. In this context, "short memory" corresponds to the case

$$\sum_{n=0}^{\infty} |a_n| < \infty \text{ and } \sum_{n=0}^{\infty} a_n \neq 0. \tag{1.6}$$

In this short memory case the conditional on $E_0(n,\varepsilon)$ law of the sequence $\big(\mathbf{1}(E_j(n,\varepsilon), \, j = 1,2,\ldots\big)$ converges weakly, as $n \to \infty$, to the law of a sequence with a.s. finitely many non-zero entries. the total number D_ε of the non-zero entries turns out to scale as ε^{-2}, and $\varepsilon^2 D_\varepsilon$ has an interesting weak limit as $\varepsilon \to 0$. We refer the reader to Chakrabarty and Samorodnitsky (2022) for details, and a minor technical condition required for the above statements.

In the present paper we are interested in the long memory case. For the moving average processes (1.1) "long memory" refers to the case when the coefficients (a_j) satisfy the square summability assumption (1.4) but not the absolute summability assumption in (1.6). A typical assumption in this is

$$(a_n) \text{ is regularly varying with exponent } -\alpha, \; 1/2 < \alpha < 1; \tag{1.7}$$

see Samorodnitsky (2016). It turns out that, in this case (under certain technical assumptions, an example of which is below), the conditional on $E_0(n,\varepsilon)$ law of the sequence $\big(\mathbf{1}(E_j(n,\varepsilon), \, j = 1,2,\ldots\big)$ converges weakly, as $n \to \infty$, to the degenerate probability measure $\delta_{(1,1,\ldots)}$. That is, once the event $E_0(n,\varepsilon)$ occurs, the events $(E_j(n,\varepsilon))$ become very likely. In order to understand their structure we concentrate on the random variables

$$I_n(\varepsilon) = \inf\left\{ j \geq 1 : E_j(n,\varepsilon) \text{ does not occur} \right\}, \, n \geq 1 \tag{1.8}$$

and establish a weak limit theorem for this sequence under a proper scaling. Interestingly, the limit turns out to be the law of the first hitting time of a randomly shifted fractional Brownian motion with drift.

The main result containing the above limit theorem and the technical assumptions it requires are in Section 2. The proof of the main theorem requires a long sequence of preliminary results, all of which are presented in that section. Finally, some useful facts needed for the proofs are collected in Section 3.

2. The assumptions and the main result. Our result on clustering of large deviation events in the long memory case will require a number of assumptions that we state next. First of all, we will replace the assumption of regular variation (1.7) by the asymptotic power function assumption

$$a_n \sim n^{-\alpha},\ 1/2 < \alpha < 1, \text{ and is eventually monotone.} \tag{2.1}$$

There is no doubt that the results of the paper hold under the more general regular variation assumption as well. The extra generality will, however, require making an already highly technical argument even more so. The potentially resulting lack of clarity makes the added generality less valuable. The same is true about the eventual monotonicity assumption.

We will need additional assumptions on the distribution of the noise variables. We will assume that some $\theta_0 > 0$,

$$\sup_{|\theta|\leq\theta_0} \int_{-\infty}^{\infty} t^2 \left| \int_{-\infty}^{\infty} e^{(it+\theta)z}\, F_Z(dz) \right| dt < \infty\,. \tag{2.2}$$

Next, let

$$\sigma_Z^2 = \int_{\mathbb{R}} z^2\, F_Z(dz) \tag{2.3}$$

be the variance of the noise. Denote

$$\kappa = \text{the smallest integer} > \frac{4\alpha - 1}{2 - 2\alpha}. \tag{2.4}$$

In other words, $\kappa = \lceil (1 + 2\alpha)/(2 - 2\alpha) \rceil$. We assume that a generic noise variable Z satisfies

$$EZ^i = EG^i \text{ for } 1 \leq i \leq \kappa, \tag{2.5}$$

where $G \sim N(0, \sigma_Z^2)$.

REMARK 2.1. It is standard to verify that (2.2) implies that the noise distribution has a twice continuously differentiable density f_Z. One the other hand, a sufficient condition for (2.2) is that the noise distribution has a four times continuously differentiable density f_Z such that

$$\int_{-\infty}^{\infty} e^{\theta_0|x|} \left| \frac{d^i}{dx^i} f_Z(x) \right| dx < \infty \text{ for } i = 1, 2, 3, 4.$$

The moment equality assumption (2.5) restricts how far the the noise distribution can be from a normal distribution. Note that in the range $1/2 < \alpha < 5/8$ we have $\kappa = 2$, in which case the assumption is vacuous. Since $\kappa \geq 2$ for all $\alpha \in (1/2, 1)$, (1.3) is implied by (2.5).

To state our main result, we need to introduce several key quantities. Let

$$\beta = \frac{4 - 4\alpha}{3 - 2\alpha} \in (0, 1) \tag{2.6}$$

and

$$H = 3/2 - \alpha \in (1/2, 1). \tag{2.7}$$

We denote by $(B_H(t) : t \geq 0)$ the standard fractional Brownian motion with Hurst index H, i.e. a zero mean Gaussian process with continuous paths and covariance function

$$E\left(B_H(s)B_H(t)\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s - t|^{2H}\right),\ s, t \geq 0\,. \tag{2.8}$$

If T_0 is a standard exponential random variable independent of the fractional Brownian motion, then

$$\tau_\varepsilon = \inf\left\{t \geq 0 :\ B_H(t) \leq (2C_\alpha)^{-1/2}\varepsilon t^{2H} - (C_\alpha/2)^{1/2}\sigma_Z^2\varepsilon^{-1}T_0\right\},\ \varepsilon > 0, \tag{2.9}$$

is an a.s. finite and strictly positive random variable. Here σ_Z^2 is the variance of the noise in (2.3) and

$$C_\alpha = \frac{\mathrm{B}(1 - \alpha, 2\alpha - 1)}{(1 - \alpha)(3 - 2\alpha)}, \tag{2.10}$$

with $\mathrm{B}(\cdot, \cdot)$ the standard Beta function.

We are now in a position to state the main result of this paper.

THEOREM 2.1. *Assume the finite exponential moment condition* (1.2), *the power-type condition* (2.1) *on the coefficients, the regularity condition* (2.2) *and the moment equality condition* (2.5). *Then for every* $\varepsilon > 0$ *the first non-occurrence times* (1.8) *satisfy*

$$P\left(n^{-\beta}I_n(\varepsilon) \in \cdot \middle| E_0(n, \varepsilon)\right) \Rightarrow P\left(\tau_\varepsilon \in \cdot\right),\ n \to \infty\,. \tag{2.11}$$

REMARK 2.2. It is worthwhile to observe that the limit law obtained in Theorem 2.1 depends on the noise distribution only through its variance σ_Z^2. This can be understood by noticing that in the long memory case considered in this paper we have $\mathrm{Var}(X_1+\cdots+X_n) \gg n$; see Lemma 2.1 below. Therefore, the events $E_j(n,\varepsilon)$ should be viewed as moderate deviation events, not large deviation events. It has been observed in many situations that moderate deviation events are influenced by the Gaussian weak limit of the quantities of interest. At the intuitive level, this explains why it is the variance of the process that appears in the limit.

For comparison, in the short memory case (1.6), we have $\mathrm{Var}(X_1+\cdots+X_n) \sim cn$ for some $c>0$, the events $E_j(n,\varepsilon)$ should be viewed as large deviation events, and their behaviour depends on much more than just the variance of the noise. See Chakrabarty and Samorodnitsky (2022) for details.

We start on the road to proving Theorem 2.1 by establishing certain basic estimates that will be used throughout the paper. Denote

$$A_j = \sum_{i=0}^{j} a_i,\ j\in\mathbb{Z}\,, \tag{2.12}$$

with the convention that a sum (or an integral) is zero if the lower limit exceeds the upper limit (so that $A_j=0$ for $j\le -1$, for example). Let

$$S_n = \sum_{i=0}^{n-1} X_i,\ n\ge 1\,, \tag{2.13}$$

and denote

$$\sigma_n^2 = \mathrm{Var}(S_n),\ n\ge 1\,. \tag{2.14}$$

In the sequel we use the following notation. We will denote by

$$\varphi_Z(t) = \log\left(\int_{\mathbb{R}} e^{tz}\,F_Z(dz)\right),\ t\in\mathbb{R} \tag{2.15}$$

the log-Laplace transform of a noise variable. We will frequently use the obvious facts

$$\varphi \text{ is convex and } \varphi_Z(x)\sim x^2\sigma_Z^2/2,\ x\to 0, \tag{2.16}$$

and

$$\varphi_Z' \text{ is continuous, nondecreasing and } \varphi_Z'(x) = x\sigma_Z^2 + O(x^2),\ x\to 0. \tag{2.17}$$

We will write G_θ for the probability measure obtained by exponentially tilting the law F_Z by $\theta \in \mathbb{R}$. That is,

$$G_\theta(dz) = \left(Ee^{\theta Z}\right)^{-1} e^{\theta z} F_Z(dz). \tag{2.18}$$

It is clear that, as $\theta \to 0$,

(2.19)
$$\int_{\mathbb{R}} z\, G_\theta(dz) \sim \theta\sigma_Z^2, \ \left| \int_{\mathbb{R}} z\, G_\theta(dz) - \theta\sigma_Z^2 \right| = O(\theta^2) \text{ and } = O(|\theta|^3) \text{ if } \kappa \geq 3,$$
$$\int_{\mathbb{R}} |z|^k\, G_\theta(dz) \to \int_{\mathbb{R}} |z|^k\, F(dz),\ k = 1, 2, \ldots.$$

Lemma 2.1. *Asymptotically we have*

$$A_j \sim (1-\alpha)^{-1} j^{1-\alpha},\ j \to \infty \tag{2.20}$$

and

$$\sigma_n^2 \sim C_\alpha \sigma_Z^2 n^{3-2\alpha},\ n \to \infty\,. \tag{2.21}$$

Furthermore, for any $t > 0$, as $n \to \infty$,

$$\sum_{i=0}^{[n^\beta t]} (A_i - A_{i-n})^2 \sim K_1 t^{3-2\alpha} n^{4-4\alpha}\,, \tag{2.22}$$

and

$$\sum_{i=n-[n^\beta t]+1}^{n} (A_i - A_{i-n})^2 \sim \sum_{i=n+1}^{n+[n^\beta t]} (A_i - A_{i-n})^2 \sim (1-\alpha)^{-2} n^{2-2\alpha+\beta} t\,, \tag{2.23}$$

with

$$K_1 = (1-\alpha)^{-2}(3-2\alpha)^{-1}\,. \tag{2.24}$$

Finally, for any $t > 0$, as $n \to \infty$,

(2.25)
$$\frac{\sigma_Z^2}{\sigma_n^2} \sum_{i=0}^{\infty} (A_i - A_{i-n}) \left(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} \right) = 1 - n^{1-2\alpha} t^{3-2\alpha} (1 + o(1))\,.$$

PROOF. The claim (2.20) is, of course, an immediate consequence of the assumption (2.1). For (2.21), first note that

$$\begin{aligned} R_n = \mathrm{Cov}(\mathrm{X_0}, \mathrm{X_n}) &\sim \sigma_Z^2 \sum_{j=1}^{\infty} j^{-\alpha}(j+n)^{-\alpha} \\ &\sim n^{1-2\alpha}\sigma_Z^2 \int_0^{\infty} x^{-\alpha}(1+x)^{-\alpha}\, dx \\ &= C_\alpha \sigma_Z^2 (1-\alpha)(3-2\alpha) n^{1-2\alpha} \end{aligned}$$

as $n \to \infty$. Therefore,

$$\begin{aligned} \sigma_n^2 = \sum_{i=-(n-1)}^{n-1} (n-|i|)R_{|i|} &\sim 2C_\alpha \sigma_Z^2 (1-\alpha)(3-2\alpha) \sum_{i=0}^{n-1} (n-i) i^{1-2\alpha} \\ \sim 2C_\alpha \sigma_Z^2 (1-\alpha)(3-2\alpha) n^{3-2\alpha} &\int_0^1 (1-x)x^{1-2\alpha}\, dx = C_\alpha \sigma_Z^2 n^{3-2\alpha}\,, \end{aligned}$$

which is (2.21). Next, for a fixed $t > 0$ and large n, by (2.20) and the fact that $\beta < 1$,

$$\sum_{i=0}^{[n^\beta t]} (A_i - A_{i-n})^2 = \sum_{i=0}^{[n^\beta t]} A_i^2 \sim (1-\alpha)^{-2} \sum_{i=1}^{[n^\beta t]} i^{2-2\alpha} \sim K_1 \left(n^\beta t\right)^{3-2\alpha},$$

proving (2.22). Similarly,

$$\sum_{i=n-[n^\beta t]+1}^{n} (A_i - A_{i-n})^2 \sim \sum_{i=n-[n^\beta t]+1}^{n} A_n^2 \sim (1-\alpha)^{-2} n^{\beta+2-2\alpha} t\,,$$

showing the first equivalence in (2.23) and the second equivalence can be shown in the same way.

For (2.25), we start by writing

$$S_n = \sum_{j=0}^{\infty} (A_j - A_{j-n}) Z_{n-1-j}\,, n \geq 1\,, \tag{2.26}$$

so that

$$\sigma_n^2 = \sigma_Z^2 \sum_{j=0}^{\infty} (A_j - A_{j-n})^2\,, n \geq 1\,. \tag{2.27}$$

Therefore, for large n,

$$\frac{\sigma_n^2}{\sigma_Z^2} - \sum_{i=0}^{\infty}(A_i - A_{i-n})(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n})$$
$$= \frac{1}{2}\left[\sum_{i=0}^{[n^\beta t]-1}(A_i - A_{i-n})^2 + \sum_{i=0}^{\infty}\Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big)^2\right]$$

(2.28)
$$= \frac{1}{2}\Bigg[\sum_{i=0}^{n-1}\Big(A_i - A_{i-[n^\beta t]}\Big)^2 + \sum_{i=n-[n^\beta t]}^{\infty}\Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big)^2\Bigg].$$

By (2.20),

$$\sum_{i=0}^{n-1}\Big(A_i - A_{i-[n^\beta t]}\Big)^2 \sim (1-\alpha)^{-2}\sum_{i=1}^{n-1}\Big(i^{1-\alpha} - (i-[n^\beta t])_+^{1-\alpha}\Big)^2$$
$$\sim n^{4-4\alpha}t^{3-2\alpha}(1-\alpha)^{-2}\int_0^\infty \big[y^{1-\alpha} - (y-1)_+^{1-\alpha}\big]^2\,dy$$

as $n\to\infty$. By (3.1) with $H = 3/2-\alpha$,

$$\int_0^\infty \big[y^{1-\alpha} - (y-1)_+^{1-\alpha}\big]^2\,dy \tag{2.29}$$
$$= [(3-2\alpha)(1-\alpha)]^{-1}\frac{\sin(\pi\alpha)}{\pi}\Gamma(2\alpha-1)\Gamma(2-\alpha)^2$$
$$= \frac{1-\alpha}{3-2\alpha}\mathrm{B}(2\alpha-1, 1-\alpha) = (1-\alpha)^2\mathrm{C}_\alpha,$$

so

$$\sum_{i=0}^{n-1}\Big(A_i - A_{i-[n^\beta t]}\Big)^2 \sim C_\alpha t^{3-2\alpha}n^{4-4\alpha},\ n\to\infty\,. \tag{2.30}$$

Since

(2.31)
$$\sum_{i=n}^{\infty}\Big(A_i - A_{i-[n^\beta t]}\Big)^2 = O\left(n^{2\beta}\sum_{i=n}^{\infty} i^{-2\alpha}\right) = O\big(n^{2\beta+1-2\alpha}\big) = o\big(n^{4-4\alpha}\big),$$

we conclude also that

$$\sum_{i=0}^{\infty}\Big(A_i - A_{i-[n^\beta t]}\Big)^2 \sim C_\alpha t^{3-2\alpha} n^{4-4\alpha},\ n\to\infty\,. \tag{2.32}$$

It follows from (2.31) and (2.32) that

$$\sum_{i=n-[n^\beta t]}^{\infty}\Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big)^2$$
$$= \sum_{j=0}^{\infty}\Big[-A_j + A_{j-[n^\beta t]} + \Big(A_{j+n} - A_{j+n-[n^\beta t]}\Big)\Big]^2 \sim C_\alpha t^{3-2\alpha} n^{4-4\alpha}\,.$$

In combination with (2.28) and (2.30) we obtain

$$\frac{\sigma_n^2}{\sigma_Z^2} - \sum_{i=0}^{\infty}(A_i - A_{i-n})(A_{i+[n^\beta t]}A_{i+[n^\beta t]-n}) \sim C_\alpha t^{3-2\alpha} n^{4-4\alpha}\,.$$

Dividing both sides by $\sigma_Z^{-2}\sigma_n^2$ and appealing to (2.21), (2.25) follows. □

We now consider certain large deviations of the partial sum S_n under a change of measure. With an eye towards a subsequent application, we allow the partial sum, given in the form (2.26), to be "corrupted". For $n \ge 1$ and $t \ge 0$ we define

$$\xi_n^1(t) = \sum_{i=1}^{[n^\beta t]} (A_i - A_{i-n})\, Z_{n-i-1}\,, \tag{2.33}$$

$$\xi_n^2(t) = \sum_{i=n-[n^\beta t]}^{n-1} (A_i - A_{i-n})\, Z_{n-i-1}\,, \tag{2.34}$$

$$\xi_n^3(t) = \sum_{i=n+1}^{n+[n^\beta t]} (A_i - A_{i-n})\, Z_{n-i-1}\,. \tag{2.35}$$

LEMMA 2.2. *Fix $t_1, t_2, t_3 > 0$ and denote*

$$\bar S_n = S_n - \sum_{i=1}^{3}\xi_n^i(t_i),\ n\ge 1\,. \tag{2.36}$$

Let (γ_n), (θ_n) and (η_n) be real sequences satisfying

$$\gamma_n = o\Big(n^{3/2-\alpha}\Big),\ \theta_n = o\Big(n^{-(1-\alpha)}\Big),\ 1 \ll \eta_n \ll n^{1/2}\,.$$

If $\tilde{S}_n$ is a random variable with the law

$$P\left(\tilde{S}_n \in dx\right) = \left(E(e^{\theta_n \bar{S}_n})\right)^{-1} e^{\theta_n x} P\left(\bar{S}_n \in dx\right),\ n \geq 1, \tag{2.37}$$

then for all $x \in \mathbb{R}$ and $h > 0$,

$$P\left(\eta_n \sigma_n^{-1}\left(\tilde{S}_n - E(\tilde{S}_n) + \gamma_n\right) \in [x, x+h]\right) \sim \eta_n^{-1}(2\pi)^{-1/2}h,\ n \to \infty. \tag{2.38}$$

Furthermore,

$$\sup_{n\geq 1,\, x\in\mathbb{R}} \eta_n P\left(\eta_n \sigma_n^{-1}\tilde{S}_n \in [x, x+1]\right) < \infty\,. \tag{2.39}$$

PROOF. Let $(\tilde{Z}_{ni},\, n \geq 1,\, i \geq 0)$ be a collection of independent random variables such that the law of $\tilde{Z}_{ni}$ is $G_{(A_i - A_{i-n})\theta_n}$ in the notation of (2.18). Then for large n,

$$\tilde{S}_n \stackrel{d}{=} A_0\tilde{Z}_{n0} + (A_n - A_0)\tilde{Z}_{nn} + \sum_{i=[n^\beta t_1]+1}^{n-[n^\beta t_2]-1} A_i\tilde{Z}_{ni} + \sum_{i=n+[n^\beta t_3]+1}^{\infty} (A_i - A_{i-n})\tilde{Z}_{ni}\,. \tag{2.40}$$

The proof applies to (2.40) the bound (3.2) in the appendix, with $n = \infty$. For any $z \in \mathbb{R}$

$$\begin{aligned}
&\left|P\left(\tilde{S}_n - E(\tilde{S}_n) \leq z\sqrt{\mathrm{Var}(\tilde{S}_n)}\right) - \Phi(z)\right| \\
&\leq C_u\left(\mathrm{Var}(\tilde{S}_n)\right)^{-3/2} \sum_{i=0}^{\infty} |A_i - A_{i-n}|^3 E\left(|\tilde{Z}_{ni} - E\tilde{Z}_{ni}|^3\right),\ n \geq 1\,.
\end{aligned} \tag{2.41}$$

It is immediate from (2.1) that

$$\sup_{i\geq 0} |A_i - A_{i-n}| = O(n^{1-\alpha})\,, \tag{2.42}$$

so that

$$\lim_{n\to\infty} \theta_n \sup_{i\geq 0} |A_i - A_{i-n}| = 0\,.$$

It follows from (2.19) that

$$E\tilde{Z}_{ni} \to 0,\ \ \mathrm{Var}(\tilde{Z}_{ni}) \to \sigma_Z^2,\ \ E\left(|\tilde{Z}_{ni} - E\tilde{Z}_{ni}|^3\right) \to \int_{-\infty}^{\infty} |z^3|\, F_Z(dz) \tag{2.43}$$

uniformly in i as $n \to \infty$. Since it is an elementary conclusion from Lemma 2.1 that for any $\kappa > 1/\alpha$,

$$\sum_{i=0}^{\infty} |A_i - A_{i-n}|^{\kappa} = O\big(n^{\kappa+1-\kappa\alpha}\big), \tag{2.44}$$

it follows from (2.41) that

$$\begin{aligned}
&\sup_{z\in\mathbb{R}} \left| P\left(\tilde{S}_n - E(\tilde{S}_n) \le z\sqrt{\operatorname{Var}(\tilde{S}_n)}\right) - \Phi(z)\right| \\
&= O\left(n^{4-3\alpha}\left(\operatorname{Var}(\tilde{S}_n)\right)^{-3/2}\right).
\end{aligned}$$

Using (2.43) again we see that

$$\operatorname{Var}(\tilde{S}_n) \sim \sigma_n^2 - \sum_{i=1}^{3} \operatorname{Var}\left(\xi_n^i(t_i)\right) \sim C_\alpha \sigma_Z^2 n^{3-2\alpha}, \tag{2.45}$$

with the second equivalence following from various claims in Lemma 2.1. Thus,

$$\sup_{z\in\mathbb{R}} \left| P\left(\tilde{S}_n - E(\tilde{S}_n) \le z\sqrt{\operatorname{Var}(\tilde{S}_n)}\right) - \Phi(z)\right| = O(n^{-1/2}) = o\left(\eta_n^{-1}\right). \tag{2.46}$$

Therefore, for $x \in \mathbb{R}$ and $h > 0$, as $n \to \infty$,

$$\begin{aligned}
&P\left(\eta_n \sigma_n^{-1}\left(\tilde{S}_n - E(\tilde{S}_n) + \gamma_n\right) \in [x, x+h]\right) \\
&= o\left(\eta_n^{-1}\right) + \int_{\mathbb{R}} \mathbf{1}\big[\operatorname{Var}(\tilde{S}_n)^{-1/2}(x\eta_n^{-1}\sigma_n - \gamma_n) \le z \\
&\qquad\qquad \le \operatorname{Var}(\tilde{S}_n)^{-1/2}((x+h)\eta_n^{-1}\sigma_n - \gamma_n)\big]\phi(z)\,dz,
\end{aligned}$$

where ϕ is the standard normal density. The assumptions on γ_n and η_n along with (2.45) imply that the integration interval shrinks towards the origin. Thus, the integral above is asymptotically equivalent to $\eta_n^{-1}\phi(0)h$, and (2.38) follows. Boundedness of ϕ in the above integral establishes (2.39). □

We now look more closely at the processes defined in (2.33), (2.34) and (2.35). The next lemma describes the limiting distribution of their increments under the same change of measure as in the previous lemma.

LEMMA 2.3. *Suppose that $\theta_n \in \mathbb{R}$ satisfies $\theta_n = o\left(n^{-(1-\alpha)}\right)$. Fix $0 \leq s < t$ and consider random variables with the laws*

$$P(U_{ni} \in dx) = c_{ni} e^{\theta_n x} P\left(\xi_n^i(t) - \xi_n^i(s) \in dx\right),\ i = 1, 2, 3,\ n \geq 1\,,$$

with appropriate c_{ni}. Then, as $n \to \infty$,

$$n^{-(2-2\alpha)}\left(U_{n1} - E(U_{n1})\right) \Rightarrow N\Big(0, K_1 \sigma_Z^2 \left(t^{3-2\alpha} - s^{3-2\alpha}\right)\Big)\,, \tag{2.47}$$

where K_1 is given in (2.24)*, and for $i = 2, 3$,*

$$n^{-(1-\alpha+\beta/2)}\left(U_{ni} - E(U_{ni})\right) \Rightarrow N\Big(0, (1-\alpha)^{-2} \sigma_Z^2 (t-s)\Big)\,. \tag{2.48}$$

PROOF. For large n,

$$U_{n1} \stackrel{d}{=} \sum_{i=[n^\beta s]+1}^{[n^\beta t]} A_i \tilde{Z}_{ni}$$

with $(\tilde{Z}_{ni})$ as in the previous lemma. That is, $U_{n1} - E(U_{n1})$ is the sum of independent zero mean random variables. By (2.43) and (2.22),

$$\mathrm{Var}(U_{n1}) \sim \sigma_Z^2 \sum_{i=[n^\beta s]+1}^{[n^\beta t]} A_i^2 \sim K_1 \sigma_Z^2 n^{4-4\alpha}\left(t^{3-2\alpha} - s^{3-2\alpha}\right)\,,$$

and a similar calculation using the third moment bound in (2.43) verifies the Lindeberg conditions of the central limit theorem. Hence (2.47) follows, and the calculations for (2.48) are similar. □

Consider the overshoot defined by

$$T_n^* = S_n - n\varepsilon,\ n \geq 1\,. \tag{2.49}$$

Conditionally on the event $E_0 = E_0(n, \varepsilon)$ in (1.5) the overshoot is nonnegative. The next lemma is a joint weak limit theorem for the joint law of the overshoot and the processes defined in (2.33), (2.34) and (2.35). The joint law is computed conditionally on E_0.

LEMMA 2.4. *Let*

$$\zeta_n = n\varepsilon / \sigma_n^2,\ \ n \geq 1\,, \tag{2.50}$$

Conditionally on E_0, as $n \to \infty$,

$$\Bigg(\zeta_n T_n^*, \left(n^{2\alpha-2}\left(\xi_n^1(t) - \sum_{i=1}^{[n^\beta t]} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz)\right), t \geq 0\right),$$
$$\left(n^{\alpha-\beta/2-1}\left(\xi_n^2(t) - \sum_{i=n-[n^\beta t]}^{n-1} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz)\right), t \geq 0\right),$$
$$\left(n^{\alpha-\beta/2-1}\left(\xi_n^3(t) - \sum_{i=n+1}^{n+[n^\beta t]} (A_i - A_{i-n}) \int_{-\infty}^{\infty} z\, G_{\zeta_n (A_i - A_{i-n})}(dz)\right), t \geq 0\right)\Bigg)$$

$$\Rightarrow \Big(T_0,\ \big(K_1^{1/2}\sigma_Z B_1(t^{3-2\alpha}),\ t \geq 0\big),$$
$$\big((1-\alpha)^{-1}\sigma_Z B_2(t),\ t \geq 0\big), \big((1-\alpha)^{-1}\sigma_Z B_3(t),\ t \geq 0\big)\Big),$$

in finite dimensional distributions, where T_0 is a standard exponential random variable independent of independent standard Brownian motions B_1, B_2, and B_3, K_1 is the constant in (2.24) *and G_θ is the exponentially tilted law in* (2.18)*.*

PROOF. Denote

$$\psi_n(s) = \frac{\sigma_n^2}{n^2} \log E\left[\exp\left(s\frac{n}{\sigma_n^2} S_n\right)\right] = \frac{\sigma_n^2}{n^2} \sum_{j=0}^{\infty} \varphi_Z\left(\sigma_n^{-2} n (A_j - A_{j-n}) s\right), \tag{2.51}$$

where the second equality follows from (2.26). By (2.16), (2.21) and (2.42) we see that

$$\lim_{n\to\infty} \psi_n(s) = s^2/2 \tag{2.52}$$

uniformly for s in a compact set. Furthermore, the sum in (2.51) can be differentiated term by term, and it follows by (2.17), (2.21) and (2.42) that

$$\lim_{n\to\infty} \psi_n'(s) = s, \tag{2.53}$$

also uniformly on compact sets. Since ψ_n' is increasing and continuous, for large n there exists a unique $\tau_n > 0$ such that

$$\psi_n'(\tau_n) = \varepsilon\,. \tag{2.54}$$

It is immediate that $\tau_n \to \varepsilon$ as $n \to \infty$. Denoting

$$\theta_n = \sigma_n^{-2} n\tau_n,\, n \geq 1\,, \tag{2.55}$$

we have

$$\left(E\left(e^{\theta_n S_n}\right)\right)^{-1} E\left(S_n e^{\theta_n S_n}\right) = n\varepsilon. \tag{2.56}$$

Fix $k \geq 1$ and for each $i = 1, 2, 3$ fix points $0 = t_{i0} < t_{i1} < \ldots < t_{ik}$. Denote

$$\bar{S}_n = S_n - \sum_{i=1}^{3} \xi_n^i(t_{ik}),\, n \geq 1\,.$$

Let U_{nij}, $n \geq 1$, $i = 1, 2, 3$, $j = 1, \ldots, k$, $\tilde{S}_n$, $n \geq 1$ be independent random variables, with

$$\begin{aligned} &P\left(U_{nij} \in dx\right) \\ &= \left(E\left(e^{\theta_n(\xi_n^i(t_{ij}) - \xi_n^i(t_{i\,j-1}))}\right)\right)^{-1} e^{\theta_n x} P\left(\xi_n^i(t_{ij}) - \xi_n^i(t_{i\,j-1}) \in dx\right)\,, \end{aligned}$$

and

$$P\left(\tilde{S}_n \in dx\right) = \left(E\left(e^{\theta_n \bar{S}_n}\right)\right)^{-1} e^{\theta_n x} P\left(\bar{S}_n \in dx\right)$$

for $n \geq 1$, $i = 1, 2, 3$ and $j = 1, \ldots, k$. Let

$$\mu_{nij} = E\left(U_{nij}\right),\; \mu_n = E(\tilde{S}_n). \tag{2.57}$$

It follows from (2.56) that

$$\mu_n + \sum_{i=1}^{3}\sum_{j=1}^{k} \mu_{nij} = n\varepsilon,\, n \geq 1\,. \tag{2.58}$$

Let $t > 0$ and $(\alpha_{ij}) \subset \mathbb{R}$. We have

$$\begin{aligned} &P\Bigg(\left\{T_n^* > t\sigma_n^2/n\varepsilon\right\} \cap \left(\bigcap_{j=1}^{k}\left\{n^{2\alpha-2}\left(\xi_n^1(t_{1j}) - \xi_n^1(t_{1\,j-1}) - \mu_{n1j}\right) > \alpha_{1j}\right\}\right) \\ &\cap \left(\bigcap_{2\leq i\leq 3,\, 1\leq j\leq k}\left\{n^{\alpha-\beta/2-1}\left(\xi_n^i(t_{ij}) - \xi_n^i(t_{i\,j-1}) - \mu_{nij}\right) > \alpha_{ij}\right\}\right)\Bigg) \end{aligned}$$

$$= \int_{\mathbb{R}^{3k+1}} \mathbf{1}\Big(x > n\varepsilon + t\sigma_n^2/n\varepsilon - \sum_{i=1}^3 \sum_{j=1}^k s_{ij}\Big)$$
$$\mathbf{1}\left(s_{1j} > n^{2-2\alpha}\alpha_{1j} + \mu_{n1j}\,, 1 \le j \le k\right)$$
$$\mathbf{1}\left(s_{ij} > n^{1-\alpha+\beta/2}\alpha_{ij} + \mu_{nij}\,, i = 2,3\,, j = 1,\dots,k\right)$$
$$P(\bar{S}_n \in dx) \prod_{i=1}^3 \prod_{j=1}^k P\big(\xi_n^i(t_{ij}) - \xi_n^i(t_{i\,j-1}) \in ds_{ij}\big)$$

$$= \int_{\mathbb{R}^{3k+1}} \mathbf{1}\Big(x > n\varepsilon + t\sigma_n^2/n\varepsilon - \sum_{i=1}^3 \sum_{j=1}^k s_{ij}\Big)$$
$$\mathbf{1}\left(s_{1j} > n^{2-2\alpha}\alpha_{1j} + \mu_{n1j}\,, 1 \le j \le k\right)$$
$$\mathbf{1}\left(s_{ij} > n^{1-\alpha+\beta/2}\alpha_{ij} + \mu_{nij}\,, i = 2,3\,, 1 \le j \le k\right)$$
$$\exp\Big(-\theta_n x - \theta_n \sum_{i=1}^3 \sum_{j=1}^k s_{ij}\Big) P\left(\tilde{S}_n \in dx\right)$$
$$E\left(e^{\theta_n S_n}\right) \prod_{i=1}^3 \prod_{j=1}^k P(U_{nij} \in ds_{ij})$$

$$= c_n \int_{\mathbb{R}^{3k}} \mathbf{1}\big(\min_{i,j}(u_{ij} - \alpha_{ij}) > 0\big) \prod_{j=1}^k P\left(n^{2\alpha-2}\big(U_{n1j} - \mu_{n1j}\big) \in du_{1j}\right)$$
$$\prod_{i=2}^3 \prod_{j=1}^k P\left(n^{\alpha-\beta/2-1}\big(U_{nij} - \mu_{nij}\big) \in du_{ij}\right)$$
$$\int_{\mathbb{R}} e^{-z} \mathbf{1}\big(z > t\theta_n\sigma_n^2/n\varepsilon\big) P\Big(\theta_n\big(\tilde{S}_n - \mu_n + \gamma_n(u_{11},\dots,u_{3k})\big) \in dz\Big)\,,$$

with

$$c_n = e^{-\theta_n n\varepsilon} E\left(e^{\theta_n S_n}\right) \tag{2.59}$$

and

$$\gamma_n(u_{11},\dots,u_{3k}) = n^{2-2\alpha} \sum_{j=1}^k u_{1j} + n^{1-\alpha+\beta/2} \sum_{i=2}^3 \sum_{j=1}^k u_{ij}\,.$$

Let θ_n be as above and $\eta_n = \sigma_n\theta_n$. For $n \geq 1$, we introduce the notation

$$\begin{aligned} &f_n(u_{11}, \dots, u_{3k}) \\ =&\eta_n \int_0^\infty e^{-z}\mathbf{1}\big(z > t\theta_n\sigma_n^2/n\varepsilon\big)P\Big(\theta_n\big(\tilde{S}_n - \mu_n + \gamma_n(u_{11}, \dots, u_{3k})\big) \in dz\Big)\,. \end{aligned}$$

Fix (u_{ij}) and let $u_{ij}^{(n)} \to u_{ij}$ as $n \to \infty$ for all i, j. Let us denote $\gamma_n = \gamma_n\big(u_{11}^{(n)}, \dots, u_{3k}^{(n)}\big)$. With θ_n and η_n already defined, we use Lemma 2.2 with this γ_n. It is elementary to check that the hypothesis of the lemma are satisfied. Since $t\theta_n\sigma_n^2/n\varepsilon \to t$, it follows from (2.38) that for all fixed $T > t$,

$$\begin{aligned} &\int_{\mathbb{R}} e^{-z}\mathbf{1}\big(t\theta_n\sigma_n^2/n\varepsilon < z \leq T\big)P\Big(\theta_n\big(\tilde{S}_n - \mu_n + \gamma_n\big) \in dz\Big) \\ &\sim \eta_n^{-1}(2\pi)^{-1/2}\int_t^T e^{-z}\,dz, \end{aligned}$$

and if follows from (2.39) that

$$\lim_{T\to\infty}\limsup_{n\to\infty}\eta_n\int_{\mathbb{R}} e^{-z}\mathbf{1}\big(z > T\big)P\Big(\theta_n\big(\tilde{S}_n - \mu_n + \gamma_n\big) \in dz\Big) = 0\,,$$

showing that

$$\lim_{n\to\infty} f_n\left(u_{11}^{(n)}, \dots, u_{3k}^{(n)}\right) = (2\pi)^{-1/2}e^{-t}\,.$$

Another application of (2.39) implies that

$$\sup_{\{u_{ij}\}\subset\mathbb{R}} f_n(u_{11}, \dots, u_{3k}) < \infty\,.$$

It follows immediately from Lemma 2.3 and bounded convergence theorem that

$$\begin{aligned} (2.60)\quad &E\Big[f\big(n^{2\alpha-2}(U_{n11} - \mu_{n11}), \dots, n^{\alpha-\beta/2-1}(U_{n3k} - \mu_{n3k})\big) \\ &\quad\mathbf{1}\Big(n^{2\alpha-2}(U_{n1j} - \mu_{n1j}) > \alpha_{1j},\ n^{\alpha-\beta/2-1}(U_{nij} - \mu_{nij}) > \alpha_{ij},\ i = 2, 3, \\ &\qquad\qquad j = 1, \dots, k,\Big)\Big] \\ &\to(2\pi)^{-1/2}P\left(T_0 > t\,, G_{ij} > \alpha_{ij} \text{ for all } i, j\right)\,, \end{aligned}$$

with T_0 standard exponential and $(G_{ij} : i = 1, 2, 3,\ j = 1, \dots, k)$ independent zero mean Gaussian random variables, independent of T_0, with

$$\mathrm{Var}(G_{1j}) = K_1\sigma_Z^2\left(t_{1j}^{3-2\alpha} - t_{1\,j-1}^{3-2\alpha}\right),\ 1 \leq j \leq k\,,$$

and for $i = 2, 3$,

$$\mathrm{Var}(G_{ij}) = (1-\alpha)^{-2}\sigma_Z^2(t_{ij} - t_{i\,,j-1}),\ 1 \le j \le k\,.$$

A simple way to verify the convergence above is to appeal to the Skorohod representation and replace the weak convergence in Lemma 2.3 by the a.s. convergence.

Notice that using (2.60) with $t = 0$ and $\alpha_{ij} = -\infty$ for all i, j tells us that

$$P(E_0) \sim (2\pi)^{-1/2} c_n/\eta_n = (2\pi)^{-1/2} e^{-\theta_n n\varepsilon} E\left(e^{\theta_n S_n}\right)/(\sigma_n\theta_n). \tag{2.61}$$

Dividing (2.60) by (2.61) gives us the statement of the lemma apart from a possibly different centring. In order to complete the proof, it suffices to show that as $n \to \infty$, for $j = 1, \dots, k$,

(2.62)

$$\mu_{n1j} = \sum_{i=[n^\beta t_{1j-1}]+1}^{[n^\beta t_{1j}]} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz) + o\left(n^{2-2\alpha}\right)\,,$$

(2.63)

$$\mu_{n2j} = \sum_{i=n-[n^\beta t_{nj}]}^{n-[n^\beta t_{nj-1}]} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz) + o\left(n^{1+\beta/2-\alpha}\right)\,,$$

(2.64)

$$\mu_{n3j} = \sum_{i=n+[n^\beta t_{nj-1}]}^{n+[n^\beta t_{nj}]} (A_i - A_{i-n}) \int_{-\infty}^{\infty} z\, G_{\zeta_n (A_i - A_{i-n})}(dz) + o\left(n^{1+\beta/2-\alpha}\right)\,.$$

For simplicity of notation we prove these statements for $j = 1$. For θ_n as in (2.55), let $(\tilde{Z}_{ni},\, n \ge 1,\, i \ge 0)$ be a collection of independent random variables such that the law of $\tilde{Z}_{ni}$ is $G_{(A_i - A_{i-n})\theta_n}$. Since both $\theta_n A_i$ and $\zeta_n A_i$ converge to zero uniformly in $i \le n^\beta t_{11}$, we can use (2.19) to write

$$\begin{aligned}
\mu_{n11} &= \sum_{i=1}^{[n^\beta t_{11}]} A_i E\left(\tilde{Z}_{ni}\right) = \sum_{i=1}^{[n^\beta t_{11}]} A_i \int_{-\infty}^{\infty} z\, G_{\theta_n A_i}(dz) \\
&= \sum_{i=1}^{[n^\beta t_{11}]} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz) + o\left(\zeta_n \sum_{i=1}^{[n^\beta t_{11}]} A_i^2\right)\,.
\end{aligned}$$

It follows from (2.21) and (2.22) that

$$\zeta_n \sum_{i=1}^{[n^\beta t_{11}]} A_i^2 = o\left(n^{2-2\alpha}\right)\,,$$

and we obtain (2.62) (for $j = 1$).

For (2.63) with $j = 1$ we notice that by (2.17),

$$E\left(\tilde{Z}_{ni}\right) = \theta_n (A_i - A_{i-n})\sigma_Z^2 + O\left(\theta_n^2 (A_i - A_{i-n})^2\right) , \tag{2.65}$$

uniformly in $i \geq 0$, as $n \to \infty$. Thus,

$$\mu_{n21} = \sigma_Z^2 \sigma_n^{-2} n \tau_n \sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i^2 + O\left(\theta_n^2 \sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i^3\right) .$$

It follows from Lemma 2.1 that

$$\theta_n^2 \sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i^3 = O\left(n^{\alpha+\beta-1}\right) = o\left(n^{1-\alpha+\beta/2}\right) .$$

Therefore,

$$\mu_{n21} = \sigma_Z^2 \sigma_n^{-2} n \tau_n \sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i^2 + o\left(n^{1-\alpha+\beta/2}\right) \tag{2.66}$$

and, similarly,

$$\sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz) = \sigma_Z^2 \zeta_n \sum_{i=n-[n^\beta t_{21}]}^{n-1} A_i^2 + o\left(n^{1-\alpha+\beta/2}\right) .$$

Another appeal to Lemma 2.1 shows that for (2.63) we only need to argue that

$$\tau_n = \varepsilon + o\left(n^{1-\alpha-\beta/2}\right) , n \to \infty . \tag{2.67}$$

However, by (2.19),

$$\psi_n'(s) = s + O\left(n\sigma_n^{-4} \sum_{j=0}^{\infty} (A_j - A_{j-n})^3\right) ,$$

uniformly for s in compact sets. Using this and (2.44), we obtain

$$\begin{aligned} \varepsilon &= \psi_n'(\tau_n) \\ &= \tau_n + O\left(n\sigma_n^{-4} \sum_{j=0}^{\infty} (A_j - A_{j-n})^3\right) \\ &= \tau_n + O(n^{\alpha-1}) = \tau_n + o\left(n^{1-\alpha-\beta/2}\right) . \end{aligned}$$

This establishes (2.67) and, hence, (2.63) with $j = 1$. The proof of (2.64) is similar. □

None of the statements proved so far required the additional assumptions stated at the beginning of this section. These assumptions start to play a role now.

The next several lemmas require additional notation designed to focus on the contribution of individual noise variables on S_n. For $n \geq 1$ and $i, j \geq 0$, $i \neq j$, we set

$$S_n'(i) = S_n - (A_i - A_{i-n})Z_{n-i-1}\,,$$

$$S_n'(i,j) = S_n - (A_i - A_{i-n})Z_{n-i-1} - (A_j - A_{j-n})Z_{n-j-1}\,,$$

and, with ζ_n given by (2.50), we let $\hat{S}_n$, $\hat{S}_{ni}$, $\hat{S}_n(i,j)$ be random variables with distributions

$$P(\hat{S}_n \in ds) \propto e^{\zeta_n s} P(S_n \in ds)\,,$$

$$P(\hat{S}_n(i) \in ds) \propto e^{\zeta_n s} P(S_n'(i) \in ds)\,,$$

$$P(\hat{S}_n(i,j) \in ds) \propto e^{\zeta_n s} P(S_n'(i,j) \in ds)\,.$$

Denote the characteristic functions of $\sigma_n^{-1}(\hat{S}_n - n\varepsilon)$, $\sigma_n^{-1}(\hat{S}_n(i) - n\varepsilon)$ and $\sigma_n^{-1}(\hat{S}_n(i,j) - n\varepsilon)$ by ϕ_n, ϕ_{ni} and ϕ_{nij}, respectively. For $\mu \in \mathbb{R}$ and $\sigma \geq 0$ we denote by $\phi_G(\mu; \sigma^2; \cdot)$ the characteristic function of $N(\mu, \sigma^2)$.

LEMMA 2.5. *Let κ be given by* (2.4) *and assume that* (2.5) *holds. Then the following statements hold uniformly in $t \in \mathbb{R}$:*

$$|\phi_n(t) - \phi_G(0;1;t)| = O\left(n^{1/2-\kappa(1-\alpha)}(1+|t|)^{\kappa+1}\right)\,, \tag{2.68}$$

$$\sup_{i \geq 0} \left|\phi_{ni}(t) - \phi_G\left(\sigma_n^{-1} n\varepsilon(\lambda_{ni} - 1); \lambda_{ni}; t\right)\right| = O\left(n^{1/2-\kappa(1-\alpha)}(1+|t|)^{\kappa+1}\right)\,, \tag{2.69}$$

$$\begin{aligned}\sup_{\substack{i,j \geq 0\\ i \neq j}} &\left|\phi_{nij}(t) - \phi_G\left(\sigma_n^{-1} n\varepsilon(\lambda_{nij} - 1); \lambda_{nij}; t\right)\right| \\ &= O\left(n^{1/2-\kappa(1-\alpha)}(1+|t|)^{\kappa+1}\right)\,,\end{aligned} \tag{2.70}$$

where for $n \geq 1$ and $i, j \geq 0$, $i \neq j$, we set

$$\lambda_{ni} = 1 - \frac{\sigma_Z^2}{\sigma_n^2}(A_i - A_{i-n})^2, \quad \lambda_{nij} = 1 - \frac{\sigma_Z^2}{\sigma_n^2}\left[(A_i - A_{i-n})^2 + (A_j - A_{j-n})^2\right]\,.$$

PROOF. It is an elementary conclusion from (2.5) that, for each $1 \leq i \leq \kappa$,

$$\left(\int_{\mathbb{R}} e^{\delta z}\, F_z(dz)\right)^{-1} \int_{\mathbb{R}} z^i e^{\delta z}\, F_z(dz) = \sigma_Z^i E\left[(G + \delta\sigma_Z)^i\right] + O\left(|\delta|^{\kappa-i+1}\right) \tag{2.71}$$

as $\delta \to 0$, where G is a standard Gaussian random variable.

Let $(\hat{Z}_{ni} : n \geq 1, i \geq 0)$ be a family of independent random variables with each $\hat{Z}_{ni} \sim G_{(A_i - A_{i-n})\zeta_n}$, so that for $n \geq 1$ and $i, j \geq 0$, $i \neq j$ we have

$$\hat{S}_n \overset{d}{=} \sum_{k=0}^{\infty} (A_k - A_{k-n})\, \hat{Z}_{nk} \,,$$

$$\hat{S}_n(i) \overset{d}{=} \sum_{k \in \{0,1,2,\dots\} \setminus \{i\}} (A_k - A_{k-n})\, \hat{Z}_{nk} \,,$$

$$\hat{S}_n(i,j) \overset{d}{=} \sum_{k \in \{0,1,2,\dots\} \setminus \{i,j\}} (A_k - A_{k-n})\, \hat{Z}_{nk} \,.$$

Let now $(G_{ni} : n \geq 1, i \geq 0)$ be a collection of independent random variables, also independent of $(\hat{Z}_{ni} : n \geq 1, i \geq 0)$, where G_{ni} follows $N\left((A_i - A_{i-n})\zeta_n \sigma_Z^2, \sigma_Z^2\right)$, for all $n \geq 1$, $i \geq 0$. It follows from Lemma 2.1 and (2.42) that (2.71) can be reformulated as

$$E\left(\hat{Z}_{nj}^i\right) - E\left(G_{nj}^i\right) = O\left(|A_j - A_{j-n}|^{\kappa - i + 1} n^{-2(1-\alpha)(\kappa - i + 1)}\right) \tag{2.72}$$

uniformly in $j \geq 0$ and $1 \leq i \leq \kappa$. For a fixed $t \in \mathbb{R}$ we use telescoping to write

(2.73)

$$\begin{aligned}
&\left| E \exp\left\{ i \left(t\sigma_n^{-1} \sum_{j=0}^{\infty} (A_j - A_{j-n}) G_{nj} \right) \right\} - E \exp\left\{ i \left(t\sigma_n^{-1} \hat{S}_n \right) \right\} \right| \\
&\leq \sum_{j=0}^{\infty} \left| E \exp\left\{ i \left(t\sigma_n^{-1} \left(\sum_{k=0}^{j-1} (A_j - A_{j-n}) \hat{Z}_{nj} + \sum_{k=j}^{\infty} (A_j - A_{j-n}) G_{nj} \right) \right) \right\} \right. \\
&\quad \left. - E \exp\left\{ i \left(t\sigma_n^{-1} \left(\sum_{k=0}^{j} (A_j - A_{j-n}) \hat{Z}_{nj} + \sum_{k=j+1}^{\infty} (A_j - A_{j-n}) G_{nj} \right) \right) \right\} \right| .
\end{aligned}$$

Fix $j \geq 0$ and denote

$$\begin{aligned}
U &= t\sigma_n^{-1} \left(\sum_{k=0}^{j-1} (A_j - A_{j-n}) \hat{Z}_{nj} + \sum_{k=j+1}^{\infty} (A_j - A_{j-n}) G_{nj} \right), \\
V &= t\sigma_n^{-1} (A_j - A_{j-n}) G_{nj} \,,
\end{aligned}$$

so that by expanding in the Taylor series around U,

$$\begin{aligned}
&E\exp\left\{i\left(t\sigma_n^{-1}\left(\sum_{k=0}^{j-1}(A_j-A_{j-n})\hat{Z}_{nj}+\sum_{k=j}^{\infty}(A_j-A_{j-n})G_{nj}\right)\right)\right\}\\
&=Ee^{i(U+V)}=\sum_{m=0}^{\kappa}\frac{i^m}{m!}E\left(V^m\right)Ee^{iU}+R_1\,,
\end{aligned}$$

with $|R_1|\le E(|V|^{\kappa+1})/(\kappa+1)!$. Similarly,

$$\begin{aligned}
&E\exp\left\{i\left(t\sigma_n^{-1}\left(\sum_{k=0}^{j}(A_j-A_{j-n})\hat{Z}_{nj}+\sum_{k=j+1}^{\infty}(A_j-A_{j-n})G_{nj}\right)\right)\right\}\\
&=\sum_{m=0}^{\kappa}\frac{i^m}{m!}E\left(W^m\right)Ee^{iU}+R_2\,,
\end{aligned}$$

with $|R_2|\le E(|W|^{\kappa+1})/(\kappa+1)!$, where

$$W=(A_j-A_{j-n})\hat{Z}_{nj}\,.$$

We conclude that

$$\begin{aligned}
&\Bigg|E\exp\left\{i\left(t\sigma_n^{-1}\left(\sum_{k=0}^{j-1}(A_j-A_{j-n})\hat{Z}_{nj}+\sum_{k=j}^{\infty}(A_j-A_{j-n})G_{nj}\right)\right)\right\}\\
&\quad-E\exp\left\{i\left(t\sigma_n^{-1}\left(\sum_{k=0}^{j}(A_j-A_{j-n})\hat{Z}_{nj}+\sum_{k=j+1}^{\infty}(A_j-A_{j-n})G_{nj}\right)\right)\right\}\Bigg|\\
&\le\sum_{i=1}^{\kappa}\frac{|t|^i}{i!}\left|(A_j-A_{j-n})^i\sigma_n^{-i}E\left(\hat{Z}_{nj}^i-G_{nj}^i\right)\right|\\
&\quad+\frac{|t|^{\kappa+1}}{(\kappa+1)!}\left|A_j-A_{j-n}\right|^{\kappa+1}\sigma_n^{-(\kappa+1)}E\left(|G_{nj}|^{\kappa+1}+|\hat{Z}_{nj}|^{\kappa+1}\right)\,. \qquad (2.74)
\end{aligned}$$

Note that by (2.44) and Lemma 2.1,

$$\begin{aligned}
&\sigma_n^{-(\kappa+1)}\sum_{j=0}^{\infty}\left|A_j-A_{j-n}\right|^{\kappa+1}E\left(|G_{nj}|^{\kappa+1}+|\tilde{Z}_{nj}|^{\kappa+1}\right)\\
&=O\left(n^{-(\kappa-1)/2}\right)=o\left(n^{1/2-\kappa(1-\alpha)}\right)\,.
\end{aligned}$$

For $1 \le i \le \kappa$ we use, in addition. (2.72) to write

$$\sigma_n^{-i} \sum_{j=0}^{\infty} \left| (A_j - A_{j-n})^i E\left(\tilde{Z}_{nj}^i - G_{nj}^i\right) \right|$$
$$= O\left(n^{-\kappa(1-\alpha)+\alpha-i(\alpha-1/2)}\right) = O\left(n^{1/2-\kappa(1-\alpha)}\right).$$

Putting these bounds into (2.74) we obtain

$$E\left(e^{\iota t \sigma_n^{-1} \tilde{S}_n}\right) = \phi_G\left(\sigma_n^{-1} n\varepsilon; 1; t\right) + O\left(n^{1/2-\kappa(1-\alpha)}\left(1 + |t|^{\kappa+1}\right)\right)$$

uniformly for $t \in \mathbb{R}$, which is equivalent to (2.68). The argument for (2.69) and (2.70) is the same. □

By the assumption (2.2), for large n, the random variables $\sigma_n^{-1}(\hat{S}_n - n\varepsilon)$, $\sigma_n^{-1}(\hat{S}_n(i) - n\varepsilon)$ and $\sigma_n^{-1}(\hat{S}_n(i,j) - n\varepsilon)$ have densities which we denote by f_n, f_{ni} and f_{nij}, correspondingly.

LEMMA 2.6. *Suppose that* (2.5) *and*(2.2) *hold. Then for large* n, *the densities* f_{ni} *and* f_{nij} *are twice differentiable. Furthermore, as* $n \to \infty$,

$$f_{ni}(0) = (2\pi)^{-1/2} + o\left(n^{1-2\alpha}\right), \tag{2.75}$$

$$f'_{ni}(0) = o\left(n^{1/2-\alpha}\right) \tag{2.76}$$

uniformly in i, *and for some* $n_0 \in \mathbb{N}$,

$$\sup\left\{\left|f''_{ni}(x)\right| : n \ge n_0,\, i \ge 0,\, x \in \mathbb{R}\right\} < \infty. \tag{2.77}$$

All three statements also hold if f_{ni} *is replaced by* f_{nij}, $i < j$. *Finally, as* $n \to \infty$,

$$\sup_{x \in \mathbb{R}} \left| f_n(x) - (2\pi)^{-1/2} e^{-x^2/2} \right| = o\left(n^{1-2\alpha}\right). \tag{2.78}$$

PROOF. We start with the proof of (2.78) which would follow from the inversion formula for densities once it is shown that

$$\int_{-\infty}^{\infty} |\phi_n(t) - \phi_G(0;1;t)|\, dt = o\left(n^{1-2\alpha}\right).$$

By Lemma 2.5 and (2.4),

$$\int_{-\log n}^{\log n} |\phi_n(t) - \phi_G(0;1;t)|\, dt = O\left(n^{1/2-\kappa(1-\alpha)}(\log n)^{\kappa+2}\right) = o\left(n^{1-2\alpha}\right).$$

Furthermore,

$$\int_{[-\log n, \log n]^c} \phi_G(0;1;t)\,dt = O\left(e^{-(\log n)^2/2}\right) = o\left(n^{1-2\alpha}\right)\,,$$

Thus, (2.78) will follow once we show that

$$\int_{[-\log n, \log n]^c} |\phi_n(t)|\,dt = o\left(n^{1-2\alpha}\right)\,. \tag{2.79}$$

With $(\hat{Z}_{ni} : n \geq 1, i \geq 0)$ as above, we set

$$U_{ni} = \sigma_n^{-1}(A_i - A_{i-n})\left[\hat{Z}_{ni} - E(\hat{Z}_{ni})\right],\, n \geq 1,\, i \geq 0\,,$$

so that

$$|\phi_n(t)| = \prod_{i=0}^{\infty} \left|E\left(e^{\iota t U_{ni}}\right)\right|\,, n \geq 1, t \in \mathbb{R}\,. \tag{2.80}$$

Set

$$H(x,t) = \left(\int_{-\infty}^{\infty} e^{xz} f_Z(z)\,dz\right)^{-1} \int_{-\infty}^{\infty} e^{(x+\iota t)z} f_Z(z)\,dz,\ (x,t) \in \mathbb{R}^2\,,$$

which is a characteristic function for any fixed x. A consequence of that is $\partial |H(x,t)|/\partial t|_{t=0} \leq 0$ for any $x \in \mathbb{R}$. Furthermore,

$$\frac{\partial^2}{\partial t^2}|H(0,t)|\Big|_{t=0} = -\sigma_Z^2 < 0$$

and by continuity of the second partial derivative we conclude that there is $\delta_0 > 0$ such that

$$\frac{\partial^2}{\partial t^2}|H(x,t)|\Big| < 0 \text{ whenever } 0 \leq |t|, |x| \leq \delta_0.$$

That means we also have

$$\frac{\partial}{\partial t}|H(x,t)|\Big| \leq 0 \text{ whenever } 0 \leq |t|, |x| \leq \delta_0. \tag{2.81}$$

We may and will choose $\delta_0 \in (0, \theta_0]$, with θ_0 as in (2.2). By (2.2) we can appeal to (3.3) to conclude that

$$\lim_{t\to\infty} \sup_{|x|\leq\delta_0} |H(x,t)| = 0\,.$$

Thus, there is $M > 0$ large enough so that

$$\sup_{t>M,|x|\le\delta_0} |H(x,t)| < 1.$$

Since by continuity of H and compactness we have

$$\sup_{\delta_0\le t\le M,|x|\le\delta_0} |H(x,t)| < 1,$$

it follows that

$$\eta = \sup_{t\ge\delta_0,|x|\le\delta_0} |H(x,t)| < 1\,.$$

The continuity argument also shows that there is $\delta_1 \in (0, \delta_0]$ such that

$$\min_{|x|\le\delta_0} |H(x,\delta_1)| \ge \eta\,.$$

Therefore, for $|x| \le \delta_0$ and $0 \le t \le \delta_1$, (2.81) implies that

$$|H(x,t)| \ge |H(x,\delta_1)| \ge \eta \ge \sup_{s\ge\delta_0} |H(x,s)|\,.$$

Since by (2.81) we also have

$$|H(x,t)| = \sup_{s\in[t,\delta_0]} |H(x,s)|\,,$$

we conclude that

$$|H(x,t)| = \sup_{s\ge t} |H(x,s)|,\ |x| \le \delta_0,\ 0 \le t \le \delta_1\,. \tag{2.82}$$

By (2.80)

$$\begin{aligned} |\phi_n(t)| &\le \left|E(e^{\iota t U_{nn}})\right| \prod_{i=[n/2]}^{n-1} \left|E(e^{\iota t U_{ni}})\right| \\ &= \left|E(e^{\iota t U_{nn}})\right| \prod_{i=[n/2]}^{n-1} \left|H\left(\zeta_n A_i, \sigma_n^{-1} A_i t\right)\right|\,. \end{aligned} \tag{2.83}$$

It follows from Lemma 2.1 that there exists $s_0 > 0$ such that for all n large enough,

$$A_i \ge s_0 \sigma_n n^{-1/2},\ [n/2] \le i \le n-1\,.$$

Thus, for n large enough and $t \geq \log n$, (2.82) implies that

$$\prod_{i=[n/2]}^{n-1} \left|H\left(\zeta_n A_i, \sigma_n^{-1} A_i t\right)\right| \leq \prod_{i=[n/2]}^{n-1} \left|H\left(\zeta_n A_i, s_0 n^{-1/2} \log n\right)\right| .$$

Since any partial derivative of H is bounded on a compact set, we can use the bound (3.4) to conclude that there exists $s_1 > 0$ such that

$$\sup_{|x|\leq \delta_0} |H(x,t)| \leq (1 - s_1 t^2)^{1/2},\ 0 \leq t \leq 1 .$$

Thus, there is $s_2 > 0$ such that for all large n and all $t \geq \log n$ we have

$$\prod_{i=[n/2]}^{n-1} \left|H\left(\zeta_n A_i, \sigma_n^{-1} A_i t\right)\right| \leq \left(1 - s_0^2 s_1 n^{-1} (\log n)^2\right)^{n/4} = O\left(e^{-s_2 (\log n)^2}\right) .$$

Using this bound in (2.83), and appealing to (2.2) we obtain

$$\begin{aligned}\int_{\log n}^{\infty} |\phi_n(t)|\, dt &= O\left(e^{-s_2(\log n)^2}\right) \int_{\log n}^{\infty} \left|E\left(e^{itU_{nn}}\right)\right| dt \\ &= O\left(n^{1/2} e^{-s_2(\log n)^2}\right) = o\left(n^{1-2\alpha}\right) .\end{aligned}$$

Since we can switch from t to $-t$, (2.79) follows, which establishes (2.78).

A similar calculation with the aid of (2.69) shows that

$$f_{ni}(0) = (2\pi\lambda_{ni})^{-1/2} \exp\left(-\sigma_n^{-2} n^2 \varepsilon^2 (\lambda_{ni} - 1)^2 / 2\lambda_{ni}\right) + o\left(n^{1-2\alpha}\right) ,$$

uniformly in $i \geq 0$. Since $\lambda_{ni} - 1 = O(1/n)$ uniformly in $i \geq 0$, it follows that

$$\begin{aligned}\lambda_{ni}^{-1/2} \exp\left(-\sigma_n^{-2} n^2 \varepsilon^2 (\lambda_{ni} - 1)^2 / 2\lambda_{ni}\right) &= 1 + O\left(n^{-1} + \sigma_n^{-2}\right) \\ &= 1 + o\left(n^{1-2\alpha}\right) ,\end{aligned}$$

uniformly for $i \geq 0$, which proves (2.75). For (2.77) we write

$$f''_{nk}(x) = -(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-itx} t^2 \phi_{nk}(t)\, dt$$

and repeat the arguments used above in the proof of (2.78), applying (2.69) and the full force of the assumption (2.2).

Finally, for (2.76) we use the identity

$$f'_{nk}(0) = -i(2\pi)^{-1/2} \int_{-\infty}^{\infty} t\phi_{nk}(t)\, dt.$$

Since

$$\left| \int_{-\infty}^{\infty} t\, \phi_G \left(\sigma_n^{-1} n\varepsilon(\lambda_{nk}-1); \lambda_{nk}; t\right) dt \right| = O\left(\sigma_n^{-1}\right) = o\left(n^{1/2-\alpha}\right) ,$$

uniformly in $k \geq 0$, (2.76) follows.

The arguments with f_{nij} replacing f_{ni} are similar. This completes the proof. □

The next lemma tackles certain expectations conditionally on E_0; its statement should be compared to (2.61).

LEMMA 2.7. *Suppose that* (2.5) *and*(2.2) *hold. Then*

(2.84)

$$E\left(Z_{n-i-1}\mathbf{1}(E_0)\right) = K_n \left[\int_{-\infty}^{\infty} z\, G_{\zeta_n(A_i - A_{i-n})}(dz) + o\left(\zeta_n^{-1}\sigma_n^{-2}|A_i - A_{i-n}|\right) \right]$$

and

$$\begin{aligned} E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right) &= K_n \Bigg(\int_{-\infty}^{\infty} z_1\, G_{\zeta_n(A_i - A_{i-n})}(dz_1) \int_{-\infty}^{\infty} z_2\, G_{\zeta_n(A_i - A_{i-n})}(dz_2) \\ &\quad + o\left(\sigma_n^{-2}|(A_i - A_{i-n})(A_j - A_{j-n})|\right) \Bigg), \ n \to \infty, \end{aligned} \tag{2.85}$$

uniformly for $i, j \geq 0$ with $i \neq j$, where

$$K_n = (2\pi)^{-1/2}\zeta_n^{-1}\sigma_n^{-1} e^{-n\varepsilon\zeta_n} E\left(e^{\zeta_n S_n}\right), \ n \geq 1 . \tag{2.86}$$

PROOF. We only prove (2.85); the proof of (2.84) is similar and easier. Write

$$\begin{aligned} &E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right) \\ &= \int_{-\infty}^{\infty} z_1\, F_Z(dz_1) \int_{-\infty}^{\infty} z_2\, F_Z(dz_2) \\ &\qquad P\left(S_n'(i,j) \geq n\varepsilon - (A_i - A_{i-n})z_1 - (A_j - A_{j-n})z_2\right) \\ &= \sigma_n^{-1} E\left(e^{\zeta_n S_n'(i,j)}\right) \int_{-\infty}^{\infty} z_1\, F_Z(dz_1) \int_{-\infty}^{\infty} z_2\, F_Z(dz_2) \\ &\qquad \int_{n\varepsilon - (A_i - A_{i-n})z_1 - (A_j - A_{j-n})z_2}^{\infty} f_{nij}\big((s - n\varepsilon)/\sigma_n\big) e^{-\zeta_n s}\, ds. \end{aligned}$$

We adopt the convention $\int_a^b \equiv -\int_b^a$, and denote

$$
\begin{aligned}
c_{nij} &= \zeta_n^{-1}\sigma_n^{-1}e^{-n\varepsilon\zeta_n}E\left(e^{\zeta_n S_n'(i,j)}\right)\\
&= K_n(2\pi)^{1/2}\left(\int_{-\infty}^{\infty} e^{\zeta_n(A_i-A_{i-n})z}F_Z(dz)\int_{-\infty}^{\infty} e^{\zeta_n(A_j-A_{j-n})z}F_Z(dz)\right)^{-1}.
\end{aligned}
$$

Changing the variable and using the fact that $EZ=0$, we obtain

$$
\begin{aligned}
&E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right)\\
&= c_{nij}\int_{-\infty}^{\infty} z_1\,F_Z(dz_1)\int_{-\infty}^{\infty} z_2\,F_Z(dz_2)\\
&\qquad \int_0^{\zeta_n(A_i-A_{i-n})z_1+\zeta_n(A_j-A_{j-n})z_2} e^x f_{nij}\big(-x/(\sigma_n\zeta_n)\big)\,dx\\
&= c_{nij}\int_{-\infty}^{\infty} z_1\,F_Z(dz_1)\int_{-\infty}^{\infty} z_2\,F_Z(dz_2)\\
&\qquad \Bigg[\int_0^{\zeta_n(A_i-A_{i-n})z_1+\zeta_n(A_j-A_{j-n})z_2} e^x f_{nij}\big(-x/(\sigma_n\zeta_n)\big)\,dx\\
&\qquad -\int_0^{\zeta_n(A_i-A_{i-n})z_1} e^x f_{nij}\big(-x/(\sigma_n\zeta_n)\big)\,dx\\
&\qquad -\int_0^{\zeta_n(A_j-A_{j-n})z_2} e^x f_{nij}\big(-x/(\sigma_n\zeta_n)\big)\,dx\Bigg].
\end{aligned}
\tag{2.87}
$$

For fixed $z_1, z_2 \in \mathbb{R}$, the expression inside the square brackets can be rewritten as

$$
\begin{aligned}
&\left(e^{\zeta_n(A_i-A_{i-n})z_1}-1\right)\int_0^{\zeta_n(A_j-A_{j-n})z_2} e^x\\
&\qquad\qquad f_{nij}\big(-(x+\zeta_n(A_i-A_{i-n})z_1)/(\sigma_n\zeta_n)\big)\,dx\\
&+\int_0^{\zeta_n(A_j-A_{j-n})z_2} e^x\\
&\qquad \Big[f_{nij}\big(-(x+\zeta_n(A_i-A_{i-n})z_1)/(\sigma_n\zeta_n)\big)-f_{nij}\big(-x/(\sigma_n\zeta_n)\big)\Big]dx.
\end{aligned}
$$

By Taylor's theorem,

$$
f_{nij}\left(-\frac{x+\zeta_n(A_i-A_{i-n})z_1}{\sigma_n\zeta_n}\right) = f_{nij}(0) - \frac{x+\zeta_n(A_i-A_{i-n})z_1}{\sigma_n\zeta_n}f_{nij}'(0)
$$

$$+O\left(\frac{(x+\zeta_n(A_i-A_{i-n})z_1)^2}{\sigma_n^2\zeta_n^2}\|f''_{nij}\|_\infty\right).$$

Using this and (2.42), straightforward algebra gives us

$$\begin{aligned}
&\int_0^{\zeta_n(A_j-A_{j-n})z_2} e^x f_{nij}\big(-(x+\zeta_n(A_i-A_{i-n})z_1)/(\sigma_n\zeta_n)\big)\,dx \\
&= f_{nij}(0)\left(e^{\zeta_n(A_j-A_{j-n})z_2}-1\right) \\
&+O\Bigg(e^{\zeta_n|A_j-A_{j-n}||z_2|}\Big(|f'_{nij}(0)|\sigma_n^{-1}\zeta_n n^{1-\alpha}|A_j-A_{j-n}||z_2|\big(|z_1|+|z_2|\big) \\
&\qquad + \|f''_{nij}\|_\infty\sigma_n^{-2}\zeta_n n^{2-2\alpha}|A_j-A_{j-n}||z_2|\big(|z_1|+|z_2|\big)^2\Big)\Bigg).
\end{aligned}$$

The obvious inequality $|e^x-1|\le|x|e^{|x|}$ for $x\in\mathbb{R}$ along with Lemma 2.6 now show that

$$\begin{aligned}
&\left(e^{\zeta_n(A_i-A_{i-n})z_1}-1\right)\int_0^{\zeta_n(A_j-A_{j-n})z_2} e^x \\
&\qquad\qquad f_{nij}\big(-(x+\zeta_n(A_i-A_{i-n})z_1)/(\sigma_n\zeta_n)\big)\,dx \\
&= f_{nij}(0)\left(e^{\zeta_n(A_i-A_{i-n})z_1}-1\right)\left(e^{\zeta_n(A_j-A_{j-n})z_2}-1\right) \\
&+o\Big(\sigma_n^{-2}\left|(A_i-A_{i-n})(A_j-A_{j-n})z_1z_2\right|\left(|z_1|+|z_2|\right)^2 \\
&\qquad\qquad e^{\zeta_n(|A_i-A_{i-n}||z_1|+|A_j-A_{j-n}||z_2|)}\Big),
\end{aligned}$$

uniformly for $i,j\ge0$ with $i\ne j$ and $z_1,z_2\in\mathbb{R}$.

Treating in a similar manner the second term, we conclude that the expression inside the square brackets in the right hand side of (2.87) equals

$$\begin{aligned}
&f_{nij}(0)\left(e^{\zeta_n(A_i-A_{i-n})z_1}-1\right)\left(e^{\zeta_n(A_j-Aj-n)z_2}-1\right) \\
&+o\Big(\sigma_n^{-2}\left|(A_i-A_{i-n})(A_j-A_{j-n})\right|(1+|z_1|^3)(1+|z_2|^3) \\
&\qquad\qquad e^{\zeta_n|(A_i-A_{i-n})z_1|+\zeta_n|(A_j-A_{j-n})z_2|}\Big),
\end{aligned}$$

uniformly for $i,j\ge0$ with $i\ne j$ and $z_1,z_2\in\mathbb{R}$, and substitution into (2.87) gives us

$$
\begin{aligned}
&E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right)\\
&= c_{nij}\Bigg[f_{nij}(0)\int_{-\infty}^{\infty} z_1 e^{\zeta_n(A_i-A_{i-n})z_1}F_Z(dz_1)\int_{-\infty}^{\infty} z_2 e^{\zeta_n(A_j-A_{j-n})z_2}F_Z(dz_2)\\
&\qquad + o\Big(\sigma_n^{-2}\left|(A_i-A_{i-n})(A_j-A_{j-n})\right|\Big)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= K_n(2\pi)^{1/2}f_{nij}(0)\int_{-\infty}^{\infty} z_1\, G_{\zeta_n(A_i-A_{i-n})}(dz_1)\int_{-\infty}^{\infty} z_2\, G_{\zeta_n(A_j-A_{j-n})}(dz_2)\\
&\qquad + c_{nij}o\Big(\sigma_n^{-2}\left|(A_i-A_{i-n})(A_j-A_{j-n})\right|\Big),
\end{aligned}
\tag{2.88}
$$

as $n\to\infty$, uniformly for $i,j\geq 0$ with $i\neq j$. Recalling that $EZ=0$, we see that

$$
\int_{-\infty}^{\infty} z_1\, G_{\zeta_n(A_i-A_{i-n})}(dz_1) = O\left(\zeta_n(A_i-A_{i-n})\right),
$$

and likewise for the second integral in (2.88). Since $K_n = O(c_{nij})$, the claim (2.85) follows from Lemma 2.6. □

The next lemma is an important step in the proof of the main result; the previous lemmas 2.5, 2.6 and 2.7 are needed for this lemma. We denote

$$
Y_{ni} = Z_{n-i-1} - \left(1+\zeta_n^{-2}\sigma_n^{-2}\right)\int_{-\infty}^{\infty} z\, G_{\zeta_n(A_i-A_{i-n})}(dz),\ i\in\mathbb{Z},\ n\geq 1\,. \tag{2.89}
$$

LEMMA 2.8. *Suppose that* (2.5) *and*(2.2) *hold. Then*

$$
\sup_{n\geq 1, i\geq 0} E\left(Y_{ni}^2\middle|E_0\right) < \infty\,, \tag{2.90}
$$

and

$$
E\left(Y_{ni}Y_{nj}\middle|E_0\right) = -\sigma_n^{-2}\sigma_Z^4\left(A_i-A_{i-n}\right)\left(A_j-A_{j-n}\right)\left(1+o(1)\right) \tag{2.91}
$$

as $n\to\infty$*, uniformly in* $i,j\geq 0$ *with* $i\neq j$.

PROOF. We prove (2.91); the proof of (2.90) is similar (and much easier). We write

$$
P(E_0) = K_n(2\pi)^{1/2}\int_0^{\infty} e^{-x} f_n\big(x/(\zeta_n\sigma_n)\big)\,dx,
$$

with K_n as in (2.86). By (2.78) and simple integration,

(2.92)
$$
\begin{aligned}
P(E_0) =& K_n (2\pi)^{1/2} \left[o\left(\zeta_n^{-2}\sigma_n^{-2}\right) + (2\pi)^{-1/2} \int_0^\infty \exp\bigl(-x - x^2/(2\zeta_n^2\sigma_n^2)\bigr)\, dx \right] \\
=& K_n \left[1 - \zeta_n^{-2}\sigma_n^{-2}\left(1+o(1)\right)\right], \; n\to\infty\,.
\end{aligned}
$$

In combination with (2.85) this means that

$$
\begin{aligned}
&E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right) P(E_0) \\
&= K_n^2 \Biggl(\left(1-\zeta_n^{-2}\sigma_n^{-2}\right) \int_{-\infty}^\infty z_1\, G_{\zeta_n(A_i-A_{i-n})}(dz_1) \int_{-\infty}^\infty z_2\, G_{\zeta_n(A_j-A_{j-n})}(dz_2) \\
&\qquad + o\left(\sigma_n^{-2}|(A_i-A_{i-n})(A_j-A_{j-n})|\right)\Biggr), \; n\to\infty,
\end{aligned}
$$

uniformly in $i,j\geq 0$ with $i\neq j$. Since by (2.84),

$$
\begin{aligned}
&E\left(Z_{n-i-1}\mathbf{1}(E_0)\right) E\left(Z_{n-j-1}\mathbf{1}(E_0)\right) \\
&= K_n^2 \int_{-\infty}^\infty z_1\, G_{\zeta_n(A_i-A_{i-n})}(dz_1) \int_{-\infty}^\infty z_2\, G_{\zeta_n(A_j-A_{j-n})}(dz_2) \\
&\qquad + o\left(K_n^2\sigma_n^{-2}|A_i-A_{i-n}||A_j-A_{j-n}|\right)\,,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&E\left(Z_{n-i-1}Z_{n-j-1}\mathbf{1}(E_0)\right) P(E_0) - E\left(Z_{n-i-1}\mathbf{1}(E_0)\right) E\left(Z_{n-j-1}\mathbf{1}(E_0)\right) \\
&= -K_n^2\zeta_n^{-2}\sigma_n^{-2} \int_{-\infty}^\infty z_1\, G_{\zeta_n(A_i-A_{i-n})}(dz_1) \int_{-\infty}^\infty z_2\, G_{\zeta_n(A_j-A_{j-n})}(dz_2) \\
&\qquad + o\left(K_n^2\sigma_n^{-2}|A_i-A_{i-n}||A_j-A_{j-n}|\right) \\
&= -K_n^2\sigma_n^{-2}\sigma_Z^4 (A_i-A_{i-n})(A_j-A_{j-n})\left(1+o(1)\right)
\end{aligned}
$$

as $n\to\infty$, uniformly in $i,j\geq 0$ with $i\neq j$. Dividing both sides by $P(E_0)^2$ and using (2.92), we obtain

$$
\begin{aligned}
(2.93)\qquad & E\left[\bigl(Z_{n-i-1}-E(Z_{n-i-1}|E_0)\bigr)\bigl(Z_{n-j-1}-E(Z_{n-j-1}|E_0)\bigr)\Big| E_0\right] \\
&= -\sigma_n^{-2}\sigma_Z^4(A_i-A_{i-n})(A_j-A_{j-n})\left(1+o(1)\right)\,,
\end{aligned}
$$

as $n\to\infty$, again uniformly for $i,j\geq 0$ with $i\neq j$. Since by (2.92) with (2.84)

$$
\begin{aligned}
E\left(Z_{n-i-1}|E_0\right) =& \left(1+\zeta_n^{-2}\sigma_n^{-2}\right) \int_{-\infty}^\infty z\, G_{\zeta_n(A_i-A_{i-n})}(dz) \\
&\qquad + o\left(\zeta_n^{-1}\sigma_n^{-2}|A_i-A_{i-n}|\right)\,,
\end{aligned}
$$

with a similar statement for Z_{n-j-1}, (2.93) implies (2.91). □

We proceed with establishing conditional distributional limits of certain truncated sums.

LEMMA 2.9. *Suppose that* (2.5) *and*(2.2) *hold. For* $0 < \delta < L$ *denote*

$$S_n(j,\delta,L) = \sum_{i=[n^\beta\delta]}^{[n^\beta L]-1} (A_{i+j} - A_i)Y_{ni} + \sum_{i=n-j}^{n-1} (A_{i+j} - A_{i+j-n} - A_i)Y_{ni} + \sum_{i=n}^{n+[n^\beta L]} (A_{i+j} - A_{i+j-n} - A_i + A_{i-n})Y_{ni},\ n \geq 1,\ j \geq 0. \tag{2.94}$$

With the overshoot T_n^* *as in* (2.49), *we have, conditionally on* E_0,

$$\Big(\zeta_n T_n^*, \Big(n^{2\alpha-2} S_n([n^\beta t],\delta,L),\ t \geq 0\Big)\Big) \Rightarrow \Bigg(T_0, \Bigg((1-\alpha)^{-1}\sigma_Z \Bigg(\int_\delta^L \big[(s+t)^{1-\alpha} - s^{1-\alpha}\big]\, dB_1(s) + \int_0^t (t-s)^{1-\alpha} dB_2(s) + \int_0^L \big[s^{1-\alpha} - (s+t)^{1-\alpha}\big]\, dB_3(s)\Bigg),\ t \geq 0\Bigg)\Bigg) \tag{2.95}$$

in finite dimensional distributions as $n \to \infty$, *where* T_0 *is a standard exponential random variable independent of independent standard Brownian motions* B_1, B_2, B_3,

PROOF. For $n \geq 1$ and $t \geq 0$ we write

$$\xi_n^{1\circ}(t) = \sum_{i=1}^{[n^\beta t]} A_i Y_{ni}, \quad \xi_n^{2\circ}(t) = \sum_{i=n-[n^\beta t]}^{n-1} A_i Y_{ni},$$

$$\xi_n^{3\circ}(t) = \sum_{i=n+1}^{n+[n^\beta t]} (A_i - A_{i-n})\, Y_{ni}.$$

It follows from Lemma 2.4 that, conditionally on E_0,

$$\Big(\zeta_n T_n^*, \big(n^{2\alpha-2}\xi_n^{1\circ}(t) : t \geq 0\big), \Big(n^{\alpha-\beta/2-1}\xi_n^{2\circ}(t) : t \geq 0\Big), \Big(n^{\alpha-\beta/2-1}\xi_n^{3\circ}(t) : t \geq 0\Big)\Big) \tag{2.96}$$

$$\Rightarrow \Big(T_0, \big(K_1^{1/2}\sigma_Z B_1(t^{3-2\alpha}) : t\geq 0\big), \big((1-\alpha)^{-1}\sigma_Z B_2(t) : t\geq 0\big), \\ \big((1-\alpha)^{-1}\sigma_Z B_3(t) : t\geq 0\big)\Big)$$

because the difference between the two processes vanishes in the limit. For example,

$$n^{2\alpha-2}\zeta_n^{-2}\sigma_n^{-2}\sum_{i=1}^{[n^\beta t]} A_i \int_{-\infty}^{\infty} z\, G_{\zeta_n A_i}(dz) = O\left(n^{1-2\alpha}\right) = o(1)\,,$$

and similarly with the other two components. Furthermore, for large n,

$$\begin{aligned} &S_n([n^\beta t],\delta,L) \\ &= \sum_{i=[n^\beta\delta]}^{[n^\beta L]-1}(A_{i+[n^\beta t]}-A_i)Y_{ni} + \sum_{i=n-[n^\beta t]}^{n-1}(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i)Y_{ni} \\ &+ \sum_{i=n}^{n+[n^\beta L]}(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i+A_{i-n})Y_{ni} =: V_n^1(t)+V_n^2(t)+V_n^3(t). \end{aligned}$$

Starting with V_n^3, we write

$$V_n^3(t) = n^{-(1-\alpha)(1-\beta)}\sum_{i=1}^{[n^\beta L]} f_n\left(n^{-\beta}i,t\right)\left(A_{n+i}-A_i\right)Y_{n,n+i}\,, \tag{2.97}$$

where for $0\leq s\leq L$,

$$f_n(s,t) = n^{(1-\alpha)(1-\beta)}\frac{A_{n+[n^\beta s]+[n^\beta t]}-A_{[n^\beta s]+[n^\beta t]}-A_{n+[n^\beta s]}+A_{[n^\beta s]}}{A_{n+[n^\beta s]}-A_{[n^\beta s]}}\,.$$

It is elementary that for fixed s,t, as $n\to\infty$,

$$\begin{aligned} A_{n+[n^\beta s]+[n^\beta t]}-A_{n+[n^\beta s]} &\ll A_{[n^\beta s]+[n^\beta t]}-A_{[n^\beta s]} \\ &\sim (1-\alpha)^{-1}n^{\beta(1-\alpha)}\left[(s+t)^{1-\alpha}-s^{1-\alpha}\right]\,, \end{aligned}$$

while $A_{n+[n^\beta s]}-A_{[n^\beta s]}\sim(1-\alpha)^{-1}n^{1-\alpha}$. Therefore,

$$\lim_{n\to\infty} f_n(s,t) = s^{1-\alpha}-(s+t)^{1-\alpha} =: f(s,t), \tag{2.98}$$

and the limit is easily seen to be uniform in $0\leq s\leq L$ and t in a compact interval. We will show that, conditionally on E_0,

$$\begin{aligned} &\big(n^{2\alpha-2}V_n^3(t),\, t\geq 0\big) \\ &\Rightarrow \left(\sigma_Z(1-\alpha)^{-1}\int_0^L\left[s^{1-\alpha}-(s+t)^{1-\alpha}\right]dB_3(s),\, t\geq 0\right) \end{aligned} \tag{2.99}$$

in finite-dimensional distributions, as $n \to \infty$. To this end, set

$$c_{nj}(k,t) = \inf_{(j-1)L/k \le s \le jL/k} f_n(s,t),\ k \ge 1,\ 1 \le j \le k\,,$$

and

$$e_{ni}(k,t) = f_n\big(n^{-\beta} i, t\big) - c_{n,\lceil L^{-1} n^{-\beta} k i\rceil}(k,t) \ge 0,\ k \ge 1,\ 1 \le i \le [n^\beta L]\,.$$

By (2.98) and monotonicity,

$$\lim_{n\to\infty} c_{nj}(k,t) = f\left((j-1)k^{-1}L, t\right),\ 1 \le j \le k\,. \tag{2.100}$$

A standard continuity argument shows that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{t\in A} \max_{1\le i\le [n^\beta L]} e_{ni}(k,t) = 0 \tag{2.101}$$

for any compact set A. We have

$$\begin{aligned}
&\sum_{i=1}^{[n^\beta L]} c_{n,\lceil L^{-1}n^{-\beta}ki\rceil}(k,t)(A_{n+i} - A_i) Y_{n,n+i} \\
&= \sum_{j=1}^{k'} c_{nj}(k,t) \sum_{i \in \left(k^{-1}Ln^\beta (j-1), k^{-1}Ln^\beta j\right] \cap \mathbb{Z}} (A_{n+i} - A_i) Y_{n,n+i} \\
&= \sum_{j=1}^{k'} c_{nj}(k,t) \Big(\xi_n^{3\circ}\big(k^{-1}Lj\big) - \xi_n^{3\circ}\big(k^{-1}L(j-1)\big)\Big) =: W_{nk}(t)\,,
\end{aligned}$$

where $k' = \lceil L^{-1} n^{-\beta} k [n^\beta L]\rceil$. This, together with (2.96) and (2.100), implies that for fixed k, as $n \to \infty$,

$$\begin{aligned}
&\big(n^{\alpha-\beta/2-1} W_{nk}(t),\ t \ge 0\big) \\
\Rightarrow &\Bigg((1-\alpha)^{-1}\sigma_Z \sum_{j=1}^{k} f\left((j-1)k^{-1}L, t\right) \left(B_3(k^{-1}jL) - B_3(k^{-1}(j-1)L)\right), \\
&\qquad t \ge 0\Bigg)
\end{aligned} \tag{2.102}$$

in finite-dimensional distributions. We have

$$\sum_{i=1}^{[n^\beta L]} f_n\big(n^{-\beta}i,t\big)\,(A_{n+i}-A_i)\,Y_{n,n+i} - W_{nk}(t)$$
$$= \sum_{i=1}^{[n^\beta L]} e_{ni}(k,t)\,(A_{n+i}-A_i)\,Y_{n,n+i}\,.$$

It follows from (2.91) that, for large n,

$$\sup_{i,j\geq 0: i\neq j} (A_i - A_{i-n})\,(A_j - A_{j-n})\,E\left(Y_{ni}Y_{nj}|E_0\right) \leq 0\,.$$

This, along with (2.90) and the non-negativity of each e_{ni}, implies that for large n,

$$E\left(\left[\sum_{i=1}^{[n^\beta L]} e_{ni}(k,t)\,(A_{n+i}-A_i)\,Y_{n,n+i}\right]^2\middle|E_0\right)$$
$$\leq \sum_{i=1}^{[n^\beta L]} \left[e_{ni}(k,t)\,(A_{n+i}-A_i)\right]^2 E(Y^2_{n,n+i}|E_0)$$
$$= O\left(\max_{1\leq j\leq [n^\beta L]} e_{nj}(k,t)^2 \sum_{i=1}^{[n^\beta L]} (A_{n+i}-A_i)^2\right)$$
$$= O\left(n^{2-2\alpha+\beta} \max_{1\leq j\leq [n^\beta L]} e_{nj}(k,t)^2\right)\,.$$

Invoking (2.101) we conclude that for any compact set A,

$$\lim_{k\to\infty}\limsup_{n\to\infty} n^{2\alpha-\beta-2}\sup_{t\in A} E\left[\left(W_{nk}(t) - \sum_{i=1}^{[n^\beta L]} f_n\big(n^{-\beta}i,t\big)\,(A_{n+i}-A_i)\,Y_{n,n+i}\right)^2\middle|E_0\right] = 0\,. \tag{2.103}$$

As $k\to\infty$, the process in the right hand side of (2.102) converges in finite-dimensional distributions to the process in the right-hand side of (2.99). Since $(2\alpha-2)-(1-\alpha)(1-\beta) = \alpha-\beta/2-1$, the claim (2.99) follows from (2.97) and (2.103) by the "convergence together" argument; see Theorem 3.2 in Billingsley (1999).

A nearly identical argument shows that, conditionally on E_0,

(2.104)
$$\begin{aligned}\big(n^{2\alpha-2}V_n^2(t),\, t\geq 0\big) &\Rightarrow \left(-\sigma_Z(1-\alpha)^{-1}\int_0^t (t-s)^{1-\alpha}dB_2(s),\, t\geq 0\right)\\ &\overset{d}{=} \left(\sigma_Z(1-\alpha)^{-1}\int_0^t (t-s)^{1-\alpha}dB_2(s),\, t\geq 0\right)\end{aligned}$$

in finite-dimensional distributions.

The situation with the term V_n^1 is, once again, similar, with a small twist. Since

$$\lim_{n\to\infty}\frac{A_{[n^\beta s]+[n^\beta t]}-A_{[n^\beta s]}}{A_{[n^\beta s]}}=\frac{(s+t)^{1-\alpha}-s^{1-\alpha}}{s^{1-\alpha}}$$

uniformly for $\delta\leq s\leq L$ and t, our argument now shows that, conditionally on E_0,

$$\big(n^{-(2-2\alpha)}V_n^1,\, t\geq 0\big) \;\Rightarrow \left(\sigma_Z K_1^{1/2}\int_\delta^L \frac{(s+t)^{1-\alpha}-s^{1-\alpha}}{s^{1-\alpha}}M(ds),\, t\geq 0\right)$$

in finite-dimensional distributions, where M is a centred Gaussian random measure with the variance measure with the density $(3-2\alpha)s^{2-2\alpha}$, $s>0$. Since the centred Gaussian random measures $(1-\alpha)^{-1}B_3(ds)$ and $K_1^{1/2}M(ds)/s^{1-\alpha}$ have the same variance measure, this means that, conditionally on E_0,

(2.105)
$$\begin{aligned}&\big(n^{2\alpha-2}V_n^2(t),\, t\geq 0\big)\\ &\Rightarrow \left(\sigma_Z(1-\alpha)^{-1}\int_\delta^L \big((s+t)^{1-\alpha}-s^{1-\alpha}\big)dB_3(s),\, t\geq 0\right)\end{aligned}$$

in finite-dimensional distributions.

Since (2.99), (2.104) and (2.105) are all consequences of (2.96), the convergence statements they contain hold jointly, and jointly with $\zeta_n T_n^*\Rightarrow T_0$. The claim (2.95) follows. □

The next lemma treats the sequence of shifts appearing due to conditioning on E_0.

LEMMA 2.10. *Define*

$$\begin{aligned}&\mu_n(t)\\ &= n^{2\alpha-2}\sum_{i=0}^\infty\Big(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i+A_{i-n}\Big)\int_{-\infty}^\infty z\, G_{\zeta_n(A_i-A_{i-n})}(dz),\end{aligned}$$

for $t \geq 0$ and $n \geq 1$. Then $\mu_n \to \mu_\infty$ as $n \to \infty$, in $D([0,\infty))$ equipped with the Skorohod J_1 topology, where $\mu_\infty(t) = -\varepsilon t^{3-2\alpha}$, $t \geq 0$.

PROOF. Writing

$$\begin{aligned}\mu_n(t) =& n^{2\alpha-2}\zeta_n \sum_{i=0}^{\infty} \Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big)\big(A_i - A_{i-n}\big) \\ &+ n^{2\alpha-2} \sum_{i=0}^{\infty} \Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big) \\ &\quad \left[\int_{-\infty}^{\infty} z\, G_{\zeta_n(A_i - A_{i-n})}(dz) - \zeta_n\big(A_i - A_{i-n}\big)\right] \\ &=: \mu_n^{(1)}(t) + \mu_n^{(2)}(t),\ t \geq 0,\end{aligned}$$

the claim of the lemma will follow once we prove that

$$\mu_n^{(1)} \to \mu_\infty \ \text{ in } \ D([0,\infty)) \tag{2.106}$$

and

$$\mu_n^{(2)}(t) \to 0 \ \text{ uniformly on compact intervals.} \tag{2.107}$$

We start by proving (2.107). Fix $L > 0$ so that $0 \leq t \leq L$. Suppose first that $1/2 < \alpha < 5/6$. By (2.19)

$$\begin{aligned}&\big|\mu_n^{(2)}(t)\big| \\ =& O\left(n^{2\alpha-2}\zeta_n^2 \sum_{i=0}^{\infty} \Big|A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big| \big(A_i - A_{i-n}\big)^2\right) \\ =& O\left(n^{2\alpha-2}\zeta_n^2 n^\beta \sum_{i=1}^{\infty} i^{-\alpha}\big(A_i - A_{i-n}\big)^2\right) = O\left(n^{2\alpha-2}\zeta_n^2 n^\beta n^{3-3\alpha}\right) \to 0\end{aligned}$$

uniformly in $0 \leq t \leq L$, showing (2.107). On the other hand, if $\alpha \geq 5/6$, then $\kappa \geq 3$ in (2.5), so by (2.19)

$$\begin{aligned}&\big|\mu_n^{(2)}(t)\big| \\ =& O\left(n^{2\alpha-2}\zeta_n^3 \sum_{i=0}^{\infty} \Big|A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big| \big(A_i - A_{i-n}\big)^3\right) \\ =& O\left(n^{2\alpha-2}\zeta_n^3 n^\beta \sum_{i=1}^{\infty} i^{-\alpha}\big(A_i - A_{i-n}\big)^3\right) = O\left(n^{2\alpha-2}\zeta_n^3 n^\beta n^{4-4\alpha}\right) \to 0\end{aligned}$$

uniformly in $0 \le t \le L$, again showing (2.107).

We now prove (2.106). The pointwise convergence is clear: for fixed t,

$$\begin{aligned}\mu_n^{(1)}(t) &= \sigma_Z^2\sigma_n^{-2}n^{2\alpha-1}\varepsilon\sum_{i=0}^{\infty}\left(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}\right)(A_i-A_{i-n})-n^{2\alpha-1}\varepsilon\\ &\to -\varepsilon t^{3-2\alpha}\end{aligned}$$

as $n\to\infty$, where we have used (2.25). Next, as in (2.28) we can write for $t\ge 0$,

$$\begin{aligned}\mu_n^{(1)}(t) =& \frac{n^{2\alpha-2}\zeta_n}{2}\left[\sum_{i=0}^{n-1}\left(A_i-A_{i-[n^\beta t]}\right)^2\right.\\ &\left.+\sum_{i=n-[n^\beta t]}^{\infty}\left(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i+A_{i-n}\right)^2\right]\\ =&: \mu_n^{(11)}(t)+\mu_n^{(12)}(t).\end{aligned}$$

The claim (2.106) will follow once we show that both $\mu_n^{(11)}$ and $\mu_n^{(12)}$ converge in $D([0,\infty))$ to continuous limits (both constant factors of μ_∞). The fact that $\mu_n^{(11)}$ converges pointwise to a constant factor of of the pointwise limit of $\mu_n^{(1)}$ is an intermediate step in the proof of (2.25). Since $\mu_n^{(11)}$ is a monotone function, its convergence in $D([0,\infty))$ follows.

We already know that $\mu_n^{(12)}$ converges pointwise to a continuous limit. Let i_0 be such that a_i is monotone for $i\ge i_0$. Write for $t\ge 0$

$$\begin{aligned}\mu_n^{(12)}(t) =& \frac{n^{2\alpha-2}\zeta_n}{2}\left[\sum_{i=n+i_0}^{\infty}\left(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i+A_{i-n}\right)^2\right.\\ &\left.-\sum_{i=n-[n^\beta t]}^{n+i_0-1}\left(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_i\right)^2\right]\\ =&: \mu_n^{(121)}(t)-\mu_n^{(122)}(t),\end{aligned}$$

so it is enough to show that both $\mu_n^{(121)}$ and $\mu_n^{(122)}$ converge in $D([0,\infty))$ to continuous limits. Splitting further, we write for $t \geq 0$,

$$\begin{aligned}
\mu_n^{(122)}(t) &= \frac{n^{2\alpha-2}\zeta_n}{2}\Bigg[\sum_{i=n-[n^\beta t]}^{n+i_0-1} A^2_{i+[n^\beta t]-n} \\
&\quad + \sum_{i=n-[n^\beta t]}^{n+i_0-1}\big(A_i - A_{i+[n^\beta t]}\big)\big(A_i - A_{i+[n^\beta t]} - 2A_{i+[n^\beta t]-n}\big)\Bigg] \\
&=: \mu_n^{(1221)}(t) + \mu_n^{(1222)}(t).
\end{aligned}$$

Clearly,

$$\mu_n^{(1221)}(t) = \frac{n^{2\alpha-2}\zeta_n}{2}\sum_{i=0}^{[n^\beta t]+i_0-1} A_i^2$$

converges pointwise to a constant factor of μ_∞. Since $\mu_n^{(1221)}$ is monotone, we conclude that $\mu_n^{(1221)}$ converges in $D([0,\infty))$ to a continuous limit. In order to prove that so does $\mu_n^{(122)}$, we will show that $\mu_n^{(1222)}(t) \to 0$ uniformly on compact intervals. Considering once again $0 \leq t \leq L$, we have

$$\begin{aligned}
&\big|\mu_n^{(1222)}(t)\big| \\
&\leq \frac{n^{2\alpha-2}\zeta_n}{2}\sum_{i=n-[n^\beta t]}^{n+i_0-1}\big(A_{i+[n^\beta t]} - A_i\big)\big[\big(A_{i+[n^\beta t]} - A_i\big) + 2A_{i+[n^\beta t]-n}\big] \\
&= O\left(n^{2\alpha-2}\zeta_n \sum_{i=n-[n^\beta t]}^{n+i_0-1} n^\beta n^{-\alpha}\big(n^\beta n^{-\alpha} + n^{\beta(1-\alpha)}\big)\right) \\
&= O\left(n^{\alpha-2}\zeta_n n^{3\beta-\beta\alpha}\right) \to 0
\end{aligned}$$

uniformly over $0 \leq t \leq L$, as required.

Finally, we already know that $\mu_n^{(121)}$ converges pointwise to a continuous limit. Furthermore, by the choice of i_0, $\mu_n^{(121)}$ is a monotone function. Therefore, it converges in $D([0,\infty))$, and the proof is complete. □

The following is the final lemma before we prove Theorem 2.1.

Lemma 2.11. *Suppose that* (2.5) *and*(2.2) *hold. Let*

$$S_n(j) = \sum_{i=j}^{j+n-1} X_i,\ j \geq 0,\ n \geq 1. \tag{2.108}$$

As $n \to \infty$, conditionally on E_0,

$$\Bigl(n^{-(2-2\alpha)}\Bigl(S_n([n^\beta t]) - n\varepsilon\Bigr),\, t \geq 0\Bigr)$$
$$\Rightarrow \Bigl((2C_\alpha)^{1/2} B_H(t) + \varepsilon^{-1} C_\alpha \sigma_Z^2 T_0 - \varepsilon t^{3-2\alpha},\, t \geq 0\Bigr)$$

in finite-dimensional distributions, where $(B_H(t) : t \geq 0)$ is the standard fractional Brownian motion (2.8) *with the Hurst exponent H given in* (2.7)*, C_α is the constant defined in* (2.10)*, and T_0 is a standard exponential random variable independent of the fractional Brownian motion.*

PROOF. It follows from (2.91) and the eventual monotonicity of the sequence (A_n) that there is $i_0 \geq 0$ such that for all large n,

$$\sup_{i_0 \leq i < j} E\left(Y_{ni} Y_{nj} | E_0\right) \leq 0\,. \tag{2.109}$$

For fixed $L, t > 0$ this and (2.90) imply that

$$\begin{aligned} &E\left[\left(\sum_{i=[n^\beta L]}^{n-[n^\beta t]-1} \Bigl(A_{i+[n^\beta t]} - A_i\Bigr) Y_{ni}\right)^2 \middle| E_0\right] \\ &= O\left(\sum_{i=[n^\beta L]}^{\infty} \Bigl(A_{i+[n^\beta t]} - A_i\Bigr)^2\right) \\ &= O\left(\sum_{j=[n^\beta L]}^{\infty} \left(\Bigl(j + [n^\beta t]\Bigr)^{1-\alpha} - j^{1-\alpha}\right)^2\right) \\ &\leq O\left(n^{4-4\alpha} \int_L^\infty \left[(x+t)^{1-\alpha} - x^{1-\alpha}\right]^2 dx\right)\,. \end{aligned}$$

Therefore, for fixed t,

$$\lim_{L\to\infty} \limsup_{n\to\infty} E\left[\left(n^{2\alpha-2} \sum_{i=[n^\beta L]}^{n-[n^\beta t]-1} \Bigl(A_{i+[n^\beta t]} - A_i\Bigr) Y_{ni}\right)^2 \middle| E_0\right] = 0\,. \tag{2.110}$$

Since the sequence (a_n) is eventually monotone, we can increase, if necessary, i_0 to guarantee that $A_{j+k} - A_j \leq A_{i+k} - A_i$ for all $i_0 \leq i \leq j$ and $k \geq 0$. By (2.109), for fixed $L, t > 0$, large n and $i, j \geq n + [n^\beta L]$,

$$\begin{aligned} &\Bigl(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Bigr) \\ &\qquad \Bigl(A_{j+[n^\beta t]} - A_{j+[n^\beta t]-n} - A_j + A_{j-n}\Bigr) E\left(Y_{ni} Y_{nj} \middle| E_0\right) \leq 0\,, \end{aligned}$$

and the same argument as above implies that

$$\lim_{L\to\infty}\limsup_{n\to\infty} E\Bigg[\Bigg(n^{2\alpha-2} \sum_{i=n+[n^\beta L]+1}^{\infty} \Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big) Y_{ni}\Bigg)^2 \Bigg| E_0\Bigg] = 0\,. \tag{2.111}$$

Similarly, for a fixed $t>0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} E\left[\left(n^{2\alpha-2}\sum_{i=i_0}^{[n^\beta\delta]-1}\Big(A_{i+[n^\beta t]} - A_i\Big) Y_{ni}\right)^2 \middle| E_0\right] = 0\,, \tag{2.112}$$

and it is elementary that for a fixed $t>0$,

$$\lim_{n\to\infty} E\left[\left(n^{2\alpha-2}\sum_{i=0}^{i_0-1}\Big(A_{i+[n^\beta t]} - A_i\Big) Y_{ni}\right)^2 \middle| E_0\right] = 0\,. \tag{2.113}$$

It follows from (2.110), (2.111), (2.112), (2.113) and Lemma 2.9 that, conditionally on E_0,

$$\begin{aligned}
&\left[\zeta_n T_n^*, \left(n^{-(2-2\alpha)}\sum_{i=0}^{\infty}\Big(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_i + A_{i-n}\Big) Y_{ni},\, t\geq 0\right)\right] \\
&\Rightarrow \Bigg[T_0, \Bigg((1-\alpha)^{-1}\sigma_Z\Bigg(\int_0^\infty \big[(s+t)^{1-\alpha} - s^{1-\alpha}\big]\, dB_1(s) \\
&\qquad + \int_0^t (t-s)^{1-\alpha}\, dB_2(s) + \int_0^\infty \big[(s+t)^{1-\alpha} - s^{1-\alpha}\big]\, dB_3(s)\Bigg),\, t\geq 0\Bigg)\Bigg]\,,
\end{aligned} \tag{2.114}$$

in finite-dimensional distributions, as $n\to\infty$. Furthermore, one can easily check the Lindeberg conditions of the central limit theorem to see that

$$\begin{aligned}
&\left(n^{-(2-2\alpha)}\sum_{i=-[n^\beta t]}^{-1} A_{i+[n^\beta t]} Z_{n-1-i},\, t\geq 0\right) \\
&\Rightarrow \left((1-\alpha)^{-1}\sigma_Z \int_0^t (t-s)^{1-\alpha}\, dB_0(s),\, t\geq 0\right)
\end{aligned} \tag{2.115}$$

in finite-dimensional distributions, as $n\to\infty$, where B_0 is a standard Brownian motion. Note that the random variables in the left hand side of (2.115) are independent of the the random variables in the left hand side of (2.114) and, in particular, independent of E_0.

Using (2.26) we conclude by (2.114) and (2.115) that, in the notation of Lemma 2.10, conditionally on E_0,

$$\begin{aligned}
&\Big[\zeta_n T_n^*, \Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t]) - S_n\Big) - \big(1+\zeta_n^{-2}\sigma_n^{-2}\big)\mu_n(t),\, t\geq 0\Big)\Big] \\
&\Rightarrow \Big[T_0, \Big((1-\alpha)^{-1}\sigma_Z\Big(\int_0^t (t-s)^{1-\alpha}\, dB_0(s) \\
&\quad + \int_0^\infty \big[(s+t)^{1-\alpha} - s^{1-\alpha}\big]\, dB_1(s) \\
&\quad + \int_0^t (t-s)^{1-\alpha}\, dB_2(s) + \int_0^\infty \big[(s+t)^{1-\alpha} - s^{1-\alpha}\big]\, dB_3(s)\Big),\, t\geq 0\Big)\Big] \\
&\overset{d}{=} \Big[T_0, \Big(2^{1/2}(1-\alpha)^{-1}\sigma_Z\int_{-\infty}^\infty \big[(t-s)_+^{1-\alpha} - (-s)_+^{1-\alpha}\big]\, dW(s),\, t\geq 0\Big)\Big]
\end{aligned}$$

in finite-dimensional distributions as $n\to\infty$, where at the intermediate step the four standard Brownian motions, B_0, B_1, B_2 and B_3 are independent (and independent of T_0), and in the final expression $(W(s),\, s\in\mathbb{R})$ is a two-sided standard Brownian motion, independent of T_0. By (2.29), this can be restated as saying that, conditionally on E_0,

$$\begin{aligned}
&\Big[\zeta_n T_n^*, \Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t]) - S_n\Big) - \mu_n(t),\, t\geq 0\Big)\Big] \\
&\Rightarrow \Big[T_0, \Big((2C_\alpha)^{1/2}B_H(t),\, t\geq 0\Big)\Big],
\end{aligned}$$

and by Lemma 2.10 also

$$\begin{aligned}
&\Big[\zeta_n T_n^*, \Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t]) - S_n\Big),\, t\geq 0\Big)\Big] \\
&\Rightarrow \Big[T_0, \Big((2C_\alpha)^{1/2}B_H(t) - \varepsilon t^{3-2\alpha},\, t\geq 0\Big)\Big]
\end{aligned}$$

in finite-dimensional distributions, as $n\to\infty$. Since

$$n^{-(2-2\alpha)}\big(S_n([n^\beta t]) - n\varepsilon\big) = n^{-(2-2\alpha)}\big(S_n([n^\beta t]) - S_n\big) + \big(n^{2\alpha-2}\zeta_n^{-1}\big)\zeta_n T_n*,$$

the claim of the lemma follows from the definition (2.50) of ζ_n and (2.22). □

Now we are in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. We will prove that

$$\Big\{P\Big[\Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t]) - n\varepsilon\Big),\, 0\leq t<\infty\Big)\in\cdot\Big| E_0\Big],\, n\geq 1\Big\} \tag{2.116}$$

is a tight family of probability measures on $D([0,\infty))$ equipped with the Skorohod J_1 topology. Assuming for a moment that this is true, it would follow from Lemma 2.11 that, conditionally on E_0,

$$\begin{aligned}&\Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t])-n\varepsilon\Big):t\geq 0\Big)\\&\Rightarrow\Big((2C_\alpha)^{1/2}B_H(t)+\varepsilon^{-1}C_\alpha\sigma_Z^2T_0-\varepsilon t^{3-2\alpha}:t\geq 0\Big)\end{aligned}$$

weakly in $D([0,\infty))$, as $n\to\infty$. Since the functional $\mathbf{x}\mapsto\inf\{t\geq 0:\,x(t)\leq 0\}$ on $D([0,\infty))$ is, clearly, a.s. continuous with respect to the law induced on that space by the limiting process, the continuous mapping theorem would imply that, conditionally on E_0,

$$\begin{aligned}n^{-\beta}I_n(\varepsilon)&=\inf\Big\{t\geq 0:n^{-(2-2\alpha)}\Big(S_n([n^\beta t])-n\varepsilon\Big)\leq 0\Big\}\\&\Rightarrow\inf\Big\{t\geq 0:(2C_\alpha)^{1/2}B_H(t)+\varepsilon^{-1}C_\alpha\sigma_Z^2T_0-\varepsilon t^{3-2\alpha}\leq 0\Big\}=\tau_\varepsilon\end{aligned}$$

as $n\to\infty$. Therefore, establishing tightness of the family (2.116) suffices to complete the proof of Theorem 2.1, and by Lemma 2.10 it is enough to prove that the family

$$\Big\{P\Big[\Big(n^{-(2-2\alpha)}\Big(S_n([n^\beta t])-n\varepsilon\Big)-\mu_n(t),\,0\leq t<\infty\Big)\in\cdot\Big|E_0\Big],\,n\geq 1\Big\}\tag{2.117}$$

is a tight family of probability measures on $D([0,\infty))$.

We have to prove tightness of the restriction of the family (2.117) to the interval $[0,L]$ for any $L>0$, so fix L. We start by showing that

$$\begin{aligned}&E\Big[\Big(S_n\Big([n^\beta t]\Big)-n^{2\alpha-2}\mu_n(t)-S_n\Big([n^\beta s]\Big)+n^{2\alpha-2}\mu_n(s)\Big)^2\Big|E_0\Big]\\&=O\Big(\Big([n^\beta t]-[n^\beta s]\Big)^{3-2\alpha}\Big),\end{aligned}\tag{2.118}$$

uniformly for $0\leq s\leq t\leq L$. We write

$$\begin{aligned}&S_n\Big([n^\beta t]\Big)-n^{2\alpha-2}\mu_n(t)-S_n\Big([n^\beta s]\Big)+n^{2\alpha-2}\mu_n(s)\\&=\sum_{i=-[n^\beta t]}^{-1}\Big(A_{i+[n^\beta t]}-A_{i+[n^\beta s]}\Big)Z_{n-i-1}\\&+\sum_{i=0}^{\infty}\Big(A_{i+[n^\beta t]}-A_{i+[n^\beta t]-n}-A_{i+[n^\beta s]}+A_{i+[n^\beta s]-n}\Big)Y_{ni}.\end{aligned}$$

Since $Z_n, Z_{n+1}, \ldots$ are independent of E_0, by Lemma 2.8,

$$\begin{aligned}
&E\left[\left(S_n\left([n^\beta t]\right) - n^{2\alpha-2}\mu_n(t) - S_n\left([n^\beta s]\right) + n^{2\alpha-2}\mu_n(s)\right)^2 \Big| E_0\right] \\
=&O\Bigg[\sum_{j=0}^{[n^\beta t]-1}\left(A_j - A_{j+[n^\beta s]-[n^\beta t]}\right)^2 \\
&\qquad + \sum_{i=0}^{\infty}\left(A_{i+[n^\beta t]} - A_{i+[n^\beta t]-n} - A_{i+[n^\beta s]} + A_{i+[n^\beta s]-n}\right)^2\Bigg] \\
=&O\left(\left([n^\beta t]-[n^\beta s]\right)^{3-2\alpha}\right)
\end{aligned}$$

uniformly for $0 \le s \le t \le L$ by (2.71) with $\kappa = 2$, and (2.118) follows.

Let now $0 \le r \le s \le t \le L$. If $t - r \le n^{-\beta}$, then

$$\begin{aligned}
E\Bigg[&\left|S_n([n^\beta s]) - \mu_n(s) - S_n([n^\beta r]) + \mu_n(r)\right| \\
&\qquad \left|S_n([n^\beta t]) - \mu_n(t) - S_n([n^\beta s]) + \mu_n(s)\right| \Big| E_0\Bigg]
\end{aligned}$$

vanishes. On the other hand, if $t-r > n^{-\beta}$, then by (2.118) and the Cauchy-Schwarz inequality, the conditional expectation can be bounded by

$$O\left(\left([n^\beta t]-[n^\beta r]\right)^{3-2\alpha}\right) = O\left(n^{4-4\alpha}(t-r)^{3-2\alpha}\right)$$

uniformly for $0 \le r \le s \le t \le L$. Since $3 - 2\alpha > 1$, the required tightness of the family in (2.117) follows, which completes the proof of Theorem 2.1. □

3. Some useful facts. We collect in this section for easy reference a number of known or easily derivable results.

The following integral evaluation follows from (2), (6) and (51) in Pickard (2011). If $H \in (0,1)$, $H \neq 1/2$, then

(3.1)
$$\int_0^\infty \left[x^{H-1/2} - (x-1)_+^{H-1/2}\right]^2 dx = \frac{\cos(\pi H)\Gamma(2-2H)}{\pi H(1-2H)}\Gamma(H+1/2)^2\,.$$

Next, we will need the following version of the Berry-Essen theorem valid for independent not necessarily identically distributed summands; see Batirov et al. (1977).

Let $X_1, \ldots, X_n$ be independent zero mean random variables with finite third moments. Denote

$$A = \sum_{i=1}^{n} E|X_i^3|, \; B = \sqrt{\sum_{i=1}^{n} E(X_i^2)}.$$

Assuming $B > 0$ we have

$$\left| P\left(\sum_{i=1}^{n} X_i \le Bz \right) - \Phi(z) \right| \le C_u A B^{-3}, \, z \in \mathbb{R}\,, \tag{3.2}$$

with C_u a universal constant, and Φ the standard normal CDF. The fact that the constant is universal means that (3.2) remains valid for $n = \infty$ as long the series in the left hand side converges and A, B are finite.

The following generalization of the Riemann-Lebesgue lemma can be proven in the same way as the original statement. If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that for some $\delta > 0$,

$$\int_{-\infty}^{\infty} e^{\theta x} |f(x)| dx < \infty \;\; \text{for all } \theta \in [-\delta, \delta]\,,$$

then

$$\lim_{t \to \infty} \sup_{|\theta| \le \delta} \left| \int_{-\infty}^{\infty} e^{(\theta + it)x} f(x)\, dx \right| = 0\,. \tag{3.3}$$

We will use a simple bound on the characteristic function ϕ of a random variable X with a finite third moment. Let X' be an independent copy of X and $Y = X - X'$. Using the bound $\cos t \le 1 - t^2/2 + |t|^3/6$ for $t \in \mathbb{R}$, we have

$$\begin{aligned} Ee^{itY} &\le 1 - t^2 E(Y^2)/2 + |t|^3 E|Y|^3/6 \\ &\le 1 - t^2 \mathrm{Var}(X) + 4|t|^3 E|X|^3/3\,. \end{aligned}$$

This implies that

$$|\phi(t)| \le \left(1 - t^2 \mathrm{Var}(X) + 4|t|^3 E|X|^3/3\right)^{1/2}, \, t \in \mathbb{R}\,. \tag{3.4}$$

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