# INTRINSIC LOCATION FUNCTIONALS OF STATIONARY PROCESSES

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Abstract. We consider a large family of measurable functionals of the sample path of a stochastic process over compact intervals (including first hitting times, leftmost location of the supremum, etc.) we call intrinsic location functionals. Despite the large variety of these functionals and their different nature, we show that for stationary processes the distribution of any intrinsic location functional over an interval is absolute continuous in the interior of the interval, and the density functions always have a version satisfying the same total variation constraints. Conversely, these total variation constraints are shown to actually characterize stationarity of the underlying stochastic process. We also show that the possible distributions of the intrinsic location functionals over an interval form a weakly closed convex set and describe its extreme points, and present applications of this description.

## 1. Introduction

We consider a large family of measurable functionals of the sample paths of a stochastic process restricted to a compact interval in the real line. The functionals are "intrinsically" connected to the sample path in the sense that they shift together with the path; this is why we call them intrinsic location functionals. They include various first/last hitting times, first/last locations of the largest value/largest jump of the process, and many others. These functionals are often highly discontinuous functions of the sample path, and for a specific process their distribution is either very difficult to derive, or else rests on a very specific property of the process, such as a Markov property.

In this paper we study the distribution of such functionals from a different point of view. Instead of looking at a specific stochastic process, we study the general question of how the stationarity of a stochastic process

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affects the distribution of an intrinsic location functional. Specifically, we show that the laws of any such functionals are absolutely continuous when restricted to the interior of the interval, and their densities have a version that satisfies very specific total variation constraints. For one very specific functional, the leftmost location of the supremum over an interval, such total variation constraints were established in Samorodnitsky and Shen (2012b), but in this paper we show that this behaviour is universal, in the sense that the constraints are shown to hold for a large variety of functionals. This universality turns out to be a characterization of stationarity. That is, given a fixed stochastic process, if for a rich enough subfamily of intrinsic location functionals, the distribution of the functional has a density within each interval that satisfies the total variation constraints, then the process has to be stationary.

We study the structure of the family of the probability distributions characterized by the total variation constraints. We determine its extreme points and show how this can be used so solve certain extremal problems.

The rest of this paper is organized as follows. In Section 2 we define the intrinsic location functionals and consider a number of examples. A description of the very specific features of the laws of intrinsic location functionals of stationary processes is stated and proved in Section 3, where we also include a discussion showing that all the defining properties of intrinsic location functionals are necessary for the conclusions of the theorem to hold. In Section 4 we discuss the structure of the set of all possible distributions of intrinsic location functionals and use it to solve certain extremal problems related to these functionals. In Section 5 we establish that the total variation constraints characterize stationarity of the process. The results of this section are refined and generalized in Section 6.

## 2. Intrinsic Location Functionals

Let $H$ be a set of functions on $\mathbb{R}$, invariant under shifts. That is, for any $f \in H$ and $c \in \mathbb{R}$ the function $\theta_c f$ defined by $\theta_c f(x) = f(x+c)$, $x \in \mathbb{R}$ belongs to $H$. We equip $H$ with its cylindrical $\sigma$-field. Let $\mathcal{I}$ be the set of all compact, non-degenerate intervals in $\mathbb{R}$: $\mathcal{I} = \{[a,b] : a < b, [a,b] \subset \mathbb{R}\}$.

**Definition 1.** *A mapping $L : H \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$ is called an intrinsic location functional, if it satisfies the following conditions.*

(1) *For every $I \in \mathcal{I}$ the map $L(\cdot, I) : H \to \mathbb{R} \cup \{\infty\}$ is measurable.*
(2) *For every $f \in H$ and $I \in \mathcal{I}$, $L(f, I) \in I \cup \{\infty\}$.*
(3) *(Shift compatibility) For every $f \in H$, $I \in \mathcal{I}$ and $c \in \mathbb{R}$,*

$$L(f, I) = L(\theta_c f, I - c) + c,$$

*where $I - c$ is the interval $I$ shifted by $-c$, and $\infty + c = \infty$.*

(4) *(Stability under restrictions) For every $f \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$,*

$$\text{if } \; L(f, I_1) \in I_2, \text{ then } \; L(f, I_2) = L(f, I_1).$$

(5) *(Consistency of existence) For every $f \in H$ and $I_1, I_2 \in \mathcal{I}$, $I_2 \subseteq I_1$,*

$$\text{if } \; L(f, I_2) \neq \infty, \text{ then } \; L(f, I_1) \neq \infty.$$

We associate the possibility of an infinite value of $L$ with "non-existence": a certain condition is never satisfied over the interval $I$ if $L(f, I) = \infty$. Otherwise, $L(f, I) \in I$. The shift compatibility requirement is the reason for the adjective "intrinsic". The stability under restrictions property asserts the global nature of $L$ over the interval $I$. Finally, the consistency of existence property says that, if a certain condition is satisfied somewhere over a small interval, it is definitely satisfied somewhere over a larger interval as well.

**Example 2.1.** Let $H$ be the space of the upper semi-continuous functions. Then the leftmost location of the supremum over the interval, defined as

$$\tau_{f,[a,b]} := \inf\{s \in [a, b] : f(s) = \sup_{t \in [a,b]} f(t)\}$$

is an intrinsic location functional. This functional was considered in detail in Samorodnitsky and Shen (2012b,a). It is an example of intrinsic location functional that does not take an infinite value. Of course, a similarly defined rightmost location of the supremum over the interval is an intrinsic location functional as well.

**Example 2.2.** Let $H$ be the space of continuous functions $\mathcal{C}(\mathbb{R})$. Then the first hitting time of certain level $l$, defined as

$$T^l_{f,[a,b]} := \inf\{s \in [a, b] : f(s) = l\}$$

is an intrinsic location functional. Replacing in this definition infimum by supremum leads to the last exit time of the level $l$, which is also an intrinsic location functional. In both cases an infinite value is a possibility.

It is easy to think of many other examples of intrinsic location functionals. A few further examples are the leftmost/rightmost point with the largest/smallest slope for $C^1$ functions, or the leftmost/rightmost location of the largest jump/the jump whose size is the closest to a given number for càdlàg functions. On the other hand, certain natural functionals fail to be intrinsic location functionals, as the following examples show.

**Example 2.3.** Let $H = \mathcal{C}(\mathbb{R})$. The first hitting time of a level $l$ after a given time point $t$:

$$T^l_{t,f,[a,b]} := \inf\{s \in [a,b], s \geq t : f(s) = l\}$$

is not an intrinsic location functional, since it involves a fixed point $t$ and, therefore, is not shift compatible.

**Example 2.4.** Let $H$ be the set of all continuous functions on $\mathbb{R}$ with separated local maxima. That is, for every $f \in H$ and compact interval $[a,b]$ there is $\delta > 0$ so that $|t_1 - t_2| \geq \delta$ for any two different local maxima $t_1, t_2$ of $f$ in $[a,b]$.

Given a function $f \in H$ and an interval $[a,b]$, denote by $A = \{t_1, t_2, ...\}$ the set of local maxima of $f$ on $[a,b]$. Then the leftmost largest local maximum

$$M^1_{f,[a,b]} := \inf\{s \in A : f(s) = \sup_{t \in A} f(t)\}$$

is an intrinsic location functional; it is just the leftmost location of supremum over the interval of Example 2.1. However, the location of the leftmost second largest local maximum

$$M^2_{f,[a,b]} := \inf\{s \in A \setminus \{M^1_{f,[a,b]}\} : f(s) = \sup_{t \in A \setminus \{M^1_{f,[a,b]}\}} f(t)\}$$

is not an intrinsic location functional, even though it is shift compatible. On a smaller interval, the second largest local maximum of the larger interval may become the largest local maximum. Therefore this functional is not stable under restrictions.

**Example 2.5.** Let $H = \mathcal{C}(\mathbb{R})$. Then the first hitting time of certain level $l$ within a fixed distance $d$ to the right endpoint of the interval, defined as

$$T^{l,d}_{f,[a,b]} := \inf\{s \in [a,b], s \geq b-d : f(s) = l\}$$

is not an intrinsic functional. Although it is both shift compatible and stable under restrictions, it does not possess consistency of existence: such a hitting time may exist on a smaller interval, but disappear on a larger interval since the original location is now too far from the right endpoint of the interval.

In the remainder of the paper $\mathbf{X} = (X(t), t \in \mathbb{R})$ is a stationary process defined on some probability space $\big(\Omega, \mathcal{F}, P\big)$, and having sample paths in $H$. For a compact interval $[a,b]$, we will denote the value of an intrinsic location functional $L$ evaluated on the process $\mathbf{X}$ on that interval by $L(\mathbf{X}, [a,b])$. Note that our assumptions imply that $L(\mathbf{X}, [a,b])$ is a well defined $[a,b] \cup \{\infty\}$-valued random variable. Stationarity of the process and shift compatibility of $L$, clearly, imply that the distribution of $L$ on an interval, relatively to its left endpoint, depends only on the length of the interval. Thus we will often study intervals of the type $[0,b]$, in which case, we will use the corresponding single variable notation $L(\mathbf{X}, b)$.

We denote by $F_{\mathbf{X},[a,b]}$ the law of $L(\mathbf{X}, [a,b])$; it is a probability measure supported on the set $[a,b] \cup \{\infty\}$. Again, if the interval is of the type $[0,b]$, the corresponding notation is $F_{\mathbf{X},b}$. We preserve the same notation for the cumulative distribution function, i.e. we will write $F_{\mathbf{X},[a,b]}(t)$ for the value $F_{\mathbf{X},[a,b]}$ assigns to the interval $[a,t]$, $a \leq t \leq b$, with the corresponding single variable notation if $a = 0$.

## 3. Properties of the distributions of intrinsic location functionals of stationary processes

The main result of this section is an extension of most parts of Theorem 3.1 in Samorodnitsky and Shen (2012b) from the special case of the leftmost location of the supremum to the general intrinsic location functionals defined in the previous section.

**Theorem 3.1.** *Let $L$ be an intrinsic location functional and $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ a stationary process. Then the restriction of the law $F_{\mathbf{X},T}$ to the interior $(0,T)$ of the interval is absolutely continuous. The density, denoted by $f_{\mathbf{X},T}$, can be taken to be equal to the right derivative of the cdf $F_{\mathbf{X},T}$, which exists at every point in the interval $(0,T)$. In this case the density is right continuous, has left limits, and has the following properties.*

*(a) The limits*

$$f_{\mathbf{X},T}(0+) = \lim_{t\to 0} f_{\mathbf{X},T}(t) \text{ and } f_{\mathbf{X},T}(T-) = \lim_{t\to T} f_{\mathbf{X},T}(t)$$

*exist.*

*(b) The density has a universal upper bound given by*

$$f_{\mathbf{X},T}(t) \leq \max\left(\frac{1}{t}, \frac{1}{T-t}\right),\ 0 < t < T\,. \tag{3.1}$$

*(c) The density has a bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,*

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) \leq \min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big) + \min\big(f_{\mathbf{X},T}(t_2),\, f_{\mathbf{X},T}(t_2-)\big)\,, \tag{3.2}$$

*where*

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) = \sup \sum_{i=1}^{n-1} \big|f_{\mathbf{X},T}(s_{i+1}) - f_{\mathbf{X},T}(s_i)\big|$$

*is the total variation of $f_{\mathbf{X},T}$ on the interval $(t_1,t_2)$, and the supremum is taken over all choices of $t_1 < s_1 < \ldots < s_n < t_2$.*

*(d) The density has a bounded positive variation at the left endpoint and a bounded negative variation at the right endpoint. Furthermore, for every $0 < \varepsilon < T$,*

$$TV^{+}_{(0,\varepsilon)}(f_{\mathbf{X},T}) \leq \min\big(f_{\mathbf{X},T}(\varepsilon), f_{\mathbf{X},T}(\varepsilon-)\big) \tag{3.3}$$

*and*

$$TV^{-}_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) \leq \min\big(f_{\mathbf{X},T}(T-\varepsilon), f_{\mathbf{X},T}(T-\varepsilon-)\big)\,, \tag{3.4}$$

*where for any interval $0 \leq a < b \leq T$,*

$$TV^{\pm}_{(a,b)}(f_{\mathbf{X},T}) = \sup \sum_{i=1}^{n-1} \big(f_{\mathbf{X},T}(s_{i+1}) - f_{\mathbf{X},T}(s_i)\big)_{\pm}$$

*is the positive (negative) variation of* $f_{\mathbf{X},T}$ *on the interval* $(a,b)$*, and the supremum is taken over all choices of* $a < s_1 < \ldots < s_n < b$*.*

*(e) The limit* $f_{\mathbf{X},T}(0+) < \infty$ *if and only if* $TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) < \infty$ *for some (equivalently, any)* $0 < \varepsilon < T$*, in which case*

$$TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) \leq f_{\mathbf{X},T}(0+) + \min\big(f_{\mathbf{X},T}(\varepsilon), f_{\mathbf{X},T}(\varepsilon-)\big)\,. \tag{3.5}$$

*Similarly,* $f_{\mathbf{X},T}(T-) < \infty$ *if and only if* $TV_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) < \infty$ *for some (equivalently, any)* $0 < \varepsilon < T$*, in which case*

$$TV_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) \leq \min\big(f_{\mathbf{X},T}(T-\varepsilon), f_{\mathbf{X},T}(T-\varepsilon-)\big) + f_{\mathbf{X},T}(T-)\,. \tag{3.6}$$

The proof of Theorem 3.1 is parallel to the proof of Theorem 3.1 in Samorodnitsky and Shen (2012b); we provide an outline here. In particular, we have to verify that the possibility of an infinite value (impossible in the earlier work) is consistent with the argument.

*Proof.* We start with a lemma that is a counterpart of Lemma 2.1 in Samorodnitsky and Shen (2012b).

**Lemma 3.1.** (i) *For any* $\Delta \in \mathbb{R}$*,*

$$F_{\mathbf{X},[\Delta,T+\Delta]}(\cdot) = F_{\mathbf{X},T}(\cdot - \Delta)\,.$$

(ii) *For any intervals* $[c,d] \subseteq [a,b]$*,*

$$F_{\mathbf{X},[a,b]}(B) \leq F_{\mathbf{X},[c,d]}(B) \text{ for any Borel set } B \subset [c,d].$$

(iii) *For any intervals* $[c,d] \subseteq [a,b]$*,*

$$F_{\mathbf{X},[a,b]}(\{\infty\}) \leq F_{\mathbf{X},[c,d]}(\{\infty\}).$$

Clearly, the three statements of Lemma 3.1 are directly implied by, respectively, shift compatibility, stability under restrictions and consistency of existence properties of intrinsic location functionals.

Choose $0 < \delta < T/2$. Using shift compatibility and stability under restrictions together with the stationarity of the process, the argument in Samorodnitsky and Shen (2012b) shows that for every $\delta \leq t \leq T-\delta$, for every $\rho > 0$ and every $0 < \varepsilon < \delta\rho/(1+\rho)$

$$P\big(t < L(\mathbf{X},T) \leq t+\varepsilon\big) \leq \varepsilon(1+\rho)\max\left(\frac{1}{t}, \frac{1}{T-t}\right)\,; \tag{3.7}$$

a possibility of an infinite value does not play a role in this argument. Obviously, (3.7) implies absolute continuity of $F_{\mathbf{X},T}$ on the interval $(\delta, T-\delta)$ and, since $\delta>0$ can be taken to be arbitrarily small, also on $(0,T)$. The version of the density given by

$$f_{\mathbf{X},T}(t) = \limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} P\big(t < L(\mathbf{X},T) \leq t+\varepsilon\big),\ 0<t<T\,,$$

automatically satisfies the bound (3.1).

The second important ingredient in the proof of the theorem is the following lemma, which is analogous to Lemma 3.1 in Samorodnitsky and Shen (2012b). Here the infinite value does play a role, so the consistency of existence property of intrinsic functionals has to be used.

**Lemma 3.2.** *Let* $0 \leq \Delta < T$. *Then for every* $0 \leq \delta \leq \Delta$, $f_{\mathbf{X},T-\Delta}(t) \geq f_{\mathbf{X},T}(t+\delta)$ *almost everywhere in* $(0, T-\Delta)$. *Furthermore, for every such* $\delta$ *and every* $\varepsilon_1, \varepsilon_2 \geq 0$, *such that* $\varepsilon_1+\varepsilon_2 < T-\Delta$,

$$\int_{\varepsilon_1}^{T-\Delta-\varepsilon_2} \big(f_{\mathbf{X},T-\Delta}(t) - f_{\mathbf{X},T}(t+\delta)\big)\, dt \tag{3.8}$$

$$\leq \int_{\varepsilon_1}^{\varepsilon_1+\delta} f_{\mathbf{X},T}(t)\, dt + \int_{T-\Delta-\varepsilon_2+\delta}^{T-\varepsilon_2} f_{\mathbf{X},T}(t)\, dt\,.$$

*Proof.* The statement $f_{\mathbf{X},T-\Delta}(t) \geq f_{\mathbf{X},T}(t+\delta)$ almost everywhere in $(0, T-\Delta)$ follows from Lemma 3.1 as in Samorodnitsky and Shen (2012b). For (3.8), we have

$$\int_{\varepsilon_1}^{T-\Delta-\varepsilon_2} \big(f_{\mathbf{X},T-\Delta}(t) - f_{\mathbf{X},T}(t+\delta)\big)\, dt$$

$$= P\big(L(\mathbf{X},T-\Delta) \in (\varepsilon_1, T-\Delta-\varepsilon_2)\big) - P\big(L(\mathbf{X},T) \in (\varepsilon_1+\delta, T-\Delta-\varepsilon_2+\delta)\big)$$

$$= P\big(L(\mathbf{X},T) \notin (\varepsilon_1+\delta, T-\Delta-\varepsilon_2+\delta)\big) - P\big(L(\mathbf{X},T-\Delta) \notin (\varepsilon_1, T-\Delta-\varepsilon_2)\big)$$

$$= P\big(L(\mathbf{X},T) \in [0, \varepsilon_1+\delta)\big) + P\big(L(\mathbf{X},T) \in (T-\Delta-\varepsilon_2+\delta, T]\big)$$

$$+P\big(L(\mathbf{X},T)=\infty\big) - P\big(L(\mathbf{X},T-\Delta) \in [0,\varepsilon_1)\big)$$

$$-P\big(L(\mathbf{X},T-\Delta) \in (T-\Delta-\varepsilon_2, T-\Delta]\big) - P\big(L(\mathbf{X},T-\Delta)=\infty\big)$$

$$= P\big(L(\mathbf{X},T) \in (\varepsilon_1, \varepsilon_1+\delta)\big) + \Big(P\big(L(\mathbf{X},T) \in [0,\varepsilon_1)\big) - P\big(L(\mathbf{X},T-\Delta) \in [0,\varepsilon_1)\big)\Big)$$

$$+P\big(L(\mathbf{X},T) \in (T-\Delta-\varepsilon_2+\delta, T-\varepsilon_2)\big) + \Big(P\big(L(\mathbf{X},T)=\infty\big) - P\big(L(\mathbf{X},T-\Delta)=\infty\big)\Big)$$

$$+\Big(P\big(L(\mathbf{X},T) \in (T-\varepsilon_2, T]\big) - P\big(L(\mathbf{X},[\Delta,T]) \in (T-\varepsilon_2, T]\big)\Big)$$

$$\leq P\big(L(\mathbf{X},T) \in (\varepsilon_1, \varepsilon_1+\delta)\big) + P\big(L(\mathbf{X},T) \in (T-\Delta-\varepsilon_2+\delta, T-\varepsilon_2)\big)$$

$$= \int_{\varepsilon_1}^{\varepsilon_1+\delta} f_{\mathbf{X},T}(t)\,dt + \int_{T-\Delta-\varepsilon_2+\delta}^{T-\varepsilon_2} f_{\mathbf{X},T}(t)\,dt\,,$$

since by Lemma 3.1, all the differences of probabilities above are non-positive.

□

Lemmas 3.1 and 3.2 are the only tools needed to complete the proof of Theorem 3.1 as in Samorodnitsky and Shen (2012b). □

Absence of even one of the three defining properties of an intrinsic location functional will, generally, void the conclusions of Theorem 3.1. To demonstrate that, we will use examples 2.3, 2.4 and 2.5 above. In all cases we will use a very simple periodic stationary process $X_{\text{per}}(t) = \sin(t+U)$, $t \in \mathbb{R}$, where $U$ is uniformly distributed between 0 and $2\pi$. We will also use a simple device to show a failure of the conclusions of Theorem 3.1: suppose that for some $0 < a < b < T$ we have $P\big(L(\mathbf{X},T) \in [a,b]\big) = 1$. Then a density with the prescribed total variation properties cannot exist. Indeed, take $0 < t_1 < a$, $b < t_2 < T$. Then the right hand side of (3.2) vanishes. On the other hand, the largest value of the density over the interval $[a,b]$ cannot be smaller than $1/(b-a)$, so the left hand side of (3.2) cannot be smaller than $2/(b-a)$.

**Example 3.2.** The first hitting time after a given time defined in Example 2.3: $T^l_{t,f,[a,b]} := \inf\{s \in [a,b], s \geq t : f(s) = l\}$ satisfies stability under restrictions and consistency of existence, but not shift compatibility. Take $l = 0$, $t > 0$ and $T > t + \pi$. Then for the periodic process $\mathbf{X}_{\text{per}}$ above, $P(T^l_{t,\mathbf{X}_{\text{per}},[0,T]} \in [t, t+\pi]) = 1$, and the conclusions of Theorem 3.1 cannot hold.

**Example 3.3.** The leftmost second largest local maximum functional $M^2_{f,[a,b]}$ of Example 2.4 satisfies shift compatibility and consistency of existence, but not stability under restrictions. Let $T > 2\pi$. For the periodic process $\mathbf{X}_{\text{per}}$ above, $P\big(M^2_{\mathbf{X}_{\text{per}},[0,T]} \in [\pi, 2\pi]\big) = 1$, so the conclusions of Theorem 3.1 cannot hold.

**Example 3.4.** The first hitting time of a level $l$ within a fixed distance $d$ to the right endpoint of the interval, $T^{l,d}_{f,[a,b]}$ of Example 2.5, satisfies shift compatibility and stability under restrictions, but not consistency of existence. Let $l=0$ and $T>d>\pi$. Then for the periodic process $\mathbf{X}_{\rm per}$ above, $P(T^{l,d}_{\mathbf{X}_{\rm per},[0,T]}\in[T-d,T-d+\pi])=1$. Once again, the conclusions of Theorem 3.1 cannot hold.

## 4. Structure of the set of all possible distributions

Theorem 3.1 of the previous section shows that the distribution of $L(\mathbf{X},T)$ for any intrinsic location functional $L$, any stationary process $\mathbf{X}$ and any positive real number $T$ is of a very special type. In this section we study the fine structure of this class of laws.

We denote by $A_T$ the class of probability measures $F$ on $[0,T]\cup\{\infty\}$ with the following properties.

(1) The restriction of $F$ to the interior $(0,T)$ of the interval is absolutely continuous.
(2) A version of the density is given by the right derivative of the cdf $F\big([0,t]\big)$, $0<t<T$, which exists at every point in the interval $(0,T)$.
(3) This density $f$ is right continuous, has left limits, and satisfies the total variation constraints (3.2), (3.3), (3.4), (3.5) and (3.6).

It is elementary to check that the total variation constraints (3.2) imply the upper bound (3.1) on the densities of all laws in $A_T$.

We endow the set $[0,T]\cup\{\infty\}$ with the topology obtained by treating the infinite point as an isolated point of the set. Let $\mathcal{P}_T$ be the collection of all probability measures on $[0,T]\cup\{\infty\}$.

**Theorem 4.1.** *The set $A_T$ is a weakly closed convex subset of $\mathcal{P}_T$. Moreover, for any $0<\varepsilon<T/2$, the restrictions of the laws in $A_T$ to the interval $(\varepsilon,T-\varepsilon)$ form a compact in total variation family of finite measures.*

*Proof.* The convexity of $A_T$ is obvious. Fix $0<\varepsilon<T/2$, and let $f$ be the version of the density of an arbitrary member of the class $A_T$ in the interior of the interval $[0,T]$ described in the definition of that class. For $x>0$ small enough, we have

$$\int_{\varepsilon}^{T-\varepsilon} \bigl|f(x+y)-f(y)\bigr|\,dy = \sum_{j=1}^{\lfloor (T-2\varepsilon)/x\rfloor} \int_{\varepsilon+(j-1)x}^{\varepsilon+jx} \bigl|f(x+y)-f(y)\bigr|\,dy$$

$$+\int_{\varepsilon+\lfloor (T-2\varepsilon)/x\rfloor x}^{T-\varepsilon} \bigl|f(x+y)-f(y)\bigr|\,dy$$

$$\leq \int_0^x \sum_{j=1}^{\lfloor (T-2\varepsilon)/x\rfloor} \bigl|f(\varepsilon+jx+y)-f(\varepsilon+(j-1)x+y)\bigr|\,dy + \max\left(\frac{1}{\varepsilon},\frac{1}{T-\varepsilon}\right)x$$

$$\leq TV_{(\varepsilon,T-\varepsilon)}(f)\,x + \max\left(\frac{1}{\varepsilon},\frac{1}{T-\varepsilon}\right)x \leq 3\max\left(\frac{1}{\varepsilon},\frac{1}{T-\varepsilon}\right)x$$

by (3.2) and (3.1). Since the final upper bound converges to 0 as $x \to 0$ uniformly over the entire class $A_T$, we conclude by Theorem 20, p. 298 in Dunford and Schwartz (1988) that the family of the densities of the laws in $A_T$ is relatively compact in $L_1(\varepsilon, T-\varepsilon)$, for each $0<\varepsilon<T/2$.

Next, let $F_n$, $n=1,2,\ldots$ be a sequence of probability measures in $A_T$ such that $F_n \Rightarrow F$ for some $F\in\mathcal{P}_T$. For $n\geq 1$ we denote by $f_n$ the version of the density of $F_n$ in the interior of the interval $[0,T]$ described in the definition of the class $A_T$. Let $0<t<T$. For $0<\varepsilon<\min(t,T-t)$ we have

$$F\bigl((t-\varepsilon,t+\varepsilon)\bigr) \leq \liminf_{n\to\infty} F_n\bigl((t-\varepsilon,t+\varepsilon)\bigr) \leq \int_{t-\varepsilon}^{t+\varepsilon} \max\left(\frac{1}{s},\frac{1}{T-s}\right) ds\,.$$

This implies that $F$ is absolutely continuous in the interior of the interval $[0,T]$ with a density $f$ satisfying

$$f(t)\leq \max\left(\frac{1}{t},\frac{1}{T-t}\right),\ \ 0<t<T\,.$$

Since for every $0<\varepsilon<T/2$ the sequence $(f_n)$ is relatively compact in $L_1(\varepsilon,T-\varepsilon)$, we conclude that

$$f_n \to f \ \text{ in } L_1(\varepsilon,T-\varepsilon). \tag{4.1}$$

Fix once again $0<\varepsilon<T/2$, and notice that, according to (4.1), there is a subsequence $(f_{n_k})$ with $n_k\to\infty$ such that

$$f_{n_k}\to f \ \text{ a.e. in } (\varepsilon,T-\varepsilon). \tag{4.2}$$

In the computations in the sequel we will identify, for typographical convenience, the subsequence $(f_{n_k})$ with the entire sequence $(f_n)$. Let $A_*$ be the set of $\varepsilon < t < T - \varepsilon$ of full measure for which the convergence in (4.2) takes place.

The next step is to show that for every $\varepsilon < t < T - \varepsilon$,

$$\lim_{s\downarrow t,\, s\in A_*} f(s) \text{ exists, and } \lim_{s\uparrow t,\, s\in A_*} f(s) \text{ exists.} \tag{4.3}$$

We will prove the first statement in (4.3); the second one is analogous. Suppose that, to the contrary, for some $\varepsilon < t < T - \varepsilon$ the limit from the right does not exist. Then there are sequences in $A_*$, $s_m \downarrow t$ and $v_m \downarrow t$, such that

$$b := \lim_{m\to\infty} f(s_m) > a := \lim_{m\to\infty} f(v_m)\,.$$

We may, of course, assume that $s_1 > v_1 > s_2 > v_2 > \ldots > t$. Let $\tau = b - a > 0$, and take $M$ so large that

$$f(s_m) > b - \tau/6,\ \ f(v_m) < a + \tau/6 \ \text{ for all } m > M. \tag{4.4}$$

Choose $K$ so large that

$$(2K-1)\tau > 6\max\left(\frac{1}{\varepsilon}, \frac{1}{T-\varepsilon}\right)\,,$$

and choose $n$ so large that

$$\bigl|f_n(v_m) - f(v_m)\bigr| \le \tau/6,\ \ \bigl|f_n(s_m) - f(s_m)\bigr| \le \tau/6 \tag{4.5}$$

for each $m = M+1, \ldots, M+K$; this is possible to achieve since each $s_m$ and each $v_m$ is in the set $A_*$. It follows from (4.4) and (4.5) that

$$f_n(s_m) > b - \tau/3,\ \ f_n(v_m) < a + \tau/3 \ \text{ for each } m = M+1, \ldots, M+K,$$

so that

$$\sum_{m=M+1}^{m+K} \bigl|f_n(s_m) - f_n(v_m)\bigr| + \sum_{m=M+1}^{m+K-1} \bigl|f_n(v_m) - f_n(s_{m+1})\bigr| > (2K-1)\tau/3\,.$$

By the choice of $K$, however, this contradicts the total variation constraint (3.2) since, by (3.1),

$$\max\Bigl(f_n(s_{M+1}),\, f_n(v_{M+K})\Bigr) \le \max\left(\frac{1}{\varepsilon}, \frac{1}{T-\varepsilon}\right)\,.$$

Therefore, (4.3) holds.

Next, we show that the set

$$B_* = \Bigl\{t \in A_* :\ f(t) \ne \lim_{s\downarrow t,\, s\in A_*} f(s)\Bigr\}$$

is, at most, countable, which will follow once we check that for any $\theta > 0$ the set

$$B_*(\theta) = \Big\{ t \in A_* : \, \Big| f(t) - \lim_{s\downarrow t,\, s\in A_*} f(s) \Big| > \theta \Big\}$$

is finite. Specifically, we will show that the cardinality of $B_*(\theta)$ does not exceed

$$\frac{6}{\theta} \max\left( \frac{1}{\varepsilon}, \frac{1}{T-\varepsilon} \right) .$$

Indeed, suppose that, to the contrary, there are points $\varepsilon < v_1 < v_2 < \ldots < v_K < T - \varepsilon$ in $B_*(\theta)$ for some

$$K > \frac{6}{\theta} \max\left( \frac{1}{\varepsilon}, \frac{1}{T-\varepsilon} \right) .$$

For each $m = 1, \ldots, K$ choose $s_m \in A_*$, $v_m < s_m < v_{m+1}$ (with $v_{K+1} = T - \varepsilon$) such that

$$\big| f(v_m) - f(s_m) \big| > \theta \, .$$

Finally, choose $n$ so large that

$$\big| f_n(v_m) - f(v_m) \big| \leq \theta/3, \; \big| f_n(s_m) - f(s_m) \big| \leq \theta/3, \; m = 1, \ldots, K \, .$$

Then for every $m = 1, \ldots, K$ we have

$$\big| f_n(v_m) - f_n(s_m) \big| > \theta/3 \, ,$$

so that by the choice of $K$,

$$\sum_{m=1}^{K} \big| f_n(s_m) - f_n(v_m) \big| > 2 \max\left( \frac{1}{\varepsilon}, \frac{1}{T-\varepsilon} \right) .$$

Once again, this is incompatible with the combination of the total variation constraint (3.2) and the upper bound (3.1). The resulting contradiction proves that the set $B_*$ is, at most, countable.

The standard diagonal argument now allows us to get rid of $\varepsilon > 0$ in the above conclusions: there is a subsequence $(f_{n_k})$ with $n_k \to \infty$ such that $f_{n_k}(t) \to f(t)$ for almost every $0 < t < T$, say, for $t \in A_*$. Furthermore, for every $0 < t < T$ (4.3) holds. Finally, the set $B_*$ (defined now for the entire interval $(0, T)$) is at most countable. We are in a position to define now

$$g(t) = \lim_{s\downarrow t,\, s\in A_*} f(s), \; 0 < t < T \, . \tag{4.6}$$

The resulting function is automatically right continuous with left limits. Moreover, $g$ coincides with $f$ on $A_* \setminus B_*$, i.e. $g$ is a version of $f$, hence a density of the limiting law $F$ in the interior of the interval $[0,T]$. The right continuity of $g$ shows that the right derivative of $F$ exists at every point in $(0,T)$ and coincides with $g$ at that point. By construction, $g$ satisfies the total variation constraints (3.2), (3.3), (3.4), (3.5) and (3.6). This proves that $A_T$ is weakly closed.

Finally, let $0 < \varepsilon < T/2$, and let $(F_n)$ be a sequence in $A_T$. By the weak compactness of $\mathcal{P}_T$, we can choose a subsequence $(F_{n_k})$ with $n_k \to \infty$ weakly converging in $\mathcal{P}_T$ to some $F$; since we already know that $A_T$ is weakly closed, $F \in A_T$. Let $f$ be some version of the density of $F$ in $(0,T)$. We have established in the course of the proof that the densities $(f_{n_k})$ of the laws $(F_{n_k})$ form a relatively compact family in $L_1(\varepsilon, T-\varepsilon)$. Since $f$ can be the only limit point, we conclude that $f_{n_k} \to f$ in $L_1(\varepsilon, T-\varepsilon)$. This, of course, means that the restrictions of the laws $(F_{n_k})$ to the interval $(\varepsilon, T-\varepsilon)$ converge in total variation to the restriction of the law $F$ to the same interval, so the last statement of the theorem has been proved. □

Note that the set of finite signed measures on $[0,T] \cup \{\infty\}$ equipped with the topology of weak convergence is a locally convex topological vector space. According to Theorem 4.1, the set $A_T$ is a compact convex subset of that space. By the Krein-Milman theorem, the set $A_T$ is equal to the closed convex hull of its extreme points; see e.g. Theorem 4, p. 440 in Dunford and Schwartz (1988). Our next result describes the extreme points of the set $A_T$.

**Theorem 4.2.** *The extreme points of the set $A_T$ are:*

*(1) the measures $\mu_t$, $t \in (0,T)$ concentrated on $(0,T)$, absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions $f_{\mu_t} = \frac{1}{t}\mathbf{1}_{(0,t)}$, $0 < t < T$;*

*(2) the measures $\nu_t$, $t \in (0,T)$ concentrated on $(0,T)$, absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions $f_{\nu_t} = \frac{1}{T-t}\mathbf{1}_{(t,T)}$, $0 < t < T$;*

*(3) the point masses $\delta_0$, $\delta_T$ and $\delta_\infty$.*

*Proof.* Since any probability measure $m$ in $A_T$ admits a unique decomposition of the type $m = \alpha_1\delta_0 + \alpha_2\delta_T + \alpha_3\delta_\infty + \beta m_{AC}$, where $\alpha_1, \alpha_2, \alpha_3, \beta \geq 0$,

$\alpha_1+\alpha_2+\alpha_3+\beta=1$, and $m_{AC}$ is an absolutely continuous measure on $(0,T)$, it is enough to prove that the first two cases in the theorem describe all the extreme points of $A_T$ that are concentrated on $(0,T)$ and are absolutely continuous there.

Let $f$ be the density of such a measure as described in the definition of the class $A_T$. We start by showing that $f$ must be monotone. To this end, define functions $f_1(t)=TV^+_{(0,t]}(f)$ and $f_2(t)=TV^-_{(t,T)}$, $t\in(0,T)$. By (3.3) and (3.4) these functions are well-defined and nonnegative. Moreover, $f_1$ is a nondecreasing càdlàg function with $f_1(0+)=0$, while $f_2$ is a nonincreasing càdlàg function with $f_2(T-)=0$. It also follows from (3.3) and (3.4) that $f(t)\ge\max\bigl(f_1(t),f_2(t)\bigr)$ for $0<t<T$.

Choose $0<t_1<T$, and note that for every $t_1<t<T$,

$$f(t)=f(t_1)+TV^+_{(t_1,t]}(f)-TV^-_{(t_1,t]}(f)\,,$$

while

$$f_1(t)=f_1(t_1)+TV^+_{(t_1,t]}(f),\quad f_2(t)=f_2(t_1)-TV^-_{(t_1,t]}(f)\,.$$

Therefore,

$$f(t)=f_1(t)+f_2(t)+\bigl(f(t_1)-f_1(t_1)-f_2(t_1)\bigr):=f_1(t)+f_2(t)+C(t_1)\,.$$

From here we immediately conclude that $C(t_1)$ is independent of $t_1$ and, hence, is equal to some constant $C$. Since $C\ge -f_1(t)$ for any $0<t<T$, we can let $t\to0$ to conclude that $C\ge0$, so we have $f=f_1+f_2'$, where $f_2'=f_2+C$. If $f$ is not monotone, then both $\int_0^T f_1(s)ds>0$ and $\int_0^T f_2'(s)ds>0$. Hence

$$f(t)=\int_0^T f_1(s)ds\cdot\frac{f_1(t)}{\int_0^T f_1(s)ds}+\int_0^T f_2'(s)ds\cdot\frac{f_2'(t)}{\int_0^T f_2'(s)ds},\ 0<t<T\,,$$

a convex combination of two monotone densities, which are automatically densities of some laws in $A_T$. That is, the law corresponding to such $f$ cannot be an extreme point of $A_T$.

Therefore, the density $f$ must be monotone. Suppose that there are points $t_1$, $t_2$ in $(0,T)$ such that $f(t_1)=a_1$, $f(t_2)=a_2$ for some $0<a_1<a_2$. Let $f_1(t)=\max\bigl(f(t)-a_1,\,0\bigr)$ and $f_2(t)=f(t)-f_1(t)$, $0<t<T$. Since $f$ is

monotone, so are both $f_1$ and $f_2$. Once again, this allows us to represent

$$f(t) = \int_0^T f_1(s)ds \cdot \frac{f_1(t)}{\int_0^T f_1(s)ds} + \int_0^T f_2(s)ds \cdot \frac{f_2(t)}{\int_0^T f_2(s)ds}, \ 0 < t < T,$$

showing that the law corresponding to such $f$ cannot be an extreme point of $A_T$.

Therefore, the density $f$ can take at most one non-zero value. In order to conclude that it must be of the form $f_{\mu_t}$ or $f_{\nu_t}$ described in (1) or (2) in the theorem, we only need to observe that the only remaining possibility, $f \equiv 1$, does not correspond to an extreme point of $A_T$ since this constant density can be written in the form $f_{\mu_{T/2}}/2 + f_{\nu_{T/2}}/2$.

It remains to prove that for each $0 < t < T$, the densities $f_{\mu_t}$ and $f_{\nu_t}$ do correspond to extreme points of $A_T$. We will consider $f_{\mu_t}$; the argument for $f_{\nu_t}$ is similar. Suppose that there are two different laws in $A_T$ that are concentrated on $(0, T)$, with the corresponding densities $g_1$ and $g_2$, as described in the definition of the class $A_T$, such that

$$f_{\mu_t}(s) = pg_1(s) + (1-p)g_2(s), \ 0 < s < T, \tag{4.7}$$

for some $0 < p < 1$. There must be a point $0 < s_i < t$ such that $g_i(s_i) > 1/t$, $i = 1, 2$. Since $g_i(t) = 0$, $i = 1, 2$, the total variation requirement forces

$$g_i(0-) \geq g_i(s_i) > \frac{1}{t}, \ i = 1, 2,$$

so that

$$pg_1(0-) + (1-p)g_2(0-) > \frac{1}{t}.$$

This means that (4.7) is violated in a neighbourhood of the left endpoint. This contradiction completes the proof. □

Knowing the set of all extreme points of the set $A_T$ allows us to obtain universal bounds on the expectation of functions of intrinsic location functionals.

**Corollary 4.3.** *Let $g$ be a bounded, or nonnegative, measurable function on $[0, T] \cup \{\infty\}$. Then for any stationary process $\mathbf{X}$ and intrinsic location* *functional* $L$,

$$\begin{aligned} & \min\Big\{g(0), g(T), g(\infty), \inf_{t\in(0,T)} \tfrac{1}{t}\int_0^t g(s)ds, \inf_{t\in(0,T)} \tfrac{1}{T-t}\int_t^T g(s)ds\Big\} \\ \leq\ & \mathbb{E}\big[g\big(L(\mathbf{X},[0,T])\big)\big] \\ \leq\ & \max\Big\{g(0), g(T), g(\infty), \sup_{t\in(0,T)} \tfrac{1}{t}\int_0^t g(s)ds, \sup_{t\in(0,T)} \tfrac{1}{T-t}\int_t^T g(s)ds\Big\}. \end{aligned}$$

The bounds obtained in Corollary 4.3 can sometimes be improved if one is interested only in certain subsets of all intrinsic location functionals. We describe now one such situation.

We call an intrinsic location functional $L : H \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$ *an earliest occurrence intrinsic location functional* if it has the following property: for every $a < b < c$ and $f \in H$,

$$\text{if } L(f,[a,b]) \in [a,b] \text{ then } L(f,[a,c]) = L(f,[a,b]).$$

The first hitting time $T^l_{f,[a,b]}$ of Example 2.2 is, clearly, an earliest occurrence intrinsic location functional.

**Proposition 4.4.** *For every $T > 0$ the distribution of $L(\mathbf{X},T)$ for any earliest occurrence intrinsic location functional $L$ and any stationary process $\mathbf{X}$ belongs to the set $A^{\mathrm{e}}_T$ consisting of all laws in $A_T$ that do not put any mass at the right endpoint of the interval, and whose density in $(0,T)$ is nonincreasing. This set is weakly closed in $\mathcal{P}_T$, and its extreme points are the point masses $\delta_0$ and $\delta_\infty$, as well as the measures $\mu_t$, $t \in (0,T]$, concentrated on $(0,T)$, absolutely continuous with respect to the Lebesgue measure on $(0,T)$, with density functions $f_{\mu_t} = \frac{1}{t}\mathbf{1}_{(0,t)}$, $0 < t \leq T$.*

**Remark 4.5.** Note that, while some of the extreme points of $A_T$ are no longer in $A^{\mathrm{e}}_T$, the latter subset of $A_T$ does have one extreme point that is not an extreme point of $A_T$, specifically the measure $\mu_T$.

*Proof of Proposition 4.4.* Let $0 < t_1 < t_2 < T$, and take $0 < \varepsilon < t_1$. Using successively the stability under restrictions, the earliest occurrence property, and the shift compatibility, together with the stationarity of $\mathbf{X}$, we have

$$P\big(L(\mathbf{X},T) \in (t_2-\varepsilon, t_2+\varepsilon)\big) \leq P\big(L(\mathbf{X},[t_2-t_1,T]) \in (t_2-\varepsilon, t_2+\varepsilon)\big)$$

$$\leq P\big(L(\mathbf{X},[t_2-t_1,T+t_2-t_1]) \in (t_2-\varepsilon, t_2+\varepsilon)\big) = P\big(L(\mathbf{X},T) \in (t_1-\varepsilon, t_1+\varepsilon)\big).$$

If $f_{\mathbf{X},T}$ is the version of the density described in the definition of the class $A_T$, we see that $f_{\mathbf{X},T}(t_2) \le f_{\mathbf{X},T}(t_1)$, so the density must be nonincreasing. Similarly,

$$P\big(L(\mathbf{X},T)=T\big) \le P\big(L(\mathbf{X},2T)=T\big)=0$$

because laws in $A_{2T}$ cannot have a mass in the interior of an interval. Therefore, no mass at the right endpoint of the interval is possible.

To see that $A_T^{\mathrm{e}}$ is weakly closed, note that by Theorem 4.1, any weakly convergent sequence in $A_T^{\mathrm{e}}$ has its limit in $A_T$. Since by the proof of Theorem 4.1 pointwise convergence of densities takes place in $(0,T)$ apart from a set of Lebesgue measure 0, and the limiting density is right continuous, the limiting density must be nonincreasing. Additionally, the density $f$ of every law $F$ in $A_T^{\mathrm{e}}$ satisfies $f(t) \le 2/T$ for every $T/2 \le t < T$ by monotonicity, so that

$$F\big((T-\varepsilon,T]\big)\big) \le 2\varepsilon/T,\ 0<\varepsilon<T/2\,,$$

and the weak limit of a sequence in $A_T^{\mathrm{e}}$ has the same property, possibly apart from a countable set of $\varepsilon$. Letting $\varepsilon \to 0$, while keeping away from the exceptional set, shows that the weak limit does not put any mass at $T$ and, hence, is in $A_T^{\mathrm{e}}$.

It remains to describe the extreme points of $A_T^{\mathrm{e}}$ and, as in Theorem 4.2, the only non-trivial case is that of the extreme points of $A_T^{\mathrm{e}}$ that are concentrated on $(0,T)$ and are absolutely continuous there. The same argument as in the proof of Theorem 4.2 shows for any such extreme point the density can take at most one non-zero value, so it has to be of the form $f_{\mu_t}$, $0<t\le T$. For $t<T$ the latter laws are extreme points of $A_T$, hence of $A_T^{\mathrm{e}}$ as well. To see that the same is true for $f_{\mu_T}$ suppose, to the contrary, that there are two different laws in $A_T^{\mathrm{e}}$ that are concentrated on $(0,T)$, with the corresponding densities $g_1$ and $g_2$, such that

$$pg_1(s)+(1-p)g_2(s)=\frac{1}{T},\ 0<s<T\,, \tag{4.8}$$

for some $0<p<1$. Once again, there are points $0<s_i<T$ such that $g_i(s_i)>1/T$, $i=1,2$, and the monotonicity of $g_1$ and $g_2$ forces

$$g_i(0-) \ge g_i(s_i) > \frac{1}{T},\ i=1,2\,.$$

Therefore, (4.8) is violated in a neighbourhood of the left endpoint of the interval. □

Proposition 4.4 immediately implies the following counterpart of Corollary 4.3.

**Corollary 4.6.** *Let $g$ be a bounded, or nonnegative, measurable function on $[0,T]\cup\{\infty\}$. Then for any stationary process $\mathbf{X}$ and earliest occurrence intrinsic location functional $L$,*

$$\begin{aligned} & \min\Big\{g(0), g(\infty), \inf_{t\in(0,T)} \tfrac{1}{t}\int_0^t g(s)ds\Big\} \\ \leq\ & \mathbb{E}\big[g\big(L(\mathbf{X},[0,T])\big)\big] \\ \leq\ & \max\Big\{g(0), g(\infty), \sup_{t\in(0,T)} \tfrac{1}{t}\int_0^t g(s)ds\Big\}. \end{aligned}$$

**Remark 4.7.** The class $A_T$ is the smallest class containing all possible distributions of $L(\mathbf{X},T)$ for any intrinsic location functional $L$ and any stationary process $\mathbf{X}$, while the class $A_T^{\mathrm{e}}$ is the smallest class containing all possible distributions of $L(\mathbf{X},T)$ for any earliest occurrence intrinsic location functional $L$ and any stationary process $\mathbf{X}$, as easy examples show. In particular, the bounds obtained in Corollaries 4.3 and 4.6 are the tightest bounds possible.

The proposition below presents one application of the bounds given in corollaries 4.3 and 4.6.

**Proposition 4.8.** *For any stationary process $\mathbf{X}$, intrinsic location functional $L$, $T>0$ and $0<c<d<T$,*

$$P(L(\mathbf{X},T)\in[c,d]) \leq \frac{d-c}{\min(T-c,d)}. \tag{4.9}$$

*If the functional is an earliest occurrence intrinsic location functional, then*

$$P(L(\mathbf{X},T)\in[c,d]) \leq \frac{d-c}{d}. \tag{4.10}$$

*Proof.* One simply uses the upper bounds in corollaries 4.3 and 4.6 with the function $g=\mathbf{1}_{[c,d]}$. □

**Remark 4.9.** It is interesting that the upper bounds in the proposition are optimal even for very specific intrinsic location functionals. For example, it follows from the results in Samorodnitsky and Shen (2012a) that the upper bound in (4.9) is optimal for the leftmost location of the supremum $\tau_{f,[a,b]}$ of Example 2.1.

On the other hand, consider the first hitting time $T^l_{f,[a,b]}$ of Example 2.2. For the continuous stationary periodic process $\mathbf{X}(t) = \sin(t\pi/d + U) + l$, $t \in \mathbb{R}$, with $U$ uniformly distributed on $[0, 2\pi]$, the first hitting time is uniformly distributed between 0 and $d$ and, hence, achieves equality in (4.10).

For certain intrinsic location functionals $L$ and certain stationary processes $\mathbf{X}$ the law of $L(\mathbf{X}, T)$ is symmetric around the mid-point of the interval $[0, T]$, i.e.

$$P\big(L(\mathbf{X}, T) \in B\big) = P\big(L(\mathbf{X}, T) \in T - B\big) \tag{4.11}$$

for any Borel subset $B$ of $[0, T/2)$. This happens, for example, when the process $\mathbf{X}$ is time reversible, i.e. if $(X(-t),\, t \in \mathbb{R}) \stackrel{d}{=} (X(t),\, t \in \mathbb{R})$, while the functional $L$ has a certain uniqueness property associated with it. The quintessential example of such a situation is the leftmost location of the supremum of Example 2.1 evaluated at a continuous stationary Gaussian process. Such a process is always time reversible and, as long as $X(t) \neq X(0)$ a.s. for $0 < t \leq T$, the supremum is achieved, with probability 1, at a unique point of the interval $[0, T]$, so that (4.11) holds; see Lemma 2.6 in Kim and Pollard (1990).

We denote by $A_T^{\mathrm{S}}$ the set of all symmetric laws in $A_T$, i.e. the set of all laws in $A_T$ satisfying (4.11). The following theorem describes the structure of this set of probability laws. Its proof is similar to that of Theorem 4.2 and is omitted.

**Theorem 4.10.** *The set $A_T^{\mathrm{S}}$ is a weakly closed convex subset of $\mathcal{P}_T$. The extreme points of this set are:*

*(1) the measures $\rho_t$, $t \in (0, T/2)$ concentrated on $(0, T)$, absolutely continuous with respect to the Lebesgue measure on $(0, T)$, with density functions*

$$f_{\rho_t}(s) = \begin{cases} \frac{1}{2t} & 0 < s < t \\ \frac{1}{2t} & T - t \leq s < T \\ 0 & \textit{otherwise} \end{cases} \;;$$

*(2) the measures $\xi_t$, $t \in (0, T/2)$ concentrated on $(0, T)$, absolutely continuous with respect to the Lebesgue measure on $(0, T)$, with density functions*

$$f_{\xi_t}(s) = \begin{cases} \frac{1}{2(T-t)} & 0 < s < t \\ \frac{1}{T-t} & t \le s < T - t \\ \frac{1}{2(T-t)} & T - t \le s < T \end{cases} ;$$

*(3) the discrete measures $(\delta_0 + \delta_T)/2$ and $\delta_\infty$.*

**Remark 4.11.** Interestingly, the uniform distribution on $(0, T)$, $\rho_{T/2} = \xi_{T/2}$, is not an extreme point of $A_T^{\mathrm{S}}$, because for every $0 < t < T/2$, the mixture

$$\frac{t}{T}\rho_t + \frac{T-t}{T}\xi_t$$

coincides with the uniform distribution.

The following corollary is a counterpart of Corollary 4.3 to the symmetric case. We restrict ourselves (without loss of generality) to symmetric functions, i.e. functions satisfying $g(T/2 - t) = g(T/2 + t)$, $0 \le t \le T/2$.

**Corollary 4.12.** *Let $g$ be a bounded, or nonnegative, measurable symmetric function on $[0, T] \cup \{\infty\}$. Then for any stationary process $\mathbf{X}$ and intrinsic location functional $L$, satisfying the symmetry assumption* (4.11),

$$\begin{aligned} & \min\Big\{g(0), g(\infty), \inf_{t\in(0,T/2)} \tfrac{1}{t}\textstyle\int_0^t g(s)ds, \\ & \qquad \inf_{t\in(0,T/2)}\Big[\tfrac{1}{T-t}\textstyle\int_0^t g(s)ds + \tfrac{2}{T-t}\int_t^{T/2} g(s)ds\Big]\Big\} \\ \le\ & \mathbb{E}\big[g\big(L(\mathbf{X}, [0, T])\big)\big] \\ \le\ & \max\Big\{g(0), g(\infty), \sup_{t\in(0,T/2)} \tfrac{1}{t}\textstyle\int_0^t g(s)ds, \\ & \qquad \sup_{t\in(0,T/2)}\Big[\tfrac{1}{T-t}\textstyle\int_0^t g(s)ds + \tfrac{2}{T-t}\int_t^{T/2} g(s)ds\Big]\Big\}. \end{aligned}$$

Corollary 4.12 implies the following upper bounds on the mass the law of an intrinsic location functional assigns, in the symmetric case, to any subinterval of $(0, T)$.

**Proposition 4.13.** *For any stationary process* $\mathbf{X}$ *and intrinsic location functional* $L$*, satisfying the symmetry assumption* (4.11)*, and* $0 < c < d < T$*,*

$$P(L(\mathbf{X},T) \in [c,d]) \leq \begin{cases} \frac{d-c}{2d} & c+2d \leq T \\ \frac{d-c}{T-c} & c+2d > T,\ c+d \leq T,\ 2d-c \leq T \\ \frac{3d-c-T}{2d} & c+d \leq T,\ 2d-c > T \\ \frac{T+d-3c}{2(T-c)} & c+d > T,\ d > 2c \\ \frac{d-c}{d} & c+d > T,\ d \leq 2c,\ 2c+d \leq 2T \\ \frac{d-c}{2(T-c)} & 2c+d > 2T \end{cases}.$$

*Proof.* One uses the upper bounds in Corollary 4.12 with the symmetrized indicator function $g = \big(\mathbf{1}_{[c,d]} + \mathbf{1}_{[T-d,T-c]}\big)/2$, and straightforward optimization over $t \in (0, T/2)$. □

**Remark 4.14.** Once again, the upper bounds we have obtained are optimal even for the leftmost location of the supremum $\tau_{f,[a,b]}$ of Example 2.1, when the supremum is unique, and the stationary process is reversible. This follows from the results in Samorodnitsky and Shen (2012a).

## 5. Characterizing stationarity

Much of the previous discussion in this paper centered around the basic property of the intrinsic location functionals of stationary processes evaluated on some interval: the fact that their law must be absolutely continuous in the interior of the interval, and have a density satisfying the total variation constraints described in Theorem 3.1. The nature of this fact is itself interesting and, intuitively at least, intimately related to the stationarity of the underlying process: an intrinsic location functional "is shifted together with the process". Since the latter is stationary, one expects a shift to have only a limited effect on the law of the functional, hence its density does not change much from point to point. For certain intrinsic location functionals one can even be forgiven for believing that the density has to be constant; this is, for instance, the situation with the leftmost location of the supremum $\tau_{f,[a,b]}$ of Example 2.1, when the supremum is unique. We know that the density does not need to be constant, but the total variation constraints on the density may be viewed as restricting how different from a constant can the density be.

In this section we make this intuition precise. It turns out that existence of a density satisfying the total variation constraints for each appropriate intrinsic location functional (or even only those in a certain subclass of intrinsic location functionals) requires stationarity of the stochastic process. The theorem below is formulated for the processes with continuous sample paths and, correspondingly, to intrinsic location functionals on the space $H = C(\mathbb{R})$. Note, however, that the proof of the fact that (2) implies (1) in that theorem is valid for any space $H$ for which the functional defined in (5.1) is measurable. This is the case, for instance for the space $H = D(\mathbb{R})$, the space of all càdlàg functions. In Section 6 we also extend the fact that (3) implies (1) to other spaces, in particular to the space $H = D(\mathbb{R})$. The following theorem is the main result of this section.

**Theorem 5.1.** *Let* $\mathbf{X}$ *be a stochastic process with continuous sample paths. The following statements are equivalent.*

(1) *The process* $\mathbf{X}$ *is stationary.*
(2) *For some (equivalently, any)* $\Delta > 0$*, any intrinsic location functional* $L : C(\mathbb{R}) \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$*, the law of* $L(\mathbf{X}, I) - a$*,* $I = [a, a + \Delta] \in \mathcal{I}$*, does not depend on* $a$*.*
(3) *For any intrinsic location functional* $L : C(\mathbb{R}) \times \mathcal{I} \to \mathbb{R} \cup \{\infty\}$*, any interval* $I = [a, b] \in \mathcal{I}$*, the law of* $L(\mathbf{X}, I)$ *is absolutely continuous on* $(a, b)$ *and has a density satisfying the total variation constraints.*

*Proof.* The fact that (1) implies (2) is obvious, while the fact that (1) implies (3) follows from the discussion in Section 3.

To see that (2) implies (1), let $\Delta > 0$. Take any $n = 1, 2, ...$, time points $0 < t_1 < ... < t_n$ and closed intervals $I_1, \dots, I_n$. Then

$$L(f, I) := \inf\{t \in I : \ X(t + t_i) \in I_i, \ i = 1, ..., n\}, \ I \in \mathcal{I}, \tag{5.1}$$

is, clearly, an intrinsic location functional on $C(\mathbb{R})$. Furthermore, for any real $a \in \mathbb{R}$,

$$P\big(X(a + t_i) \in I_i, \ i = 1, ..., n\big) = P\big(L(\mathbf{X}, [a, a + \Delta]) = a\big)$$

$$= P\big(L(\mathbf{X}, [a, a + \Delta]) - a = 0\big),$$

which is independent of $a$ by the assumption. We conclude that

$$\big(X(a+t_i),\ i=1,...,n\big)\overset{d}{=}\big(X(t_i),\ i=1,...,n\big)$$

for all real $a$. Since this is true for all $n=1,2,\ldots$ and $0<t_1<...<t_n$, the process $\mathbf{X}$ is stationary.

In the remainder of the proof we show that (3) implies (1). Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. For $n=1,2,...$, $h\in\mathbb{R}$, $d\geq 0$, $\mathbf{t}=(t_1,...,t_n)$ such that $0<t_1<...<t_n$ and a collection of open intervals $\mathbf{I}=\big(I_1,...,I_n\big)$, define a set of points by

$$A^{h,d}_{\mathbf{t},\mathbf{I}}(f)=\{s\in\mathbb{R}: f(s)=h,\ \inf\{r>s: f(r)=h\}>s+d, \quad f(s+t_i)\in I_i,\ i=1,...,n\}. \tag{5.2}$$

We start with recording a simple fact.

**Lemma 5.1.** *Let $\mathbf{X}$ be a continuous stochastic process. If (3) is satisfied, then for any $h,\mathbf{t},\mathbf{I}$, and $d>0$,*

$$\mathbb{P}\big(a\in A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\big)=0$$

*for any $a\in\mathbb{R}$.*

*Proof.* The functional $L(f,[a,b])=\inf\big(A^{h,d}_{\mathbf{t},\mathbf{I}}(f)\cap[a,b]\big)$ is easily seen to be an intrinsic location functional on $C(\mathbb{R})$. Since by the definition, if $a\in A^{h,d}_{\mathbf{t},\mathbf{I}}(f)$, then

$$A^{h,d}_{\mathbf{t},\mathbf{I}}(f)\cap(a,a+d]=\emptyset\,,$$

we obtain

$$\mathbb{P}\big(a\in A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\big)\leq\mathbb{P}\big(L(\mathbf{X},[a-d,a+d]=a\big)=0$$

by the absolute continuity property in (3). □

The following lemma shows a feature of the random sets $A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})$ implied by the total variation property; note that *if we knew that the process* $\mathbf{X}$ *was stationary*, its statement would follow from the ergodic decomposition.

**Lemma 5.2.** *Let $\mathbf{X}$ be a continuous stochastic process. If (3) is satisfied, then for any $h,d,\mathbf{t}$ and $\mathbf{I}$, with probability 1, $A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})$ is either the empty set or both $\inf\big(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\big)=-\infty$ and $\sup\big(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\big)=\infty$.*

*Proof of Lemma 5.2.* It is easy to check that the collections of outcomes we are discussing are measurable. Suppose, for example, that, to the contrary, there exist $h, d, \mathbf{t}$ and $\mathbf{I}$ such that, on event of positive probability, $-\infty < \inf\bigl(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\bigr) < \infty$; the supremum can be dealt with similarly. The functional $L(f,[a,b]) = \inf\bigl(A^{h,d}_{\mathbf{t},\mathbf{I}}(f) \cap [a,b]\bigr)$ is, again, an intrinsic location functional on $C(\mathbb{R})$. Choose an interval $[a_1,b_1]$, such that $\mathbb{P}\bigl[\inf\bigl(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\bigr) \in [a_1,b_1]\bigr] =: c > 0$. For any $a < a_1$ and $b > b_1$, $\inf\bigl(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\bigr) \in [a_1,b_1]$ implies $L(\mathbf{X},[a,b]) = \inf\bigl(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})\bigr)$, so $\mathbb{P}(L(\mathbf{X},[a,b]) \in [a_1,b_1]) \geq c$. However, the upper bounds (3.1) on the density (following from the assumption (3)) require that probability to converge to zero as $a \to \infty$ and $b \to \infty$. The resulting contradiction proves the lemma. $\square$

For any $h \in \mathbb{R}$ the set

$$\mathcal{C}_h = \bigl\{f \in C(\mathbb{R}) : \inf\bigl(A^{h,0}(f)\bigr) = -\infty,\ \sup\bigl(A^{h,0}(f)\bigr) = \infty\bigr\}$$

(meaning that the vector $\mathbf{t}$ is empty) is a cylindrical set. Let $h_i, i = 1,2,\ldots$ be an enumeration of the rationals in $\mathbb{R}$, and construct inductively a subsequence $h_{i_j}, j = 1,2,\ldots$ according to the rule $i_1 = \inf\bigl\{i \geq 1 : \mathbb{P}\bigl(\mathbf{X} \in \mathcal{C}_{h_i}\bigr) > 0\bigr\}$, while for $j \geq 2$,

$$i_j = \inf\Bigl\{i > i_{j-1} : \mathbb{P}\Bigl(\mathbf{X} \in \mathcal{C}_{h_i} \setminus \bigl(\cup_{k=0}^{j-1}\mathcal{C}_{h_{i_k}}\bigr)\Bigr) > 0\Bigr\}.$$

Let $\mathcal{C}'_1 = \mathcal{C}_{h_{i_1}}$, $\mathcal{C}'_j = \mathcal{C}_{h_{i_j}} \setminus \bigl(\cup_{k=0}^{j-1}\mathcal{C}_{h_{i_k}}\bigr)$, $j \geq 2$. If the process $\mathbf{X}$ is, with positive probability, a constant process, we also define $\mathcal{C}'_0$ to be the collection of constant functions in $C(\mathbb{R})$. Then the sets $\bigl(\mathcal{C}'_j\bigr)$ are disjoint, $\mathbb{P}\bigl(\mathbf{X} \in \mathcal{C}'_j\bigr) > 0$ for each $j$, while by Lemma 5.2, $\mathbb{P}\bigl(\mathbf{X} \notin \cup_j \mathcal{C}'_j\bigr) = 0$. Let $\mathbf{X}_j$ be a continuous stochastic process whose law is the conditional law of $\mathbf{X}$ given $\mathbf{X} \in \mathcal{C}'_j$. Note that, if each $\mathbf{X}_j$ is a stationary process, then so is $\mathbf{X}$ itself, as a mixture of stationary processes (note that each set $\mathcal{C}'_j$ is shift invariant). Since a constant process is, obviously, stationary, we only need to establish stationarity of each process $\mathbf{X}_j$, $j \geq 1$. We also claim that the statement (3) of the theorem is satisfied for each one of these processes. To see that, let $L$ be any intrinsic location functional on $C(\mathbb{R})$. Define

$$L_j(f,[a,b]) = \begin{cases} L(f,[a,b]) & \text{if } f \in \mathcal{C}'_j \\ \infty & \text{if } f \notin \mathcal{C}'_j \end{cases}.$$

Then $L_j$ is also an intrinsic location functional on $C(\mathbb{R})$. Further, for any $a < c < d < b$ we have

$$\mathbb{P}\big(L(\mathbf{X}_j, [a,b]) \in [c,d]\big) = \frac{1}{\mathbb{P}\big(\mathbf{X} \in \mathcal{C}'_j\big)} \mathbb{P}\big(L_j(\mathbf{X}, [a,b]) \in [c,d]\big) .$$

Therefore, the restriction of the law of $L(\mathbf{X}_j, [a,b])$ to the interior of the interval differs only by a multiplicative constant from the restriction of the law of $L_j(\mathbf{X}, [a,b])$ to the interior of that interval. It follows that the statement (3) of the theorem is satisfied for the process $\mathbf{X}_j$.

In the remainder of the proof, therefore, we will establish stationarity of the process $\mathbf{X}_j,\ j \geq 1$. For notational convenience we will still call it $\mathbf{X}$, with the understanding that the process has its sample paths in $\mathcal{C}_h$ for some $h \in \mathbb{R}$. Fixing such $h$, we denote for $a \in \mathbb{R}$ and $\Delta > 0$,

$$p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X}) = \mathbb{P}\big(A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X}) \cap [a, a+\Delta] \neq \emptyset\big). \tag{5.3}$$

The proof of the theorem will be completed by the next two lemmas.

**Lemma 5.3.** *If for any $\Delta > 0$, $d \geq 2\Delta$, $\mathbf{t}$ and $\mathbf{I}$, the probability $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ is independent of $a$, then the process $\mathbf{X}$ is stationary.*

*Proof.* We note that, by Lemma 5.1, the probability $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ does not change if either or both of the endpoints of the interval $[a, a+\Delta]$ are removed.

Fix $\mathbf{t} = (t_1, ..., t_n)$ and intervals $\mathbf{I} = \big(I_1, ..., I_n\big)$ such that both

$$P\big(X(t_j) \in \partial(I_j)\big) = 0 \ \text{ and } h \notin \bar{I}_j,\ j = 1, \ldots, n\, . \tag{5.4}$$

For $m = 1, 2, ...$, let $a^m_j = t_1 - (j+1)2^{-m}$ be the left endpoint of the interval $T^m_j = [t_1 - (j+1)2^{-m}, t_1 - j2^{-m}),\ j = 0, 1, ...,$ of the length $\Delta_m = 2^{-m}$. Consider the sum

$$S_m(\mathbf{t}, \mathbf{I}) := \sum_{j=0}^{\infty} p^{h,(j+2)\Delta_m}_{\mathbf{t}-a^m_j,\mathbf{I},a^m_j,\Delta_m}(\mathbf{X}),$$

with $\mathbf{t} - a^m_j = (t_1 - a^m_j, ..., t_n - a^m_j)$. Notice that the values of $d = (j+2)\Delta_m$ are chosen in such a way that this sum is the sum of probabilities of disjoint events. Moreover,

$$\Big\{A^{h,(j+2)\Delta_m}_{\mathbf{t}-a^m_j,\mathbf{I}}(\mathbf{X}) \cap T^m_j \neq \emptyset \ \text{ for some } j\Big\}$$
$$\subseteq \Big\{X(t_i + \delta) \in I_i, i = 1, \ldots, n, \text{ for some } \delta \in [0, \Delta_m].\Big\}.$$

That is,

$$S_m(\mathbf{t},\mathbf{I}) \le \mathbb{P}\Big(X(t_i+\delta)\in I_i, i=1,\ldots,n, \text{ for some } \delta\in[0,\Delta_m].\Big).$$

Taking limit on both sides gives us

$$\begin{aligned} \limsup_{m\to\infty} S_m(\mathbf{t},\mathbf{I}) &\le \mathbb{P}\big(X(t_i)\in \bar{I}_i,\ i=1,2,\ldots,n\big) \\ &= \mathbb{P}\big(X(t_i)\in I_i,\ i=1,2,\ldots,n\big) \end{aligned} \tag{5.5}$$

by (5.4).

On the other hand, consider the event

$$A=\big\{X(t_i)\in I_i,\ i=1,2,\ldots,n\big\}\,.$$

On this event we can define three random variables as follows. First, let $l=l(\omega):=\sup\{s<t_1 : X(s)=h, \inf\{t>s : X(t)=h\}>s\}$ and $r=r(\omega):=\inf\{s>t_1 : X(s)=h, \inf\{t>s : X(t)=h\}>s\}$. Next, let

$$\epsilon_0=\epsilon_0(\omega):=\inf\big\{\theta>0:\ X(t_i+\theta)\in\partial(I_i),\ \text{some } i=1,\ldots,n.\big\}$$

For $\omega\in A$ take any $m$ satisfying $\Delta_m<\min\{\epsilon_0,\frac{r-t_1}{2}\}$, and let $j=j(m,\omega)$ be such that $l\in T_j^m$. It follows from (5.4) that $l$ and $r$ are consecutive points in the set $\{s : X(s)=h, \inf\{t>s : X(t)=h\}>s\}$. Therefore, for some $M(\omega)<\infty$, for all $m>M(\omega)$,

$$l(\omega)\in A^{h,(j(m,\omega)+2)\Delta_m}_{\mathbf{t}-a^m_{j(m,\omega)},\mathbf{I}}\cap T^m_{j(m,\omega)}.$$

We conclude that

$$A\subseteq \liminf_{m\to\infty}\bigcup_{j=0}^{\infty}\Big\{A^{h,(j+2)\Delta_m}_{\mathbf{t}-a^m_j,\mathbf{I}}(\mathbf{X})\cap T^m_j\neq\emptyset\Big\}$$

and, hence,

$$\begin{aligned} &\mathbb{P}\big(X(t_i)\in I_i,\ i=1,2,\ldots,n\big)=\mathbb{P}(A) \\ \le\ & \liminf_{m\to\infty}\mathbb{P}\left(\bigcup_{j=0}^{\infty}\Big\{A^{h,(j+2)\Delta_m}_{\mathbf{t}-a^m_j,\mathbf{I}}(\mathbf{X})\cap T^m_j\neq\emptyset\Big\}\right) \\ =\ & \liminf_{m\to\infty} S_m(\mathbf{t},\mathbf{I})\,. \end{aligned}$$

Together with (5.5) this proves that

$$\mathbb{P}\big(X(t_i)\in I_i,\ i=1,2,\ldots,n\big)=\lim_{m\to\infty}S_m(\mathbf{t},\mathbf{I})\,. \tag{5.6}$$

Let now $u \in \mathbb{R}$, and impose an extra assumption on the intervals:

$$P\big(X(t_j+u) \in \partial(I_j)\big) = 0,\ j = 1,\dots,n\,. \tag{5.7}$$

Then (5.6) implies that

$$\mathbb{P}\big(X(t_i+u) \in I_i,\ i = 1,2,\dots,n\big) = \lim_{m\to\infty} S_m(\mathbf{t}+u,\mathbf{I})\,.$$

However, by the assumptions of the lemma,

$$S_m(\mathbf{t}+u,\mathbf{I}) = \sum_{j=0}^{\infty} p^{h,(j+2)\Delta_m}_{\mathbf{t}-a_j^m,\mathbf{I},a_j^m+u,\Delta_m}(\mathbf{X}) = S_m(\mathbf{t},\mathbf{I})\,,$$

$m = 1,2,\dots$. Therefore

$$\mathbb{P}\big(X(t_i+u) \in I_i,\ i = 1,2,\dots,n\big) = \mathbb{P}\big(X(t_i) \in I_i,\ i = 1,2,\dots,n\big)$$

for all open intervals $I_1,\dots,I_n$ satisfying (5.4) and (5.7). This implies that

$$\big(X(t_1+u),\dots,X(t_n+u)\big) \overset{d}{=} \big(X(t_1),\dots,X(t_n)\big)\,.$$

Since this is true for all $n \geq 1$, $t_1 < \ldots < t_n$ and $u \in \mathbb{R}$, the process $\mathbf{X}$ is stationary. □

The following lemma shows that condition (3) of the theorem implies the key assumption of Lemma 5.3.

**Lemma 5.4.** *Suppose that (3) of the theorem is satisfied. Then $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ is independent of $a$ for any $\Delta, \mathbf{t}, \mathbf{I}$ and $d \geq 2\Delta$.*

*Proof.* We start by showing that $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ is a non-increasing function of $a$. If this is not the case, then there are $a_1 < a_2$ such that

$$p^{h,d}_{\mathbf{t},\mathbf{I},a_1,\Delta}(\mathbf{X}) < p^{h,d}_{\mathbf{t},\mathbf{I},a_2,\Delta}(\mathbf{X})\,. \tag{5.8}$$

By splitting the interval $[a_1, a_2]$ into two intervals of equal length, and repeating the procedure as many times as necessary, we can achieve the above inequality with $0 < a_2 - a_1 < \Delta$, so we simply assume that this constraint already holds. In this case, $[a_1, a_1+\Delta] \cap [a_2, a_2+\Delta] = [a_2, a_1+\Delta] \neq \emptyset$, $[a_1, a_1+\Delta] \cup [a_2, a_2+\Delta] = [a_1, a_2+\Delta]$, and the length of this union $a_2 + \Delta - a_1 < d$.

Recall the distance between any two points in set $A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})$ must be at least $d$. Therefore, any interval of the length smaller than $d$, contains at most one point of $A^{h,d}_{\mathbf{t},\mathbf{I}}(\mathbf{X})$. We call this "self-excluding" property. As a result of this property, inside any interval with length not exceeding $d$, the probability $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ is, actually, additive. Specifically, let $I = [a, a+\Delta]$, $\Delta \le d$, and let $I_1 = [a_1, a_1+\Delta_1]$, $I_2 = [a_2, a_2+\Delta_2]$, ... be disjoint subintervals of $I$. Then

$$p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X}) = \sum_{i=1}^{\infty} p^{h,d}_{\mathbf{t},\mathbf{I},a_i,\Delta_i}(\mathbf{X}).$$

Therefore, by (5.8),

$$\begin{aligned} 0 &< p^{h,d}_{\mathbf{t},\mathbf{I},a_2,\Delta}(\mathbf{X}) - p^{h,d}_{\mathbf{t},\mathbf{I},a_1,\Delta}(\mathbf{X}) \\ &= \Big(p^{h,d}_{\mathbf{t},\mathbf{I},a_2,a_1+\Delta-a_2}(\mathbf{X}) + p^{h,d}_{\mathbf{t},\mathbf{I},a_1+\Delta,a_2-a_1}(\mathbf{X})\Big) \\ &- \Big(p^{h,d}_{\mathbf{t},\mathbf{I},a_1,a_2-a_1}(\mathbf{X}) + p^{h,d}_{\mathbf{t},\mathbf{I},a_2,a_1+\Delta-a_2}(\mathbf{X})\Big) \\ &= p^{h,d}_{\mathbf{t},\mathbf{I},a_1+\Delta,a_2-a_1}(\mathbf{X}) - p^{h,d}_{\mathbf{t},\mathbf{I},a_1,a_2-a_1}(\mathbf{X}), \end{aligned}$$

so that

$$p^{h,d}_{\mathbf{t},\mathbf{I},a_1,a_2-a_1}(\mathbf{X}) < p^{h,d}_{\mathbf{t},\mathbf{I},a_1+\Delta,a_2-a_1}(\mathbf{X}).$$

Consider again the intrinsic location functional

$$L(f, I) := \inf\{t : t \in A^{h,d}_{\mathbf{t},\mathbf{I}}(f) \cap I\}.$$

Take $I = [a_1, a_1 + D]$ for $D > d$. By the self-excluding property,

$$\mathbb{P}(L(\mathbf{X}, I) \in [a, a+\delta]) = p^{h,d}_{\mathbf{t},\mathbf{I},a,\delta}(\mathbf{X})$$

for any $a$ and $\delta$ satisfying $a \ge a_1$ and $a + \delta \le a_1 + d$. In particular, the density of the law of $L(\mathbf{X}, I)$ in $(a, a+\delta)$, which exists by the condition (3), can be chosen independent of the length $D$ of the interval $I$. Since

$$\mathbb{P}(L(\mathbf{X}, I) \in [a_1, a_2]) = p^{h,d}_{\mathbf{t},\mathbf{I},a_1,a_2-a_1}(\mathbf{X})$$

$$< p^{h,d}_{\mathbf{t},\mathbf{I},a_1+\Delta,a_2-a_1}(\mathbf{X}) = \mathbb{P}(L(\mathbf{X}, I) \in [a_1+\Delta, a_2+\Delta]),$$

there are $s_1 \in [a_1, a_2]$ and $s_2 \in [a_1+\Delta, a_2+\Delta]$, independent of $D$, such that $c := f_{\mathbf{X},I}(s_2) - f_{\mathbf{X},I}(s_1) > 0$. By the total mass considerations, there is $t = t_D \in (a_1 + D/2, a_1 + D)$ such that $f_{\mathbf{X},I}(t) \le 2/D$, so that by the total variation constraint on the interval $[s_1, t]$ we have

$$\big(f_{\mathbf{X},I}(s_2) - f_{\mathbf{X},I}(t)\big) + \big(f_{\mathbf{X},I}(s_2) - f_{\mathbf{X},I}(s_1)\big) \le TV_{[s_1,t]}(f_{\mathbf{X},I}) \le f_{\mathbf{X},I}(s_1) + f_{\mathbf{X},I}(t).$$

Rearranging the terms gives us

$$\frac{2}{D} \ge f_{\mathbf{X},I}(t) \ge f_{\mathbf{X},I}(s_2) - f_{\mathbf{X},I}(s_1) = c > 0.$$

This relation, however, cannot hold for $D$ large enough. The resulting contradiction proves that $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}(\mathbf{X})$ is a non-increasing function of $a$.

We can repeat the above argument by considering instead the intrinsic location functional,

$$L_1(f,I) := \sup\{t : t \in A^{h,d}_{\mathbf{t},\mathbf{I}}(f) \cap I\}.$$

This will show that $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}$ is a non-decreasing function of $a$. It follows that $p^{h,d}_{\mathbf{t},\mathbf{I},a,\Delta}$ must be independent of $a$. □

The combination of the last two lemmas, obviously, completes the proof of Theorem 5.1. □

## 6. Intrinsic locations sets

The arguments in the proof of Theorem 5.1 establishing the stationarity of the process $\mathbf{X}$ used only intrinsic location functionals of a special kind. In this section we concentrate on these special functionals. This will allow us both to relax the assumptions of Theorem 5.1 and to extend its statement to stochastic processes with sample paths in certain spaces different from the space of continuous functions.

Let $\mathcal{V}$ denote the collection of all subsets of $\mathbb{R}$. We equip $\mathcal{V}$ with the $\sigma$-field $\mathcal{F}_{\mathcal{V}}$ generated by the sets

$$\Big\{A \subseteq \mathbb{R} : \ A \cap I = \emptyset\Big\}, \ I = [a,b], \ -\infty < a < b < \infty \, .$$

**Definition 2.** *Let $H$ be a set of functions on $\mathbb{R}$, invariant under shifts, equipped with its cylindrical $\sigma$-field. An **intrinsic location set** $A$ is a measurable mapping from $H$ to $\mathcal{V}$ that satisfies*

$$A(\theta_c f) = A(f) - c$$

*for every $c \in \mathbb{R}$.*

The following example shows that the set that played the crucial role in the proof of Theorem 5.1 is an intrinsic location set on $C(\mathbb{R})$.

**Example 6.1.** Let $H = C(\mathbb{R})$, and consider the set $A^{h,d}_{\mathbf{t},\mathbf{I}}$ defined in (5.2). We will check that it is an intrinsic location set. Clearly, only measurability needs to be checked. To this end, for $h, r \in \mathbb{R}$ define a map $\tau_{h,r}:\ C(\mathbb{R}) \to [-\infty, r]$ by

$$\tau_{h,r}(f) = \sup\bigl\{t \le r:\ f(t) = h\bigr\},\ f \in C(\mathbb{R}).$$

Since for each $t \in (-\infty, r]$,

$$\begin{aligned}\bigl\{f \in C(\mathbb{R}):\ \tau_{h,r}(f) \ge t\bigr\} &= \bigl\{f \in C(\mathbb{R}):\ f(s) = h \text{ for some } t \le s \le r\bigr\}\\ &= \bigcap_{k=1}^{\infty} \bigcup_{q \in [t,r]\cap\mathbb{Q}} \bigl\{f \in C(\mathbb{R}):\ |f(q) - h| \le 1/k\bigr\},\end{aligned}$$

where $\mathbb{Q}$ is the set of rational numbers, the map $\tau_{h,r}$ is measurable. Since for any $-\infty < a < b < \infty$

$$\Bigl\{f \in C(\mathbb{R}):\ A^{h,d}_{\mathbf{t},\mathbf{I}}(f) \cap [a,b] \neq \emptyset\Bigr\}$$

$$= \bigcup_{\substack{r \in [a,b]\cap\mathbb{Q}\\ \text{or } r = a \text{ or } r = b}} \Bigl\{f \in C(\mathbb{R}):\ \tau_{h,r}(f) \ge a,\ f(t) \neq h \text{ on } \bigl(\tau_{h,r}(f), \tau_{h,r}(f) + d + \varepsilon\bigr)$$

$$\text{for some } \varepsilon > 0,\ f\bigl(\tau_{h,r}(f) + t_i\bigr) \in I_i,\ i = 1, \ldots, n\Bigr\},$$

the measurability of $A^{h,d}_{\mathbf{t},\mathbf{I}}$ will follow once we check that every set (say, $B_r$) in the union is measurable. To see that this last statement is true, denote for an interval $I = (c_1, c_2)$ and $\delta > 0$, $I_\delta = (c_1 + \delta, c_2 - \delta)$ if $c_1 + \delta < c_2 - \delta$, and set $I_\delta = \emptyset$ otherwise. Then

$$B_r = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{m=M}^{\infty} \bigcup_{i=1}^{m}$$

$$\Bigl\{f \in C(\mathbb{R}):\ \tau_{h,r}(f) \in \bigl[a + (i-1)(r-a)/m, a + i(r-a)/m\bigr],$$
$$f(t) \neq h \text{ on } \bigl[a + i(r-a)/m, a + i(r-a)/m + d + 1/j\bigr],$$
$$f\bigl(a + i(r-a)/m + t_l\bigr) \in (I_l)_{1/k},\ l = 1, \ldots, n\Bigr\},$$

which makes it clear that $B_r$ is a cylindrical set.

Therefore, $A^{h,d}_{\mathbf{t},\mathbf{I}}$ is indeed an intrinsic location set.

Given an intrinsic location set $A$ on $H$, the functionals

$$L_1(f, I) := \inf\{t \in I \cap A(f)\},\ \ L_2(f, I) := \sup\{t \in I \cap A(f)\},\ f \in H, \tag{6.1}$$

turn out to be intrinsic location functionals. Indeed, only their measurability is not immediately clear. However, if $I = [a, b]$, then

$$\bigl\{f \in H:\ L_1(f, I) = \infty\bigr\} = \bigl\{f \in H:\ A(f) \cap [a,b] = \emptyset\bigr\},$$

while for $a < c < b$,

$$\{f \in H : L_1(f, I) \in (c, b] \cup \{\infty\}\} = \bigcup_{k=1}^{\infty} \{f \in H : A(f) \cap [a, c + 1/k] = \emptyset\}.$$

Since these subsets of $H$ are measurable, $L_1(\cdot, I)$ is measurable. Measurability of $L_2(\cdot, I)$ can be established in a similar way.

Notice that the functional $L_1(\cdot, I)$ is an earliest occurrence intrinsic location functional in the sense introduced in Section 4 . Similarly, the functional $L_2(\cdot, I)$ is *a latest occurrence intrinsic location functional*, i.e. a functional with the following property: for every $a < b < c$ and $f \in H$,

$$\text{if } L(f, [b, c]) \in [b, c] \text{ then } L(f, [a, c]) = L(f, [b, c]).$$

We already know that, if the process is stationary, then the distribution of $L(X, T)$ for any earliest occurrence intrinsic location functional $L$ does not put any mass at the right endpoint of the interval, and its density in $(0, T)$ is nonincreasing (Proposition 4.4). This applies, in particular, to the functionals $L_1$ in (6.1). In a similar way we can show that distribution of $L(X, T)$ for any latest occurrence intrinsic location functional L does not put any mass at the left endpoint of the interval, and its density in $(0, T)$ is nondecreasing. This applies, in particular, to the functionals $L_2$ in (6.1). Moreover, only functionals of the type (6.1) were used in the proof of Theorem 5.1. We obtain, therefore, the following corollary.

**Corollary 6.2.** *Suppose that for any intrinsic location functional $L$ on the space $C(\mathbb{R})$, of the type* (6.1)*, any interval $I = [a, b] \in \mathcal{I}$, the law of $L(\mathbf{X}, I)$ is absolutely continuous on $(a, b)$ and has a density satisfying the total variation constraints. Then the process $\mathbf{X}$ is stationary.*

We can use the idea of the intrinsic location set to extend the results of Theorem 5.1 and Corollary 6.2 to processes with sample paths in certain spaces other than the space of continuous functions. We will call a set $H$ of functions on $\mathbb{R}$ an *LI* set (from *locally integrable*) if it has following properties:

- $H$ is invariant under shifts;
- $H$ is equipped with its cylindrical $\sigma$-field $\mathcal{C}_H$;
- the map $H \times \mathbb{R} \to \mathbb{R}$ defined by $(f, t) \to f(t)$ is measurable;

- any $f \in H$ is locally integrable.

An example of an *LI* set is the space $D(\mathbb{R})$ of càdlàg functions on $\mathbb{R}$. Note that, by Fubini's theorem, for any $\delta > 0$ and an *LI* set $H$, the map $T_\delta : H \to C(\mathbb{R})$, defined by

$$T_\delta(f) = \int_t^{t+\delta} f(s)\, ds,\ t \in \mathbb{R} \tag{6.2}$$

is $\mathcal{C}_H/\mathcal{C}_{C(\mathbb{R})}$-measurable.

Let now $A : C(\mathbb{R}) \to \mathcal{V}$ be an intrinsic location set on $C(\mathbb{R})$. If $H$ is an *LI* set, then for $\delta > 0$ we can define a mapping $B_\delta : H \to \mathcal{V}$ by $B_\delta = A \circ T_\delta$. By definition, $B_\delta$ is measurable. Further, for any $c \in \mathbb{R}$, $T_\delta(\theta_c f) = \theta_c(T_\delta f)$ for any $f \in H$ (using the same notation for the shift operator on different spaces). Therefore, for $f \in H$,

$$B_\delta(\theta_c f) = A(T_\delta(\theta_c f)) = A(\theta_c(T_\delta f))$$

$$= A(T_\delta f) - c = B_\delta(f) - c\,,$$

so that $B_\delta$ is an intrinsic location set on $H$.

Let $\mathbf{X}$ be a stochastic process with sample paths in $H$. For every $\delta > 0$, we view $\mathbf{Y}_\delta = T_\delta \mathbf{X}$ as a stochastic process with sample paths in $C(\mathbb{R})$. For any intrinsic location set $A$ on $C(\mathbb{R})$ we have, in the above notation

$$\inf\{t \in I \cap A(\mathbf{Y}_\delta)\} = \inf\{t \in I \cap B_\delta(\mathbf{X})\}$$

for any interval $I$ and, similarly, with the functionals of the type $L_2$ in (6.1). Therefore, if we assume that for any intrinsic location functional $L$ on the space $H$, of the type (6.1), any interval $I = [a, b] \in \mathcal{I}$, the law of $L(\mathbf{X}, I)$ is absolutely continuous on $(a, b)$ and has a density satisfying the total variation constraints, then, for every $\delta > 0$, the continuous process $\mathbf{Y}_\delta$ satisfies the assumptions of Corollary 6.2 and, hence, is stationary.

By Fubini's theorem we know that there is a Borel subset $\mathbb{R}_0$ of $\mathbb{R}$ of Lebesgue measure zero, such that for any $t \notin \mathbb{R}_0$,

$$nY_{1/n}(t) \to X(t) \ \text{ a.s..}$$

Combining this with the stationarity of $\mathbf{Y}_\delta$ for each $\delta > 0$ tells us that

$$\big(X(t_1 + h), \ldots, X(t_k + h)\big) \overset{d}{=} \big(X(t_1), \ldots, X(t_k)\big) \tag{6.3}$$

for any $k = 1, 2 \ldots$ and $t_1, \ldots, t_k$ and $h$ such that none of the times in (6.3) is in the null set $\mathbb{R}_0$. For certain spaces $H$ this implies stationarity of the process $\mathbf{X}$; by the right continuity this is, certainly, true for the space $D(\mathbb{R})$. Hence, we obtain the following result.

**Proposition 6.3.** *Let $H$ be an LI set, and let $\mathbf{X}$ be a stochastic process with sample paths in $H$. Suppose that, for any intrinsic location functional $L$ on the space $H$, of the type* (6.1)*, any interval $I = [a, b] \in \mathcal{I}$, the law of $L(\mathbf{X}, I)$ is absolutely continuous on $(a, b)$ and has a density satisfying the total variation constraints. If for any process with sample paths in the set $H$,* (6.3) *implies stationarity, then $\mathbf{X}$ is stationary. This is the case, in particular, if $H = D(\mathbb{R})$.*

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