March 13, 2012

FRACTIONAL MOMENTS OF SOLUTIONS TO STOCHASTIC RECURRENCE EQUATIONS

THOMAS MIKOSCH, GENNADY SAMORODNITSKY, AND LALEH TAFAKORI

Abstract. In this paper we study the fractional moments of the stationary solution to the stochastic recurrence equation $X_t = A_t\,X_{t-1} + B_t$, $t \in \mathbb{Z}$, where $((A_t, B_t))_{t\in\mathbb{Z}}$ is an iid bivariate sequence. We derive recursive formulas for the fractional moments $E|X_0|^p$, $p \in \mathbb{R}$. Special attention is given to the case when B_t has an Erlang distribution. We provide various approximations to the moments $E|X_0|^p$ and show their performance in a small numerical study.

1. Introduction

We consider the stochastic recurrence equation

$$X_t = A_t\,X_{t-1} + B_t\,, \quad t \in \mathbb{Z}\,, \tag{1.1}$$

for an iid sequence $((A_t, B_t))_{t\in\mathbb{Z}}$ of pairs (A_t, B_t) with values in $[0,\infty) \times \mathbb{R}$. We will write $A, B, C, \ldots$ for a generic variable of the stationary sequences $(A_t), (B_t), (C_t), \ldots$, respectively. A unique causal stationary solution to (1.1) exists if $E \log A < 0$ and $E \log B^+ < \infty$ (see Kesten [28]), and the solution can be written in the form

$$X_t = B_t + \sum_{i=-\infty}^{t-1} A_{i+1} \cdots A_t\,B_i\,, \quad t \in \mathbb{Z}\,. \tag{1.2}$$

In what follows, we always assume that the stationary solution (1.2) exists.

The stochastic recurrence equation (1.1) and its solution (1.2) have attracted significant attention in the literature; see e.g. Kesten [28], Vervaat [36], Goldie [21], Diaconis and Freedman [13] and the references mentioned therein. This attention is due to the numerous applications of the model (1.1). Among the most popular ones are the ARCH(1) and GARCH(1, 1) processes in financial time series analysis introduced by Engle [19] and Bollerslev [7], respectively; see Example 3.8 below for further details. Another recent application is the modeling of the TCP in telecommunications networks; see e.g. Dumas et al. [18], Guillemin et al. [23] and Löpker and van Leeuwaarden [31]. Boxma et al. [8] consider (1.1) in the context of growth-collapse processes with renewal collapse epochs. The stochastic

1991 *Mathematics Subject Classification.* Primary 60G70; Secondary 60K99.

Key words and phrases. Moment, stochastic recurrence equation, GARCH, Erlang distribution, numerical approximation.

Thomas Mikosch's research is partly supported by the Danish Natural Science Research Council (FNU) Grants 09-072331 "Point process modelling and statistical inference" and 10-084172 "Heavy tail phenomena: Modeling and estimation". This paper was written when Laleh Tafakori visited the Department of Mathematics of the University of Copenhagen in 2011. She takes pleasure to thank the host institution for its hospitality. Gennady Samorodnitsky's research was partially supported by the ARO grant W911NF-07-1-0078, NSF grant DMS-1005903 and NSA grant H98230-11-1-0154 at Cornell University.

recurrence equation (1.1) has also been used in the context of insurance risk models; see Gjessing and Paulsen [20] and Collamore [12]. Moreover, this equation is closely related to exponential functionals of Lévy processes; see e.g. Carmona et al. [11], Behme et al. [4], Brockwell and Lindner [10], Maulik and Zwart [32], and Hirsch and Yor [25].

The distributional properties of the stationary solution to (1.1) are rather sophisticated. This fact is highlighted by a famous result of Kesten [28] concerning the tails of X; see also Goldie [21]. Write

$$f(\kappa) = EA^\kappa\,, \quad \kappa \in \mathbb{R}\,.$$

Under the assumptions that there exists a positive α such that $f(\alpha) = 1$, $EA^\alpha \log A$ and $E|B|^\alpha$ are both finite, the law of $\log A$ is non-arithmetic and for every x, $P(A_1 x + B_1 = x) < 1$, there exist constants $c_+, c_- \geq 0$ such that $c_+ + c_- > 0$ and

$$P(X > x) \sim c_+\, x^{-\alpha} \quad \text{and} \quad P(X \leq -x) \sim c_-\, x^{-\alpha}\,, \quad x \to \infty\,. \tag{1.3}$$

Because of convexity of f and since $f(0) = 1$ one necessarily has $f(\kappa) < 1$ for $\kappa \in (0, \alpha)$. Goldie [21] gave an alternative proof of (1.3) and determined the explicit form of the constants c_+ and c_-. In particular, for $A, B \geq 0$ he proved that

$$c_+ = \frac{E[(A_1 X_0 + B_1)^\alpha - (A_1 X_0)^\alpha]}{\alpha\, EA^\alpha \log A}\,. \tag{1.4}$$

Notice that, due to the tail behavior (1.3), $EX^\alpha = \infty$, hence both $E(A_1 X_0 + B_1)^\alpha$ and $E(A_1 X_0)^\alpha$ are infinite while the nominator in the previous formula is finite.

If $A \leq 1$ a.s., $P(0 < A < 1) > 0$ and $E\mathrm{e}^{\,r|B|} < \infty$ for some $r > 0$, Goldie and Grübel [22] showed that (1.3) does not remain true. In this case, the tails of X decay exponentially fast, thus ensuring the existence of all moments of X. If we formally set $\alpha = \infty$, we have $f(\kappa) = EA^\kappa < 1$ for $\kappa < \alpha$, just as in the Kesten-Goldie case. Under mild conditions on A, B, Alsmeyer et al. [1] showed that $A \leq 1$ a.s. and the existence of exponential moments of B are conditions which are necessary to ensure that exponential moments of X exist. In the same paper, the authors quote a technical report of Kellerer (1992) who proved for $A, B \geq 0$ a.s. that $E\mathrm{e}^{\,rX} < \infty$ for some $r > 0$ if and only if $A \leq 1$ a.s. and $E\mathrm{e}^{\,rB} < \infty$.

The results on the asymptotic tail behavior of X triggered research on the limit behavior of the partial maxima $M_n = \max(X_1, \ldots, X_n)$ of the process (X_t). In the Kesten-Goldie situation, i.e. with power law tails (1.3), de Haan et al. [24] proved that the scaled maxima M_n, $n \geq 1$, converge in distribution to a Fréchet distributed random variable. In the Goldie-Grübel situation, i.e. with exponential tails for X, it was recently proved by Hitczenko [26] that the normalized sequence (M_n) has a Gumbel distributional limit.

In some special cases the distribution of X can be calculated explicitly; see e.g. Alsmeyer et al. [1], Dufresne [14, 15, 16, 17], Goldie [21], Pitman and Yor [33], Boxma et al. [8], and the references therein. Then, in principle, one could also calculate all moments of X, both the integer and the fractional ones although the fractional moments are in general not easily derived. In this paper, we are concerned with the calculation of the fractional moments of the solution X_t, $t \in \mathbb{Z}$, to the stochastic recurrence equation (1.1). From the literature and the calculations in the present paper it is clear that only in a few cases one is able to derive explicit expressions for the fractional moments of X.

The paper is organized as follows. We start in Section 2 by considering some recursive formulas for the moments of X. In Section 3 we consider special cases. Our main focus is on cases which are related to exponential random variables B_t. We use the results and techniques proved in Guillemin et al. [23] and Carmona et al. [11]. We also consider the case when $0 \le A < 1$ and B is bounded and derive explicit formulas for EX^p.

2. Recursive formulas for the moments

The positive integer moments of X can be calculated by using the recursive argument given in Vervaat [36]. First, observe that from (1.1) for $n \ge 1$ integer,

$$X^n \stackrel{d}{=} (A_1X_0 + B_1)^n ,$$

where X_0 and (A_1, B_1) are independent. Then, assuming $E|X|^p$ finite and $f(p) < 1$ for some $p \ge 1$, an application of the binomial formula yields a recursive relation for the moments EX^l, $1 \le l \le n = [p]$, given by

$$EX^l = (\phi(l))^{-1} \sum_{k=0}^{l-1} \binom{l}{k} E(A_1^k B_1^{l-k}) EX^k . \tag{2.1}$$

Here and in what follows,

$$\phi(p) = 1 - f(p) , \quad p \in \mathbb{R} .$$

Notice that $f(l) < 1$, $l \le n$ if $p < \alpha$ for the value $\alpha \le \infty$ introduced in Section 1. It follows from Alsmeyer et al. [1], Theorem 1.4, that $E|X|^p < \infty$ if and only if $EA^p < 1$ and $E|B|^p < \infty$ for any $p > 0$, provided the mild conditions $A \ne 0$ a.s., $P(B = 0) < 1$ and $P(A_1x + B_1 = x) < 1$ hold.

For the fractional moments the following simple observation is useful. We mention that a similar formula was applied for calculating the moments of exponential functionals of Lévy processes in Carmona et al. [11], Guillemin et al. [23] and Maulik and Zwart [32]. In what follows, we write F_Z for the distribution function af any random variable Z and $\overline{F}_Z = 1 - F_Z$ for its right tail.

Lemma 2.1. *Let $p \ne 0$ be any real number. Assume that $A, B \ge 0$ a.s. are independent. Then*

$$E[(A_1X_0 + B_1)^p - (A_1X_0)^p] = p \int_0^\infty E(A_1X_0 + u)^{p-1} \overline{F}_B(u)\, du \quad , \tag{2.2}$$

where both sides are finite or infinite at the same time.

Remark 2.2. If $0 < EB < \infty$, (2.2) can be written in the form

$$E[(A_1X_0 + B_1)^p - (A_1X_0)^p] = p\, EB\, E(A_1X_0 + B^*)^{p-1} ,$$

where B^* is independent of A_1, X_0 and has the integrated tail distribution of B given by

$$F_B^*(b) = \frac{\int_0^b \overline{F}_B(u)\, du}{EB} , \quad b > 0 . \tag{2.3}$$

Proof. We observe that for any $p \in \mathbb{R}$,

$$(A_1X_0 + B_1)^p - (A_1X_0)^p = p \int_0^{B_1} (A_1X_0 + u)^{p-1}\, du . \tag{2.4}$$

Hence, by independence of A_1X_0 and B_1,

$$\begin{aligned} E[(A_1X_0+B_1)^p-(A_1X_0)^p] &= p\,E\Big[\int_0^{B_1}(A_1X_0+u)^{p-1}\,du\Big] \\ &= p\int_0^\infty\Big[\int_0^b E(A_1X_0+u)^{p-1}\,du\Big]\,F_B(db) \\ &= p\int_0^\infty E(A_1X_0+u)^{p-1}\overline{F}_B(u)\,du\,. \end{aligned}$$

Then the statement of the lemma follows. □

Remark 2.3. If $EX^p<\infty$, $EB<\infty$ and $f(p)\neq 1$ then the lemma yields the formula

$$EX^p=\frac{p\,EB}{\phi(p)}E(A_1X_0+B^*)^{p-1}\,. \tag{2.5}$$

Following the argument after (2.1), we conclude that $f(p)<1$ is a necessary condition for $EX^p<\infty$ provided $p>0$ and some mild conditions on A,B are satisfied. Since f is convex and $f(0)=1$, $f(p)>1$ for $p<0$. Hence $f(p)\neq 1$ is satisfied for all $p\neq 0$ such that $f(p)$ is finite.

The idea of the proof of the lemma can be applied iteratively. We explain the approach via an example. Assume the conditions of the lemma are satisfied. Write F_B^{n*} for the distribution function of a random variable B^{n*} which is obtained by n times applying the integrated tail operation (2.3) and assume that B^{n*} is independent of (A_1,X_0). Then, assuming that all moments involved are finite and $p\neq 0$,

$$\begin{aligned} EX^p &= \frac{p\,EB}{\phi(p)}[(E(A_1X_0+B^*)^{p-1}-E(A_1X_0)^{p-1}+E(A_1X_0)^{p-1}] \\ &= \frac{p\,EB}{\phi(p)}[(p-1)EB^*E(A_1X_0+B^{2*})^{p-2}+E(A_1X_0)^{p-1}] \\ &= \frac{p\,(p-1)\,EB}{\phi(p)}\Big[EB^*E(A_1X_0+B^{2*})^{p-2}+\frac{f(p-1)EB}{\phi(p-1)}E(A_1X_0+B^*)^{p-2}\Big]\,. \end{aligned}$$

Remark 2.4. In Section 1 we mentioned that, according to the Kesten-Goldie theory [28, 21], the stationary solution to (1.1) satisfies the relation $P(X>x)\sim c_+x^{-\alpha}$ as $x\to\infty$ provided $f(\kappa)=1$, $\kappa>0$, has a positive solution α (which is then unique) and some further conditions are satisfied. The constant c_+ is then given by the finite value (1.4), in particular,

$$\begin{aligned} c_+EA^\alpha\log A &= \alpha^{-1}E[(A_1X_0+B_1)^\alpha-(A_1X_0)^\alpha] \\ &= EB\,E(A_1X_0+B^*)^{\alpha-1}<\infty\,, \end{aligned} \tag{2.6}$$

and $EX^\alpha=\infty$. If B is standard exponential, $B^*\overset{d}{=}B$ and therefore $E(A_1X_0+B^*)^{\alpha-1}=EX^{\alpha-1}$; see also Example 3.1 below. Then, if c_+ and $EA^\alpha\log A$ are known one can calculate the moments $EX^{\alpha-k}$, $k=1,2,\ldots$, iteratively since, by (2.6), $EX^{\alpha-1}=c_+EA^\alpha\log A$ and

$$EX^{\alpha-k-1}=(\alpha-k)^{-1}\phi(\alpha-k)EX^{\alpha-k}\,,\quad k=1,2,\ldots\,,$$

provided the quantities on both sides are finite.

A similar remark applies in the case when B has the mixture distribution $P(B \le x) = (1-a) + a(1-\mathrm{e}^{-x})$, $x > 0$, for some $a \in (0,1)$. Then B^* is standard exponential and the moments $EX^{\alpha-k}$, $k = 1, 2, \ldots$, can be calculated by using the recursion (3.11).

3. Special cases

3.1. **The distribution of B is of Erlang-type.** It is in general difficult to evaluate EX^p by using the formulas above. However, if B has an Erlang distribution the calculations simplify. We start by considering the case of exponential B.

Example 3.1. Assume that B has a standard exponential distribution, i.e. $\overline{F}_B(x) = \mathrm{e}^{-x}$, $x > 0$. Then $B^* \stackrel{d}{=} B$ and $A_1 X_0 + B^* \stackrel{d}{=} X$. Multiple use of (2.5) yields

$$EX^p = \frac{p \cdots (p-n+1)}{\phi(p) \cdots \phi(p-n+1)} \, EX^{p-n}\,, \quad n \ge 1\,. \tag{3.1}$$

The case of exponential B has been studied in the literature for some time. As a matter of fact, the stationary solution to (1.1) has the following series representation in law

$$X \stackrel{d}{=} \sum_{i=1}^{\infty} B_i A_1 \cdots A_{i-1} = \sum_{i=1}^{\infty} B_i \mathrm{e}^{S_{i-1}}\,,$$

where $S_i = \sum_{k=1}^{i} \log A_i$, $i \ge 0$, with the convention that $S_0 = 0$. Since $E \log A < 0$ the random walk (S_i) has negative drift. Writing $N_t = \#\{i \ge 1 : B_1 + \cdots + B_i \le t\}$, $t \ge 0$, for the unit rate Poisson process generated by the iid exponential sequence (B_i), we have

$$X \stackrel{d}{=} \int_0^{\infty} \mathrm{e}^{\xi_t}\, dt\,, \tag{3.2}$$

where $(\xi_t) = (S_{N_t})$ is the compound Poisson process generated by the sequence $(\log A_i)$. Carmona et al. [11] studied exponential functionals of the type (3.2) for Lévy processes (ξ_t) more general than compound Poisson processes. They derived (3.1) for positive p. Related results were obtained by Behme et al. [4], Brockwell and Lindner [10], Maulik and Zwart [32]; see also the references therein. Hirsch and Yor [25] provide a survey of results on exponential functionals (3.2) for subordinators ξ.

Exponential functionals were also studied in Guillemin et al. [23]. They considered the stochastic recurrence equation (1.1) for independent $(A_t), (B_t)$ under the assumptions that $A_t = \beta^{Y_t}$, $t \in \mathbb{Z}$, $\beta \in (0,1)$, for an iid sequence (Y_t) such that $Y > 0$ a.s. and for exponential B. Under these conditions, using Mellin transforms, they derived (3.1) also for negative values p and proved that

$$EX^p = \Gamma(p+1) \prod_{k=1}^{\infty} \frac{\phi(p+k)}{\phi(k)}\,, \tag{3.3}$$

provided $p \in \mathbb{R}$, $-p \notin \mathbb{N}$, $p \neq 0$, $f(p+1) < \infty$ and $E[(1-A)^{-1}] < \infty$. An inspection of the proof shows that the latter conditions are not needed if $p > 0$.

If $Y \equiv 1$ a.s. (X_t) is an autoregressive process of order one and $X_t \stackrel{d}{=} \sum_{i=0}^{\infty} \beta^i B_i$. Then $f(p) = \beta^p < \infty$ for any real p, $E[(1-A)^{-1}] = (1-\beta)^{-1} < \infty$ and (3.3) turns into

$$EX^p = \Gamma(p+1) \prod_{k=1}^{\infty} \frac{1-\beta^{p+k}}{1-\beta^k}\,, \quad p \in \mathbb{R}\,, p \neq 0\,, -p \notin \mathbb{N}\,. \tag{3.4}$$

The same formula can be found in Bertoin et al. [5], Theorem 1, and the authors also derived the density and the Laplace transform of X in this case. A plot of the moments EX^p as a function of $\beta \in (0,1)$ and $p \in (0,3]$ is given in Figure 3.1. Due to the presence of the factor $\Gamma(p+1)$ the moments EX^p increase very fast when p increases and therefore we restricted p to small values in this graph.

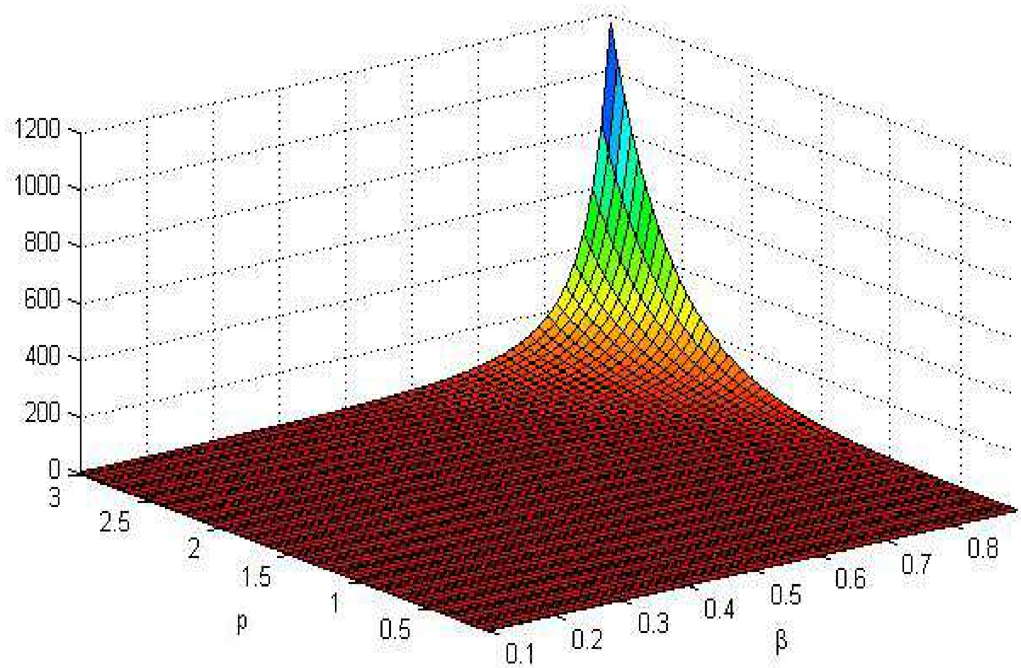


Figure 3.2. *The moments EX^p for $p \in [0,3]$, $A = \beta \in (0,1)$ and B standard exponential.*

Next consider the case of a geometric distribution for Y, i.e. $P(Y = k) = (1-q)q^{k-1}$, $k = 1, 2, \ldots$, for some $q \in (0,1)$. Then $f(p) = (1-q)\beta^p/(1-\beta^p q) < \infty$ for $p \neq 0$ and it is easy to see that $E[(1-A)^{-1}] < \infty$. Then (3.3) holds for $p \in \mathbb{R}\,, p \neq 0\,, -p \notin \mathbb{N}$.

If Y has an exponential distribution with mean λ^{-1} for some $\lambda > 0$ then $f(p) = \lambda(\lambda - p\log\beta)^{-1} < \infty$ for $p > \lambda/\log\beta$, but $E[(1-A)^{-1}] = \infty$. Then (3.3) holds for $p > 0$.

Next we consider the case of general Erlang distributed B.

Example 3.3. Assume that $B_1 = \Gamma_n$ where $\Gamma_n = E_1 + \cdots + E_n$ and (E_i) is an iid standard exponential sequence independent of (A_t). We claim that, for real p,

$$\begin{aligned} \phi(p) EX^p &= E(A_1X_0 + \Gamma_n)^p - E(A_1X_0)^p \\ &= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} EX^{p-k} p\,(p-1)\cdots(p-k+1)\,, \end{aligned} \tag{3.5}$$

where we assume that all moments in this formula are finite. For $n = 1$ this is just (3.1). We observe for general $l \geq 1$ (with the convention that $\Gamma_0 = 0$ a.s.)

$$\begin{aligned} (A_1X_0 + \Gamma_l)^p - (A_1X_0 + \Gamma_{l-1})^p &= p \int_{\Gamma_{l-1}}^{\Gamma_l} (A_1X_0 + u)^{p-1}\,du \\ &= p \int_0^{E_l} (A_1X_0 + \Gamma_{l-1} + u)^{p-1}\,du\,. \end{aligned}$$

Taking expectations on both sides and proceeding as in the proof of Lemma 2.1, we obtain

$$E(A_1X_0 + \Gamma_l)^p - E(A_1X_0 + \Gamma_{l-1})^p = p\,E(A_1X_0 + \Gamma_l)^{p-1}\,. \tag{3.6}$$

For $l = n$, $A_1X_0 + \Gamma_n = A_1X_0 + B_1 \stackrel{d}{=} X$. Therefore,

$$E(A_1X_0 + \Gamma_{n-1})^p = EX^p - pEX^{p-1}\,. \tag{3.7}$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned} &E(AX + \Gamma_{n-1})^p - E(AX + \Gamma_{n-2})^p \\ &= p\,E(AX + \Gamma_{n-1})^{p-1} = p\,EX^{p-1} - p\,(p-1)\,EX^{p-2}\,. \end{aligned}$$

Induction yields, for $l = 1, \ldots, n$,

$$E(A_1X_0 + \Gamma_l)^p - E(A_1X_0 + \Gamma_{l-1})^p = \sum_{j=0}^{n-l} (-1)^j \binom{n-l}{j} p(p-1)\cdots(p-j) EX^{p-j-1}\,.$$

Finally, the telescoping sum representation

$$E(A_1X_0 + \Gamma_n)^p - E(A_1X_0)^p = \sum_{l=1}^{n} \Big[E(A_1X_0 + \Gamma_l)^p - E(A_1X_0 + \Gamma_{l-1})^p \Big]$$

leads to (3.5).

Relation (3.5) yields a recursive relation for EX^p in terms of the lower moments EX^{p-1}, $\ldots, EX^{p-n}$. In contrast to the case $n = 1$ (see (3.3)), we could not find an explicit solution to equation (3.5). In the Kesten-Goldie setting, the right-hand side of (3.5) remains valid for the tail index $p = \alpha$ and if the left-hand side is replaced by $E[(A_1X_0 + \Gamma_n)^\alpha - (A_1X_0)^\alpha]$. The resulting formula yields an expression for Goldie's constant c_+ in terms of the moments $EX^{\alpha-1}, \ldots, EX^{\alpha-n}$.

We conducted a small numerical study based on the recursion (3.5) for $n = 2$ and A uniformly distributed in $(0, 0.5)$. Then (3.5) for $p \neq 0$ reads as

$$EX^p = (\phi(p))^{-1} \Big(2\,pEX^{p-1} - p(p-1)EX^{p-2} \Big)\,. \tag{3.8}$$

We calculate EX^i, $i=0,\dots,k$, for integer $k>2$ by using (2.1). Then we determine the fractional moments EX^p for non-integer $p\in[0,k]$ by Monte Carlo simulation (with 30 000 repetitions, solid line). This simulation has two purposes. First, we use the simulated moments EX^p, $p\in[0,2]$, as initial values for calculating the fractional moments for $p\in(2,k]$ based on the recursion (3.8) with grid size 0.5×10^{-3} (dotted line). Second, we use the simulated moments EX^p, $p\in(2,k]$, to compare with the approximative moments obtained by the recursive formula. The results of the approximation are visualized in Figure 3.4 for $k=4$ (top) and for $k=10$ on logarithmic scale (bottom). They are very satisfactory: The numerical approximations follow the moments EX^p closely; the two curves can hardly be distinguished.

Example 3.5. A particular case of (1.1) has attracted some attention:

$$X_t = A_t\,(X_{t-1}+C_t)\,,\quad t\in\mathbb{R}\,, \tag{3.9}$$

for independent sequences (A_t) and (C_t) of iid non-negative random variables. In this case, $B_t=A_tC_t$, and A_t and B_t are dependent for every t with the exception of constant A. The marginal distribution of the solution to (3.9) is known in some particular cases when A, C have gamma- or beta-like distributions; see Dufresne [14, 15, 16, 17], Carmona et al. [11] and Boxma et al. [8].

Assume that C is standard exponential and $EX^p<\infty$ for some $p>0$. Then, using the same arguments as above,

$$\begin{aligned} EX^p &= f(p)\,E(X_0+C_1)^p \\ &= f(p)\,[E(X_0+C_1)^p - EX^p + EX^p] \\ &= f(p)\,[p\,E(X_0+C_1)^{p-1} + EX^p]\,. \end{aligned}$$

Hence

$$E(X_0+C_1)^p = \frac{p}{\phi(p)}E(X_0+C_1)^{p-1}\,.$$

Then one is in the situation of Example 3.1. Replacing in (3.1) the moments EX^{p-k}, $k=0,1,\dots$, by the corresponding moments of X_0+C_1, one can use the idea of the proof of Proposition 7 in Guillemin et al. [23] to conclude that

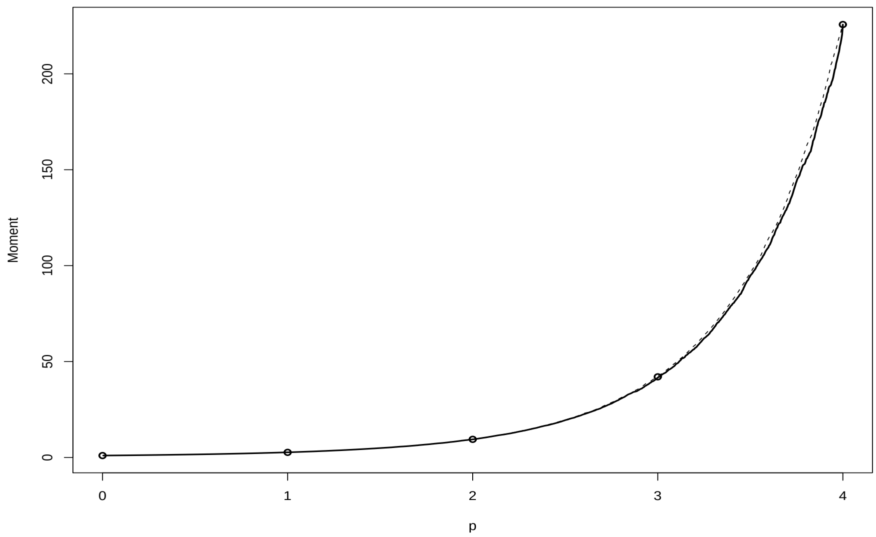
$$E(X_0+C_1)^p = \Gamma(p+1)\prod_{k=1}^{\infty}\frac{\phi(p+k)}{\phi(k)}\,,$$

if $(A_t)=(\beta^{Y_t})$ for an iid sequence (Y_t) such that $Y>0$ and $\beta\in(0,1)$, provided $p\in\mathbb{R}$, $-p\notin\mathbb{N}$, $p\neq 0$, $f(p+1)<\infty$ and $E(1-A)^{-1}<\infty$. The latter conditions are not needed for $p>0$. Finally,

$$EX^p = f(p)E(X_0+C_1)^p = f(p)\,\Gamma(p+1)\prod_{k=1}^{\infty}\frac{\phi(p+k)}{\phi(k)}\,.$$

Example 3.6. Here we consider the case when B has the distribution

$$P(B=0)=1-a \text{ for some } a\in(0,1) \text{ and } \quad P(B>x)=a\,\mathrm{e}^{-x}\,,\quad x>0\,.$$



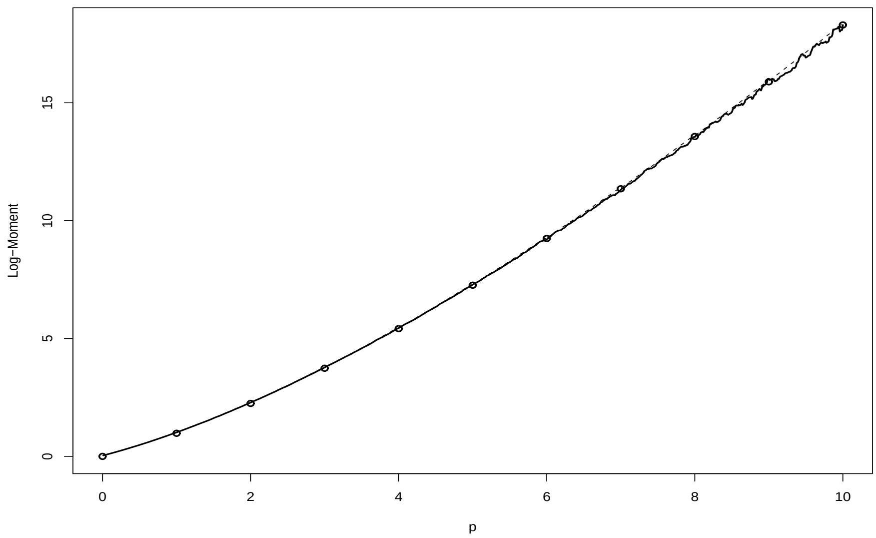


Figure 3.4. *The approximate moments EX^p for $p \in [2, k]$ (dotted line), A uniform on $(0, 0.5)$ and B with a $\Gamma(2, 1)$ distribution. Top: $k = 4$. Bottom: $k = 10$, the log-moments. The exact values EX^p, $p \in [0, k]$, are shown as solid line.*

Then B^* has a standard exponential distribution. We also assume that $A_t = \beta^{Y_t}$ for an iid sequence (Y_t) of positive random variables and some $\beta \in (0, 1)$ and that (A_t) and (B_t) are independent. An application of (2.5) yields

$$EX^p = \frac{p\,a}{\phi(p)} E(A_1X_0 + B^*)^{p-1}\,. \tag{3.10}$$

The moment on the right-hand side is finite for $p > 0$. For $p \geq 1$ this is elementary, and for $p \in (0,1)$ we have

$$E(A_1X_0 + B^*)^{p-1} \leq E(B^*)^{p-1} = \int_0^\infty x^{-(1-p)}\,\mathrm{e}^{-x}\,dx < \infty\,.$$

Under the assumption

$$(1-a)EA^{p-1} < 1\,, \quad p \in (0,1)$$

we also have $EX^{p-1} < \infty$ for $p \in (0,1)$ as we will show next. We have the representation $B_i = E_i r_i$, $i \in \mathbb{Z}$, where (E_i) is iid standard exponential independent of the iid sequence (r_i), where $P(r_i = 1) = 1 - P(r_i = 0) = a$. Now consider $T_0 = 0$ and $T_1 = \min\{i \geq 1 : r_i = 1\}$. This random variable is geometrically distributed with success probability a: $P(T_1 = k) = a(1-a)^{k-1}$, $k \geq 1$. By induction, define $T_{i+1} = \min\{k > T_i : r_k = 1\}$, $i \geq 1$. The sequence $(T_{i+1} - T_i)$ is iid. Using the representation $X \stackrel{d}{=} \sum_{i=0}^\infty B_{i+1}\Pi_i$, where $\Pi_i = \prod_{k=1}^i A_k$ with the convention that $\Pi_0 = 1$, we also have $X \stackrel{d}{=} \sum_{k=1}^\infty E_{T_k}\Pi_{T_k-1}$. Hence

$$\begin{aligned} EX^{p-1} &\leq E(E_{T_1}\Pi_{T_1})^{p-1} \\ &= \sum_{k=1}^\infty a(1-a)^{k-1}E(E_k\Pi_k)^{p-1} \\ &= \sum_{k=1}^\infty a(1-a)^{k-1}E(B^*)^{p-1}(EA^{p-1})^k \\ &= aEA^{p-1}E(B^*)^{p-1}(1-(1-a)EA^{p-1})^{-1}\,. \end{aligned}$$

Relation (3.10) can be written in the form

$$\begin{aligned} EX^p &= \frac{p}{\phi(p)}\big[a\,E(A_1X_0 + B^*)^{p-1} + (1-a)E(A_1X_0)^{p-1}\big] - (1-a)E(A_1X_0)^{p-1} \\ &= \frac{p}{\phi(p)}\big[1-(1-a)EA^{p-1}\big]EX^{p-1} = \frac{p}{g(p)}\,EX^{p-1}\,, \quad p > 0\,, \end{aligned} \tag{3.11}$$

We will verify that the following relation holds

$$\begin{aligned} EX^p &= \Gamma(p+1)\lim_{n\to\infty}\prod_{k=1}^n \frac{g(p+k)}{g(k)} \\ &= \Gamma(p+1)\prod_{k=1}^\infty \frac{\phi(p+k)}{\phi(k)}\,\frac{1-(1-a)EA^{k-1}}{1-(1-a)EA^{p+k-1}}\,, \quad p > 0\,, \end{aligned} \tag{3.12}$$

if $E[(1-A)^{-1}] < \infty$. For the convergence of $\prod_{k=1}^\infty g(p+k)$, $p \geq 0$, to a finite positive limit we verify that

$$\sum_{k=1}^\infty |g(p+k) - 1| < \infty. \tag{3.13}$$

However, for some constant $c > 0$

$$\begin{aligned}\sum_{k=1}^{\infty} |g(p+k)-1| &= \sum_{k=1}^{\infty} \frac{|-f(p+k)+(1-a)f(p+k-1)|}{1-(1-a)f(p+k-1)} \\ &\le c\sum_{k=0}^{\infty} f(p+k) \le c\,E[(1-A)^{-1}] < \infty\,.\end{aligned}$$

Therefore (3.13) is satisfied and the infinite products in (3.12) have finite positive limits. Now we proceed as in the proof of Proposition 7 in Guillemin et al. [23]. Consider the function

$$\psi(p) = E(X^{p-1}) \prod_{k=1}^{\infty} \frac{g(k)}{g(p+k-1)}\,.$$

In view of (3.11) it satisfies the relations $\psi(p+1) = p\psi(p)$ for $p > 0$ and also $\psi(1) = 1$. We will show that ψ is the gamma function. As in [23] we will use the Bohr-Mollerup theorem (see Andrews et al. [6]), according to which it remains to verify that $\log\psi$ is convex on $(0,\infty)$. This can be done by showing that the second derivative $(\log\psi(p))''$ is non-negative. We have

$$\log\psi(p) = \log\Big(\prod_{k=1}^{\infty} g(k)/g(p+k-1)\Big) + \log EX^{p-1}\,.$$

As in [23], one calculates the second derivative of $\log EX^{p-1}$ which is non-negative and direct calculation shows that $\sum_{k=1}^{\infty}\log(g(k)/g(p+k-1))$ has a non-negative second derivative.

Now, in order to indicate that (B_t) and (X_t) depend on a we write $(B_t^{(a)})$ and $(X_t^{(a)})$. Of course, $B^{(a)} \stackrel{d}{\to} B^{(1)}$ as $a \uparrow 1$, and the limiting random variable has a standard exponential distribution. We also have

$$X^{(a)} \stackrel{d}{\to} X^{(1)} \quad \text{and} \quad E|X^{(a)}|^p \to E|X^{(1)}|^p \tag{3.14}$$

for $p > 0$, as we show next. Recall the Mallows metric $d_p(R,S) = \inf(E|R-S|^p)^{\min(1,1/p)}$, $p > 0$, where the infimum is taken over all joint distributions of the bivariate vectors (R,S) with fixed marginals and pth finite moments. It is well known (e.g. Rachev [34]) that d_p metrizes convergence in distribution and L^p convergence. Construct the sequences $(X_t^{(a)})$, $a \in (0,1]$, on the same probability space in such a way that

$$X_t^{(a)} = B_t^{(a)} + \sum_{i=-\infty}^{t-1} A_{i+1}\cdots A_t\,B_i^{(a)}\,, \quad t \in \mathbb{Z}\,.$$

where $B_t^{(a)} = F_a^{\leftarrow}(U_t)$, F_a is the distribution function of $B^{(a)}$, $F_a^{\leftarrow}$ is the quantile function of F_a and (U_t) is an iid sequence with a uniform distribution on $(0,1)$. For $p \le 1$ we have

$$\begin{aligned} d_p(X_t^{(a)}, X_t^{(1)}) &\le E|X_t^{(a)} - X_t^{(1)}|^p \\ &= E\Big|\sum_{i=-\infty}^{t} A_{i+1}\cdots A_t\,[B_i^{(a)} - B_i^{(1)}]\Big|^p \\ &\le \sum_{i=-\infty}^{t} (EA^p)^{t-i} E|B^{(a)} - B^{(1)}|^p\,. \end{aligned} \tag{3.15}$$

But $\sum_{i=-\infty}^{t}(EA^p)^{t-i} < \infty$ since $EA^p < 1$, and $E|B^{(a)} - B^{(1)}|^p \to 0$ as $a \uparrow 1$. Then (3.14) follows. For $p > 1$ we can use the same idea of proof but in (3.15) one has to use the Minkowski inequality of order p. Of course, $E|X^{(a)}|^p \to E|X|^p$ as $a \uparrow 1$. This fact can also be checked directly by using (3.12).

Example 3.7. Assume that

$$B_n = E_n^{1/\alpha}\, C_n\,, \quad n \in \mathbb{Z}\,, \tag{3.16}$$

(E_n) is an iid standard exponential sequence, for some $0 < \alpha \le 2$, (C_n) is an iid sequence of strictly α-stable random variables (cf. Samorodnitsky and Taqqu [35]), $A > 0$ a.s. and (A_n), (E_n), (C_n) are independent. We mention that (3.16) defines a strictly geometric α-stable random variable. Indeed, a random variable B is strictly geometric α-stable, $\alpha \in (0,2]$, if it has representation in law $B \stackrel{d}{=} E_1^{1/\alpha}\, D$, where D is strictly α-stable independent of E_1; see Klebanov et al. [29] and Kozubowski [30]. Let Z be an α-stable Lévy motion on $[0,\infty)$ such that $Z_1 \stackrel{d}{=} C_1$ and $N_t = \#\{i \ge 1 : \Gamma_n \le t\}$, $t \ge 0$, $\Gamma_0 = 0$, $\Gamma_n = E_1 + \cdots + E_n$, $n \ge 1$, be the Poisson process generated by (E_n), $\xi_t = \sum_{i=1}^{N_t} \log A_i$, $t \ge 0$. Also assume that Z, (E_n) and (A_n) are independent. The stationary solution to $X_n = A_n X_{n-1} + B_n$, $n \in \mathbb{Z}$, has representation in law

$$X \stackrel{d}{=} \int_0^\infty \mathrm{e}^{\,\xi_t} dZ_t\,.$$

Indeed, recalling that $Z_{E_1} \stackrel{d}{=} E_1^{1/\alpha} C_1$, we have

$$\begin{aligned} \int_0^\infty \mathrm{e}^{\,\xi_t} dZ_t &= \sum_{k=1}^{\infty} \mathrm{e}^{\,\sum_{t=1}^{k} \log A_i}\,(Z_{\Gamma_k} - Z_{\Gamma_{k-1}}) \\ &\stackrel{d}{=} \sum_{k=1}^{\infty} A_1 \cdots A_k\, E_k^{1/\alpha} C_k\,. \end{aligned}$$

Write $Y_0 = \int_0^\infty \mathrm{e}^{\,\alpha\xi_t}\, dt$ and assume Y_0 and C_1 independent. By strict α-stability of Z,

$$X \stackrel{d}{=} C_1\, Y_0^{1/\alpha}\,,$$

and then for $p < \alpha$,

$$E|X|^p = E|C|^p\, EY_0^{p/\alpha}\,,$$

where $E|C|^p$ is a known constant (see Zolotarev [37], Chapter 2) while according to the discussion in Example 3.1,

$$Y_0 \stackrel{d}{=} A_1^\alpha Y_0 + E_1\,,$$

where A_1, E_1, Y_0 are independent. Now the moments of Y_0 can be determined by using (3.3). In a similar way, one can determine the moments of the positive and negative parts of X.

3.2. **The case when B is a constant.**

Example 3.8. Engle [19] and Bollerslev [7] introduced the ARCH and GARCH processes, respectively. These models have become major building blocks in financial time series analysis. Interestingly, these models have a close relationship with the stochastic recurrence equation (1.1) although one needs to consider multivariate versions of this equation in order to be able to handle the general GARCH case; see e.g. Basrak et al. [2, 3]. We focus on the GARCH$(1,1)$ model which fits into the 1-dimensional stochastic recurrence equation case. A GARCH$(1,1)$ process is a stationary process (Y_t) given by $Y_t = \sigma_t\, Z_t$, where the multiplicative noise (Z_t) is an iid sequence with $EZ = 0$, $\mathrm{var}(Z) = 1$ and the volatility sequence (σ_t) is the stationary non-negative causal solution of the stochastic recurrence equation $\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\,\sigma_{t-1}^2$, $t \in \mathbb{Z}$. Here $\alpha_0, \alpha_1, \beta_1 > 0$, and the latter coefficients have to satisfy some additional conditions in order to ensure the existence of a unique stationary solution (σ_t), in particular $E\log(\alpha_1 Z^2 + \beta_1) < 0$. For example, an application of Jensen's inequality shows that the latter condition holds if $\alpha_1 + \beta_1 < 1$. In most applications, Z is supposed to be standard normal or student distributed with $\beta > 2$ degrees of freedom and standardized to unit variance. Then one is in the Kesten-Goldie framework, i.e. there exist constants $\alpha, c_0 > 0$ such that $P(\sigma > x) \sim c_0\, x^{-2\alpha}$. Writing $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$ and $B_t = \alpha_0$, the constant α is given as the positive solution to the equation $f(\kappa) = 1$, $\kappa > 0$, and c_0 is given in (1.4) (with X_0 replaced by σ_0^2). Then, by a slight modification of a result of Breiman [9] which can be found e.g. in Jessen and Mikosch [27], it follows that

$$P(Y > x) \sim EZ_+^{2\alpha}\, P(\sigma > x) \quad \text{and} \quad P(Y \le -x) \sim EZ_-^{2\alpha}\, P(\sigma > x)\,,$$

provided $E|Z|^{2\alpha} < \infty$. The fractional moments of Y are given by $E|Y|^p = E\sigma^p\, E|Z|^p$, $p > 0$. Thus the moments of Y crucially depend on those of the noise and those of the solution to (1.1) with constant B.

Example 3.8 yields some motivation for studying the moments of (1.1) for a positive constant B. We assume without loss of generality that $B \equiv 1$ a.s. This case can be understood as a limiting one when B is Erlang distributed and the parameter of the Erlang distribution tends to infinity. Indeed, consider the sequence of stochastic recurrence equations $X_t^{(n)} = A_t X_{t-1}^{(n)} + B_t^{(n)}$, $t \in \mathbb{Z}$, where $(B_t^{(n)})$ is an iid sequence with $B^{(n)} \stackrel{d}{=} n^{-1}(E_1 + \cdots + E_n)$ for an iid sequence (E_t) with standard exponential marginal distribution. We also assume that (A_t) and $(B_t^{(n)})$ are independent. By the strong law of large numbers, $B_t^{(n)} \stackrel{\text{a.s.}}{\to} 1$ as $n \to \infty$ for every t. By (X_t) we denote the solution to the stochastic recurrence equation

$X_t = A_t X_{t-1} + 1$, $t \in \mathbb{Z}$. Again using the Mallows metric, we have for $p \le 1$,

$$\begin{aligned} d_p(X_t^{(n)}, X_t) &\le E|X_t^{(n)} - X_t|^p \\ &= E\Big|\sum_{i=-\infty}^{t} A_{i+1}\cdots A_t\,[B_i^{(n)} - 1]\Big|^p \\ &\le \sum_{i=-\infty}^{t} (EA^p)^{t-i} E|B^{(n)} - 1|^p\,. \end{aligned} \tag{3.17}$$

Recall that $EA^p < 1$ is necessary for $E|X|^p < \infty$. Since $E|B^{(n)} - 1|^p \to 0$ for every $p > 0$ we have $d_p(X_t^{(n)}, X_t) \to 0$, also ensuring $E|X^{(n)}|^p \to E|X|^p$. The latter limit remains valid for $p > 1$: then one can use Minkowski's inequality of order p in (3.17). Of course, the argument above remains valid for any choice of iid sequences $(B_t^{(n)})$, $n = 1, 2, \ldots$, for which one can ensure that $E|B^{(n)} - 1|^p \to 0$.

In some cases it is possible to express the fractional moments EX^p, $p > 0$, in terms of the integer moments EX^n, $n = 1, 2, \ldots$. In turn, these can be calculated by Vervaat's recursion (2.1). We assume that $0 \le A \le a$ a.s. for some constant $a < 0.5$ and $B = 1$. Write $\Pi_j = A_1 \cdots A_j$, $j \ge 1$. Then

$$X \stackrel{d}{=} 1 + A_1 X_0 \stackrel{d}{=} 1 + \sum_{j=1}^{\infty} \Pi_j \le 1 + a(1-a)^{-1} < 2\,.$$

A Taylor expansion at zero yields

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, x^n, \quad 0 \le x < 1\,.$$

Now, replacing x by $A_1 X_0$, taking expectations and using (2.1), we arrive at

$$\begin{aligned} EX^p &= \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, f(n) EX^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} \frac{f(n)}{\phi(n)} \sum_{k=0}^{n-1} \binom{n}{k} f(k)\, EX^k\,. \end{aligned}$$

To simplify notation, set $f(0)/\phi(0) = 1$ and interpret sums over void index sets as zero. Then iteration of (2.1) yields

$$\begin{aligned} EX^p &= \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} \times \\ &\quad \times \frac{f(n)}{\phi(n)} \sum_{k_1=0}^{n-1} \binom{n}{k_1} \frac{f(k_1)}{\phi(k_1)} \sum_{k_2=0}^{k_1-1} \binom{k_1}{k_2} \frac{f(k_2)}{\phi(k_2)} \cdots \sum_{k_n=0}^{k_{n-1}-1} \binom{k_{n-1}}{k_n} \frac{f(k_n)}{\phi(k_n)} \\ &= \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} \times \\ &\quad \times \frac{f(n)}{\phi(n)} \sum_{k_1=0}^{n-1} \frac{f(k_1)}{\phi(k_1)} \sum_{k_2=0}^{k_1-1} \frac{f(k_2)}{\phi(k_2)} \cdots \sum_{k_n=0}^{k_{n-1}} \frac{f(k_n)}{\phi(k_n)} \times \\ &\quad \times \binom{n}{k_1-k_2\ k_2-k_3\ \cdots k_n-k_{n-1}\ k_n}. \end{aligned}$$

A change of the indices yields an alternative expression:

$$\begin{aligned} EX^p &= \sum_{j=1}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_j=1}^{\infty} \frac{p(p-1)\cdots(p-(n_1+\cdots+n_j)+1)}{\prod_{d=1}^{j} n_d!} \\ &\quad \frac{f(n_1)}{\phi(n_1)} \frac{f(n_1+n_2)}{\phi(n_1+n_2)} \cdots \frac{f(n_1+\cdots+n_j)}{\phi(n_1+\cdots+n_j)}. \end{aligned} \tag{3.18}$$

Using the same idea as above, one can get similar formulas for the recursion (1.1) with bounded A and B.

Proposition 3.9. *Assume A, B independent and $0 \le A \le a < 1$ and $0 \le B \le b$ a.s. for some positive constants a, b. Write $f(p) = EA^p$ and $g(p) = EB^p$, $p > 0$. Then the following relation holds for $p > 0$:*

$$\begin{aligned} EX^p &= \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} \frac{f(n)g(p-n)}{\phi(n)} \times \\ &\quad \times \sum_{k_1=0}^{n-1} \frac{f(k_1)g(n-k_1)}{\phi(k_1)} \sum_{k_2=0}^{k_1-1} \frac{f(k_2)g(k_1-k_2)}{\phi(k_2)} \cdots \sum_{k_n=0}^{k_{n-1}-1} \frac{f(k_n)g(k_{n-1}-k_n)}{\phi(k_n)} \times \\ &\quad \times \binom{n}{k_1-k_2\ k_2-k_3\ \cdots k_{n-1}-k_n\ k_n}. \end{aligned} \tag{3.19}$$

Proof. We start by observing that $A_1 X_0 \le a\, b(1-a)^{-1}$. Thus, if $a < (b+1)^{-1}$ a Taylor expansion of $(B_1 + x)^p$ for $0 \le x \le ab(1-a)^{-1} < 1$ yields

$$E(B_1 + A_1 X_0)^p = \sum_{n=0}^{\infty} \frac{p\,(p-1)\cdots(p-n+1)}{n!} f(n) EB^{p-n} EX^n .$$

Then multiple use of (2.1) yields (3.19).

For any $c>0$, write $B_t(c)=B_t/c$, $X_t(c)=X_t/c$. The stochastic recurrence equation (1.1) is equivalent to

$$X_t(c)=A_t\,X_{t-1}(c)+B_t(c)\,,\quad t\in\mathbb{Z}\,, \tag{3.20}$$

and $EX^p=c^pE(X(c))^p$. Since $B(c)\le b/c$ a.s. and $X(c)\le a(b/c)(1-a)^{-1}$ the above calculations are valid for the new equation (3.20) under the condition $a<((b/c)+1)^{-1}$. Thus, if c is sufficiently large the value a may be arbitrarily close to one and all moment calculations can be extended for arbitrary $a<1$ and $b>0$. □

Remark 3.10. The condition $a<1$ is crucial for the above calculations. If $P(A=1)>0$ all moments EX^p, $p>0$, can still be finite. For example, assume that $q=P(A=0)=1-P(A=1)\in(0,1)$ and $B\equiv 1$ a.s. Then X has a geometric distribution with parameter q and the moments are given by $EX^p=(1-q)\sum_{n=1}^\infty q^{n-1}n^p$, $p>0$.

The latter example extends to the case when $q=P(A=0)=1-P(A=a)\in(0,1)$ for some $a>0$. Then $X=\sum_{n=1}^N a^n$ where N is geometric with success probability q and the moments are given by

$$EX^p=(1-q)\sum_{n=1}^\infty q^{n-1}\Big(\sum_{k=1}^n a^k\Big)^p\,.$$

For $a\ne 1$ this turns into

$$EX^p=(1-q)\frac{a^p}{|1-a|^p}\sum_{n=1}^\infty q^{n-1}|1-a^n|^p$$

which is finite for $qa^p<1$.

The formula (3.18) can easily be evaluated and converges very fast. We illustrate this aspect for A with a uniform distribution on $(0,a)$, $a<1$. We obtain the following expression:

$$\begin{aligned}&\sum_{j=1}^\infty\sum_{n_1=1}^\infty\cdots\sum_{n_j=1}^\infty \frac{p(p-1)\cdots(p-(n_1+\cdots+n_j)+1)}{\prod_{d=1}^j n_d!}\times\\ &\times\frac{a^{n_1}}{(n_1+1)-a^{n_1}}\frac{a^{n_1+n_2}}{(n_1+n_2+1)-a^{n_1+n_2}}\cdots\frac{a^{n_1+\cdots+n_j}}{(n_1+\cdots+n_j+1)-a^{n_1+\cdots+n_j}}\,.\end{aligned} \tag{3.21}$$

In Figure 3.2 we approximate the latter expression for $p=0.6$ and values $a\in(0,1)$ by replacing all infinite sums in (3.21) by the sums truncated at some integer $M\ge 1$. The value M is indicated on the x-axis. The corresponding absolute change of the approximated moment from M to $M+1$ is tabulated in Table 3.12.

a	Δ_{12}	Δ_{23}	Δ_{34}	Δ_{45}	Δ_{56}	Δ_{67}	Δ_{78}
.1	4×10^{-4}	1×10^{-5}	6×10^{-7}	4×10^{-8}	2×10^{-9}	1×10^{-10}	1×10^{-11}
.3	5×10^{-3}	3×10^{-4}	5×10^{-5}	8×10^{-6}	1×10^{-6}	3×10^{-7}	6×10^{-8}
.5	2×10^{-2}	2×10^{-3}	3×10^{-4}	7×10^{-5}	2×10^{-5}	6×10^{-6}	2×10^{-6}
.7	4×10^{-2}	4×10^{-3}	9×10^{-4}	3×10^{-4}	9×10^{-5}	4×10^{-5}	3×10^{-5}

Table 3.12. *Absolute values of the differences of the approximated moments for $p=0.6$. The symbol $\Delta_{M,M+1}$ describes the absolute value of the change of the approximation from M to $M+1$.*

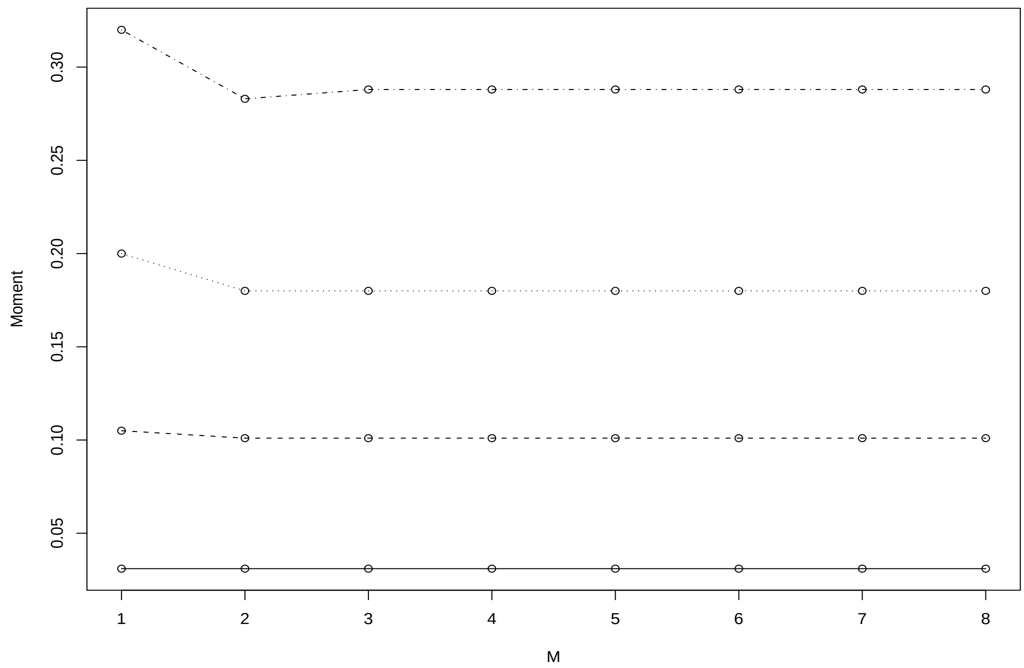


Figure 3.11. *Approximation of $EX^{0.6}$ for $B = 1$, and A uniform on $(0, a)$, taking into account only the first M (indicated on the x-axis) summands in each of the infinite series in* (3.21). *The curves from bottom to top correspond to $a = 0.1, 0.3, 0.5, 0.7$.*

References

[1] Alsmeyer, G., Iksanov, A. and Rösler, U. (2009) On distributional properties of perpetuities. *J. Theor. Probab.* **22**, 666–682.

[2] Basrak, B., Davis, R.A. and Mikosch, T. (2002) A characterization of multivariate regular variation. *Ann. Appl. Probab.* **12**, 908–920.

[3] Basrak, B., Davis, R.A. and Mikosch, T. (2002) Regular variation of GARCH processes. *Stoch. Proc. Appl.* **99**, 95–116.

[4] Behme, A., Lindner, A. and Maller, R.A. (2011) Stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. *Stoch. Proc. Appl.* **121**, 91–108.

[5] Bertoin, J., Biane, P. and Yor, M. (2004) Poissonian exponential functionals, q-series, q-integrals, and the moment problem for log-normal distributions. In: *Seminar on Stochastic Analysis, Random Fields and Applications IV. Progr. Probab.* **58**, pp. 45–56. Birkhäuser, Basel.

[6] Andrews, G.E., Askey, R. and Roy, R. (1999) *Special Functions.* Cambrigde University Press, Cambridge (UK).

[7] Bollerslev, T. (1986) Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31**, 307–327.

[8] Boxma, O., Kella, O. and Perry, D. (2011) On some tractable gowth-collapse processes with renewal collapse epochs. In: Glynn, P., Mikosch, T. and Rolski, T. (Eds.) *New Frontiers in Applied Probability. A Festschrift for Søren Asmussen.J. Appl. Probab.* Special volume **48A**, 217–234.
[9] Breiman, L. (1965) On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.* **10**, 323–331.
[10] Brockwell, P.J. and Lindner, A. (2009) Existence and uniqueness of stationary Lévy-driven CARMA processes. *Stoch. Proc. Appl.* **119** 2660–2681.
[11] Carmona, P., Petit, F. and Yor, M. (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. In: Yor, M. (Ed.) *Exponential Functionals and Principle Values Related to Brownian motion.* Bibl. Rev. Mat. Iberoamericana, Revista Matematica Iberoamericana, Madrid, pp. 73–121.
[12] Collamore, J.F. (2009) Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.* **19**, 1404–1458.
[13] Diaconis, P. and Freedman, D. (1999) Iterated random functions. *SIAM Review* **41**, 45–76.
[14] Dufresne, D. (1990) The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.* 1990, 39–79.
[15] Dufresne, D. (1996) On the stochastic equation $L(X) = L[B(X + C)]$ and a property of gamma distributions. *Bernoulli* **2**, 287-291.
[16] Dufresne, D. (1998) Algebraic properties of beta and gamm distributions, and applications. *Adv. Appl. Math.* **20**, 285–299.
[17] Dufresne, D. (2010) G-distributions and the beta-gamma algebra. *Electron. J. Probab.* **15**, 2163-2199.
[18] Dumas, V., Guillemin, F. and Robert, P. (2002) A Markovian analysis of additive-increase multiplicative-decrease algorithms. *Adv. Appl. Probab.* **34**, 85–111.
[19] Engle, R.F. (1982) Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50**, 987–1007.
[20] Gjessing, H.K. and Paulsen, J. (1997) Present value distributions with applications to ruin theory and stochastic equations. *Stoch. Proc. Appl.* **71**, 123-144.
[21] Goldie, C.M. (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.
[22] Goldie, C.M. and Grübel, R. (1996) Perpetuities with thin tails. *Adv. Appl. Probab.* **28** , 463–480.
[23] Guillemin, F., Robert, P. and Zwart, B. (2004) AIMD algorithms and exponential functionals. *Ann. Appl. Probab.* **14**, 90–117.
[24] Haan, L. de, Resnick, S.I., Rootzén, H. and Vries, C.G. de (1989) Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. *Stoch. Proc. Appl.* **32**, 213-224.
[25] Hirsch, F. and Yor, M. (2011) On the Mellin transforms of the perpetuity and the remainder variables associated to a subordinator. Technical report.
[26] Hitczenko, P. (2011) Convergence to type I distribution of the extremes of sequences defined by random difference equation. *Stoch. Proc. Appl.* **121**, 2231–2242.
[27] Jessen, A.H. and Mikosch, T. (2006) Regularly varying functions. *Publ. Inst. Math.* Nouvelle Série **80(94)**, 171–192.
[28] Kesten, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.
[29] Klebanov, L.B., Maniya, G.M. and Melamed, J.A. (1984) A problem of Zolotarev and analogs of infinitely divisible and stable distributions in a scheme for summing a random number of random variables. *Th. Probab. Appl.* **29**, 791–794.
[30] Kozubowski, T.J. (2000) Exponential mixture representation of geometric stable distributions. *Ann. Inst. Statist. Math.* **52**, 231–238.

[31] Löpker, A.H. and Leeuwaarden, van J.S.H. (2008) Transient moments of the TCP window size process. *J. Appl. Probab.* **45**, 163–175.

[32] Maulik, K. and Zwart, B. (2006) Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Proc. Appl.* **116**, 156–177.

[33] Pitman, J. and Yor, M. (2003) Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.* **55**, 292-330.

[34] Rachev, S.T. (1991) *Probability Metrics and the Stability of Stochastic Models.* Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Ltd., Chichester.

[35] Samorodnitsky, G. and Taqqu, M. (1994) *Stable Non-Gaussian Random Processes.* Chapman & Hall, New York.

[36] Vervaat, W. (1979) On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Adv. Appl. Probab.* **11**, 750–783.

[37] Zolotarev, V.M. (1886) *One-Dimensional Stable Distributions. AMS Translations of Mathematical Monographs* **65**, AMS, Providence, Rhode Island.

T. Mikosch, University of Copenhagen, Department of Mathematics, Universitetsparken 5, DK-2100 Copenhagen, Denmark
E-mail address: mikosch@math.ku.dk

G. Samorodnitsky, School of Operations Research and Industrial Engineering, Cornell University, 220 Rhodes Hall, Ithaca, NY 14853, U.S.A.
E-mail address: gennady@orie.cornell.edu

L. Tafakori, University of Shiraz, Department of Statistics, College of Sciences, Shiraz, 71454, Iran
E-mail address: tafakori@shirazu.ac.ir