FUNCTIONAL CENTRAL LIMIT THEOREM FOR HEAVY TAILED STATIONARY INFINITELY DIVISIBLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

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Abstract. We establish a new class of functional central limit theorems for partial sum of certain symmetric stationary infinitely divisible processes with regularly varying Lévy measures. The limit process is a new class of symmetric stable self-similar processes with stationary increments, that coincides on a part of its parameter space with a previously described process. The normalizing sequence and the limiting process are determined by the ergodic theoretical properties of the flow underlying the integral representation of the process. These properties can be interpreted as determining how long is the memory of the stationary infinitely divisible process. We also establish functional convergence, in a strong distributional sense, for conservative pointwise dual ergodic maps preserving an infinite measure.

1. Introduction

Let $\mathbf{X} = (X_1, X_2, \ldots)$ be a discrete time stationary stochastic process. A (functional) central limit theorem for such a process is a statement of the type

$$\left(\frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} X_k - h_n t,\, 0 \le t \le 1 \right) \Rightarrow \Big(Y(t),\, 0 \le t \le 1 \Big)\,. \tag{1.1}$$

Here (c_n) is a positive sequence growing to infinity, (h_n) a real sequence, and $\Big(Y(t),\, 0 \le t \le 1\Big)$ is a non-degenerate (i.e. non-deterministic) process. Convergence in (1.1) is at least in finite dimensional distributions, but preferably it is a weak convergence in the space $D[0,1]$ equipped with an appropriate topology. Not every stochastic process satisfies a central limit theorem, and for those that do, it is well known that both the rate of growth of the scaling constant c_n and the nature of the limiting process $\mathbf{Y} = \big(Y(t),\, 0 \le t \le 1\big)$ are determined both by the marginal tails of the stationary process $\mathbf{X}$ and its dependence structure. The limiting process (under very minor assumptions) is necessarily self-similar with stationary increments; this is known as the Lamperti theorem; see Lamperti (1962).

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If, say, X_1 has a finite second moment, and $\mathbf{X}$ is an i.i.d. sequence then, clearly, one can choose $c_n = n^{1/2}$, and then $\mathbf{Y}$ is a Brownian motion. With equally light marginal tails, if the memory is sufficiently short, then one expects the situation to remain, basically, the same, and this turns out to be the case. When the variance is finite, the basic tool to measure dependence is, obviously, the correlations, which have to decay fast enough. It is well known, however, that a fast decay of correlations is alone not sufficient for this purpose, and, in general, certain strong mixing conditions have to be assumed. See for example Rosenblatt (1956) and, more recently, Merlevéde et al. (2006). If the memory is not sufficiently short, then both the rate of growth of c_n can be different from $n^{1/2}$, and the limiting process can be different from the Brownian motion. In fact, the limiting process may fail to be Gaussian at all; see e.g. Dobrushin and Major (1979) and Taqqu (1979).

If the marginal tails of the process are heavy, which, in this case, means that X_1 is in the domain of attraction of an α-stable law, $0 < \alpha < 2$, and $\mathbf{X}$ is an i.i.d. sequence then, clearly, one can choose c_n to be the inverse of the marginal tail (this makes c_n vary regularly with exponent $1/\alpha$), and then $\mathbf{Y}$ is an α-stable Lévy motion. Again, one expects the situation to remain similar if the memory is sufficiently short. Since correlations do not exist under heavy tails, statements of this type have been established for special models, often for moving average models; see e.g. Davis and Resnick (1985), Avram and Taqqu (1992) and Paulauskas and Surgailis (2008). Once again, as the memory gets longer, then both the rate of growth of c_n can be different from that obtained by inverting the marginal tail, and the limiting process will no longer have independent increments (i.e. be an α-stable Lévy motion). It is here, however, that the picture gets more interesting than in the case of light tails. First of all, in absence of correlations there is no canonical way of measuring how much longer the memory gets. Even more importantly, certain types of memory turn out to result in the limiting process $\mathbf{Y}$ being a self-similar α-stable process with stationary increments of a canonical form, the so-called Linear Fractional Stable motion; see e.g. Maejima (1983) for an example of such a situation, and Samorodnitsky and Taqqu (1994) for information on self-similar processes. However, when the memory gets even longer, Linear Fractional Stable motions disappear as well, and even more "unusual" limiting processes $\mathbf{Y}$ may appear. This phenomenon may qualify as change from short to long memory; see Samorodnitsky (2006).

In this paper we consider a functional central limit theorem for a class of heavy tailed stationary processing exhibiting long memory in this sense. It is particularly interesting both because of the manner in which memory in the process is measured, and because the limiting process $\mathbf{Y}$ that happens to be an extension of a very recently discovered self-similar stable process with stationary increments. Specifically, we will assume that $\mathbf{X}$ is a stationary infinitely divisible process (satisfying certain assumptions, described in details in Section 2). That is, all finite dimensional distributions of $\mathbf{X}$ are infinitely divisible; we refer the reader to Rajput and Rosiński (1989) for more information on infinitely divisible processes and their integral representations we will work with in the sequel.

The class of central limit theorems we consider involves a significant interaction of probabilistic and ergodic theoretical ideas and tools. To make the discussion more transparent, we will only consider symmetric infinitely divisible processes without a Gaussian component (but there is no doubt that results of this type will hold in a greater generality as well). The law of such a process is determined by its (function level) Lévy measure. This is a (uniquely determined) symmetric measure κ on $\mathbb{R}^{\mathbb{N}}$ satisfying

$$\kappa\Big(\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \, x_j = 0 \ \text{ for all } j \in \mathbb{N}\Big) = 0$$

and

$$\int_{\mathbb{R}^{\mathbb{N}}} \min(1, x_j^2) \, \kappa(d\mathbf{x}) < \infty \ \text{ for each } j \in \mathbb{N},$$

such that for each finite subset $\{j_1, \ldots, j_k\}$ of $\mathbb{N}$, the k-dimensional Lévy measure of the infinitely divisible random vector $\big(X_{j_1}, \ldots, X_{j_k}\big)$ is given by the projection of κ on the appropriate coordinates of $\mathbf{x}$. See Maruyama (1970).

Because of the stationarity of the process $\mathbf{X}$, its Lévy measure μ is invariant under the left shift θ on $\mathbb{R}^{\mathbb{N}}$,

$$\theta(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \, .$$

It has been noticed in the last several years that the ergodic-theoretical properties of the shift operator with respect to the Lévy measure have a profound effect on the memory of the stationary process $\mathbf{X}$. The Lévy measure of the process is often described via an integral representation of the process, and in some cases the shift operator with respect to the Lévy measure can be related to an operator acting on the space on which the integrals are taken. Thus, Rosiński and Samorodnitsky (1996) and Samorodnitsky (2005) dealt with the ergodicity and mixing of stationary stable processes, while Roy (2008) dealt with general stationary infinitely divisible processes. The effect of the ergodic-theoretical properties of the shift operator with respect to the Lévy measure on the partial maxima of stationary stable processes was discussed in Samorodnitsky (2004).

In the present paper we consider stationary symmetric infinitely divisible processes without a Gaussian component given via an integral representation described in Section 2. This representation naturally includes a measure-preserving operator on a measurable space, and we related its ergodic-theoretical properties to the kind of central limit theorem the process satisfies. We consider the so-called conservative operators, that turn out to lead to non-standard limit theorems of the type that, to the best of our knowledge, have not been observed before.

We describe our setup in Section 2. In Section 3 we introduce the limiting symmetric α-stable (henceforth, SαS) process self-similar processes with stationary increments and discuss its properties. In Section 4 we present the ergodic-theoretical notions that we use in the paper. The exact assumptions in the central limit theorem are stated in Section 5. In this section we also present the statement of the theorem and several examples. The proof of the theorem uses several distributional ergodic-theoretical results we present and prove in Section 6. These results may be of independent interest in ergodic theory. Finally, the proof of the central limit theorem is completed in Section 7.

2. The setup

We consider infinitely divisible processes of the form

$$X_n = \int_E f_n(x)dM(x), \quad n = 1, 2, \dots , \tag{2.1}$$

where M is an infinitely divisible random measure on a measurable space $(E, \mathcal{E})$, and the functions f_n, $n = 1, 2, \dots$ are deterministic functions of the form

$$f_n(x) = f \circ T^n(x) = f\big(T^n x\big),\ x \in E,\ n = 1, 2, \dots , \tag{2.2}$$

where $f : E \to \mathbb{R}$ is a measurable function, and $T : E \to E$ a measurable map. The (independently scattered) infinitely divisible random measure M is assumed to be a homogeneous symmetric infinitely divisible random measure without a Gaussian component, with control measure μ and local Lévy measure ρ. That is, μ is σ-finite measure on E, which we will assume to be infinite. Further, ρ is a symmetric Lévy measure on $\mathbb{R}$, and for every $A \in \mathcal{E}$ with $\mu(A) < \infty$, $M(A)$ is a (symmetric) infinitely divisible random variable such that

$$Ee^{iuM(A)} = \exp\left\{ -\mu(A) \int_{\mathbb{R}} \big(1 - \cos(ux)\big)\, \rho(dx) \right\} \quad u \in \mathbb{R}. \tag{2.3}$$

It is clear that, in order for the process $\mathbf{X}$ to be well defined, the functions f_n, $n = 1, 2, \dots$ have to satisfy certain integrability assumptions; the assumptions we will impose below will be sufficient for that. Once the process $\mathbf{X}$ is well defined, it is, automatically, symmetric and infinitely divisible, without a Gaussian component, with the function level Lévy measure given by

$$\kappa = (\rho \times \mu) \circ K^{-1}\,, \tag{2.4}$$

with $K : \mathbb{R} \times E \to \mathbb{R}^N$ given by $K(x, s) = x\big(f_1(s), f_2(s), \dots\big)$, $s \in E$, $x \in \mathbb{R}$. For details see Rajput and Rosiński (1989).

We will assume that the measurable map T preserves the control measure μ. It follows immediately from (2.4) and the form of the functions (f_n) given in (2.2) that the Lévy measure κ is invariant under the left shift θ and, hence, the process $\mathbf{X}$ is stationary. We intend to relate the ergodic-theoretical properties of the map T to the dependence properties of the process $\mathbf{X}$ and, subsequently, to the kind of central limit theorem the process satisfies. We refer the reader to Aaronson (1997) for more details on the ergodic-theoretical notions used in the sequel. A short review of what we need will be given in Section 4 below.

Our basic assumption is that the map T is conservative. This property has already been observed to be related to long memory in the process $\mathbf{X}$; see e.g. Samorodnitsky (2004) and Roy (2008). We will quantify the resulting length of memory by assuming, further, that the map T is ergodic and pointwise dual ergodic, with a regularly varying normalizing sequence. We will see that the exponent of regular variation plays a major role in the central limit theorem.

The second major "player" in the central limit theorem is the heaviness of the marginal tail of the process $\mathbf{X}$. We will assume that the local Lévy measure ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$. That is,

$$\rho(\cdot, \infty) \in RV_{-\alpha} \ \text{ at infinity.} \tag{2.5}$$

With a proper integrability assumption on the function f in (2.2), the process $\mathbf{X}$ has regularly varying marginal (and even finite-dimensional) distributions, with the same tail exponent $-\alpha$; see Rosiński and Samorodnitsky (1993). That is, all the finite-dimensional distributions of the process are in the domain of attraction of a SαS law.

This leads to a rather satisfying picture, in which the kind of the central limit theorem that holds for the process $\mathbf{X}$ depends both on the marginal tails of the process and on the length of memory in it, and both are clearly parametrized.

In fact, in order to obtain the central limit theorem for the process $\mathbf{X}$, we will need to impose more specific assumptions on the map T. We will also, clearly, need specific integrability assumptions on the kernel in the integral representation of the process. These assumptions are presented in Section 5.

We proceed, first, with a description of the limiting process we will eventually obtain.

3. The limiting process

In this section, we will introduce a class of self-similar SαS processes with stationary increments. These processes will later appear as weak limits in the central limit theorem. We will see this process is an extension (to a wider range of parameters) of a class recently introduced by Dombry and Guillotin-Plantard (2009). Before introducing this process we need to some preliminary work.

For $0 < \beta < 1$, let $\big(S_\beta(t), t \geq 0\big)$ be a β-stable subordinator, i.e. a Lévy process with increasing sample paths, satisfying $Ee^{-\theta S_\beta(t)} = \exp\{-t\theta^\beta\}$ for $\theta \geq 0$ and $t \geq 0$; see e.g. Chapter III of Bertoin

(1996). Define its inverse process by

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\{u \geq 0 : \; S_\beta(u) \geq t\}, \; t \geq 0 . \tag{3.1}$$

Recall that the marginal distributions of the process $\big(M_\beta(t), \, t \geq 0\big)$ are the Mittag-Leffler distributions, with the Laplace transform

$$E \exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \quad \theta \in \mathbb{R}; \tag{3.2}$$

see Proposition 1(a) in Bingham (1971). We will call this process *the Mittag-Leffler process.* This process has a continuous and non-decreasing version; we will always assume that we are working with such a version. It follows from (3.2) (or simply from the definition) that the Mittag-Leffler process is self-similar with exponent β. Further, all of its moments are finite. Recall, however, that this process has neither stationary nor independent increments; see e.g. Meerschaert and Scheffler (2004).

We are now ready to introduce the new class of self-similar SαS processes with stationary increments announced at the beginning of this section. Let $0 < \alpha < 2$ and $0 < \beta < 1$, and let $(\Omega', \mathcal{F}', P')$ be a probability space. We define

$$Y_{\alpha,\beta}(t) = \int_{\Omega' \times [0,\infty)} M_\beta\big((t-x)_+, \omega'\big) dZ_{\alpha,\beta}(\omega', x), \quad t \geq 0, \tag{3.3}$$

where $Z_{\alpha,\beta}$ is a SαS random measure on $\Omega' \times [0,\infty)$ with control measure $P' \times \nu$, with ν a measure on $[0,\infty)$ given by $\nu(dx) = (1-\beta)x^{-\beta}\, dx$, $x > 0$. Here M_β is a Mittag-Leffler process defined on $(\Omega', \mathcal{F}', P')$. The random measure $Z_{\alpha,\beta}$ itself and, hence, also the process $Y_{\alpha,\beta}$, are defined on some generic probability space $(\Omega, \mathcal{F}, P)$. We refer the reader to Samorodnitsky and Taqqu (1994) for more information on integrals with respect to stable random measures.

In Theorem 3.1 below we prove that the process $\big(Y_{\alpha,\beta}(t), \, t \geq 0\big)$ is a well defined self-similar SαS processes with stationary increments. We call it *the β-Mittag-Leffler* (or β-ML) *fractional SαS motion.*

Theorem 3.1. *The β-ML fractional SαS motion is a well defined self-similar SαS processes with stationary increments. It is also self-similar with exponent of self-similarity $H = \beta + (1-\beta)/\alpha$.*

Proof. By the monotonicity of the process M_β we have, for any $t \geq 0$,

$$\int_{[0,\infty)} \int_{\Omega'} M_\beta((t-x)_+, \omega')^\alpha P'(d\omega)\nu(dx) \leq t^\beta E' M_\beta(t)^\alpha < \infty \, ,$$

which proves that the process $\big(Y_{\alpha,\beta}(t),\, t \ge 0\big)$ is well defined. Further, by the β-self-similarity of the process M_β, we have for any $k \ge 1$, $t_1 \dots t_k \ge 0$, and $c > 0$, for all real $\theta_1, \dots, \theta_k$,

$$E \exp\left\{ i \sum_{j=1}^k \theta_j Y_{\alpha,\beta}(ct_j) \right\} = \exp\left\{ -\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j M_\beta((ct_j - x)_+)\Big|^\alpha (1-\beta) x^{-\beta} dx \right\}$$

$$= \exp\left\{ -\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j c^H M_\beta((t_j - y)_+)\Big|^\alpha (1-\beta) y^{-\beta} dy \right\} = E \exp\left\{ i \sum_{j=1}^k \theta_j c^H Y_{\alpha,\beta}(t_j) \right\},$$

which shows the H-self-similarity of the β-ML fractional SαS motion.

For the proof of stationary increment property, it suffices to check that

$$E \exp\left\{ i \sum_{j=1}^k \theta_j \big(Y_{\alpha,\beta}(t_j + s) - Y_{\alpha,\beta}(s)\big) \right\} = E \exp\left\{ i \sum_{j=1}^k \theta_j Y_{\alpha,\beta}(t_j) \right\}$$

for all $k \ge 1$, $t_1 \dots t_k \ge 0$, $s \ge 0$, and $\theta_1 \dots \theta_k \in \mathbb{R}$. This is equivalent to verifying the equality in

$$\int_0^\infty E'\Big|\sum_{j=1}^k \theta_j \{M_\beta((t_j + s - x)_+) - M_\beta((s-x)_+)\}\Big|^\alpha x^{-\beta} dx$$

$$= \int_0^\infty E'\Big|\sum_{j=1}^k \theta_j M_\beta((t_j - x)_+)\Big|^\alpha x^{-\beta} dx\,.$$

Changing variable by $r = s - x$ in the left hand side and rearranging the terms shows that we need to check the equality in

$$\int_0^s E'\Big|\sum_{j=1}^k \theta_j (M_\beta(t_j + r) - M_\beta(r))\Big|^\alpha (s-r)^{-\beta} dr \tag{3.4}$$

$$= \int_0^\infty E'\Big|\sum_{j=1}^k \theta_j M_\beta((t_j - x)_+)\Big|^\alpha (x^{-\beta} - (s+x)^{-\beta}) dx.$$

Let $\delta_r = S_\beta\big(M_\beta(r)\big) - r$ be the overshoot of the level $r > 0$ by the β-stable subordinator $\big(S_\beta(t), t \ge 0\big)$ related to $\big(M_\beta(t), t \ge 0\big)$ by (3.1). The law of δ_r is known to be given by

$$P(\delta_r \in dx) = \frac{\sin \beta\pi}{\pi} r^\beta (r+x)^{-1} x^{-\beta}\, dx,\ x > 0\,; \tag{3.5}$$

see e.g. Exercise 5.6 in Kyprianou (2006). Further, by the strong Markov property of the stable subordinator we have

$$\big(M_\beta(t+r) - M_\beta(r),\, t \ge 0\big) \overset{d}{=} \big(M_\beta((t - \delta_r)_+),\, t \ge 0\big)\,,$$

with the understanding that M_β and δ_r in the right hand side are independent. We conclude that

$$\int_0^s E'| \sum_{j=1}^k \theta_j (M_\beta(t_j + r) - M_\beta(r))|^\alpha (s-r)^{-\beta} dr \tag{3.6}$$

$$= \frac{\sin \beta\pi}{\pi} \int_0^\infty \int_0^s E' \Big| \sum_{j=1}^k \theta_j M_\beta((t_j - x)_+) \Big|^\alpha r^\beta (r+x)^{-1} x^{-\beta} (s-r)^{-\beta} dr dx.$$

Using the integration formula

$$\int_0^1 \left(\frac{t}{1-t}\right)^\beta \frac{1}{t+y}\, dt = \frac{\pi}{\sin \beta\pi} \left[1 - \left(\frac{y}{1+y}\right)^\beta\right], \; y > 0\,,$$

given on p. 338 of Gradshteyn and Ryzhik (1994), shows that (3.6) is equivalent to (3.4). This completes the proof. □

Recall that, when $0 < \beta \le 1/2$, the Mittag-Leffler process of (3.1) is distributionally equivalent to the local time at zero of a symmetric stable Lévy process with index of stability $\hat\beta = (1-\beta)^{-1}$. Specifically, let $\big(W_{\hat\beta}(t),\, t \ge 0\big)$ be a symmetric $\hat\beta$-stable Lévy process, such that $Ee^{irW_{\hat\beta}(t)} = \exp\{-t|r|^{\hat\beta}\}$ for $r \in \mathbb{R}$ and $t \ge 0$. This process has a jointly continuous local time process, $L_t(x)$, $t \ge 0$, $x \in \mathbb{R}$; see e.g. Getoor and Kesten (1972). Then

$$\big(M_\beta(t),\, t \ge 0\big) \overset{d}{=} \big(c_\beta L_t(0),\, t \ge 0\big) \tag{3.7}$$

for some $c_\beta > 0$; see Section 11.1.1 in Marcus and Rosen (2006). Therefore, in the range $0 < \beta \le 1/2$, the β-ML fractional SαS motion (3.3) can be represented in law as

$$Y_{\alpha,\beta}(t) = c_\beta \int_{\Omega' \times [0,\infty)} L_{(t-x)_+}\big(0, \omega'\big) dZ_{\alpha,\beta}(\omega', x), \quad t \ge 0, \tag{3.8}$$

where $\big(L_t(x)\big)$ is the local time of a symmetric $\hat\beta$-stable Lévy process defined on $(\Omega', \mathcal{F}', P')$. Recall also the $\hat\beta$-stable local time fractional SαS motion introduced in Dombry and Guillotin-Plantard (2009) (see also Cohen and Samorodnitsky (2006)). That process can be defined by

$$\hat Y_{\alpha,\beta}(t) = \int_{\Omega' \times \mathbb{R}} L_t\big(x, \omega'\big) d\hat Z_\alpha(\omega', x), \quad t \ge 0, \tag{3.9}$$

where $\hat Z_\alpha$ is a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $P' \times$ Leb. We claim that, in fact, if $0 < \beta \le 1/2$,

$$\big(Y_{\alpha,\beta}(t)\;\; t \ge 0\big) \overset{d}{=} c_\beta^{(1)} \big(\hat Y_{\alpha,\beta}(t)\;\; t \ge 0\big)\,, \tag{3.10}$$

for some multiplicative constant $c_\beta^{(1)}$. Therefore, one can view the ML fractional SαS motion as an extension of the $\hat\beta$-stable local time fractional SαS motion from the range $1 < \hat\beta \le 2$ to the range $1 < \hat\beta < \infty$. It is interesting to note that the central limit theorem in Section 5 is of a very different type from the random walk in random scenery situation of Cohen and Samorodnitsky (2006) and Dombry and Guillotin-Plantard (2009).

To check (3.10), let

$$H_x = \inf\big\{t \ge 0 : \; W_{\hat\beta}(t) = x\big\}, \; x \in \mathbb{R}\,.$$

Since $1 < \hat\beta \le 2$, H_x is a.s. finite for any $x \in \mathbb{R}$; see e.g. Remark 43.12 in Sato (1999). Further, by the strong Markov property, for every $x \in \mathbb{R}$, the conditional law of $\bigl(L_{H_x+t}(x),\, t \ge 0\bigr)$ given $\mathcal{F}'_{H_x}$, coincides a.s. with the law of $\bigl(L_t(0),\, t \ge 0\bigr)$. We conclude that for any $k \ge 1$, $t_1 \dots t_k \ge 0$, and real $\theta_1, \dots, \theta_k$,

$$- \log E \exp\Bigl\{\sum_{j=1}^k \theta_j \hat Y_{\alpha,\beta}(t_j)\Bigr\} = \int_{\mathbb{R}} E' \Bigl|\sum_{j=1}^k \theta_j L_{t_j}(x)\Bigr|^\alpha dx$$

$$= \int_{\mathbb{R}} \int_0^\infty E' \Bigl|\sum_{j=1}^k \theta_j L_{(t_j-y)_+}(0)\Bigr|^\alpha F_x(dy)\, dx\,,$$

where F_x is the law of H_x. Using the obvious fact that $H_x \stackrel{d}{=} |x|^{\hat\beta} H_1$, an easy calculation shows that the mixture $\int_{\mathbb{R}} F_x\, dx$ is, up to a multiplicative constant, equal to the measure ν in (3.3). Therefore, for some constant $c^{(1)}_\beta$,

$$- \log E \exp\Bigl\{\sum_{j=1}^k \theta_j c^{(1)}_\beta \hat Y_{\alpha,\beta}(t_j)\Bigr\} = - \log E \exp\Bigl\{\sum_{j=1}^k \theta_j Y_{\alpha,\beta}(t_j)\Bigr\}\,,$$

and (3.10) follows.

Remark 3.2. It is interesting to observe that, for a fixed $0 < \alpha < 2$, the range of the exponent of self-similarity $H = \beta + (1-\beta)/\alpha$ of the β-ML fractional SαS motion, as β varies between 0 and 1, is a proper subset of the feasible range of the exponent of self-similarity of stationary increment self-similar SαS processes, which is $0 < H \le \max(1, 1/\alpha)$; see Samorodnitsky and Taqqu (1994).

It was shown in Dombry and Guillotin-Plantard (2009) that the stable local time fractional SαS motion is Hölder continuous. We extend this statement to the ML fractional SαS motion.

Theorem 3.3. *The β-ML fractional SαS motion satisfies, with probability 1,*

$$\sup_{0 \le s < t \le 1/2} \frac{\bigl|Y_{\alpha,\beta}(t) - Y_{\alpha,\beta}(s)\bigr|}{(t-s)^\beta \bigl|\log(t-s)\bigr|^{1-\beta}} < \infty$$

if $0 < \alpha < 1$, and

$$\sup_{0 \le s < t \le 1/2} \frac{\bigl|Y_{\alpha,\beta}(t) - Y_{\alpha,\beta}(s)\bigr|}{(t-s)^\beta \bigl|\log(t-s)\bigr|^{3/2-\beta}} < \infty$$

if $1 \le \alpha < 2$.

Proof. The statement of the theorem follows from Lemma 3.4 and the argument in Theorem 5.1 in Cohen and Samorodnitsky (2006); see also Theorem 1.5 in Dombry and Guillotin-Plantard (2009). $\square$

The next lemma establishes Hölder continuity of the Mittag-Leffler process (3.1). The statement might be known, but we could not find a reference, so we present a simple argument. In the case $0 < \beta \le 1/2$ (most of) the statement is in Theorem 2.1 in Ehm (1981), through the relation with the local time (3.7).

Lemma 3.4. *For $B > 0$ let*

$$K = \sup_{0 \le s < t < s+1/2 \le B} \frac{\bigl|M_\beta(t) - M_\beta(s)\bigr|}{(t-s)^\beta \bigl|\log(t-s)\bigr|^{1-\beta}} .$$

Then K is an a.s. finite random variable with all finite moments.

Proof. Because of the self-similarity of the Mittag-Leffler process it is enough to consider $B = 1/2$. In the course of the proof we will use the notation $c(\beta)$ for a finite positive constant that may depend on β, and that may change from one appearance to another. Recall the lower tail estimate of a positive β-stable random variable:

$$P\bigl(S_\beta(1) \le \theta\bigr) \le \exp\bigl\{-c(\beta)\theta^{-\beta/(1-\beta)}\bigr\}, \ 0 < \theta \le 1 ; \tag{3.11}$$

see Zolotarev (1986). Let $\lambda \ge 1$. We have

$$P(K > \lambda) \le \sum_{n=1}^\infty P\Bigl(\sup_{\substack{0 \le s < t \le 1/2 \\ 2^{-(n+1)} \le t-s \le 2^{-n}}} M_\beta(t) - M_\beta(s) > c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Bigr) := \sum_{n=1}^\infty q_n(\lambda) .$$

For $n = 1, 2, \ldots$ we use the following decomposition:

$$q_n(\lambda) \le P\bigl(S_\beta(\lambda \log n) \le 1/2\bigr)$$
$$+P\Bigl[\text{for some } 0 < t \le \lambda \log n, \ S_\beta\Bigl(t + c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Bigr) - S_\beta(t) \le 2^{-n}\Bigr] := q_n^{(1)}(\lambda) + q_n^{(2)}(\lambda) .$$

Using (3.11) and self-similarity of the stable subordinator, we obtain

$$\sum_{n=1}^\infty q_n^{(1)}(\lambda) \le c(\beta)^{-1} \exp\bigl\{-c(\beta)\lambda^{1/(1-\beta)}\bigr\} .$$

On the other hand,

$$q_n^{(2)}(\lambda) \le P\Bigl(S_\beta\Bigl(2^{-1}(i+1)c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Bigr) - S_\beta\Bigl(2^{-1} i c(\beta)\lambda n^{1-\beta} 2^{-n\beta}\Bigr) \le 2^{-n}, \text{ some } i = 0, \ldots, K_n\Bigr) ,$$

with $K_n \le 2c(\beta)^{-1} n^{\beta-1} 2^{n\beta} \log n$. Switching to the complements, and using once again (3.11) together with the independence of the increments and self-similarity of the stable subordinator, we conclude, after some straightforward calculus, that for all $\lambda \ge \lambda(\beta) \in (0, \infty)$,

$$\sum_{n=1}^\infty q_n^{(2)}(\lambda) \le c(\beta)^{-1} \exp\bigl\{-c(\beta)\lambda^{1/(1-\beta)}\bigr\} .$$

The resulting bound on the tail probability $P(K > \lambda)$ is sufficient for the statement of the lemma. □

Recall that the only self-similar Gaussian process with stationary increments is the Fractional Brownian motion (FBM), whose law is, apart from the scale, uniquely determined by the self-similarity parameter $H \in (0,1)$; see Samorodnitsky and Taqqu (1994). This parameter of self-similarity also determines the dependence properties of the increment process of the FBM, the so-called Fractional Gaussian noise, with the case $H > 1/2$ regarded as the long memory case. In contrast, the self-similarity parameter almost never determines the dependence properties of the increment processes of stable self-similar processes with stationary increments; see Samorodnitsky (2006). Therefore, it is interesting and important to discuss the memory properties of the increment process

$$V_n^{(\alpha,\beta)} = Y_{\alpha,\beta}(n+1) - Y_{\alpha,\beta}(n), \quad n = 0,1,2,\ldots,. \tag{3.12}$$

We refer the reader to Rosiński (1995) and Samorodnitsky (2005) for some of the notions used in the statement of the following theorem.

Theorem 3.5. *The stationary process* $\big(V_n^{(\alpha,\beta)}\big)$ *is generated by a conservative null flow and is mixing.*

Proof. Note that the increment process has the integral representation

$$V_n^{(\alpha,\beta)} = \int_{\Omega'\times[0,\infty)} \big(M_\beta\big((n+1-x)_+,\omega'\big) - M_\beta\big((n-x)_+,\omega'\big)\big)\, dZ_{\alpha,\beta}(\omega',x),\ n = 0,1,2,\ldots .$$

Since for every $x > 0$, on a set of P' probability 1, by the strong Markov property of the stable subordinator we have

$$\limsup_{n\to\infty} M_\beta\big((n+1-x)_+\big) - M_\beta\big((n-x)_+\big) > 0\,,$$

we see that

$$\sum_{n=1}^{\infty}\big(M_\beta\big((n+1-x)_+,\omega'\big) - M_\beta\big((n-x)_+,\omega'\big)\big)^\alpha = \infty \quad P'\times\nu \text{ a.e..}$$

By Corollary 4.2 in Rosiński (1995) we conclude that the increment process is generated by a conservative flow.

It remains to prove that the increment process is mixing, since mixing implies ergodicity which, in turns, implies that the increment process is generated by a null flow; see Samorodnitsky (2005). By Theorem 5 of Rosiński and Żak (1996), it is enough to show that for every $\epsilon > 0$,

$$\begin{aligned}&(P'\times\nu)\{(\omega',x) : M_\beta((1-x)_+,\omega') > \epsilon,\, M_\beta((n+1-x)_+,\omega') - M_\beta((n-x)_+,\omega') > \epsilon\}\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

However, an obvious upper bound on the expression in the left hand side is

$$\begin{aligned}
&\int_0^1 P'\big(M_\beta(n+1-x) - M_\beta(n-x) > \epsilon\big)(1-\beta)x^{-\beta}\, dx \\
&= \int_0^1 P'\big(M_\beta((1-\delta_{n-x})_+) > \epsilon\big)(1-\beta)x^{-\beta}\, dx\,,
\end{aligned}$$

where for $r > 0$, δ_r is a random variable, independent of the Mittag-Leffler process, with the distribution given by (3.5). Since δ_r converges weakly to infinity as $r \to \infty$, by the dominated convergence theorem, the above expression converges to zero as $n \to \infty$. □

Remark 3.6. Two extreme cases deserve mentioning. A formal substitution of $\beta = 0$ into (3.2) leads to a well-defined process $M_0(0) = 0$ and $M_0(t) = E$, the same standard exponential random variable for all $t > 0$. This process is no longer the inverse of a stable subordinator. It can, however, be used in (3.3). It is elementary to see that the resulting SαS process $Y_{\alpha,0}$ is, in fact, a SαS Lévy motion.

On the other hand, a formal substitution of $\beta = 1$ into (3.2) leads to the degenerate process $M_1(t) = t$ for all $t \geq 0$ (which can be viewed as the inverse of the degenerate 1-stable subordinator $S_1(t) = t$ for $t \geq 0$.) Once again, this process can be used in (3.3), if one interprets the measure ν as the unit point mass at the origin. The resulting SαS process $Y_{\alpha,1}$ is now the degenerate process $Y_{\alpha,1}(t) = tY_{\alpha,1}(1)$ for all $t \geq 0$, where $Y_{\alpha,1}(1)$ is a SαS random variable.

Both limiting cases, $Y_{\alpha,0}$ and $Y_{\alpha,1}$, are processes of a very different nature from the β-ML fractional SαS motion with $0 < \beta < 1$.

4. Some Ergodic Theory

In this section we present some elements of ergodic theory used in this paper. The main reference for these notions is Aaronson (1997); see also Zweimüller (2009).

Let $\big(E, \mathcal{E}, \mu\big)$ be a σ-finite measure space. We will often use the notation $A = B \mod \mu$ for $A, B \in \mathcal{E}$ when $\mu(A \triangle B) = 0$.

Let $T : E \to E$ be a measurable map that preserves the measure μ. When the entire sequence $T, T^2, T^3, \ldots$ of iterates of T is involved, we will sometimes refer to it as *a flow*. The map T is called *ergodic* if the only sets A in $\mathcal{E}$ for which $A = T^{-1}A \mod \mu$ are those for which $\mu(A) = 0$ or $\mu(A^c) = 0$. The map T is called *conservative* if

$$\sum_{n=1}^{\infty} \mathbf{1}_A \circ T^n = \infty \text{ a.e. on } A$$

for every $A \in \mathcal{E}$ with $\mu(A) > 0$. If T is ergodic, then the qualification "on A" above is not needed.

The dual operator $\widehat{T}$ is an operator $L^1(\mu) \to L^1(\mu)$ defined by

$$\widehat{T}f = \frac{d(\nu_f \circ T^{-1})}{d\mu},$$

with ν_f a signed measure on $\big(E, \mathcal{E}\big)$ given by $\nu_f(A) = \int_A f \, d\mu$, $A \in \mathcal{E}$. The dual operator satisfies the relation

$$\int_E \widehat{T}f \cdot g \, d\mu = \int_E f \cdot g \circ T \, d\mu \tag{4.1}$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$. For any nonnegative measurable function f on E a similar definition gives a nonnegative measurable function $\widehat{T}f$, and (4.1) holds for any two nonnegative measurable functions f and g.

An ergodic conservative measure preserving map T is called *pointwise dual ergodic* if there is a sequence of positive constants $a_n \to \infty$ such that

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k f \to \int_E f \, d\mu \ \text{ a.e.} \tag{4.2}$$

for every $f \in L^1(\mu)$. If the measure μ is infinite, pointwise dual ergodicity rules out invertibility of the map T; in fact no factor of T can be invertible, see p. 129 of Aaronson (1997).

Sometimes the convergence of the type described in the definition (4.2) of pointwise dual ergodicity is uniform on certain sets. Let $A \in \mathcal{E}$ be a set with $0 < \mu(A) < \infty$. We say that A is *a Darling-Kac set* for an ergodic conservative measure preserving map T if for some sequence of positive constants $a_n \to \infty$,

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k \mathbf{1}_A \to \mu(A) \ \text{ uniformly, a.e. on } A \tag{4.3}$$

(that is, the convergence in (4.3) is uniform on a measurable subset B of A with $\mu(B) = \mu(A)$). By Proposition 3.7.5 of Aaronson (1997), existence of a Darling-Kac set implies pointwise dual ergodicity of T, so it is legitimate to use the same sequence (a_n) in (4.2) and (4.3).

Given a set $A \in \mathcal{E}$, the map $\varphi : E \to \mathbb{N} \cup \{\infty\}$ defined by $\varphi(x) = \inf\{n \geq 1 : T^n x \in A\}$, $x \in E$ is called *the first entrance time to* A. If T is conservative and ergodic (in addition to being measure preserving), and $\mu(A) > 0$, then $\varphi < \infty$ a.e. on E. It is natural to measure how often the set A is visited by the flow (T^n) by *the wandering rate* sequence

$$w_n = \mu\left(\bigcup_{k=0}^{n-1} T^{-k} A\right), \ n = 1, 2, \dots .$$

There are several alternative expressions for the wandering rate sequence, the last two following from the fact that T is measure preserving.

$$w_n = \sum_{k=0}^{n-1} \mu(A_k) = \sum_{k=0}^{n-1} \mu\big(A \cap \{\varphi > k\}\big) = \sum_{k=1}^{\infty} \min(k,n)\mu\big(A \cap \{\varphi = k\}\big)\,. \tag{4.4}$$

Here $A_0 = A$ and $A_k = A^c \cap \{\varphi = k\}$ for $k \geq 1$. If μ is an infinite measure, T is conservative and ergodic, and $0 < \mu(A) < \infty$, then it follows from (4.4) that

$$w_n \sim \mu(\varphi < n) \quad \text{as } n \to \infty\,. \tag{4.5}$$

Let T be a conservative ergodic measure preserving map. If a set A is a Darling-Kac set, then there is a precise connection between the return sequence (w_n) and the normalizing sequence (a_n) in (4.3) (and, hence, also in (4.2)), assuming regular variation. Specifically, if either $(w_n) \in RV_{1-\beta}$ or $(a_n) \in RV_\beta$ for some $\beta \in [0,1]$, then

$$a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{w_n} \quad \text{as } n \to \infty\,. \tag{4.6}$$

Proposition 3.8.7 in Aaronson (1997) gives one direction of this statement, but the argument is easily reversed.

We will also have an opportunity to use a variation of the notion of a Darling-Kac set. Let T be an ergodic conservative measure preserving map. A set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$ is said to be a uniform set for a nonnegative function $g \in L^1(\mu)$ if

$$\frac{1}{a_n} \sum_{k=1}^{n} \widehat{T}^k g \to \int_E g\, d\mu \ \text{ uniformly, a.e. on } A. \tag{4.7}$$

If $g = \mathbf{1}_A$, then a uniform set is just a Darling-Kac set.

5. Central Limit Theorem Associated with Conservative Null Flows

In this section we state and discuss a functional central limit theorem for stationary infinitely divisible processes generated by certain conservative flows. Throughout, T is an ergodic conservative measure preserving map on an infinite σ-finite measure space $\big(E, \mathcal{E}, \mu\big)$, and M a symmetric homogeneous infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and local Lévy measure ρ, satisfying the regular variation with index $-\alpha$, $0 < \alpha < 2$ at infinity condition (2.5). We will impose an extra assumption on the lower tail of the local Lévy measure: for some $p_0 < 2$

$$x^{p_0} \rho(x, \infty) \to 0 \text{ as } x \to 0. \tag{5.1}$$

Let $f : E \to \mathbb{R}$ be a measurable function. We will assume that f is supported by a set of finite μ-measure, and has the following integrability properties:

$$f \in \begin{cases} L^{1\vee p}(\mu) \text{ for some } \ p > p_0 & \text{if } 0 < \alpha < 1 \\ L^\infty(\mu) & \text{if } \alpha = 1 \\ L^2(\mu) & \text{if } 1 < \alpha < 2 \end{cases} . \tag{5.2}$$

We will, further, assume that

$$\mu(f) = \int_E f(s)\,\mu(ds) \neq 0\,. \tag{5.3}$$

We consider a stochastic process $\mathbf{X} = \big(X_1, X_2, \ldots\big)$ of the form (2.1) - (2.2). The integral is well defined under the condition

$$\int_E \int_{\mathbb{R}} \min\big(1, x^2 f_n(s)^2\big)\,\rho(dx)\,\mu(ds) < \infty\,.$$

It is not difficult to verify that this condition holds due to the assumptions on the Lévy measure ρ and the integrability conditions (5.2) on f. Therefore, the process $\mathbf{X}$ is a well defined infinitely divisible stochastic process. It is automatically stationary. The Lévy measure of each X_n is given by $\nu_{\rm marg} = (\rho \times \mu) \circ H^{-1}$, where $H : \mathbb{R} \times E \to \mathbb{R}$ is given by $H(x, s) = xf(s)$. The assumptions on the Lévy measure ρ and the integrability conditions (5.2) on f imply that

$$\nu_{\rm marg}(\lambda, \infty) \sim \Big(\int_E |f(s)|^\alpha\,\mu(ds)\Big)\rho(\lambda, \infty)$$

as $\lambda \to \infty$. It follows that the marginal tail of the process itself is the same:

$$P(X_n > \lambda) \sim \Big(\int_E |f(s)|^\alpha\,\mu(ds)\Big)\rho(\lambda, \infty)$$

as $\lambda \to \infty$; see Rosiński and Samorodnitsky (1993). In particular, the marginal distributions of the process $\mathbf{X}$ are in the domain of attraction of a SαS law; its memory is determined by the operator T through (2.2).

We will assume that the operator T has a Darling-Kac set A (recall (4.3)), and that the normalizing sequence (a_n) is regularly varying with exponent $\beta \in (0, 1)$. We will also assume that the function f is supported by A. We will add an extra assumption on the set A. We will assume that there exists a measurable function $K : E \to \mathbb{R}_+$ such that, in the notation of (4.4),

$$\frac{\widehat{T}^n \mathbf{1}_{A_n}}{\mu(A_n)} \to K \quad \text{uniformly, a.e. on } A. \tag{5.4}$$

This condition is an extension of the property shared by certain operators T, the so-called Markov shifts (see Chapter 4 in Aaronson (1997)), to a more general class of operators. See examples 5.5 and 5.6 below.

Let $\rho^{\leftarrow}(y) = \inf\{x \geq 0 : \rho(x, \infty) \leq y\}$, $y > 0$ be the left continuous inverse of the tail of the local Lévy measure. The regular variation of the tail implies that $\rho^{\leftarrow} \in RV_{1/\alpha}$ at infinity. Define

$$c_n = \Gamma(1+\beta)\, C_\alpha^{-1/\alpha}\, a_n\, \rho^{\leftarrow}(1/w_n),\ n = 1, 2, \dots , \tag{5.5}$$

where C_α is the α-stable tail constant (see Samorodnitsky and Taqqu (1994)), (a_n) is the normalizing sequence in the Darling-Kac property (4.3) (or, equivalently, in the pointwise dual ergodicity property (4.2)), and (w_n) is the wandering rate sequence for the set A (related to the sequence (a_n) via (4.6)). It follows immediately that

$$c_n \in RV_{\beta+(1-\beta)/\alpha}. \tag{5.6}$$

The sequence (c_n) is the normalizing sequence in the functional central limit theorem below. We will see that under the conditions of that theorem we have the asymptotic relation

$$\rho\,(c_n/a_n, \infty) \sim C_\alpha \big(C_{\alpha,\beta}/\Gamma(1+\beta)\big)^\alpha |\mu(f)|^\alpha a_n^\alpha \left(\int_E |\sum_{k=1}^n f \circ T^k(x)|^\alpha \mu(dx) \right)^{-1} \quad \text{as } n \to \infty , \tag{5.7}$$

with

$$C_{\alpha,\beta} = \Gamma(1+\beta)\Big((1-\beta)B(1-\beta, 1+\alpha\beta)E\,(M_\beta(1))^\alpha\Big)^{1/\alpha}. \tag{5.8}$$

Here B is the standard beta function, and M_β the Mittag-Leffler process defined in (3.1). The following is our functional central limit theorem.

Theorem 5.1. *Let T be an ergodic conservative measure preserving map on an infinite σ-finite measure space $\big(E, \mathcal{E}, \mu\big)$, possessing a Darling-Kac set A whose normalizing sequence (a_n) is regularly varying with exponent $\beta \in (0,1)$. Assume that* (5.4) *holds. Let M be a symmetric homogeneous infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and local Lévy measure ρ, satisfying the regular variation with index $-\alpha$, $0 < \alpha < 2$ at infinity condition* (2.5)*. Assume, further, that* (5.1) *holds for some $p_0 < 2$.*

Let f be a measurable function supported by A and satisfying (5.2) *and* (5.3)*. If $1 < \alpha < 2$, assume further that either*
(i) A is a uniform set for $|f|$, or
(ii) f is bounded.
Then the stationary infinitely divisible stochastic process $\mathbf{X} = \big(X_1, X_2, \dots\big)$ given by (2.1) *and* (2.2) *satisfies*

$$\frac{1}{c_n} \sum_{k=1}^{\lceil n\cdot \rceil} X_k \Rightarrow |\mu(f)| Y_{\alpha,\beta} \quad \text{in } D[0,\infty)\,, \tag{5.9}$$

where (c_n) is defined by (5.5)*, and $\{Y_{\alpha,\beta}\}$ is the β-Mittag-Leffler fractional $S\alpha S$ motion defined by (3.3).*

Remark 5.2. The type of the limiting process obtained in Theorem 5.1 is an indication of the long memory in the process $\mathbf{X}$. On the other hand, the Darling-Kac assumption (4.3) and the duality relation(4.1) imply that

$$\frac{1}{a_n}\sum_{k=1}^{n}\mu(A\cap T^{-k}A)=\frac{1}{a_n}\sum_{k=1}^{n}\int_E \mathbf{1}_A\cdot\mathbf{1}_A\circ T^k\,d\mu=\int_A\frac{1}{a_n}\sum_{k=1}^{n}\widehat{T}^k\mathbf{1}_A\,d\mu\to\mu(A)^2\in(0,\infty)$$

as $n\to\infty$. Since $a_n=o(n)$, and f is supported by A, we see that for every $\epsilon>0$,

$$\frac{1}{n}\sum_{k=1}^{n}\mu\big\{x\in E:|f(x)|>\epsilon,|f\circ T^k(x)|>\epsilon\big\}\le\frac{1}{n}\sum_{k=1}^{n}\mu(A\cap T^{-k}A)\to 0,$$

and it follows immediately, e.g. from Theorem 2 in Rosiński and Żak (1997), that the process $\mathbf{X}$ is ergodic.

Under certain additional assumptions on the map T, one can check that the process $\mathbf{X}$ is, in fact, mixing. We skip the details. See, however, examples 5.5 and 5.6 below.

Remark 5.3. The statement of Theorem 5.1 makes sense in the limiting cases $\beta=0$ and $\beta=1$ of Remark 3.6 (in the case $\beta=1$ the constant $C_{\alpha,1}$ needs to be interpreted as $C_\alpha^{1/\alpha}$). Most of the argument in the proof of Theorem 5.1 automatically works in these cases. The limiting processes would then turn out to be, correspondingly, a SαS Lévy motion and the straight line process; see Remark 3.6. This case $\beta=0$ corresponds to short memory in the process $\mathbf{X}$, while the case $\beta=1$ corresponds to extremely long memory.

Remark 5.4. When $0<\alpha<1$, the argument we will use in the proof of Theorem 5.1 can be used to establish a "positive" version of the theorem. Specifically, assume now that the local Lévy measure ρ is concentrated on $(0,\infty)$, and that the function f is nonnegative. Then

$$\frac{1}{c_n}\sum_{k=1}^{\lceil n\cdot\rceil}X_k\Rightarrow\mu(f)Y^+_{\alpha,\beta}\quad\text{in } D[0,\infty)\,, \tag{5.10}$$

where $\{Y^+_{\alpha,\beta}\}$ is a positive β-Mittag-Leffler fractional α-stable motion defined as in (3.3), but with SαS random measure $Z_{\alpha,\beta}$ replaced by a positive α-stable random measure with the same control measure.

We finish this section with two examples of different situations where Theorem 5.1 applies. The first example is close to the heart of a probabilist.

Example 5.5. Consider an irreducible null recurrent Markov chain with state space $\mathbb{Z}$ and transition matrix $P=(p_{ij})$. Let $\{\pi_j,j\in\mathbb{Z}\}$ be the unique invariant measure of the Markov chain that satisfies $\pi_0=1$. We define a σ-finite measure on $(E,\mathcal{E})=\big(\mathbb{Z}^{\mathbb{N}},\mathcal{B}(\mathbb{Z}^{\mathbb{N}})\big)$ by

$$\mu(\cdot)=\sum_{i\in\mathbb{Z}}\pi_iP_i(\cdot)\,,$$

with the usual notation of $P_i(\cdot)$ being the probability law of the Markov chain starting in state $i \in \mathbb{Z}$. Since $\sum_j \pi_j = \infty$, μ is an infinite measure.

Let $T : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ be the left shift map $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$ for $\{x_k, k = 0, 1, \dots\} \in \mathbb{Z}^{\mathbb{N}}$. Obviously, T preserves the measure μ. Since the Markov chain is irreducible and null recurrent, the flow $\{T^n\}$ is conservative and ergodic; see Harris and Robbins (1953).

Consider the set $A = \{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\}$ and the corresponding first entrance time $\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$. Assume that

$$\sum_{k=1}^{n} P_0(\varphi \geq k) \in RV_{1-\beta} \tag{5.11}$$

for some $\beta \in (0, 1)$. Since $\mu(\varphi = k) = P_0(\varphi \geq k)$ for $k \geq 1$ (see Lemma 3.3 in Resnick et al. (2000)), we see that $\mu(\varphi \leq n) \in RV_{1-\beta}$ and, hence, by (4.5), the wandering rates (w_n) have the same property,

$$w_n \in RV_{1-\beta}\,. \tag{5.12}$$

In this example,

$$\widehat{T}^k \mathbf{1}_A(x) = P_0(x_k = 0), \quad \text{constant for } x \in A;$$

see Section 4.5 in Aaronson (1997). In particular, the set A is a Darling-Kac set, and by (5.12) and (4.6), we see that the corresponding normalizing sequence (a_n) is regularly varying with exponent β. The assumption (5.4) is easily seen to hold in this example. Indeed, applying the explicit expression for the dual operator given on p. 156 in Aaronson (1997) to the function

$$f(x_0, x_1, \ldots) = \mathbf{1}\big(x_j \neq 0,\, j = 0, \ldots, n-1,\, x_n = 0\big)\,,$$

we see that

$$\widehat{T}^n \mathbf{1}_{A_n}(x_0, x_1, \ldots) = \mathbf{1}(x_0 = 0) \sum_{i_0 \neq 0} \pi_{i_0} \sum_{i_1 \neq 0} p_{i_0 i_1} \cdots \sum_{i_{n-1} \neq 0} p_{i_{n-2} i_{n-1}} p_{i_{n-1} 0}$$

is constant on A and vanishes outside of A. Therefore, the ratio in (5.4) is identically equal to 1 on A.

We conclude that Theorem 5.1 applies in this case if we choose any measurable function f supported by A and satisfying the conditions of the theorem.

It is easy to see that the stationary infinitely divisible process $\mathbf{X}$ in this example is mixing. Indeed, by Theorem 5 of Rosiński and Żak (1996) it is enough to check that

$$\mu\{x : |f(x)| > \epsilon,\, |f \circ T^n(x)| > \epsilon\} \to 0$$

for every $\epsilon > 0$. However, since f vanishes outside of A, null recurrence implies that as $n \to \infty$,

$$\mu\{x : |f(x)| > \epsilon,\, |f \circ T^n(x)| > \epsilon\} \leq \mu(A \cap T^{-n} A) = P_0(x_n = 0) \to 0.$$

The next example is less familiar to probabilists, but is well known to ergodic theorists.

Example 5.6. We start with a construction of the so-called *AFN-system*, studied in, e.g., Zweimüller (2000) and Thaler and Zweimüller (2006). Let E be the union of a finite family of disjoint bounded open intervals in $\mathbb{R}$ and let $\mathcal{E}$ be the Borel σ-field on E. Let λ be the one-dimensional Lebesgue measure.

Let ξ be a (possibly, infinite) collection of nonempty disjoint open subintervals (of the intervals in E) such that $\lambda\big(E \setminus \bigcup_{Z\in\xi} Z\big) = 0$. Let $T : E \to E$ be a map that is twice differentiable on (each interval of) E. We assume that T is strictly monotone on each $Z \in \xi$.

The map T is further assumed to satisfy the following three conditions, (A), (F), and (N), (giving rise to the name AFN-system).

(A) *Adler's condition*:

$$T''/(T')^2 \ \text{ is bounded on } \bigcup_{Z\in\xi} Z\,.$$

(F) *Finite image condition*:

$$\text{the collection } T\xi = \{TZ : Z \in \xi\} \ \text{ is finite.}$$

(N) *A possibility of non-uniform expansion*: there exists a finite subset $\zeta \subseteq \xi$ such that each $Z \in \zeta$ has *an indifferent fixed point* x_Z as one of its end points. That is,

$$\lim_{x\to x_Z, x\in Z} Tx = x_Z \quad \text{and} \quad \lim_{x\to x_Z, x\in Z} T'x = 1.$$

Moreover, we suppose, for each $Z \in \zeta$,

$$\text{either } T' \text{ decreases on } (-\infty, x_Z)\cap Z, \ \text{ or } T' \text{ increases on } (x_Z, \infty)\cap Z\,,$$

depending on whether x_Z is the left endpoint or the right endpoint of Z. Finally, we assume that T is uniformly expanding away from $\{x_Z : Z \in \zeta\}$, i.e. for each $\epsilon > 0$, there is $\rho(\epsilon) > 1$ such that

$$|T'| \geq \rho(\epsilon) \text{ on } E \setminus \bigcup_{Z\in\zeta}\Big(\big(x_Z - \epsilon, x_Z + \epsilon\big)\cap Z\Big).$$

If the conditions (A), (F), and (N) are satisfied, the triplet (E, T, ξ) is called an AFN-system, and the map T is called an AFN-map. If T is also conservative and ergodic with respect to λ, and the collection ζ is nonempty, then the AFN-map T is said to be *basic*; we will assume this property in the sequel. Finally, we will assume that T admits *nice expansions* at the indifferent fixed points. That is, for every $Z \in \zeta$ there is $0 < \beta_Z < 1$ such that

$$Tx = x + a_Z|x - x_Z|^{1/\beta_Z+1} + o\big(|x - x_Z|^{1/\beta_Z+1}\big) \quad \text{as } x \to x_Z \text{ in } Z, \tag{5.13}$$

for some $a_Z \neq 0$.

It is shown in Zweimüller (2000) that every basic AFN-map has an infinite invariant measure $\mu \ll \lambda$ with the density given by $d\mu/d\lambda(x) = h_0(x)G(x)$, $x \in E$, where

$$G(x) = \begin{cases} (x - x_Z)\big(x - (T|_Z)^{-1}(x)\big)^{-1} & \text{if } x \in Z \in \zeta, \\ 1 & \text{if } x \in E \setminus \bigcup_{Z\in\zeta} Z\,, \end{cases}$$

and h_0 is a function of bounded variation bounded away from both 0 and infinity. We view T as a conservative ergodic measure-preserving map on the infinite measure space $(E, \mathcal{E}, \mu)$.

An example of a basic AFN-map is Boole's transformation placed on $E = (0, 1/2) \cup (1/2, 1)$, defined by

$$T(x) = \frac{x(1-x)}{1-x-x^2}, \ x \in (0, 1/2), \quad T(x) = 1 - T(1-x), \ x \in (1/2, 1)\,.$$

It admits nice expansions at the indifferent fixed points $x_Z = 0$ and $x_Z = 1$ with $\beta_Z = 1/2$ in both cases. The invariant measure μ satisfies

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}, \ x \in E\,.$$

See Thaler (2001).

Let T be a basic AFN-map. We put

$$A = E \setminus \bigcup_{Z\in\zeta} \Big((x_Z - \epsilon, x_Z + \epsilon) \cap Z \Big)$$

for some $\epsilon > 0$ small enough so that the set A is non-empty. Since $\lambda(\partial A) = 0$ and A is bounded away from the indifferent fixed points $\{x_Z : Z \in \zeta\}$, it follows from Corollary 3 of Zweimüller (2000) that A is a Darling-Kac set. Moreover, the corresponding normalizing sequence (a_n) is regularly varying with exponent $\beta = \min_{Z\in\zeta} \beta_Z$ in the notation of (5.13); see Theorems 3 and 4 in Zweimüller (2000). The assumption (5.4) also holds; see (2.6) in Thaler and Zweimüller (2006).

Once again, Theorem 5.1 applies if we choose any measurable function f supported by A and satisfying the conditions of Theorem 5.1. Note that, by Theorem 9 in Zweimüller (2000), Riemann integrability of $|f|$ on A suffices for the uniformity of the set A for $|f|$.

The stationary infinitely divisible process $\mathbf{X}$ in this example is also mixing. Indeed, the basic AFN-map T is *exact*, i.e. the σ-field $\cap_{n=1}^{\infty} T^{-n}\mathcal{B}$ is trivial; see e.g. p. 1522 in Zweimüller (2000) . The exactness of T implies that

$$\mu(A \cap T^{-n}A) = \int_A \widehat{T}^n \mathbf{1}_A \, d\mu \to 0$$

as $n \to \infty$; see p. 12 in Thaler (2001). Now mixing of the process $\mathbf{X}$ follows from the fact that f is supported by A, as in Example 5.5.

6. Distributional Results in Ergodic Theory

In this section we prove two distributional ergodic theoretical results that will be used in the proof of Theorem 5.1. These results may be of interest on their own as well. We call our first result a generalized Darling-Kac theorem, because the first result of this type was proved in Darling and Kac (1957) as a distributional limit theorem for the occupation times of Markov processes and chains under a certain uniformity assumption on the transition law. The limiting law is the Mittag-Leffler distribution described in (3.2). Under the same setup and assumptions, Bingham (1971) extended the result to weak convergence in the space $D[0,\infty)$ endowed with the Skorohod J_1 topology, and the limiting process is the Mittag-Leffler process defined in (3.1).

The result of Darling and Kac (1957) was put into ergodic-theoretic context by Aaronson (1981) who established the one-dimensional convergence for abstract conservative infinite measure preserving maps under the assumption of pointwise dual ergodicity, i.e. dispensing with a condition of uniformity. Furthermore, Aaronson proves convergence in a *strong distributional sense*, a stronger mode of convergence than weak convergence. The same strong distributional convergence was established later in Thaler and Zweimüller (2006), with the assumption of pointwise dual ergodicity replaced by an averaged version of (5.4). The latter assumption was further weakened in Zweimüller (2007a). Our result, Theorem 6.1 below, extends Aaronson's result to the space $D[0,\infty)$, under the assumption of pointwise dual ergodicity.

We start with defining strong distributional convergence. Let Y be a separable metric space, equipped with its Borel σ-field. Let $\big(\Omega_1,\mathcal{F}_1,m\big)$ be a measure space and $\big(\Omega_2,\mathcal{F}_2,P_2\big)$ a probability space. We say that a sequence of measurable maps $R_n:\Omega_1\to Y$, $n=1,2,\ldots$ converges strongly in distribution to a measurable map $R:\Omega_2\to Y$ if $P_1\circ R_n^{-1}\Rightarrow P_2\circ R^{-1}$ in Y for any probability measure $P_1\ll m$ on $\big(\Omega_1,\mathcal{F}_1\big)$. That is,

$$\int_{\Omega_1} g(R_n)\,dP_1 \to \int_{\Omega_2} g(R)\,dP_2$$

for any such P_1 and a bounded continuous function g on Y. We will use the notation $R_n \overset{\mathcal{L}(m)}{\Rightarrow} R$ when strong distributional convergence takes place.

Theorem 6.1. ***(Generalized Darling-Kac Theorem)***

Let T be an ergodic conservative measure preserving map on an infinite σ-finite measure space $\big(E,\mathcal{E},\mu\big)$. Assume that T is pointwise dual ergodic with a normalizing sequence (a_n) that is regularly varying with exponent $\beta\in(0,1)$. Let $f\in L^1(\mu)$ be such that $\mu(f)\neq 0$, and denote $S_n(f)=\sum_{k=1}^n f\circ T^k$, $n=1,2,\ldots$. Then

$$\frac{1}{a_n}S_{\lceil n\cdot\rceil}(f) \overset{\mathcal{L}(\mu)}{\Rightarrow} \mu(f)\Gamma(1+\beta)M_\beta(\cdot) \quad \text{in } D[0,\infty)\,, \tag{6.1}$$

where M_β is the Mittag-Leffler process, and $D[0,\infty)$ is equipped with the J_1 topology.

Proof. It is shown in Corollary 3 of Zweimüller (2007b) that proving weak convergence in (6.1) for one fixed probability measure on $\big(E,\mathcal{E}\big)$, that is absolutely continuous with respect to μ, already guarantees the full strong distributional convergence. We choose and fix an arbitrary set $A\in\mathcal{E}$ with $0<\mu(A)<\infty$, and prove weak convergence in (6.1) with respect to $\mu_A(\cdot)=\mu(\cdot\cap A)/\mu(A)$.

It turns out that we only need to consider one particular function $f=\mathbf{1}_A$ and to establish the appropriate finite-dimensional convergence, i.e. to show that

$$\left(\frac{1}{a_n}S_{\lceil nt_i\rceil}(\mathbf{1}_A)\right)_{i=1}^k \Rightarrow \big(\mu(A)\Gamma(1+\beta)M_\beta(t_i)\big)_{i=1}^k \quad \text{in } \mathbb{R}^k \tag{6.2}$$

for all $k\geq 1$, $0\leq t_1<\cdots<t_k$, when the law of the random vector in the left hand side is computed with respect to μ_A.

Indeed, suppose that (6.2) holds. By Hopf's ergodic theorem (also sometimes called a ratio ergodic theorem; see Theorem 2.2.5 in Aaronson (1997)), the finite-dimensional convergence immediately extends to the corresponding finite-dimensional convergence with any function $f\in L^1(\mu)$ such that $\mu(f)\neq 0$. Next, write $f=f_+-f_-$, the difference of the positive and negative parts. Since the process $\big(S_{\lceil nt\rceil}(f_+),\, t\geq 0\big)$ has, for each n, nondecreasing sample paths, Theorem 3 in Bingham (1971) tells us that the convergence of the finite-dimensional distributions, and the continuity in probability of the limiting Mittag-Leffler process already imply weak convergence, hence tightness, of this sequence of processes. Similarly, the sequence of the processes $\big(S_{\lceil nt\rceil}(f_-),\, t\geq 0\big)$, $n=1,2,\ldots$ is tight as well. Since both converge to a continuous limit, their sum, $\big(S_{\lceil nt\rceil}(f),\, t\geq 0\big)$, $n=1,2,\ldots$, is tight as well, because in this case the uniform modulus of continuity can be used instead of the J_1 modulus of continuity; see e.g. Billingsley (1999).

This will give us the required weak convergence and, hence, finish the proof of the theorem.

It remains to show (6.2). We will use a strategy similar to the one used in Bingham (1971). We start with defining a continuous version of the process $\big(S_{\lceil nt\rceil}(\mathbf{1}_A),\, t\geq 0\big)$ given by the linear interpolation

$$\tilde S_n(t)=\big((i+1)-nt\big)S_i(\mathbf{1}_A)+(nt-i)S_{i+1}(\mathbf{1}_A) \;\text{ if }\; \frac{i}{n}\leq t\leq\frac{i+1}{n},\; i=0,1,2,\ldots. \tag{6.3}$$

With the implicit argument $x\in E$ viewed as random (with the law μ_A), each $\tilde S_n$ defines a random Radon measure on $[0,\infty)$. Therefore, for any $k\geq 1$ the k-tuple product $\tilde S_n^k=\tilde S_n\times\ldots\times\tilde S_n$ is a random Radon measure on $[0,\infty)^k$. By Fubini's theorem,

$$\tilde m_n^{(k)}(B)=\int_A \tilde S_n^k(B)(x)\,\mu_A(dx),\;\; B\subseteq[0,\infty)^k, \;\; \text{Borel},$$

is a Radon measure on $[0,\infty)^k$. We define, similarly, S_n, S_n^k and $m_n^{(k)}$, starting with $S_n(t) = S_{\lceil nt\rceil}(\mathbf{1}_A)$, $t\geq 0$. Finally, we perform the same operation on the limiting process and define $M_{\beta,A}$ by $\mu(A)\Gamma(1+\beta)M_\beta$, and then construct $M_{\beta,A}^k$ and $m_{\beta,A}^{(k)} = EM_{\beta,A}^k$.

Note that $\tilde m_n^{(k)}$ is absolutely continuous with respect to the k-dimensional Lebesgue measure, and

$$\frac{d^k\tilde m_n^{(k)}}{dt_1\dots dt_k} = n^k\int_A\prod_{j=1}^k \mathbf{1}_A\circ T^{i_j}(x)\,\mu_A(dx)\ \text{ on }\ \frac{i_j}{n}\leq t_j<\frac{i_j+1}{n},\ i_j=0,1,\dots,\ j=1,\dots,k\,.$$

We will prove that for all $k\geq 1$, $\theta_1,\dots,\theta_k\geq 0$,

$$\frac{1}{a_n^k}\int_0^\infty\!\!\dots\int_0^\infty e^{-\sum_{j=1}^k\theta_jt_j}\,\tilde m_n^{(k)}(dt_1,\dots dt_k)\to\int_0^\infty\!\!\dots\int_0^\infty e^{-\sum_{j=1}^k\theta_jt_j}\,m_{\beta,A}^{(k)}(dt_1,\dots dt_k) \tag{6.4}$$

as $n\to\infty$. We claim that this will suffice for (6.2).

Indeed, suppose that (6.4) holds. Convergence of the joint Laplace transforms implies that

$$a_n^{-k}\tilde m_n^{(k)}\xrightarrow{v} m_{\beta,A}^{(k)}$$

(vaguely) in $[0,\infty)^k$. Since the rectangles are, clearly, compact continuity sets with respect to the limiting measure $m_{\beta,A}^{(k)}$, we conclude that for every $k=1,2,\dots$ and $t_j\geq 0$, $j=1,\dots,k$, we have

$$\int_A\prod_{j=1}^k a_n^{-1}\tilde S_n(t_j)(x)\,\mu_A(dx) = a_n^{-k}\tilde m_n^{(k)}\Big(\prod_{j=1}^k[0,t_j]\Big)$$

$$\to m_{\beta,A}^{(k)}\Big(\prod_{j=1}^k[0,t_j]\Big) = E\Big[\prod_{j=1}^k\mu(A)\Gamma(1+\beta)M_\beta(t_j)\Big]$$

as $n\to\infty$. Since for every fixed $\varepsilon>0$ and $n>1/\varepsilon$,

$$\tilde S_n(t)\leq S_n(t)\leq\tilde S_n(t+\varepsilon)$$

for each $t\geq 0$, we conclude by monotonicity and continuity of the Mittag-Leffler process that

$$\int_A\prod_{j=1}^k a_n^{-1}S_n(t_j)\,\mu_A(dx)\to E\Big[\prod_{j=1}^k\mu(A)\Gamma(1+\beta)M_\beta(t_j)\Big]\,. \tag{6.5}$$

We claim that (6.5) implies (6.2). By taking linear combinations with nonnegative weights, we see that it is enough to show that the distribution of such a linear combination,

$$\sum_{j=1}^k\theta_jM_\beta(t_j),\ \theta_j>0,\ j=1,\dots,k\,,$$

is determined by its moments, and by the Carleman sufficient condition it is enough to check that

$$\sum_{m=1}^\infty\left(\frac{1}{E\big(\sum_{j=1}^k\theta_jM_\beta(t_j)\big)^m}\right)^{1/(2m)}=\infty\,.$$

A simple monotonicity and scaling argument shows that it is sufficient to verify only that

$$\sum_{m=1}^{\infty}\left(\frac{1}{E\bigl(M_\beta(1)\bigr)^m}\right)^{1/(2m)} = \infty\,. \tag{6.6}$$

However, the moments of $M_\beta(1)$ can be read off (3.2), and Stirling's formula together with elementary algebra imply (6.6). Hence (6.2) follows.

It follows that we need to prove (6.4). Taking into account the form of the density of $\tilde m_n^{(k)}$ with respect to the k-dimensional Lebesgue measure, we can write the left hand side of (6.4) as

$$\sum_\pi F_{n,A}(\theta_{\pi(1)}\dots\theta_{\pi(k)})\,,$$

where

$$F_{n,A}(\theta_1\dots\theta_k) = \left(\frac{n}{a_n}\right)^k \int\cdots\int_{0<t_1<\cdots<t_k} e^{-\sum_{j=1}^k \theta_j t_j}\mu_A\left(\bigcap_{j=1}^k T^{-\lceil nt_j\rceil}A\right)dt_1\dots dt_k\,,$$

and π runs through the permutations of the sets $\{1,\dots,k\}$. To establish (6.4), it is enough to verify that

$$F_{n,A}(\theta_1\dots\theta_k) \to \bigl(\mu(A)\Gamma(1+\beta)\bigr)^k\bigl((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\dots\theta_k\bigr)^{-\beta} \tag{6.7}$$

as $n\to\infty$, because Lemma 3 in Bingham (1971) shows that summing up the expression in the right hand side of (6.7) over all possible permutations $(\theta_{\pi(1)}\dots\theta_{\pi(k)})$ produces the expression in the right hand side of (6.4).

Given $0<\varepsilon<1$, we use repeatedly pointwise dual ergodicity and Egorov's theorem to construct a nested sequence of measurable subsets of E, with $A_0=A$, and for $i=0,1,\dots$, $A_{i+1}\subseteq A_i$, and $\mu(A_{i+1})\ge(1-\varepsilon)\mu(A_i)$, while

$$\frac{1}{a_n}\sum_{k=1}^{n}\widehat T^k \mathbf{1}_{A_i} \to \mu(A_i) \ \text{ uniformly on } A_{i+1}. \tag{6.8}$$

It is elementary to see that with $v_1=\theta_1+\theta_2+\cdots+\theta_k$, $v_2=\theta_2+\cdots+\theta_k,\dots,v_k=\theta_k$,

$$\begin{aligned}F_{n,A}(\theta_1\dots\theta_k) &\sim \frac{1}{a_n^k}\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty} e^{-n^{-1}\sum_{j=1}^k v_j m_j}\mu_A\left(\bigcap_{j=1}^k T^{-(m_1+\ldots+m_j)}A\right)\\ &= \frac{1}{a_n^k}\int_A\left[\left(\sum_{m_1=0}^{\infty}\widehat T^{m_1}\mathbf{1}_A\,e^{-v_1m_1/n}\right)\prod_{j=2}^k\left(\sum_{m_j=0}^{\infty}\mathbf{1}_A\circ T^{m_2+\ldots+m_j}\,e^{-v_jm_j/n}\right)\right]d\mu_A\\ &\ge \frac{1}{a_n^k}\int_{A_1}\bigl(\cdots\bigr)\,,\end{aligned} \tag{6.9}$$

where the equality is due to the duality relation (4.1). Note that by (6.8) with $i=0$,

$$\sum_{m_1=0}^{\infty}\widehat T^{m_1}\mathbf{1}_A\,e^{-v_1m_1/n} = \bigl(1-e^{-v_1/n}\bigr)\sum_{i=0}^{\infty}\left(\sum_{m_1=0}^{i}\widehat T^{m_1}\mathbf{1}_{A_0}\right)e^{-v_1 i/n} \tag{6.10}$$

$$\sim \frac{\mu(A_0)v_1}{n}\sum_{i=0}^{\infty} a_i e^{-v_1 i/n}$$

uniformly on A_1 as $n \to \infty$. Therefore,

$$\begin{aligned}
F_{n,A}(\theta_1 \dots \theta_k) &\geq \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\mu(A_0)v_1}{n}\sum_{i=0}^{\infty} a_i e^{-v_1 i/n} \\
&\quad \times \int_{A_1}\prod_{j=2}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_A\circ T^{m_2+\ldots+m_j}\, e^{-v_j m_j/n}\right) d\mu_A \\
&= \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\mu(A_0)v_1}{n}\sum_{i=0}^{\infty} a_i e^{-v_1 i/n}
\end{aligned}$$

$$\times \int_A \left[\left(\sum_{m_2=0}^{\infty}\widehat{T}^{m_2}\mathbf{1}_{A_1}\, e^{-v_2 m_2/n}\right)\prod_{j=3}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_A\circ T^{m_3+\ldots+m_j}\, e^{-v_j m_j/n}\right)\right] d\mu_A$$

$$\geq \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\mu(A_0)v_1}{n}\sum_{i=0}^{\infty} a_i e^{-v_1 i/n}\int_{A_2}\big(\cdots\big)\,.$$

Using now repeatedly (6.8) with larger and larger i, together with the same argument as in (6.10), we conclude that

$$\begin{aligned}
F_{n,A}(\theta_1 \dots \theta_k) &\geq \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\mu(A_0)\mu(A_1)v_1v_2}{n^2}\sum_{i_1=0}^{\infty} a_{i_1} e^{-v_1 i_1/n}\sum_{i_2=0}^{\infty} a_{i_2} e^{-v_2 i_2/n} \\
&\quad \times \int_{A_2}\prod_{j=3}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_A\circ T^{m_3+\ldots+m_j}\, e^{-v_j m_j/n}\right) d\mu_A \\
&\geq \cdots \geq \big(1-o(1)\big)\frac{1}{a_n^k}\frac{\prod_{j=0}^{k-1}\mu(A_j)v_{j+1}}{n^k}\prod_{j=1}^{k}\left(\sum_{i=0}^{\infty} a_i e^{-v_j i/n}\right)\frac{\mu(A_k)}{\mu(A)} \\
&\geq \big(1-o(1)\big)(1-\varepsilon)^{k(k+1)/2}\left(\frac{\mu(A)}{na_n}\right)^k (v_1\dots v_k)\prod_{j=1}^{k}\left(\sum_{i=0}^{\infty} a_i e^{-v_j i/n}\right).
\end{aligned}$$

Extending the sequence (a_n) into a piece-wise constant regular varying function of real variable $\big(a(x),\, x>0\big)$ and using Karamata's Tauberian Theorem (see e.g. Section 3.6 in Aaronson (1997)), we conclude that for every $j = 1, \dots, k$,

$$\sum_{i=0}^{\infty} a_i e^{-v_j i/n} \sim \Gamma(1+\beta)\frac{n}{v_j}a(n/v_j), \quad n \to \infty\,.$$

It follows that

$$F_{n,A}(\theta_1 \dots \theta_k) \geq \big(1-o(1)\big)(1-\varepsilon)^{k(k+1)/2}\Big(\mu(A)\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}\frac{a(n/v_j)}{a_n}$$

$$\to (1-\varepsilon)^{k(k+1)/2}\Big(\mu(A)\Gamma(1+\beta)\Big)^k \prod_{j=1}^k v_j^{-\beta}$$

by the regular variation. Since this is true for every $0<\varepsilon<1$, we have obtained the lower bound

$$\liminf_{n\to\infty} F_{n,A}(\theta_1\dots\theta_k) \geq \big(\mu(A)\Gamma(1+\beta)\big)^k \big((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\dots\theta_k\big)^{-\beta}. \tag{6.11}$$

The lower bound (6.11) is valid for any measurable set A with $0<\mu(A)<\infty$. We will now show that for any $k\geq 1$ and $0<\theta<1$ there is a measurable set $A_{k,\theta}\subseteq A$ such that

$$\mu\big(A_{k,\theta}\big) \geq (1-\theta)\mu(A)\,, \tag{6.12}$$

and such that

$$\limsup_{n\to\infty} F_{n,A_{k,\theta}}(\theta_1\dots\theta_k) \leq \big(\mu(A_{k,\theta})\Gamma(1+\beta)\big)^k \big((\theta_1+\cdots+\theta_k)(\theta_2+\cdots+\theta_k)\dots\theta_k\big)^{-\beta}. \tag{6.13}$$

We know that (6.11) and (6.13) together imply (6.7), hence that (6.2) holds for the set $A_{k,\theta}$. We claim that this implies that (6.2) for every measurable A with $0<\mu(A)<\infty$.

Indeed, suppose that, to the contrary, (6.2) fails for some measurable A with $0<\mu(A)<\infty$, some $k\geq 1$ and some $0<t_1<\ldots<t_k$. By the one-dimensional result of Aaronson (1981), the k components in the left hand side of (6.2), individually, converge weakly. Therefore, the sequence of the laws of the k-dimensional vectors in the left hand side of (6.2) is tight, and so there is a sequence of integers $n_l\uparrow\infty$ and a random vector $(Y_1,\dots,Y_k)$ with

$$(Y_1,\dots,Y_k) \overset{d}{\neq} \mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)\,, \tag{6.14}$$

such that

$$\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_A),\dots,S_{\lceil n_l t_k\rceil}(\mathbf{1}_A)\big) \Rightarrow (Y_1,\dots,Y_k)\,, \tag{6.15}$$

when the law of the random vector in the left hand side is computed with respect to μ_A. It follows from (6.14) that there is a Borel set $B\subset\mathbb{R}^k$ such that, for each $b>0$, bB is a continuity set for both $(Y_1,\dots,Y_k)$ and $\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)$ and (abusing the notation a bit by using the same letter P),

$$P\Big(\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)\in B\Big) > (1+\rho)P\Big((Y_1,\dots,Y_k)\in B\Big) \tag{6.16}$$

for some $\rho>0$. In fact, since the law of a Mittag-Leffler random variable is atomless, such a B can be taken to be either a “SW corner” of the type $B=\prod_{j=1}^k(-\infty,x_j]$ for some $(x_1,\dots,x_k)\in\mathbb{R}^k$, or its complement.

Choose now $0<\theta<1$ so small that

$$(1-\theta)(1+\rho) > 1\,, \tag{6.17}$$

and consider the set $A_{k,\theta}$. It follows from (6.15) and Hopf's ergodic theorem that

$$\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}),\dots,S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big) \Rightarrow \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}(Y_1,\dots,Y_k)\,,$$

when the law of the random vector in the left hand side is still computed with respect to μ_A. However, since (6.2) holds for the set $A_{k,\theta}$, we see that

$$\begin{aligned}
P\Big((Y_1,\dots,Y_k)\in B\Big) &= \lim_{l\to\infty}\mu_A\Big(\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}),\dots,S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big)\in\frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}B\Big)\\
&= \frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}\lim_{l\to\infty}\mu_{A_{k,\theta}}\Big(\frac{1}{a_{n_l}}\big(S_{\lceil n_l t_1\rceil}(\mathbf{1}_{A_{k,\theta}}),\dots,S_{\lceil n_l t_k\rceil}(\mathbf{1}_{A_{k,\theta}})\big)\in\frac{\mu\big(A_{k,\theta}\big)}{\mu(A)}B\Big)\\
&\geq (1-\theta)P\Big(\mu(A)\Gamma(1+\beta)\big(M_\beta(t_1)\dots M_\beta(t_k)\big)\in B\Big)\\
&> P\big((Y_1,\dots,Y_k)\in B\big)\,,
\end{aligned}$$

where the last inequality follows from (6.16) and (6.17). This contradiction shows that, once we prove (6.13), this will establish (6.2) for every measurable A with $0<\mu(A)<\infty$.

We call a nested sequence $(A_0,A_1,\dots)$ of sets in (6.8) an ε-sequence starting at A_0. Its finite subsequence $(A_0,A_1,\dots,A_k)$ will be called an ε-sequence of length $k+1$ starting at A_0 and ending at A_k. Let A be a measurable set with $0<\mu(A)<\infty$. Fix $0<\theta<1$. Let $0<r<1$ be a small number, to be specified in the sequel. We construct a nested sequence of sets as follows.

Let $B_0=A$. Construct an r-sequence of length $k+1$ starting at B_0, and ending at some set $B_1\subseteq B_0$. Next, construct an r^2-sequence of length $k+1$ starting at B_1, and ending at some set $B_2\subseteq B_1$. Proceeding this way we obtain a nested sequence of measurable sets $A=B_0\supseteq B_1\supseteq B_2\supseteq\dots$, such that

$$\mu(B_n)\geq\prod_{i=1}^n(1-r^i)^k\,\mu(A),\ n=1,2,\dots\,.$$

The sets (B_n) decrease to some set $A_{k,\theta}$ with

$$\mu(A_{k,\theta})\geq\prod_{i=1}^\infty(1-r^i)^k\,\mu(A)\,.$$

Notice that, by choosing $0<r<1$ small enough, we can ensure that (6.12) holds. Note, further, that by construction, for every $d=1,2,\dots$,

$$\mu(A_{k,\theta})\geq f_d\,\mu(B_d),\quad\text{with}\quad f_d=\prod_{i=d+1}^\infty(1-r^i)^k\,.$$

Clearly, $f_d\uparrow 1$ as $d\to\infty$. Starting with the first line in (6.9), we see that

$$F_{n,A_{k,\theta}}(\theta_1\dots\theta_k)\leq\big(1+o(1)\big)\frac{1}{a_n^k}\sum_{m_1=0}^\infty\dots\sum_{m_k=0}^\infty e^{-n^{-1}\sum_{j=1}^k v_jm_j}\mu_{B_d}\left(\bigcap_{j=1}^k T^{-(m_1+\ldots+m_j)}B_d\right)\frac{\mu(B_d)}{\mu\big(A_{k,\theta}\big)}$$

$$\le \big(1+o(1)\big)\frac{1}{f_d}\frac{1}{a_n^k}\int_{B_d}\left[\left(\sum_{m_1=0}^{\infty}\widehat{T}^{m_1}\mathbf{1}_{B_{d-1}}\, e^{-v_1 m_1/n}\right)\prod_{j=2}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_{B_d}\circ T^{m_2+\ldots+m_j}\, e^{-v_j m_j/n}\right)\right] d\mu_{B_d}\,.$$

Using repeatedly uniform convergence as in (6.10) above, we conclude, as in the case of the corresponding lower bound calculation, that

$$F_{n,A_{k,\theta}}(\theta_1\ldots\theta_k) \le \big(1+o(1)\big)\frac{1}{f_d}\frac{1}{a_n^k}\frac{\mu\big(B_{d-1}\big)v_1}{n}\sum_{i=0}^{\infty}a_i e^{-v_1 i/n}$$

$$\times\int_{B_d}\left[\left(\sum_{m_2=0}^{\infty}\widehat{T}^{m_2}\mathbf{1}_{B_{d-1}}\, e^{-v_2 m_2/n}\right)\prod_{j=3}^{k}\left(\sum_{m_j=0}^{\infty}\mathbf{1}_{B_d}\circ T^{m_3+\ldots+m_j}\, e^{-v_j m_j/n}\right)\right] d\mu_{B_d}$$

$$\le\cdots\le \big(1+o(1)\big)\frac{1}{f_d}\Big(\frac{\mu(B_{d-1})}{na_n}\Big)^k(v_1\ldots v_k)\prod_{j=1}^{k}\Big(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\Big)$$

$$\le \big(1+o(1)\big)\frac{1}{f_d f_{d-1}^k}\Big(\frac{\mu(A_{k,\theta})}{na_n}\Big)^k(v_1\ldots v_k)\prod_{j=1}^{k}\Big(\sum_{i=0}^{\infty}a_i e^{-v_j i/n}\Big)\,.$$

As in the case of the lower bound, Karamata's Tauberian theorem shows that

$$F_{n,A_{k,\theta}}(\theta_1\ldots\theta_k) \le \big(1+o(1)\big)\frac{1}{f_d f_{d-1}^k}\Big(\mu(A_{k,\theta})\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}\frac{a(n/v_j)}{a_n}$$

$$\to \frac{1}{f_d f_{d-1}^k}\Big(\mu(A_{k,\theta})\Gamma(1+\beta)\Big)^k\prod_{j=1}^{k}v_j^{-\beta}$$

as $n\to\infty$. Since this is true for every $d\ge 1$, we can let now $d\to\infty$ to obtain (6.12), and the proof of the theorem is complete. □

Remark 6.2. It follows immediately from Theorem 6.1 and continuity of the limiting Mittag-Leffler process that for the continuous process $(\tilde{S}_n)$ defined in (6.3), strong distributional convergence as in (6.1) also holds, either in $D[0,\infty)$ or in $C[0,\infty)$.

We use the strong distributional convergence obtained in Theorem 6.1 in the following proposition.

Proposition 6.3. *Under the assumptions of Theorem 6.1, let A be a measurable set with $0<\mu(A)<\infty$, such that (5.4) is satisfied, and suppose that the function f is supported by A. Define a probability measure on E by $\mu_n(\cdot)=\mu(\cdot\cap\{\varphi\le n\})/\mu(\{\varphi\le n\})$, where φ is the first entrance time of A. Let $0\le t_1<\cdots<t_H$, $H\ge 1$, and fix $L\in\mathbb{N}$ with $t_H\le L$. Then under μ_{nL}, the sequence $\big(S_{\lceil nt_h\rceil}(f)/a_n\big)_{h=1}^H$ converges weakly in $\mathbb{R}^H$ to the random vector $\big(\mu(f)\Gamma(1+\beta)M_\beta(t_h-T_\infty^{(L)})_+\big)_{h=1}^H$, where $T_\infty^{(L)}$ is a random variable independent of the Mittag-Leffler process M_β, with $P\big(T_\infty^{(L)}\le x\big)=(x/L)^{1-\beta},\ 0\le x\le L$.*

Proof. Since T preserves measure μ, for the duration of the proof we may and will modify the definition of S_n to $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$, $n = 1, 2, \ldots$. Fix $\theta_1, \ldots, \theta_H \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$. Since f is supported by A, we have, as $n \to \infty$,

$$\begin{aligned}
&\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right) \\
&\sim \mu_{nL}\left(A^c \cap \left\{\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right\}\right) \\
&= \mu(\varphi \le nL)^{-1}\sum_{m=1}^{nL}\mu\left(A_m \cap \left\{\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right\}\right) \\
&\sim \mu(\varphi \le nL)^{-1}\sum_{m=1}^{nL}\mu\left(A_m \cap T^{-m}\left\{\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{(\lceil nt_h\rceil - m)_+}(f) > \lambda\right\}\right) \\
&= \int_A \frac{1}{\mu(\varphi \le nL)}\sum_{m=1}^{nL}\widehat{T}^m I_{A_m} \cdot \mathbf{1}_{\{\sum_{h=1}^{H}\theta_h S_{(\lceil nt_h\rceil - m)_+}(f) > \lambda a_n\}} d\mu.
\end{aligned}$$

Note that the measure on E defined by $\eta(\cdot) = \int_\cdot K d\mu$ with K in (5.4) is necessarily a probability measure. We conclude by (5.4) that

$$\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right) \sim \sum_{m=1}^{nL}\eta\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{(\lceil nt_h\rceil - m)_+}(f) > \lambda\right) p_n(m)\,, \tag{6.18}$$

where $p_n(j) = \mu(A_j)/\sum_{m=1}^{nL}\mu(A_m)$, $j = 1, \ldots, nL$, is a probability mass function. Let $T_n^{(L)}$ be a discrete random variable with this probability mass function, independent of $S_{\lceil n\cdot\rceil}(f)$, which is, in turn, governed by the probability measure η. If we declare that $T_n^{(L)}$ is defined on some probability space $\big(\Omega_n, \mathcal{F}_n, P_n\big)$, then the right hand side of (6.18) becomes

$$\big(\eta \times P_n\big)\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{(\lceil nt_h\rceil - T_n^{(L)})_+}(f) > \lambda\right).$$

Since η is a probability measure absolutely continuous with respect to μ, it follows from the strong distributional convergence in Theorem 6.1 that

$$\frac{1}{a_n}S_{\lceil n\cdot\rceil}(f) \Rightarrow \mu(f)\Gamma(1+\beta)M_\beta(\cdot) \quad \text{in } D[0, L]\,, \tag{6.19}$$

when the law in the left hand side is computed with respect to η. On the other hand, by the regular variation of the wandering rate sequence and (4.5), for $x \in [0, L]$,

$$P_n\left(\frac{T_n^{(L)}}{n} \le x\right) = \sum_{m=1}^{\lceil nx\rceil} p_n(m) \sim \frac{w_{\lceil nx\rceil}}{w_{nL}} \sim \left(\frac{x}{L}\right)^{1-\beta}, \tag{6.20}$$

which is precisely the law of $T_\infty^{(L)}$. We can put together (6.19), (6.20), and independence between S_n and $T_n^{(L)}$ to obtain

$$\mu_{nL}\left(\frac{1}{a_n}\sum_{h=1}^{H}\theta_h S_{\lceil nt_h\rceil}(f) > \lambda\right) \to P\left(\mu(f)\Gamma(1+\beta)\sum_{h=1}^{H}\theta_h M_\beta((t_h - T_\infty^{(L)})_+) > \lambda\right)$$

for all continuity points λ of the right hand side, and all $\theta_1 \dots \theta_H \in \mathbb{R}$ by, e.g., Theorem 13.2.2 in Whitt (2002). This proves the proposition. □

7. Proof of the Main Theorem

In this section we prove Theorem 5.1. We start with several preliminary results. The first lemma explains the asymptotic relation (5.7).

Lemma 7.1. *Under the assumptions of Proposition 6.3, assume, additionally, that the set A supporting f is a Darling-Kac set. Let $0 < \alpha < 2$. If $1 < \alpha < 2$, assume, additionally, that $f \in L^2(\mu)$, and that either*
(i) A is a uniform set for $|f|$, or
(ii) f is bounded.
Then

$$\left(\int_E |S_n(f)|^\alpha d\mu\right)^{1/\alpha} \sim |\mu(f)|\, C_{\alpha,\beta}\, a_n w_n^{1/\alpha} \quad \text{as } n \to \infty, \tag{7.1}$$

and (5.7) holds.

Proof. It is an elementary calculation to check that (7.1) implies (5.7), so in the sequel we concentrate on checking (7.1). It follows from (4.5) and the fact that f is supported by A, that

$$\left(\int_E |S_n(f)|^\alpha d\mu\right)^{1/\alpha} = a_n\big(\mu(\varphi \le n)\big)^{1/\alpha} A_n^{(\alpha)} \sim a_n w_n^{1/\alpha} A_n^{(\alpha)}\,, \tag{7.2}$$

where $A_n^{(\alpha)} = (\int_E |S_n(f)/a_n|^\alpha d\mu_n)^{1/\alpha}$. Therefore, proving (7.1) reduces to checking that

$$A_n^{(\alpha)} \to |\mu(f)|\, C_{\alpha,\beta} \ \text{ as } n \to \infty. \tag{7.3}$$

If $\alpha = 1$ and f is nonnegative, then this follows by direct calculation, using the definition of $C_{\alpha,\beta}$. If f is not necessarily nonnegative, we can use the obvious bound $-S_n(|f|) \le S_n(f) \le S_n(|f|)$ together with the so-called Pratt lemma; see Pratt (1960), or Problem 16.4 (a) in Billingsley (1995).

It remains to consider the case $\alpha \in (0,1) \cup (1,2)$. Proposition 6.3 shows that $\big(A_n^{(\alpha)}\big)$ is the sequence of the α-norms of a weakly converging sequence, and the expression in the right hand side of (7.3) is easily seen to be the α-norm of the weak limit. Therefore, our statement will follow once we show that this weakly convergent sequence is uniformly integrable, which we proceed now to do.

Suppose first that $0 < \alpha < 1$. Recalling the relation (4.6) and the fact that T preserves measure μ, we see that

$$
\begin{aligned}
\sup_{n\geq 1} \int_E \left| \frac{S_n(f)}{a_n} \right| d\mu_n &= \sup_{n\geq 1} \frac{1}{a_n \mu(\varphi \leq n)} \int_E |S_n(f)| d\mu \\
&\leq \sup_{n\geq 1} \frac{n}{a_n \mu(\varphi \leq n)} \int_E |f| d\mu < \infty \,, \qquad (7.4)
\end{aligned}
$$

which proves uniformly integrability in this case.

Finally, we consider the case $1 < \alpha < 2$, when it is sufficient to prove that

$$
\sup_{n\geq 1} \int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n < \infty. \qquad (7.5)
$$

Under the assumption (i), since f is supported by A, we can use the duality relation (4.1) to write

$$
\begin{aligned}
\int_E S_n(f)^2 d\mu &= n \int_E f^2 d\mu + \sum_{k=1}^{n} \sum_{l=1, k\neq l}^{n} \int_E f \circ T^k f \circ T^l d\mu \\
&= n \int_E f^2 d\mu + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j f \cdot f d\mu \,,
\end{aligned}
$$

so that

$$
\int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n \leq \frac{n}{a_n^2 \mu(\varphi \leq n)} \int_E f^2 d\mu + \frac{2}{a_n^2 \mu(\varphi \leq n)} \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j |f| \cdot |f| d\mu.
$$

Clearly, $n / \big(a_n^2 \mu(\varphi \leq n)\big) \to 0$. Further, since A is uniform for $|f|$,

$$
\begin{aligned}
\frac{1}{a_n^2 \mu(\varphi \leq n)} \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \int_A \widehat{T}^j |f| \cdot |f| d\mu &\leq \frac{n}{a_n \mu(\varphi \leq n)} \int_A \frac{1}{a_n} \sum_{j=1}^{n} \widehat{T}^j |f| \cdot |f| d\mu \\
&\sim \mu(|f|)^2 \frac{n}{a_n \mu(\varphi \leq n)} \,.
\end{aligned}
$$

Using (4.6), we see that (7.5) follows. On the other hand, under the assumption (ii), the ratio $S_n(f)/S_n(\mathbf{1}_A)$ is bounded, hence for some finite $C > 0$,

$$
\sup_{n\geq 1} \int_E \left(\frac{S_n(f)}{a_n} \right)^2 d\mu_n \leq C \sup_{n\geq 1} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^2 d\mu_n.
$$

However, the Darling-Kac property of A means that it is uniform for $\mathbf{1}_A$, and so we are, once again, under the assumption (i). $\square$

In preparation for the proof of Theorem 5.1, we introduce a useful decomposition of the process $\mathbf{X}$ given in (2.1). We begin by decomposing the local Lévy measure ρ into a sum of two parts, corresponding to "large jumps" and "small jumps". Let

$$
\begin{aligned}
\rho_1(\cdot) &= \rho\big(\cdot \cap \{|x| > 1\}\big), \\
\rho_2(\cdot) &= \rho\big(\cdot \cap \{|x| \leq 1\}\big) \,,
\end{aligned}
$$

and let M_1, M_2 be independent homogeneous symmetric infinitely divisible random measures, without a Gaussian component, with the same control measure μ and local Lévy measures ρ_1, ρ_2 accordingly. Under the integrability assumptions (5.2), the stochastic processes $X_n^{(i)} = \int_E f \circ T^n(x) dM_i(x)$, $n = 1, 2, \ldots$, for $i = 1, 2$, are independent stationary infinitely divisible processes, and $X_n = X_n^{(1)} + X_n^{(2)}$, $n = 1, 2, \ldots$.

Our final lemma shows that, from the point of view of the central limit behavior in the case $0 < \alpha < 1$, the contribution of the process $\big(X_n^{(2)}\big)$, corresponding to the "small jumps", is negligible.

Lemma 7.2. *If* $0 < \alpha < 1$, *then*

$$\frac{1}{c_n} \sum_{k=1}^{n} X_k^{(2)} \xrightarrow{p} 0\,. \tag{7.6}$$

Proof. By Chebyshev's inequality, for any $\epsilon > 0$,

$$P\left(|\sum_{k=1}^{n} X_k^{(2)}| > \epsilon c_n \right) \leq \frac{n}{\epsilon c_n} E|X_1^{(2)}| \to 0$$

(since $c_n \in RV_{\beta+(1-\beta)/\alpha}$ implies $n/c_n \to 0$ in the case $0 < \alpha < 1$) as long as the expectation $E|X_1^{(2)}|$ is finite. Since for every $p_1 > p_0$ in (5.1) and $p_1 \geq 1$,

$$\int_E \int_{\mathbb{R}} |xf(s)| \mathbf{1}\big(|xf(s)| > 1\big)\, \rho_2(dx)\, \mu(ds) \leq \int_{-1}^{1} |x|^{p_1}\, \rho(dx) \int_E |f(s)|^{p_1}\, \mu(ds)\,,$$

the expectation is finite because, by (5.2), we can find p_1 as above such that $\int_E |f|^{p_1}\, d\mu < \infty$. □

Proof of Theorem 5.1. We start with proving the finite dimensional weak convergence, for which it enough to show the convergence

$$\frac{1}{c_n} \sum_{h=1}^{H} \theta_h \sum_{k=1}^{\lceil nt_h \rceil} X_k \Rightarrow |\mu(f)| \sum_{h=1}^{H} \theta_h Y_{\alpha,\beta}(t_h)$$

for all $H \geq 1$, $0 \leq t_1 < \cdots < t_H$, and $\theta_1 \ldots \theta_H \in \mathbb{R}$. Conditions for weak convergence of infinitely divisible random variables (see e.g. Theorem 15.14 in Kallenberg (2002)) simplify in this one-dimensional symmetric case to

$$\begin{aligned} &\int_E \left(\frac{1}{c_n} \sum_{h=1}^{H} \theta_h S_{\lceil nt_h \rceil}(f) \right)^2 \int_0^{rc_n/|\sum \theta_h S_{\lceil nt_h \rceil}(f)|} x\rho(x,\infty)\, dx\, d\mu \\ &\qquad \to \frac{r^{2-\alpha} C_\alpha}{2-\alpha} |\mu(f)|^\alpha \int_{[0,\infty)} \int_{\Omega'} \left| \sum_{h=1}^{H} \theta_h M_\beta((t_h - x)_+, \omega') \right|^\alpha P'(d\omega')\nu(dx) \end{aligned} \tag{7.7}$$

and

$$\int_E \rho \left(rc_n |\sum_{h=1}^{H} \theta_h S_{\lceil nt_h \rceil}(f)|^{-1}, \infty \right) d\mu \tag{7.8}$$

$$\to r^{-\alpha} C_\alpha \, |\mu(f)|^\alpha \int_{[0,\infty)} \int_{\Omega'} \left| \sum_{h=1}^H \theta_h M_\beta((t_h - x)_+, \omega') \right|^\alpha P'(d\omega')\nu(dx)$$

for every $r > 0$. Fix $L \in \mathbb{N}$ with $t_H \le L$ and $r > 0$.

Since the argument for (7.7) and the argument for (7.8) are very similar, we only prove (7.7). By Proposition 6.3 and Skorohod's embedding theorem, there is some probability space $(\Omega^*, \mathcal{F}^*, P^*)$ and random variables Y, Y_n, $n = 1, 2, \dots$ defined on that space such that, for every n, the law of Y_n coincides with the law of $a_n^{-1} \sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f)$ under μ_{nL}, the law of Y coincides with the law of $\mu(f)\Gamma(1+\beta) \sum_{h=1}^H \theta_h M_\beta((t_h - T_\infty^{(L)})_+)$ under P', and $Y_n \to Y$ P^*-a.s.

Introduce a function

$$\psi(y) = y^{-2} \int_0^{ry} x\rho(x, \infty)\, dx, \ y > 0\,,$$

so that the expression in the left hand side of (7.7) becomes

$$\int_E \psi\left(\frac{c_n}{|\sum_{h=1}^H \theta_h S_{\lceil nt_h \rceil}(f)|} \right) d\mu = \mu(\varphi \le nL)\, E^* \left[\psi\left(\frac{c_n}{a_n |Y_n|} \right) \right] .$$

By Karamata's theorem (see e.g. Theorem 0.6 in Resnick (1987)),

$$\psi(y) \sim \frac{r^2}{2-\alpha} \rho(ry, \infty) \quad \text{as } y \to \infty\,,$$

so that, as $n \to \infty$,

$$\begin{aligned} &\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|} \right) \\ &\sim \frac{r^2}{2-\alpha} \mu(\varphi \le nL) |Y_n|^\alpha \rho(rc_n a_n^{-1}, \infty) \\ &+ \frac{r^2}{2-\alpha} \mu(\varphi \le nL) \rho(rc_n a_n^{-1}, \infty) \left(\frac{\rho(rc_n a_n^{-1}|Y_n|^{-1}, \infty)}{\rho(rc_n a_n^{-1}, \infty)} - |Y_n|^\alpha \right). \end{aligned} \tag{7.9}$$

By (5.7), Lemma 7.1 and (4.5),

$$\rho(rc_n a_n^{-1}, \infty) \sim r^{-\alpha} C_\alpha \big(\Gamma(1+\beta)\big)^{-\alpha} \big(\mu(\varphi \le n)\big)^{-1} \quad \text{as } n \to \infty\,. \tag{7.10}$$

This, together with the basic properties of regularly varying functions of a negative index (see e.g. Proposition 0.5 Resnick (1987)), shows that the second term in the right hand side of (7.9) converges to 0. Therefore,

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|} \right) \to \frac{r^{2-\alpha}}{2-\alpha} C_\alpha L^{1-\beta} \left(\frac{|Y|}{\Gamma(1+\beta)} \right)^\alpha .$$

Integrating the limit yields

$$E^* \left[\frac{r^{2-\alpha}}{2-\alpha} C_\alpha\, L^{1-\beta} \left(\frac{|Y|}{\Gamma(1+\beta)} \right)^\alpha \right] = \frac{r^{2-\alpha}}{2-\alpha} C_\alpha\, L^{1-\beta}\, |\mu(f)|^\alpha E' \left[\sum_{h=1}^H \theta_h M_\beta((t_h - T_\infty^{(L)})_+) \right]^\alpha$$

$$= \frac{r^{2-\alpha} C_\alpha}{2-\alpha} |\mu(f)|^\alpha \int_{[0,\infty)} \int_{\Omega'} \left(\sum_{h=1}^{H} \theta_h M_\beta((t_h - x)_+, \omega') \right)^\alpha P'(d\omega')\nu(dx),$$

which is exactly the right hand side of (7.7). Therefore, in order to complete the proof of (7.7), we only need to justify taking the limit inside the integral. For this purpose we use, once again, Pratt's lemma. We need to exhibit random variables G_n, $n = 0, 1, 2, \ldots$ on $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le G_n \quad P^*\text{-a.s.,} \tag{7.11}$$

$$G_n \to G_0 \quad P^*\text{-a.s.,} \tag{7.12}$$

$$E^*G_n \to E^*G_0 \in [0, \infty). \tag{7.13}$$

We start with writing (using (7.10))

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_1 \frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n > a_n|Y_n|\}} + C_1 \frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n \le a_n|Y_n|\}},$$

where $C_1 > 0$ is a constant. Suppose first that $1 \le \alpha < 2$, and choose $0 < \xi < 2 - \alpha$. Then by the Potter bounds (see Proposition 0.8 in Resnick (1987)), for some constant $C_2 > 0$,

$$\frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n > a_n|Y_n|\}} \le C_2(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi})$$

for all n large enough. Further, since $y^2\psi(y) \to 0$ as $y \downarrow 0$, we have, for some constant $C_3 > 0$,

$$\frac{\psi\big(c_n a_n^{-1}|Y_n|^{-1}\big)}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n \le a_n|Y_n|\}} \le C_3 \left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})},$$

hence, for some constant $C_4 > 0$,

$$\mu(\varphi \le nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \le C_4 \left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})} \right) \tag{7.14}$$

for all n (large enough) and all realizations. We take

$$G_n = C_4 \left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \left(\frac{a_n}{c_n}\right)^2 \frac{|Y_n|^2}{\psi(c_n a_n^{-1})} \right) \quad n = 1, 2, \ldots,$$

$$G_0 = C_4(|Y|^{\alpha-\xi} + |Y|^{\alpha+\xi}).$$

Then (7.11) holds by construction, while (7.12) follows from the fact that

$$\left(\frac{a_n}{c_n}\right)^2 \frac{1}{\psi(c_n a_n^{-1})} \in RV_{(1-\beta)(1-2/\alpha)},$$

and $(1-\beta)(1-2/\alpha) < 0$. Keeping this in mind, and recalling that, by (7.5) (which holds also for $\alpha = 1$ under the assumptions of the theorem), $\sup_{n\ge1} E^*Y_n^2 < \infty$, we obtain the uniform integrability implying (7.13). This proves (7.7) in the case $1 \le \alpha < 2$.

If $0 < \alpha < 1$, then Lemma 7.2 allows us to assume, without loss of generality, that $\rho(x : |x| \leq 1) = 0$. Then ψ is bounded on $(0, 1]$, so that for some $C_5 > 0$,

$$\frac{\psi(c_n a_n^{-1}|Y_n|^{-1})}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n \leq a_n |Y_n|\}} \leq C_5 \frac{a_n}{c_n} \frac{|Y_n|}{\psi(c_n a_n^{-1})} ,$$

and the upper bound (7.14) is replaced with

$$\mu(\varphi \leq nL)\psi\left(\frac{c_n}{a_n|Y_n|}\right) \leq C_6 \left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + \frac{a_n}{c_n} \frac{|Y_n|}{\psi(c_n a_n^{-1})}\right) ,$$

for some $C_6 > 0$, where we now choose $0 < \xi < 1 - \alpha$. Since

$$\frac{a_n}{c_n} \frac{1}{\psi(c_n a_n^{-1})} \in RV_{(1-\beta)(1-1/\alpha)}$$

with $(1 - \beta)(1 - 1/\alpha) < 0$ and $\sup_{n\geq 1} E^*|Y_n| < \infty$ by (7.4), an argument similar to the case $1 \leq \alpha < 2$ applies here as well. A similar argument proves, in the case $0 < \alpha < 1$, the "positive" version described in Remark 5.4.

It remains to prove that the laws in the left hand side of (5.9) are tight in $D[0, L]$ for any fixed $L > 0$. By Theorem 13.5 of Billingsley (1999), it is enough to show that there exist $\gamma_1 > 1$, $\gamma_2 \geq 0$ and $B > 0$ such that

$$P\Big[\min\Big(\Big|\sum_{k=1}^{\lceil ns\rceil} X_k - \sum_{k=1}^{\lceil nr\rceil} X_k\Big|, \Big|\sum_{k=1}^{\lceil nt\rceil} X_k - \sum_{k=1}^{\lceil ns\rceil} X_k\Big|\Big) \geq \lambda c_n\Big] \leq \frac{B}{\lambda^{\gamma_2}}(t - r)^{\gamma_1}$$

for all $0 \leq r \leq s \leq t \leq L$, $n \geq 1$ and $\lambda > 0$. We start with a simple observation that, in the case $0 < \alpha < 1$, we may assume that the function f is bounded. To see that, note that we can always write $f = f\mathbf{1}_{|f|>M} + f\mathbf{1}_{|f|\leq M}$, and use the finite-dimensional convergence in (5.10) and the fact that $\mu\big(f\mathbf{1}_{|f|>M}\big) \to 0$ as $M \to \infty$.

Next, for any $0 < \alpha < 2$, if $0 < t - r < 1/n$, then the probability in the left hand side vanishes. If $X_n = X_n^{(1)} + X_n^{(2)}$, $n = 1, 2, \ldots$ be the decomposition described prior to Lemma 7.2. We start with the part corresponding to the "small jumps". Note that, by Lemma 7.2, this part is negligible if $0 < \alpha < 1$ (since we can apply the lemma to the supremum of the process). Therefore, we only consider the case $1 \leq \alpha < 2$, and prove that there exist $\gamma_1 > 1$, $\gamma_2 \geq 0$ and $B > 0$ such that for all $0 \leq s \leq t \leq L$, $n \geq 1$, $|t - s| \geq 1/n$ and $\lambda > 0$,

$$P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)}\Big| \geq \lambda c_n\Big) \leq \frac{B}{\lambda^{\gamma_2}}(t - s)^{\gamma_1} . \tag{7.15}$$

Note that the Lévy-Itô decomposition yields

$$\begin{aligned}\sum_{k=1}^{\lceil nt\rceil} X_k^{(2)} - \sum_{k=1}^{\lceil ns\rceil} X_k^{(2)} &\stackrel{d}{=} \int_E S_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dM_2 \\ &\stackrel{d}{=} \iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\leq\lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, d\bar{N}_2 + \iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n} xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\, dN_2 ,\end{aligned}$$

where N_2 is a Poisson random measure on $\mathbb{R}\times E$ with mean measure $\rho_2\times\mu$ and $\bar N_2 \equiv N_2-\big(\rho_2\times\mu\big)$. Therefore,

$$
\begin{aligned}
&P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil}X_k^{(2)}-\sum_{k=1}^{\lceil ns\rceil}X_k^{(2)}\Big|\geq\lambda c_n\Big)\\
&\leq P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\leq\lambda c_n}xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\,d\bar N_2\Big|\geq\lambda c_n\Big)+P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n}xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\,dN_2\Big|>0\Big). \qquad (7.16)
\end{aligned}
$$

It follows from (5.1) that for some constant $C_1>0$,

$$
\begin{aligned}
P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\leq\lambda c_n}xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\,d\bar N_2\Big|\geq\lambda c_n\Big)&\leq\frac{1}{\lambda^2c_n^2}E\left|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\leq\lambda c_n}xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\,d\bar N_2\right|^2\\
&=\frac{1}{\lambda^2c_n^2}\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|\leq\lambda c_n}\big|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\big|^2\,\rho_2(dx)\,d\mu\\
&\leq 4\int_E\left(\frac{S_{\lceil nt\rceil-\lceil ns\rceil}(f)}{\lambda c_n}\right)^2\int_0^{\lambda c_n/|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|}x\rho_2(x,\infty)\,dx\,d\mu\\
&\leq\frac{C_1}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu\,.
\end{aligned}
$$

Similarly, for some constant $C_2>0$,

$$
\begin{aligned}
P\Big(\Big|\iint_{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n}xS_{\lceil nt\rceil-\lceil ns\rceil}(f)\,dN_2\Big|>0\Big)&\leq P\big(N_2\{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n\}\geq1\big)\\
&\leq EN_2\{|xS_{\lceil nt\rceil-\lceil ns\rceil}(f)|>\lambda c_n\}\\
&=2\int_E\rho_2\big(\lambda c_n|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{-1},\infty\big)\,d\mu\\
&\leq\frac{C_2}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu\,,
\end{aligned}
$$

so that, in the notation of (7.3),

$$
P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil}X_k^{(2)}-\sum_{k=1}^{\lceil ns\rceil}X_k^{(2)}\Big|\geq\lambda c_n\Big)\leq\frac{C_1+C_2}{\lambda^{p_0}}\frac{1}{c_n^{p_0}}\int_E|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}\,d\mu
$$

$$
=\frac{C_1+C_2}{\lambda^{p_0}}\frac{\mu(\varphi\leq\lceil nt\rceil-\lceil ns\rceil)}{\mu(\varphi\leq n)}\left(\frac{a_{\lceil nt\rceil-\lceil ns\rceil}}{a_n}\right)^{p_0}\frac{(A^{(p_0)}_{\lceil nt\rceil-\lceil ns\rceil})^{p_0}}{c_n^{p_0}\mu(\varphi\leq n)^{-1}a_n^{-p_0}}\,.
$$

It follows from (7.5) that

$$
\sup_{n\geq1,0\leq s\leq t\leq L}A^{(p_0)}_{\lceil nt\rceil-\lceil ns\rceil}<\infty.
$$

Next, we may, if necessary, increase p_0 in (5.1) to achieve $p_0>\alpha$. In that case, the sequence $c_n^{p_0}\mu(\varphi\leq n)^{-1}a_n^{-p_0}\in RV_{(1-\beta)(p_0/\alpha-1)}$ diverges to infinity, so for some constant $C_3>0$,

$$
\frac{1}{c_n^{p_0}}\int_E|S_{\lceil nt\rceil-\lceil ns\rceil}(f)|^{p_0}d\mu\leq C_3\frac{\mu(\varphi\leq\lceil n(t-s)\rceil)}{\mu(\varphi\leq n)}\left(\frac{a_{\lceil n(t-s)\rceil}}{a_n}\right)^{p_0}.
$$

By the regular variation and the constraint $t-s\geq 1/n$, for every $0<\eta<\min(\beta,1-\beta)$, there is $C_4>0$, such that

$$\frac{\mu(\varphi\leq \lceil n(t-s)\rceil)}{\mu(\varphi\leq n)}\leq C_4\left(\frac{\lceil n(t-s)\rceil}{n}\right)^{1-\beta-\eta}\leq 2^{1-\beta-\eta}\,C_4\,(t-s)^{1-\beta-\eta}\,,$$

$$\frac{a_{\lceil n(t-s)\rceil}}{a_n}\leq 2^{\beta-\eta}\,C_4\,(t-s)^{\beta-\eta}\,.$$

Therefore, for some constant $C_5>0$,

$$P\Big(\Big|\sum_{k=1}^{\lceil nt\rceil}X_k^{(2)}-\sum_{k=1}^{\lceil ns\rceil}X_k^{(2)}\Big|\geq\lambda c_n\Big)\leq C_5\frac{1}{\lambda^{p_0}}(t-s)^{1+(p_0-1)\beta-(1+p_0)\eta}.$$

Since $p_0>\alpha\geq 1$, we can choose $\eta>0$ so small that $1+(p_0-1)\beta-(1+p_0)\eta>0$. This establishes (7.15).

Next, we take up the process $\big(X_n^{(1)}\big)$. Lévy-Itô decomposition and the symmetry of the Lévy measure ρ_1 allow us to to write, for any $K>0$,

$$\begin{aligned}\frac{1}{c_n}\sum_{k=1}^{\lceil nt\rceil}X_k^{(1)}&\overset{d}{=}\frac{1}{c_n}\sum_{k=1}^{\lceil nt\rceil}\iint_{|xf_k|\leq Kc_na_n^{-1}}xf_k\,d\bar N_1+\frac{1}{c_n}\sum_{k=1}^{\lceil nt\rceil}\iint_{|xf_k|>Kc_na_n^{-1}}xf_k\,dN_1\\&:=Z_n^{(1,K)}(t)+Z_n^{(2,K)}(t),\end{aligned}$$

where N_1 and $\bar N_1$ are as above. Here we first show that or any $\epsilon>0$,

$$\lim_{K\to\infty}\limsup_{n\to\infty}P\big(\sup_{0\leq t\leq L}\big|Z_n^{(2,K)}(t)\big|\geq\epsilon\big)=0\,. \tag{7.17}$$

Consider first the case $1<\alpha<2$. Choose $0<\tau\leq 2-\alpha$, and define

$$\kappa(w)=\begin{cases}1 & \text{if } 0\leq w<1\\ w^{-(\alpha+\tau)} & \text{if } w\geq 1,\end{cases}$$

$$g(w)=\big((w+1)\kappa(w)\big)^{-1},\quad w\geq 0\,.$$

Since $2g(w)/g(u)\geq 1$ for $0\leq u\leq w$, we have

$$\begin{aligned}P\big(\sup_{0\leq t\leq L}\big|Z_n^{(2,K)}(t)\big|\geq\epsilon\big)&\leq P\left(\iint_{\mathbb{R}\times E}|x|\sum_{k=1}^{nL}|f|\circ T^k\,\mathbf{1}\big(|x||f|\circ T^k>Kc_na_n^{-1}\big)\,dN_1\geq\epsilon c_n\right)\\&=P\left(2\iint_{\mathbb{R}\times E}|x|\sum_{k=1}^{nL}|f|\circ T^k\,g\big(|f|\circ T^k\big)\frac{1}{g\big(Kc_na_n^{-1}/|x|\big)}\,dN_1\geq\epsilon c_n\right)\\&\leq\frac{2}{\epsilon}c_n^{-1}E\left(\iint_{\mathbb{R}\times E}|x|\sum_{k=1}^{nL}|f|\circ T^k\,g\big(|f|\circ T^k\big)\frac{1}{g\big(Kc_na_n^{-1}/|x|\big)}\,dN_1\right)\\&\leq C_1\,nc_n^{-1}\int_1^\infty x\big(Kc_na_n^{-1}/x+1\big)\kappa\big(Kc_na_n^{-1}/x\big)\,\rho(dx)\,,\end{aligned}$$

where $C_1>0$ is another constant. It is now straightforward to check that for some constant $C_2>0$,

$$\limsup_{n\to\infty}P\big(\sup_{0\leq t\leq L}\big|Z_n^{(2,K)}(t)\big|\geq\epsilon\big)\leq C_2K^{-(\alpha-1)}\,.$$

This implies (7.17).

On the other hand, let $0 < \alpha \le 1$. Recall that we are assuming that the function f is now bounded. We have

$$\begin{aligned} P\big(\sup_{0\le t\le L}|Z_n^{(2,K)}(t)|\ge\epsilon\big) &\le P\big(\max_{k=1,\dots,nL} N_1\big\{(x,s)\,:\,|xf_k(s)|>Kc_na_n^{-1}\big\}\ge 1\big)\\ &\le EN_1\big\{(x,s)\,:\,|x|\max_{k=1,\dots,nL}|f_k|>Kc_na_n^{-1}\big\}\\ &=2\int_E\rho_1\left(\frac{Kc_na_n^{-1}}{\max_{k=1,\dots,nL}|f_k|},\infty\right)d\mu\,. \end{aligned}$$

If we denote $\| f \| = \sup_{x\in E}|f(x)|<\infty$, then we can use once again Potter's bounds to see that for some constant $C_1>0$ and $0<\xi<\alpha$,

$$\frac{\rho_1\left(Kc_na_n^{-1}(\max_k|f_k|)^{-1},\infty\right)}{\rho_1(c_na_n^{-1},\infty)}\le C_1\left(\left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha-\xi}+\left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha+\xi}\right).$$

Therefore by (4.5), (5.7) and the fact that f is supported by A, for some constant $C_2>0$,

$$\begin{aligned} P\big(\sup_{0\le t\le L}|Z_n^{(2,K)}(t)|\ge\epsilon\big) &\le 2C_1\rho_1(c_na_n^{-1},\infty)\int_E\left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha-\xi}+\left(\frac{1}{K}\max_{k=1,\dots,nL}|f_k|\right)^{\alpha+\xi}d\mu\\ &\le 2C_1\rho_1(c_na_n^{-1},\infty)\left(\left(\frac{\| f \|}{K}\right)^{\alpha-\xi}+\left(\frac{\| f \|}{K}\right)^{\alpha+\xi}\right)\mu(\varphi\le nL)\\ &\le C_2\left(\left(\frac{\| f \|}{K}\right)^{\alpha-\xi}+\left(\frac{\| f \|}{K}\right)^{\alpha+\xi}\right), \end{aligned}$$

and (7.17) follows.

It remains to consider the processes $\{Z_n^{(1,K)}(t), 0\le t\le L\}$, $n=1,2,\dots$ for a fixed $K>0$. In the sequel we drop the superscript K for notational convenience. We will show that exist $\gamma_1>1$, and $B>0$ such that for all $0\le s<t\le L$, $n\ge 1$, $t-s\ge 1/n$ and $\lambda>0$,

$$P(|Z_n^{(1)}(t)-Z_n^{(1)}(s)|\ge\lambda)\le\frac{B}{\lambda^2}(t-s)^{\gamma_1}\,. \tag{7.18}$$

Indeed, by Chebyshev's inequality and the fact that f is supported by A, we see that

$$\begin{aligned} P\big(|Z_n^{(1)}(t)-Z_n^{(1)}(s)|\ge\lambda\big) &\le\frac{1}{\lambda^2c_n^2}E\left|\sum_{k=1}^{\lceil nt\rceil-\lceil ns\rceil}\iint_{|xf_k|\le Kc_na_n^{-1}}xf_k\,d\bar N_1\right|^2\\ &\le\frac{2}{\lambda^2c_n^2}\sum_{k=1}^{\lceil n(t-s)\rceil}\sum_{l=1}^{\lceil n(t-s)\rceil}\int_E|f_kf_l|\int_0^{Kc_na_n^{-1}/|f_k|\vee|f_l|}x^2\,\rho_1(dx)\,d\mu\,. \end{aligned}$$

It follows from the Potter bounds and the fact that ρ_1 does not assigns mass to the interval $(0,1)$ that for any $0<\xi<2-\alpha$ there is $C>0$ such that for all $a>0$ large enough and all $r>0$,

$$\frac{\int_0^{ra}x^2\,\rho_1(dx)}{\int_0^a x^2\,\rho_1(dx)}\le C\big(r^{2-\alpha-\xi}\vee r^{2-\alpha+\xi}\big)\,.$$

Therefore, for all n large enough, for some constant $C_1 > 0$,

$$P\big(|Z_n^{(1)}(t) - Z_n^{(1)}(s)| \geq \lambda\big) \leq \frac{C_1}{\lambda^2 c_n^2} \sum_{k=1}^{\lceil n(t-s)\rceil} \sum_{l=1}^{\lceil n(t-s)\rceil} \int_E \frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{2-\alpha-\xi}}\, d\mu \int_0^{c_n a_n^{-1}} x^2\, \rho_1(dx)$$
$$+ \frac{C_1}{\lambda^2 c_n^2} \sum_{k=1}^{\lceil n(t-s)\rceil} \sum_{l=1}^{\lceil n(t-s)\rceil} \int_E \frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{2-\alpha+\xi}}\, d\mu \int_0^{c_n a_n^{-1}} x^2\, \rho_1(dx)\,.$$

Note that by Karamata's theorem, (4.5) and the definition (5.5) of the normalizing sequence (c_n), there is $C_2 > 0$ such that

$$\int_0^{c_n a_n^{-1}} x^2\, \rho_1(dx) \leq C_2\, \frac{c_n^2}{n a_n}\,.$$

If $1 < \alpha < 2$, we impose also the constraint $\xi < \alpha - 1$, and use the relation

$$\frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{2-\alpha\pm\xi}} = \big(|f_k| \wedge |f_l|\big)\big(|f_k| \vee |f_l|\big)^{\alpha-1\mp\xi}\,, \tag{7.19}$$

so that

$$\frac{1}{c_n^2} \sum_{k=1}^{\lceil n(t-s)\rceil} \sum_{l=1}^{\lceil n(t-s)\rceil} \int_E \frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{2-\alpha\pm\xi}}\, d\mu \int_0^{c_n a_n^{-1}} x^2\, \rho_1(dx)$$
$$\leq C_2 \frac{1}{n a_n} \sum_{k=1}^{\lceil n(t-s)\rceil} \sum_{l=1}^{\lceil n(t-s)\rceil} \int_E \big(|f_k| \wedge |f_l|\big)\big(|f_k| \vee |f_l|\big)^{\alpha-1\mp\xi}\, d\mu$$
$$\leq 2C_2 \frac{1}{n a_n} \Bigg[\lceil n(t-s)\rceil \int_E |f|^{\alpha\mp\xi}\, d\mu$$
$$+ \sum_{k=1}^{\lceil n(t-s)\rceil-1} \sum_{l=k+1}^{\lceil n(t-s)\rceil} \left(\int_E |f_l||f_k|^{\alpha-1\mp\xi}\, d\mu + \int_E |f_k||f_l|^{\alpha-1\mp\xi}\, d\mu\right)\Bigg]$$
$$:= J_n(1) + J_n(2) + J_n(3)\,.$$

The fact that $t - s > 1/n$ and (a_n) is regularly varying with the positive exponent β, shows that for any $1 < \gamma_1 < 1 + \beta$ there is some constant $C_3 > 0$, such that for all $n = 1, 2, \ldots$,

$$J_n(1) \leq C_3 (t-s)^{\gamma_1}\,.$$

Next, by the duality relation (4.1),

$$J_n(2) \leq \frac{4C_2}{a_n}(t-s) \sum_{k=1}^{\lceil n(t-s)\rceil} \int_E |f_k||f|^{\alpha-1\mp\xi}\, d\mu$$
$$= \frac{4C_2}{a_n}(t-s) \int_A |f| \left(\sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k |f|^{\alpha-1\mp\xi}\right) d\mu\,.$$

If f is bounded, then by the Darling-Kac property of the set A we have, for some constants C_4, $C_5 > 0$,

$$J_n(2) \leq C_4 (t-s) \frac{a_{\lceil n(t-s)\rceil}}{a_n} \mu(|f|) \leq C_5 (t-s)^{\gamma_1}, \quad 1 < \gamma_1 < 1 + \beta\,,$$

by the regular variation of (a_n). If, on the other hand, A is a uniform set for $|f|$, then we can write

$$\sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k |f|^{\alpha-1\mp\xi} \leq \sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k \mathbf{1}_A + \sum_{k=1}^{\lceil n(t-s)\rceil} \widehat{T}^k |f|\,,$$

and obtain the same bound on J_2 by using both the Darling-Kac property and the uniform property of the set A. A similar argument shows that, for some constant $C_6 > 0$ we also have

$$J_n(3) \leq C_6 (t-s)^{\gamma_1}, \quad 1 < \gamma_1 < 1+\beta\,,$$

which proves (7.18) in the case $1 < \alpha < 2$.

Finally, for $0 < \alpha \leq 1$ the same argument works, if we replace the relation (7.19) by

$$\frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{1+\xi}} \leq \big(|f_k| \wedge |f_l|\big)^{1-\xi}, \quad \frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{1-\xi}} = \big(|f_k| \wedge |f_l|\big)\big(|f_k| \vee |f_l|\big)^{\xi}\,,$$

respectively if $\alpha = 1$, and

$$\frac{|f_k f_l|}{(|f_k| \vee |f_l|)^{2-\alpha\pm\xi}} \leq \big(|f_k| \wedge |f_l|\big)^{\alpha\pm\xi}$$

if $0 < \alpha < 1$. This proves (7.18) in all cases and, hence, completes the proof of the theorem. □

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