

**FUNCTIONAL CENTRAL LIMIT THEOREM FOR NEGATIVELY
DEPENDENT HEAVY-TAILED STATIONARY INFINITELY DIVISIBLE
PROCESSES GENERATED BY CONSERVATIVE FLOWS**

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ABSTRACT. We prove a functional central limit theorem for partial sums of symmetric stationary long range dependent heavy tailed infinitely divisible processes with a certain type of negative dependence. Previously only positive dependence could be treated. The negative dependence involves cancellations of the Gaussian second order. This leads to new types of limiting processes involving stable random measures, due to heavy tails, Mittag-Leffler processes, due to long memory, and Brownian motions, due to the Gaussian second order cancellations.

1. INTRODUCTION

Let $\mathbf{X} = (X_1, X_2, \dots)$ be a discrete time stationary stochastic process; depending on notational convenience we will sometimes allow the time index to extend to the entire \mathbb{Z} . Assume that \mathbf{X} is symmetric (i.e. that $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$) and that the marginal law of X_1 is in the domain of attraction of an α -stable law, $0 < \alpha < 2$. That is,

$$(1.1) \quad P(|X_1| > \cdot) \in RV_{-\alpha} \text{ at infinity};$$

see Feller (1971) or Resnick (1987). Here and elsewhere in this paper we use the notation RV_p for the set of functions of regular variation with exponent $p \in \mathbb{R}$. If the process satisfies a functional central limit theorem, then a statement of the type

$$(1.2) \quad \left(\frac{1}{c_n} \sum_{k=1}^{\lfloor nt \rfloor} X_k, 0 \leq t \leq 1 \right) \Rightarrow \left(Y(t), 0 \leq t \leq 1 \right)$$

holds, with (c_n) a positive sequence growing to infinity, and $\mathbf{Y} = \left(Y(t), 0 \leq t \leq 1 \right)$ a non-degenerate (non-deterministic) process. The convergence is either weak convergence in the appropriate topology on $D[0, 1]$ or just convergence in finite dimensional distributions. The heavy tails in (1.1) will necessarily affect the order of magnitude of the normalizing sequence (c_n) and the nature of the limiting process \mathbf{Y} . The latter process is, under mild assumptions, self-similar, with stationary increments; see Lamperti (1962) and Embrechts and Maejima (2002). If the process \mathbf{X} is long range dependent, then both the sequence (c_n) and the limiting process \mathbf{Y} may be affected by the length of the memory as well.

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A new class of central limit theorems for long range dependent stationary processes with heavy tails was introduced in Owada and Samorodnitsky (2015). In that paper the process \mathbf{X} was a stationary infinitely divisible process given in the form

$$(1.3) \quad X_n = \int_E f \circ T^n(s) dM(s), \quad n = 1, 2, \dots,$$

where M is a symmetric homogeneous infinitely divisible random measure on a measurable space (E, \mathcal{E}) , without a Gaussian component, with control measure μ , $f : E \rightarrow \mathbb{R}$ is a measurable function, and $T : E \rightarrow E$ a measurable map, preserving the measure μ ; precise definitions of these and following notions are below. The regularly varying tails, in the sense of (1.1), of the process \mathbf{X} are due to the random measure M , while the long memory is due to the ergodic-theoretical properties of the map T , assumed to be conservative and ergodic. In the model considered in Owada and Samorodnitsky (2015) the length of the memory could be quantified by a single parameter $0 \leq \beta \leq 1$ (the larger β is, the longer the memory). Under the crucial assumption that

$$(1.4) \quad \mu(f) := \int_E f(s) \mu(ds) \neq 0$$

(with the integral being well defined), it turns out that the normalizing sequence (c_n) is regularly varying with exponent $H = \beta + (1 - \beta)/\alpha$, and the limiting process \mathbf{Y} is, up to a multiplicative factor of $\mu(f)$, the β -Mittag-Leffler fractional symmetric α -stable (SaS) motion defined by

$$(1.5) \quad Y_{\alpha, \beta}(t) = \int_{\Omega' \times [0, \infty)} M_\beta((t-s)_+, \omega') dZ_{\alpha, \beta}(\omega', s), \quad t \geq 0,$$

where $Z_{\alpha, \beta}$ is a SaS random measure on $\Omega' \times [0, \infty)$ with control measure $\mathbf{P}' \times \nu_\beta$. Here ν_β is a measure on $[0, \infty)$ given by $\nu_\beta(dx) = (1 - \beta)x^{-\beta} dx$, $x > 0$, and M_β is a Mittag-Leffler process defined on a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ (all the notions will be defined momentarily). The random measure $Z_{\alpha, \beta}$ and the process $Y_{\alpha, \beta}$, are defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

The β -Mittag-Leffler fractional SaS motion is a self-similar process with Hurst exponent H as above. Note that

$$H \in \begin{cases} (1, 1/\alpha] & \text{if } 0 < \alpha < 1, \\ \{1\} & \text{if } \alpha = 1, \\ (1/\alpha, 1) & \text{if } 1 < \alpha < 2, \end{cases}$$

which is the top part of the feasible region

$$H \in \begin{cases} (0, 1/\alpha] & \text{if } 0 < \alpha < 1, \\ (0, 1] & \text{if } \alpha = 1, \\ (0, 1) & \text{if } 1 < \alpha < 2 \end{cases}$$

for the Hurst exponent of a self-similar SaS process with stationary increments; see Samorodnitsky and Taqqu (1994). This is usually associated with positive dependence both in the increments of the process \mathbf{Y} itself and the original process \mathbf{X} in the functional central limit theorem (1.2); the best-known example is that of the Fractional Brownian motion, the Gaussian self-similar process with stationary increments. For the latter process the range of H is the interval $(0, 1)$, and positive dependence corresponds to the range $H \in (1/2, 1)$.

In the Gaussian case of the Fractional Brownian motion, negative dependence ($0 < H < 1/2$) is often related to ‘‘cancellations’’ between the observations; the statement

$$\sum_{n=-\infty}^{\infty} \text{Cov}(X_0, X_n) = 0$$

is trivially true if the process \mathbf{X} is the increment process of the Fractional Brownian motion with $H < 1/2$, and the same is true in most of the situations in (1.2), when the limit process is the Fractional Brownian motion with $H < 1/2$.

In the infinite variance case considered in Owada and Samorodnitsky (2015), “cancellations” appear when the integral $\mu(f)$ in (1.4) vanishes. It is the purpose of the present paper to take a first step towards understanding this case, when the long memory due to the map T interacts with the negative dependence due to the cancellations. We use the cautious formulation above because with the integral $\mu(f)$ vanishing, the second order behaviour of f becomes crucial, and in this paper we only consider a Gaussian type of second order behaviour. Furthermore, even in this case our assumptions on the space E and map T in (1.3) are more restrictive than those in Owada and Samorodnitsky (2015). Nonetheless, we still obtain an entirely new class of functional limit theorems and limiting fractional S α S motions.

This paper is organized as follows. In Section 2 we provide the necessary background on infinitely divisible and stable processes and integrals, and related notions, used in this paper. In Section 3 we describe a new class of self-similar S α S processes with stationary increments, some of which will appear as limits in the functional central limit theorem proved later. Certain facts on general state space Markov chains, needed to define and treat the model considered in the paper, are in Section 4. The main result of the paper is stated and proved in Section 5. Finally, Section 6 is an appendix containing bounds on fractional moments of infinitely divisible random variables needed elsewhere in the paper.

We will use several common abbreviations throughout the paper: *ss* for “self-similar”, *sssi* for “self-similar, with stationary increments”, and *S α S* for “symmetric α -stable”.

2. BACKGROUND

In this paper we will work with symmetric infinitely divisible processes defined as integrals of deterministic functions with respect to homogeneous symmetric infinitely divisible random measures, the symmetric stable processes and measures forming a special case. Let (E, \mathcal{E}) be a measurable space. Let μ be a σ -finite measure on E , it will be assumed to be infinite in most of the paper, but at the moment it is not important. Let ρ be a one-dimensional symmetric Lévy measure, i.e. a σ -finite measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} \min(1, x^2) \rho(dx) < \infty.$$

If $\mathcal{E}_0 = \{A \in \mathcal{E} : \mu(A) < \infty\}$, then a homogeneous symmetric infinitely divisible random measure M on (E, \mathcal{E}) with control measure μ and local Lévy measure ρ is a stochastic process $(M(A), A \in \mathcal{E}_0)$ such that

$$\mathbf{E} e^{iuM(A)} = \exp \left\{ -\mu(A) \int_{\mathbb{R}} (1 - \cos(ux)) \rho(dx) \right\} \quad u \in \mathbb{R}$$

for every $A \in \mathcal{E}_0$. The random measure M is independently scattered and σ -additive in the usual sense of random measures; see Rajput and Rosiński (1989). The random measure is symmetric α -stable (S α S), $0 < \alpha < 2$, if

$$\rho(dx) = \alpha |x|^{-(\alpha+1)} dx.$$

If M has a control measure μ and a local Lévy measure ρ , and $g : E \rightarrow \mathbb{R}$ is a measurable function such that

$$(2.1) \quad \int_E \int_{\mathbb{R}} \min(1, x^2 g(s)^2) \rho(dx) \mu(ds) < \infty,$$

then the integral $\int_E g dM$ is well defined and is a symmetric infinitely divisible random variable. In the α -stable case the integral is a S α S random variable and the integrability condition (2.1) reduces to the L^α condition

$$(2.2) \quad \int_E |g(s)|^\alpha \mu(ds) < \infty.$$

We remark that in the α -stable case it is common to use the α -stable version of the control measure; it is just a scaled version $C_\alpha \mu$ of the control measure μ , with C_α being the α -stable tail constant given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx \right)^{-1} = \begin{cases} (1-\alpha)/(\Gamma(2-\alpha) \cos(\pi\alpha/2)) & \text{if } \alpha \neq 1, \\ 2/\pi & \text{if } \alpha = 1. \end{cases}$$

See Rajput and Rosiński (1989) for this and the subsequent properties of infinitely divisible processes and integrals.

We will consider symmetric infinitely divisible stochastic processes (without a Gaussian component) \mathbf{X} given in the form

$$X(t) = \int_E g(t, s) M(ds), \quad t \in \mathcal{T},$$

where \mathcal{T} is a parameter space, and $g(t, \cdot)$ is, for each $t \in \mathcal{T}$, a measurable function satisfying (2.1). The (function level) Lévy measure of the process \mathbf{X} is given by

$$(2.3) \quad \kappa_{\mathbf{X}} = (\rho \times \mu) \circ K^{-1},$$

with $K : \mathbb{R} \times E \rightarrow \mathbb{R}^{\mathcal{T}}$ given by $K(x, s) = x(g(t, s), t \in \mathcal{T})$, $s \in E$, $x \in \mathbb{R}$.

An important special case for us is that of $\mathcal{T} = \mathbb{N}$ and

$$(2.4) \quad g(n, s) = f \circ T^n(s), \quad n = 1, 2, \dots,$$

where $f : E \rightarrow \mathbb{R}$ is a measurable function satisfying (2.1), and $T : E \rightarrow E$ a measurable map, preserving the control measure μ . In this case we obtain the process exhibited in (1.3). It is elementary to check that in this case the Lévy measure $\kappa_{\mathbf{X}}$ in (2.3) is invariant under the left shift θ on $\mathbb{R}^{\mathbb{N}}$,

$$\theta(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

In particular, the process \mathbf{X} is, automatically, stationary. There is a close relation between certain ergodic-theoretical properties of the shift operator θ with respect to the Lévy measure $\kappa_{\mathbf{X}}$ (or of the map T with respect to the control measure μ) and certain distributional properties of the stationary process \mathbf{X} ; we will discuss these below.

Switching gears a bit, we now recall a crucial notion needed for the main result of this paper as well as for the presentation of the new class of fractional S α S noises in the next section. For $0 < \beta < 1$, let $(S_\beta(t))$ be a β -stable subordinator, a Lévy process with increasing sample paths, satisfying $\mathbf{E}e^{-\theta S_\beta(t)} = \exp\{-t\theta^\beta\}$ for $\theta \geq 0$ and $t \geq 0$. The *Mittag-Leffler process* is its inverse process given by

$$(2.5) \quad M_\beta(t) := S_\beta^{\leftarrow}(t) = \inf\{u \geq 0 : S_\beta(u) \geq t\}, \quad t \geq 0.$$

It is a continuous process with nondecreasing sample paths. Its marginal distributions are the Mittag-Leffler distributions, whose Laplace transform is finite for all real values of the argument and is given by

$$(2.6) \quad \mathbf{E} \exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \quad \theta \in \mathbb{R};$$

see Proposition 1(a) in Bingham (1971). Using (2.6), the definition of the Mittag-Leffler process can be naturally extended to the boundary cases $\beta = 0$ and $\beta = 1$. We do this by setting $M_0(0) = 0$ and $M_0(t) = E_{\text{st}}, t > 0$, with E_{st} a standard exponential random variable, and $M_1(t) = t, t \geq 0$.

The Mittag-Leffler process is self-similar with exponent β . It has neither stationary nor independent increments (apart from the degenerate case $\beta = 1$).

3. A NEW CLASS OF SELF-SIMILAR S α S PROCESSES WITH STATIONARY INCREMENTS

In this section we introduce a new class of self-similar S α S processes with stationary increments. A subclass of these processes will appear as a weak limit in the functional central limit theorem in Section 5, but the entire class has intrinsic interest. Furthermore, we anticipate that other members of the class will appear in other limit theorems. The processes in this class are defined, up to a scale factor, by 3 parameters, α, β and γ :

$$0 < \alpha < \gamma \leq 2, \quad 0 \leq \beta \leq 1.$$

We proceed with a setup similar to the one in (1.5). Define a σ -finite measure on $[0, \infty)$ by

$$\nu_\beta(dx) = \begin{cases} (1 - \beta)x^{-\beta} dx, & 0 \leq \beta < 1, \\ \delta_0(dx), & \beta = 1 \end{cases}$$

(δ_0 being the point mass at zero). Let $(\Omega', \mathcal{F}', \mathbf{P}')$ be a probability space, and let $(S_\gamma(t, \omega'))$ be a S γ S Lévy motion and $(M_\beta(t, \omega'))$ be an independent β -Mittag-Leffler process, both defined on $(\Omega', \mathcal{F}', \mathbf{P}')$. We define

$$(3.1) \quad Y_{\alpha, \beta, \gamma}(t) := \int_{\Omega' \times [0, \infty)} S_\gamma(M_\beta((t-x)_+, \omega'), \omega') dZ_{\alpha, \beta}(\omega', x), \quad t \geq 0,$$

where $Z_{\alpha, \beta}$ is a S α S random measure on $\Omega' \times [0, \infty)$ with control measure $\mathbf{P}' \times \nu_\beta$; we use the α -stable version of the control measure (see the remark following (2.2)).

Remark 3.1. The boundary cases $\beta = 0$ and $\beta = 1$ are somewhat special. In the case $\beta = 0$ we interpret the process in (3.1) as

$$Y_{\alpha, 0, \gamma}(t) = \int_{\Omega' \times [0, \infty)} S_\gamma(E_{\text{st}}(\omega'), \omega') \mathbf{1}(x < t) dZ_{\alpha, 0}(\omega', x), \quad t \geq 0,$$

where E_{st} is a standard exponential random variable defined on $(\Omega', \mathcal{F}', \mathbf{P}')$, independent of $(S_\gamma(t, \omega'))$, while $Z_{\alpha, 0}$ is a S α S random measure on $\Omega' \times [0, \infty)$ with control measure $\mathbf{P}' \times \text{Leb}$. It is elementary to see that this process is a S α S Lévy motion itself, and the dependence on γ is only through a multiplicative constant, equal to

$$(\mathbf{E}' |S_\gamma(1)|^\alpha \mathbf{E}'(E_{\text{st}}^{\alpha/\gamma}))^{1/\alpha}.$$

In the second boundary case $\beta = 1$ the variable x in the integral becomes redundant, and we interpret the process in (3.1) as

$$Y_{\alpha, 1, \gamma}(t) = \int_{\Omega'} S_\gamma(t, \omega') dZ_{\alpha, 1}(\omega'), \quad t \geq 0,$$

where $Z_{\alpha, 1}$ is a S α S random measure on Ω' with control measure \mathbf{P}' . This process is, distributionally, sub-stable, with an alternative representation

$$Y_{\alpha, 1, \gamma}(t) = W^{1/\gamma} S_\gamma(t), \quad t \geq 0,$$

with W a positive strictly α/γ -stable random variable independent of the S γ S Lévy motion (S_γ) , both of which are now defined on $(\Omega, \mathcal{F}, \mathbf{P})$; see Section 3.8 in Samorodnitsky and Taqqu (1994).

Proposition 3.2. $(Y_{\alpha,\beta,\gamma}(t))$ is a well defined H -sssi $S\alpha S$ process with

$$(3.2) \quad H = \frac{\beta}{\gamma} + \frac{1-\beta}{\alpha}.$$

Proof. The boundary cases $\beta = 0$ and $\beta = 1$ are discussed in Remark 3.1, so we will consider now the case $0 < \beta < 1$.

The argument is similar to that of Theorem 3.1 in Owada and Samorodnitsky (2015). To see that $(Y_{\alpha,\beta,\gamma}(t))$ is well defined, notice that for $t > 0$, by self-similarity of (S_γ) ,

$$\begin{aligned} & \mathbf{E}' \int_0^\infty |S_\gamma(M_\beta((t-x)_+))|^\alpha (1-\beta)x^{-\beta} dx \\ &= \mathbf{E}' |S_\gamma(1)|^\alpha \mathbf{E}' \int_0^\infty M_\beta((t-x)_+)^{\alpha/\gamma} (1-\beta)x^{-\beta} dx \\ &\leq \mathbf{E}' |S_\gamma(1)|^\alpha \mathbf{E}' M_\beta(t)^{\alpha/\gamma} t^{1-\beta} < \infty; \end{aligned}$$

the finiteness of $\mathbf{E}' |S_\gamma(1)|^\alpha$ follows since $\alpha < \gamma$. Next, let $c > 0$, $t_1, \dots, t_k > 0$, and $\theta_1, \dots, \theta_k \in \mathbb{R}$. We have

$$\begin{aligned} \mathbf{E}' \exp \left\{ i \sum_{j=1}^k \theta_j Y_{\alpha,\beta,\gamma}(ct_j) \right\} &= \exp \left\{ - \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((ct_j-x)_+)) \right|^\alpha (1-\beta)x^{-\beta} dx \right\} \\ &= \exp \left\{ -c^{1-\beta} \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta(c(t_j-y)_+)) \right|^\alpha (1-\beta)y^{-\beta} dy \right\}, \end{aligned}$$

where we substituted $x = cy$ in the last step. Because of the self-similarity of (M_β) and (S_γ) , the above equals

$$\begin{aligned} & \exp \left\{ -c^{1-\beta} \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j c^{\beta/\gamma} S_\gamma(M_\beta((t_j-y)_+)) \right|^\alpha (1-\beta)y^{-\beta} dy \right\} \\ &= \exp \left\{ -c^{\alpha H} \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((t_j-y)_+)) \right|^\alpha (1-\beta)y^{-\beta} dy \right\} \\ &= \exp \left\{ i \sum_{j=1}^k \theta_j c^H Y_{\alpha,\beta,\gamma}(t_j) \right\}. \end{aligned}$$

This shows that $Y_{\alpha,\beta,\gamma}$ is H -ss with H given by (3.2).

Finally, we check stationarity of the increments of $Y_{\alpha,\beta,\gamma}$. We must check that for any $s > 0$, $t_1, \dots, t_k > 0$, and $\theta_1, \dots, \theta_k \in \mathbb{R}$,

$$\begin{aligned} & \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j [S_\gamma(M_\beta((t_j+s-x)_+)) - S_\gamma(M_\beta((s-x)_+))] \right|^\alpha x^{-\beta} dx \\ &= \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((t_j-x)_+)) \right|^\alpha x^{-\beta} dx. \end{aligned}$$

Split the integral in the left-hand side according to the sign of $s - x$ and use the substitutions $r = s - x$ and $-r = s - x$ to get

$$\begin{aligned} & \int_0^s \mathbf{E}' \left| \sum_{j=1}^k \theta_j [S_\gamma(M_\beta(t_j + r)) - S_\gamma(M_\beta(r))] \right|^\alpha (s - r)^{-\beta} dr \\ & + \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((t_j - r)_+)) \right|^\alpha (s + r)^{-\beta} dr. \end{aligned}$$

Rearranging terms, we are left to check

$$\begin{aligned} & \int_0^s \mathbf{E}' \left| \sum_{j=1}^k \theta_j [S_\gamma(M_\beta(t_j + r)) - S_\gamma(M_\beta(r))] \right|^\alpha (s - r)^{-\beta} dr \\ (3.3) \quad & = \int_0^\infty \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((t_j - x)_+)) \right|^\alpha \left(x^{-\beta} - (s + x)^{-\beta} \right) dx. \end{aligned}$$

However, by the stationarity of the increments of (S_γ) , the left-hand side of (3.3) reduces to

$$\int_0^s \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta(t_j + r) - M_\beta(r)) \right|^\alpha (s - r)^{-\beta} dr.$$

Let $\delta_r = S_\beta(M_\beta(r)) - r$ be the overshoot of the level $r > 0$ by the β -stable subordinator $(S_\beta(t))$ related to $(M_\beta(t))$ by (2.5). The law of δ_r is given by

$$P(\delta_r \in dx) = \frac{\sin \beta \pi}{\pi} r^\beta (r + x)^{-1} x^{-\beta} dx, \quad x > 0;$$

see Exercise 5.6 in Kyprianou (2006). Further, by the strong Markov property of the stable subordinator we have

$$(M_\beta(t + r) - M_\beta(r), t \geq 0) \stackrel{d}{=} (M_\beta((t - \delta_r)_+), t \geq 0),$$

with the understanding that M_β and δ_r in the right-hand side are independent. We conclude that

$$\begin{aligned} (3.4) \quad & \int_0^s \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta(t_j + r) - M_\beta(r)) \right|^\alpha (s - r)^{-\beta} dr \\ & = \frac{\sin \beta \pi}{\pi} \int_0^\infty \int_0^s \mathbf{E}' \left| \sum_{j=1}^k \theta_j S_\gamma(M_\beta((t_j - x)_+)) \right|^\alpha r^\beta (r + x)^{-1} x^{-\beta} (s - r)^{-\beta} dr dx. \end{aligned}$$

Using the integration formula

$$\int_0^1 \left(\frac{t}{1-t} \right)^\beta \frac{1}{t+y} dt = \frac{\pi}{\sin \beta \pi} \left[1 - \left(\frac{y}{1+y} \right)^\beta \right], \quad y > 0,$$

given on p. 338 in Gradshteyn and Ryzhik (2004), shows that (3.4) is equivalent to (3.3). \square

Remark 3.3. The increment process

$$V_n^{(\alpha, \beta, \gamma)} = Y_{\alpha, \beta, \gamma}(n+1) - Y_{\alpha, \beta, \gamma}(n), \quad n = 0, 1, 2, \dots,$$

is a stationary SaS process. The argument in Theorem 3.5 in Owada and Samorodnitsky (2015) can be used to check that, in the case $0 < \beta < 1$, this process corresponds to a conservative null operator T in the sense of (2.4). Furthermore, this process is mixing. See Rosiński (1995) and Samorodnitsky (2005) for details. On the other hand, in the case $\beta = 0$, the increment process is an i.i.d. sequence and, hence, corresponds to a dissipative operator T ; recall Remark 3.1. Furthermore, in the case $\beta = 1$, the increment process is sub-stable, hence corresponds to a positive operator T . In particular, it is not even an ergodic process.

4. SOME MARKOV CHAIN THEORY

The class of stationary infinitely divisible processes for which we will prove a functional central limit theorem, is based on a dynamical system related to Example 5.5 in Owada and Samorodnitsky (2015). In the present paper we allow the Markov chains involved in the construction to take values in a space more general than \mathbb{Z} .

We follow the setup of Chen (2000) and prove additional auxiliary results we will need in the sequel. Let (Z_n) be an irreducible Harris recurrent Markov chain (or simply *Harris chain* in the sequel) on state space $(\mathbb{X}, \mathcal{X})$ with transition probability $P(x, A)$ and invariant measure $\pi(A)$. Our general reference for such processes is Meyn and Tweedie (2009). As usually, we assume that the σ -field \mathcal{X} is countably generated. We denote by \mathbf{P}_ν the probability law of (Z_n) with initial distribution ν , and by \mathbf{E}_ν the expectation with respect to \mathbf{P}_ν .

The collections of sets of finite, and of finite and positive π -measure are denoted by

$$\mathcal{X}^+ := \{A \in \mathcal{X} \text{ such that } \pi(A) > 0\}, \quad \mathcal{X}_0^+ := \{A \in \mathcal{X} \text{ such that } 0 < \pi(A) < \infty\}.$$

Since the Markov chain is Harris, for any set $A \in \mathcal{X}^+$ and any initial distribution ν , on an event of full probability with respect to \mathbf{P}_ν , the sequence of return times to A defined by $\tau_A(0) = 0$ and

$$(4.1) \quad \tau_A(k) = \inf\{n > \tau_A(k-1) : Z_n \in A\} \quad \text{for } k \geq 1,$$

is a well defined finite sequence. An alternative name for $\tau_A(1)$ is simply τ_A .

For a set $A \in \mathcal{X}_0^+$ we denote by

$$(4.2) \quad a_n(\nu, A) = \pi(A)^{-1} \sum_{k=1}^n \int_{\mathbb{X}} P^k(x, A) \nu(dx), \quad n \geq 1$$

the mean number of visits to the set starting from initial distribution ν , up to time n , relative to its π -measure. When needed, we extend the domain of a to $[1, \infty)$ by rounding the argument down to the nearest integer.

An *atom* \mathfrak{a} of the Markov chain is a subset of \mathbb{X} such that $P(x, \cdot) = P(y, \cdot)$ for all $x, y \in \mathfrak{a}$. For an atom the notation $\mathbf{P}_\mathfrak{a}$ and $\mathbf{E}_\mathfrak{a}$ makes an obvious sense. A finite union of atoms is a set of the form

$$(4.3) \quad D = \bigcup_{i=1}^q \mathfrak{a}_i, \quad q < \infty,$$

where $\mathfrak{a}_j \in \mathcal{X}_0^+$, $j = 1, \dots, q$ are atoms. Any such set is a special set, otherwise known as a D -set, see Definition 5.4 in Nummelin (1984) and Orey (1971), p. 29. The importance of this fact is that

for any two special sets (and, hence, for any two finite unions of atoms) D_1 and D_2 ,

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{a_n(\nu_1, D_1)}{a_n(\nu_2, D_2)} = 1$$

for any two initial distributions ν_1 and ν_2 ; see Theorem 2 in Chapter 2 of Orey (1971) or Theorem 7.3 in Nummelin (1984) (without the assumption of “speciality” there might be π -small exceptional sets of initial states). This fact allows us to use the notation $a_t := a_t(\nu, D)$ for any arbitrary fixed special set D and initial distribution ν when only the limiting behaviour as $t \rightarrow \infty$ of this function is important. For concreteness, we fix a special set D and use $\nu(dx) = \pi(D)^{-1}1_D(x)\pi(dx)$.

A Harris chain is said to be β -regular, $0 \leq \beta \leq 1$, if the function (a_t) is regularly varying at infinity with exponent β , i.e.

$$\lim_{t \rightarrow \infty} a_{ct}/a_t = c^\beta \quad \text{for any } c > 0.$$

Let $f : \mathbb{X} \rightarrow \mathbb{R}$ be a measurable function. For a set $A \in \mathcal{X}_0^+$ the sequence

$$\xi_k(A) = \sum_{j=\tau_A(k-1)+1}^{\tau_A(k)} f(Z_j), \quad k = 1, 2, \dots$$

is a well defined sequence of random variables under any law \mathbf{P}_ν . It is a sequence of i.i.d. random variables under \mathbf{P}_ν if A is an atom and ν is concentrated on A .

The following two conditions on a function f will be imposed throughout the paper.

$$(4.5) \quad f \in L^1(\pi) \cap L^2(\pi) \quad \text{and} \quad \int_{\mathbb{X}} f(x) \pi(dx) = 0,$$

$$(4.6) \quad \sum_{k=1}^{\infty} f(\cdot) P^k f(\cdot) \text{ converges in } L^1(\mathbb{X}, \mathcal{X}, \pi).$$

It follows that

$$(4.7) \quad \sigma_f^2 := \int_{\mathbb{X}} f^2(x) \pi(dx) + 2 \sum_{k=1}^{\infty} \int_{\mathbb{X}} f(x) P^k f(x) \pi(dx) < \infty.$$

We refer the reader to Chen (1999b) and Chen (2000) for a discussion and examples of functions f satisfying conditions (4.5) and (4.6). An important implication of the above assumptions is the following result, proven in Lemma 2.3 of Chen (1999b): if $\mathbf{a} \in \mathcal{X}_0^+$ is an atom such that $\inf_{x \in \mathbf{a}} |f(x)| > 0$ (i.e. an f -atom), then

$$(4.8) \quad \mathbf{E}_{\mathbf{a}}(\xi_1(\mathbf{a}))^2 = \frac{1}{\pi(\mathbf{a})} \sigma_f^2.$$

We prove next a functional version of Theorem 1.3 in Chen (2000). In infinite ergodic theory related results are known as Darling-Kac theorems; see Aaronson (1981), Thaler and Zweimüller (2006), and Owada and Samorodnitsky (2015). The result below can be viewed as a mean-zero functional Darling-Kac theorem for Harris chains.

Theorem 4.1. *Suppose that (Z_n) is a β -regular Harris recurrent chain with $0 < \beta \leq 1$, and suppose f satisfies conditions (4.5) and (4.6). Set for $nt \in \mathbb{N}$*

$$S_{nt}(f) = \sum_{k=1}^{nt} f(Z_k),$$

and for all other $t > 0$ define $S_{nt}(f)$ by linear interpolation. Then for any initial distribution ν , under \mathbf{P}_ν ,

$$(4.9) \quad \left(\frac{1}{\sqrt{a_n}} S_{nt}(f), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left((\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t)), t \geq 0 \right),$$

weakly in $\mathcal{C}[0, \infty)$, where B is a standard Brownian motion, which is independent of the Mittag-Leffler process M_β . If $\beta = 0$, then the same convergence holds in finite dimensional distributions.

As in Chen (2000), the proof of Theorem 4.1 proceeds in three steps: regeneration, the split chain method of Nummelin (1978), and finally the use of a geometrically sampled approximation, also known as a resolvent approximation (see Chapter 5 in Meyn and Tweedie (2009)).

In the first step we derive a functional version of a part of Lemma 2.3 in Chen (2000), assuming existence of an atom $\mathbf{a} \in \mathcal{X}_0^+$. For a related result see Theorem 5.1 in Kasahara (1984).

We define the discrete local time at \mathbf{a} by

$$\ell_{\mathbf{a}}(n) := \max\{k \geq 0 : \tau_{\mathbf{a}}(k) \leq n\}$$

Then the sequence

$$(4.10) \quad ((\xi_k(\mathbf{a}), \tau_{\mathbf{a}}(k) - \tau_{\mathbf{a}}(k-1)), k = 1, 2, \dots) \text{ is i.i.d under } \mathbf{P}_{\mathbf{a}}.$$

Lemma 4.2. *The convergence (4.9) holds under the assumptions of Theorem 4.1, with the additional assumption that (Z_n) has an f -atom \mathbf{a} .*

Proof. Unless stated otherwise, all the distributional statements below are understood to be under \mathbf{P}_ν , for an arbitrary fixed initial distribution ν . Let

$$\phi(n) = \sum_{k=1}^n P^k(\mathbf{a}, \mathbf{a}), \quad n = 1, 2, \dots,$$

and note that by (4.4),

$$(4.11) \quad \lim_{n \rightarrow \infty} \phi(n)/a_n = \pi(\mathbf{a}).$$

Let ϕ^{-1} denote an asymptotic inverse of ϕ . By Lemma 3.4 in Chen (1999a), as $n \rightarrow \infty$, the stochastic process

$$T_n(t) := \tau_{\mathbf{a}}(\lfloor nt \rfloor) / \phi^{-1}(n), \quad t \geq 0$$

converges weakly in the J_1 -topology on $\mathcal{D}[0, \infty)$ to a β -stable subordinator with the Laplace transform $\exp\{-t\theta^\beta/\Gamma(\beta+1)\}$ for $0 < \beta < 1$, and to a line with slope one when $\beta = 1$ (we will deal with $\beta = 0$ in a moment). Setting

$$(4.12) \quad W_n(t) = n^{-1/2} \sum_{k=1}^{nt} \xi_k(\mathbf{a}), \quad t \geq 0$$

(defined for fractional values of nt by linear interpolation), the laws of $\{(W_n(t), t \geq 0), (T_n(t), t \geq 0)\}_{n \in \mathbb{N}}$ are tight in $\mathcal{C}[0, \infty) \times \mathcal{D}[0, \infty)$ since the marginal laws converge weakly in the corresponding spaces as $n \rightarrow \infty$. By (4.10) every subsequential limit is a bivariate Lévy process, with one marginal process a Brownian motion, and the other marginal process a subordinator. By the Lévy-Itô decomposition, the Brownian and subordinator components must be independent, so all subsequential limits coincide, and the entire bivariate sequence converges weakly in $\mathcal{C}[0, \infty) \times \mathcal{D}[0, \infty)$ to a bivariate Lévy process with independent marginals.

If $0 < \beta < 1$, weak convergence of $(T_n(t))$ is easily seen to imply, by inversion, finite dimensional convergence, as $n \rightarrow \infty$, of the sequence $(\ell_{\mathbf{a}}(nt)/\phi(n))$ (defined for fractional nt by linear

interpolation) to $\Gamma(\beta + 1)M_\beta$. Since the paths are increasing, and the limit is continuous, this guarantees convergence in $\mathcal{D}[0, \infty)$; see Bingham (1971) and Yamazato (2009). Similarly, we obtain weak convergence to a line with slope one for $\beta = 1$.

In the case $\beta = 0$, by Theorem 2.3 of Chen (1999a) and (4.11), we see that the sequence of processes $(\ell_a(nt)/\phi(n))$ converges in finite-dimensional distributions to a limit, equal to zero at $t = 0$ and consisting of the same standard exponential random variable repeated for all $t > 0$. Moreover, by Lemma 2.3 in Chen (2000), the exponential random variable is independent of the limiting Brownian motion in (4.12), when we perform a subsequential limit scheme for the bivariate process $\{((W_n(t), t \geq 0), (\ell_a(nt)/\phi(n)))\}_{n \in \mathbb{N}}$, similarly to the above. This time the convergence is in finite-dimensional distributions.

Suppose now that $0 < \beta \leq 1$. If $\mathcal{D}_+[0, \infty)$ denotes the subset of $\mathcal{D}[0, \infty)$ consisting of nonnegative functions, then the composition map $(x, y) \rightarrow x \circ y$ from $\mathcal{C}[0, \infty) \times \mathcal{D}_+[0, \infty)$ to $\mathcal{D}[0, \infty)$ is continuous at a point (x, y) if y is continuous. It follows, therefore, by the continuous mapping theorem that

$$(4.13) \quad \left(\frac{1}{\sqrt{\phi(n)}} \sum_{k=1}^{\ell_a(\lfloor nt \rfloor)} \xi_k(\mathbf{a}), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left((\Gamma(\beta + 1) \mathbf{E}_a \xi_1(\mathbf{a})^2)^{1/2} B(M_\beta(t)), t \geq 0 \right)$$

in $\mathcal{D}[0, \infty)$, with $(M_\beta(t))$ independent of $(B(t))$ in the right hand side. By (4.11) we obtain

$$\left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{\ell_a(\lfloor nt \rfloor)} \xi_k(\mathbf{a}), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left((\pi(\mathbf{a}) \Gamma(\beta + 1) \mathbf{E}_a \xi_1(\mathbf{a})^2)^{1/2} B(M_\beta(t)), t \geq 0 \right).$$

Recalling (4.8), we have shown that

$$(4.14) \quad \left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{\ell_a(\lfloor nt \rfloor)} \xi_k(\mathbf{a}), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left((\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t)), t \geq 0 \right)$$

in $\mathcal{D}[0, \infty)$.

We now proceed to relate (4.14) to the statement of the lemma. First of all, Corollary 3 in Zweimüller (2007) allows us to simplify the situation and assume that the chain starts at the f -atom \mathbf{a} . We can write

$$(4.15) \quad \left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{nt} f(Z_k), t \geq 0 \right) = \left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{\ell_a(\lfloor nt \rfloor)} \xi_k(\mathbf{a}) \right. \\ \left. + \frac{1 - nt + \lfloor nt \rfloor}{\sqrt{a_n}} \sum_{k=\tau_a(\ell_a(\lfloor nt \rfloor)+1)}^{\lfloor nt \rfloor} f(Z_k) + \frac{nt - \lfloor nt \rfloor}{\sqrt{a_n}} \sum_{k=\tau_a(\ell_a(\lfloor nt \rfloor)+1)}^{\lfloor nt \rfloor+1} f(Z_k), t \geq 0 \right).$$

Since the convergence in (4.14) occurs in the J_1 topology on $\mathcal{D}[0, \infty)$ and the limit is continuous (recall that we are considering the case $0 < \beta \leq 1$), in order to prove convergence of the processes in (4.15) in $\mathcal{C}[0, \infty)$, we need only show that the second and the third terms in the right hand side of (4.15) are negligible in $\mathcal{C}[0, \infty)$. We treat in details the second term; the third term can be treated

similarly. Restricting ourselves to $\mathcal{C}[0, 1]$, we will prove that

$$\frac{1}{\sqrt{a_n}} \sup_{0 \leq t \leq 1} \left| \sum_{k=\tau_{\mathbf{a}}(\ell_{\mathbf{a}}(\lfloor nt \rfloor))+1}^{\lfloor nt \rfloor} f(Z_k) \right| \xrightarrow{n \rightarrow \infty} 0$$

in probability. To this end we rewrite this expression as

$$\frac{1}{\sqrt{a_n}} \max_{m=0, \dots, n} \left| \sum_{k=\tau_{\mathbf{a}}(\ell_{\mathbf{a}}(m))+1}^m f(Z_k) \right| \leq \frac{1}{\sqrt{a_n}} \max_{j=0, \dots, \ell_{\mathbf{a}}(n)} \max_{\tau_{\mathbf{a}}(j)+1, \dots, \tau_{\mathbf{a}}(j+1)} \left| \sum_{k=\tau_{\mathbf{a}}(j)+1}^m f(Z_k) \right|.$$

Letting W_j denote the inner maximum, we must show that for any $\epsilon > 0$, choosing n large enough implies

$$\mathbf{P}_{\mathbf{a}} \left(\max_{j=0, \dots, \ell_{\mathbf{a}}(n)} W_j > \epsilon \sqrt{a_n} \right) < \epsilon.$$

Since the sequence $(\ell_{\mathbf{a}}(n)/a_n)$ converges weakly, it is tight, and there exists M_{ϵ} such that for all n , $\mathbf{P}_{\mathbf{a}}(\ell_{\mathbf{a}}(n) > M_{\epsilon} a_n) < \epsilon/2$. Thus we need only check that for n large enough

$$(4.16) \quad \begin{aligned} \mathbf{P}_{\mathbf{a}} \left(\max_{0 \leq j \leq M_{\epsilon} a_n} W_j > \epsilon \sqrt{a_n} \right) &= 1 - \mathbf{P}_{\mathbf{a}} \left(\max_{0 \leq j \leq M_{\epsilon} a_n} W_j \leq \epsilon \sqrt{a_n} \right) \\ &= 1 - (1 - \mathbf{P}_{\mathbf{a}}(W_1^2 > \epsilon^2 a_n))^{[M_{\epsilon} a_n]+1} < \epsilon/2. \end{aligned}$$

To see this we use Lemma 2.1 in Chen (2000) which states that under our assumptions we have $\mathbf{E}_{\mathbf{a}} W_1^2 < \infty$. Thus,

$$\mathbf{P}_{\mathbf{a}}(W_1^2 > \epsilon^2 a_n) = o(a_n^{-1}) \text{ as } n \rightarrow \infty,$$

which verifies (4.16). This proves the lemma in the case $0 < \beta \leq 1$.

If $\beta = 0$, then the same argument starting with (4.13) works. The argument is easier in this case since we only need to prove convergence in finite-dimensional distributions. We omit the details. \square

We are now ready to prove Theorem 4.1 in general.

Proof of Theorem 4.1. As before, the distributional statements below are understood to be under \mathbf{P}_{ν} , for an arbitrary fixed initial distribution ν . The split chain method of Nummelin (1978) allows us to extend the result from the situation in Lemma 4.2, where we assumed the existence of an f -atom, to the case where we only assume that there exists a $C \in \mathcal{X}_0^+$ such that

$$(4.17) \quad \inf_{x \in C} |f(x)| > 0 \quad \text{and} \quad P(x, A) \geq b 1_C(x) \pi_C(A) \quad x \in \mathbb{X}, A \in \mathcal{X}$$

for some $0 < b \leq 1$ where $\pi_C(\cdot) := \pi(C)^{-1} \pi(C \cap \cdot)$.

A very brief outline of the split chain method is as follows (see Nummelin (1978) or Chapter 5 in Meyn and Tweedie (2009) for more details). One can enlarge the probability space in order to obtain an extra sequence of Bernoulli random variables (Y_n) . The (Y_n) are chosen so that the *split chain* (Z_n, Y_n) is a Harris chain on $\mathbb{X} \times \{0, 1\}$ and such that $C \times \{1\}$ is an f -atom (where f is extended to $\mathbb{X} \times \{0, 1\}$ in the natural way). Moreover, this can be done so that conditions (4.5) and (4.6) continue to hold for the split chain, and also (4.7) holds for \tilde{P} and $\tilde{\pi}$ (the transition kernel and invariant measure for the split chain.) Then an application of Lemma 4.2 to the split chain proves the claim of the theorem under the assumption (4.17).

The final step is to get rid of assumption (4.17), so we no longer assume that (4.17) holds to start with. We follow the usual procedure which, for (a small) $p > 0$ uses the renewal process

$$N(t) := \max\{n \geq 1 : \Gamma_n \leq t\}, \quad t \geq 0,$$

with i.i.d. renewal intervals $(\Gamma_{n+1} - \Gamma_n)$, which have the geometric distribution

$$(4.18) \quad \mathbf{P}(\Gamma_1 = k) := (1-p)p^{k-1}, \quad k = 1, 2, \dots$$

The idea is to approximate the original chain (Z_n) by its resolvent chain (Z_{Γ_k}) , where (Γ_k) are as in (4.18), and independent of (Z_n) . We then let $p \rightarrow 0$.

The resolvent chain is just (Z_n) observed at the negative binomial renewal times (Γ_k) (this chain is also called a geometrically sampled chain). Its transition kernel is

$$P_p(x, A) := (1-p) \sum_{k=1}^{\infty} p^{k-1} P^k(x, A),$$

and, clearly, π is still an invariant measure. For the resolvent chain, the assumptions (4.5) and (4.6) allow one to define, similarly to (4.7),

$$\begin{aligned} \sigma_{p,f}^2 &:= \int_{\mathbb{X}} f^2(x) \pi(dx) + 2 \sum_{k=1}^{\infty} \int_{\mathbb{X}} f(x) P_p^k f(x) \pi(dx) \\ &= \int_{\mathbb{X}} f^2(x) \pi(dx) + 2(1-p) \sum_{k=1}^{\infty} \int_{\mathbb{X}} f(x) P^k f(x) \pi(dx). \end{aligned}$$

Furthermore, the resolvent chain is β -regular if the original chain is, and the sequence $(a_n^{(p)})$ corresponding to the resolvent chain (see the discussion following (4.4)) satisfies

$$a_n^{(p)} \sim (1-p)^{1-\beta} a_n, \quad n \rightarrow \infty;$$

see (4.26) in Chen (2000). The latter paper also shows that the resolvent chain (Z_{Γ_k}) satisfies (4.17) (see also Theorem 5.2.1 in Meyn and Tweedie (2009)).

Suppose that $0 < \beta \leq 1$. Since we have already proved the theorem under the assumption (4.17), we can use Theorem 2.15(c) in Jacod and Shiryaev (1987), and the ‘‘converging together lemma’’ in Proposition 3.1 of Resnick (2007) to obtain

$$\begin{aligned} &\left[\left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{nt} f(Z_{\Gamma_k}), \quad t \geq 0 \right), \left(\frac{1}{n} N(nt), \quad t \geq 0 \right) \right] \xrightarrow{n \rightarrow \infty} \\ &\left[\left(\sqrt{(1-p)^{1-\beta} \Gamma(\beta+1)} \sigma_{p,f} B(M_\beta(t)), \quad t \geq 0 \right), \left((1-p)t, \quad t \geq 0 \right) \right] \end{aligned}$$

in $\mathcal{C}[0, \infty) \times \mathcal{D}[0, \infty)$. As before, it is legitimate to apply the continuous mapping theorem, to obtain

$$(4.19) \quad \left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{N(nt)} f(Z_{\Gamma_k}), \quad t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left(\sqrt{(1-p)^{1-\beta} \Gamma(\beta+1)} \sigma_{p,f} B(M_\beta((1-p)t)), \quad t \geq 0 \right)$$

in $\mathcal{D}[0, \infty)$. Repeating the same argument with the continuous version of the counting process $(N(t))$, given by

$$N_c(t) = N(t) + \frac{t - \Gamma_n}{\Gamma_{n+1} - \Gamma_n} \quad \text{for } \Gamma_n \leq t \leq \Gamma_{n+1}, \quad n = 0, 1, \dots,$$

leads to

$$(4.20) \quad \left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{N_c(nt)} f(Z_{\Gamma_k}), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left(\sqrt{(1-p)^{1-\beta} \Gamma(\beta+1)} \sigma_{p,f} B(M_\beta((1-p)t)), t \geq 0 \right),$$

this time in $\mathcal{C}[0, \infty)$. We now show that the convergence statement in (4.20) is sufficiently close to the required convergence statement in (4.9), and for this purpose the $\mathcal{D}[0, \infty)$ version in (4.19) will be useful.

We will restrict ourselves to the interval $[0, 1]$. Since

$$\sqrt{1-p} \sigma_{p,f} \longrightarrow \sigma_f \quad \text{as } p \rightarrow 0,$$

the second converging together theorem (see Theorem 3.5 in Resnick (2007)), says that (4.9) will follow once we check that for any $\epsilon > 0$,

$$(4.21) \quad \lim_{p \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{N_c(nt)} f(Z_{\Gamma_k}) - \sum_{k=1}^{nt} f(Z_k) \right| > \epsilon \right) = 0.$$

To this end we bound the probability in the left hand side of (4.21) by a sum of 3 probabilities:

$$(4.22) \quad \begin{aligned} & \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{N_c(nt)} f(Z_{\Gamma_k}) - \sum_{k=1}^{N(nt)} f(Z_{\Gamma_k}) \right| > \frac{\epsilon}{3} \right) + \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{N(nt)} f(Z_{\Gamma_k}) - \sum_{k=1}^{\lfloor nt \rfloor} f(Z_k) \right| > \frac{\epsilon}{3} \right) \\ & + \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{\lfloor nt \rfloor} f(Z_k) - \sum_{k=1}^{nt} f(Z_k) \right| > \frac{\epsilon}{3} \right). \end{aligned}$$

Keep for a moment $0 < p < 1/2$ fixed. Note that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{N_c(nt)} f(Z_{\Gamma_k}) - \sum_{k=1}^{N(nt)} f(Z_{\Gamma_k}) \right| > \frac{\epsilon}{3} \right) \\ & \leq \limsup_{n \rightarrow \infty} \mathbf{P}_\nu \left(\frac{1}{\sqrt{a_n}} \max_{k \leq 2n} |f(Z_k)| > \frac{\epsilon}{3} \right) = 0 \end{aligned}$$

because, eventually, $N_c(n) \leq 2n$ and the convergence in (4.19) is to a continuous limit. A similar argument shows that for a fixed $0 < p < 1/2$,

$$\limsup_{n \rightarrow \infty} \mathbf{P}_\nu \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{\lfloor nt \rfloor} f(Z_k) - \sum_{k=1}^{nt} f(Z_k) \right| > \frac{\epsilon}{3} \right) = 0.$$

It remains to handle the middle probability in (4.22). Let $\delta_k = 1$ at the renewal times and 0 otherwise. By the self-similarity of the Brownian motion and the Mittag-Leffler process, the convergence statement in (4.19) can be rewritten as

$$\left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{\lfloor nt \rfloor} \delta_k f(Z_k), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} \left(\sqrt{(1-p) \Gamma(\beta+1)} \sigma_{p,f} B(M_\beta(t)), t \geq 0 \right).$$

Replacing p by $1 - p$ and, hence, each δ_k by $1 - \delta_k$, gives us also

$$\left(\frac{1}{\sqrt{a_n}} \sum_{k=1}^{\lfloor nt \rfloor} (1 - \delta_k) f(Z_k), t \geq 0 \right) \xrightarrow{n \rightarrow \infty} (\sqrt{p\Gamma(\beta+1)} \sigma_{1-p,f} B(M_\beta(t)), t \geq 0).$$

Both of these weak convergence statements take place in the J_1 topology on $\mathcal{D}[0, \infty)$. Therefore,

$$\mathbf{P} \left(\sup_{0 \leq t \leq 1} \frac{1}{\sqrt{a_n}} \left| \sum_{k=1}^{N(nt)} f(Z_{\Gamma_k}) - \sum_{k=1}^{\lfloor nt \rfloor} \delta_k f(Z_k) \right| > \frac{\epsilon}{3} \right) \xrightarrow{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq t \leq 1} \sqrt{p\Gamma(\beta+1)} \sigma_{1-p,f} |B(M_\beta(t))| > \frac{\epsilon}{3} \right),$$

which goes to 0 as $p \rightarrow 0$. This completes the proof in the case $0 < \beta \leq 1$. Once again, the case $\beta = 0$ is similar but easier, since we are only claiming finite-dimensional weak convergence. \square

Remark 4.3. If in Theorem 4.1 the function f is supported by a finite union of atoms D , then we also have

$$(4.23) \quad \sup_{n \geq 1} \mathbf{E}_\nu \sup_{0 \leq t \leq L} \left(\frac{1}{\sqrt{a_n}} S_{nt}(f) \right)^2 < \infty$$

for the initial distribution $\nu(dx) = \pi(D)^{-1} 1_D(x) \pi(dx)$ and any $0 < L < \infty$. To see this, it is enough to consider the case where the initial distribution ν is given, instead, by $\nu(dx) = \pi(\mathbf{a})^{-1} 1_{\mathbf{a}}(x) \pi(dx)$, where \mathbf{a} is a single atom of positive measure, forming a part of D . We use the notation in Lemma 4.2. It is elementary to check that for each $n \geq 1$,

$$\hat{\ell}_{\mathbf{a}}(n) := \min\{k \geq 0 : \tau_{\mathbf{a}}(k) > n\} = \ell_{\mathbf{a}}(n) + 1$$

is a stopping time with respect to the discrete time filtration

$$\mathcal{F}_k = \sigma(\xi_j(\mathbf{a}), \tau_{\mathbf{a}}(j), j = 1, \dots, k), \quad k = 1, 2, \dots,$$

while the process

$$\sum_{j=1}^k \xi_j(\mathbf{a}), \quad k = 1, 2, \dots$$

is a martingale with respect to the same filtration. Increasing L , if necessary, to make it an integer, we see that

$$(4.24) \quad \mathbf{E}_{\mathbf{a}} \sup_{0 \leq t \leq L} \left(\frac{1}{\sqrt{a_n}} S_{nt}(f) \right)^2 \leq \frac{2}{a_n} \mathbf{E}_{\mathbf{a}} \max_{m=1, \dots, \hat{\ell}_{\mathbf{a}}(\lfloor nL \rfloor)} \left(\sum_{j=1}^m \xi_j(\mathbf{a}) \right)^2.$$

By Doob's inequality and the optional stopping theorem, this can be further bounded by

$$\frac{8}{a_n} \mathbf{E}_{\mathbf{a}} \left(\sum_{j=1}^{\hat{\ell}_{\mathbf{a}}(\lfloor nL \rfloor)} \xi_j(\mathbf{a}) \right)^2 = 8 \mathbf{E}_{\mathbf{a}} (\xi_1(\mathbf{a}))^2 \frac{\mathbf{E}_{\mathbf{a}} \hat{\ell}_{\mathbf{a}}(\lfloor nL \rfloor)}{a_n}.$$

Since

$$\mathbf{E}_{\mathbf{a}} (\xi_1(\mathbf{a}))^2 < \infty \quad \text{and} \quad \sup_{n \geq 1} \frac{\mathbf{E}_{\mathbf{a}} \hat{\ell}_{\mathbf{a}}(n)}{a_n} < \infty$$

by the assumption and the discussion at the beginning of the proof of Lemma 4.2, the claim (4.23) follows.

We will also need a version of Theorem 4.1, in which the initial distribution is not fixed but, rather, diffuses, with n , over the set $\{\tau_D \leq n\}$. We will only consider the case of a finite union of atoms $D = \bigcup_{i=1}^q \mathbf{a}_i \in \mathcal{X}_0^+$.

We first reformulate our Markovian setup in the language of standard infinite ergodic theory. Let $E = \mathbb{X}^{\mathbb{N}}$ be the path space corresponding to the Markov chain. We equip E with the usual cylindrical σ -field $\mathcal{E} = \mathcal{X}^{\mathbb{N}}$. Let T be the left shift operator on the path space E , i.e. $T(\mathbf{x}) = (x_2, x_3, \dots)$ for $\mathbf{x} = (x_1, x_2, \dots) \in E$. Note that T preserves the measure μ on E defined by

$$(4.25) \quad \mu(A) := \int_{\mathbb{X}} \mathbf{P}_x(A) \pi(dx), \text{ for events } A \in \mathcal{E}$$

(as usually, the notation \mathbf{P}_x refers to the initial distribution $\nu = \delta_x$, $x \in \mathbb{X}$.) Notice that the measure μ is infinite if the invariant measure π is. This is, of course, always the case if $0 \leq \beta < 1$. In the sequel we will usually assume that π is infinite even when $\beta = 1$.

We will need certain ergodic-theoretical properties of the quadruple (E, \mathcal{E}, μ, T) . As shown in Aaronson et al. (1979), T is *conservative* and *ergodic*; this implies that

$$\sum_{n=1}^{\infty} 1_A \circ T^n = \infty \quad \mu\text{-a.e. on } E$$

for every $A \in \mathcal{E}$ with $\mu(A) > 0$. For a finite union of atoms $D \in \mathcal{X}_0^+$ as in (4.3), let

$$(4.26) \quad \tilde{D} := \{\mathbf{x} \in E : x_1 \in D\}$$

be the set of paths which start in D . The first return time to D as defined in (4.1) can then be viewed as a function on the product space E via

$$\tau_D(\mathbf{x}) = \inf\{n \geq 0 : T^n \mathbf{x} \in \tilde{D}\} = \inf\{n \geq 0 : x_n \in D\}, \quad \mathbf{x} = (x_1, x_2, \dots) \in E.$$

The *wandering rate sequence* (corresponding to the set \tilde{D}) is the sequence $\mu(\tau_D \leq n)$, $n = 1, 2, \dots$. Since T is measure-preserving, this is a finite sequence. Since the Markov chain is β -regular, this sequence turns out to be regularly varying as well, as we show below. For the ergodic-theoretical notions used in the proof see Aaronson (1997) and Zweimüller (2009).

Lemma 4.4. *Suppose that (Z_n) is a β -regular Harris recurrent chain with $0 \leq \beta \leq 1$, with an infinite invariant measure π . Let D be a finite union of atoms. Then the wandering rate $\mu(\tau_D \leq n)$ is a regularly varying sequence of exponent $1 - \beta$. More precisely,*

$$\mu(\tau_D \leq n) \sim \frac{1}{\Gamma(1 + \beta)\Gamma(2 - \beta)} \frac{n}{a_n} \quad \text{as } n \rightarrow \infty.$$

Proof. Let Q be a Markov semigroup on \mathbb{X} which is dual to P with respect to the measure π . That is, for every $k = 1, 2, \dots$ and every bounded measurable function $f : \mathbb{X}^k \rightarrow \mathbb{R}$,

$$\begin{aligned} & \int_{\mathbb{X}} \pi(dx_1) \int_{\mathbb{X}} P(x_1, dx_2) \dots \int_{\mathbb{X}} P(x_{k-1}, dx_k) f(x_1, \dots, x_k) \\ &= \int_{\mathbb{X}} \pi(dx_k) \int_{\mathbb{X}} Q(x_k, dx_{k-1}) \dots \int_{\mathbb{X}} Q(x_2, dx_1) f(x_1, \dots, x_k). \end{aligned}$$

Since π is an invariant measure for P , it is also invariant for Q . Define a σ -finite measure on (E, \mathcal{E}) analogous to μ in (4.25), but using Q instead of P , i.e.

$$\hat{\mu}(A) := \int_{\mathbb{X}} \mathbf{Q}_x(A) \pi(dx), \text{ for events } A \in \mathcal{E}.$$

We claim that the set \tilde{D} given in (4.26) is a Darling-Kac set for the shift operator T on the space $(E, \mathcal{E}, \hat{\mu})$. According to the definition of a Darling-Kac set (see Chapter 3 in Aaronson (1997)), it is enough to show that

$$(4.27) \quad \frac{1}{a_n} \sum_{k=1}^n \hat{T}_Q^k 1_{\tilde{D}}(\mathbf{x}) \rightarrow \hat{\mu}(\tilde{D}) = \pi(D) \quad \text{uniformly } \hat{\mu}\text{-a.e. on } \tilde{D},$$

where $\hat{T}_Q : L^1(\hat{\mu}) \rightarrow L^1(\hat{\mu})$ is the dual operator defined by

$$\hat{T}_Q g(\mathbf{x}) := \frac{d(\hat{\mu}_g \circ T^{-1})}{d\hat{\mu}}(\mathbf{x})$$

with

$$\hat{\mu}_g(A) := \int_A g(\mathbf{x}) \hat{\mu}(d\mathbf{x}), \quad A \in \mathcal{E}$$

a signed measure on (E, \mathcal{E}) , absolutely continuous with respect to $\hat{\mu}$. Note that the dual operator \hat{T}_Q satisfies $\hat{T}_Q^k 1_{\tilde{D}}(\mathbf{x}) = P^k(x_1, D)$; see Example 2 in Aaronson (1981). Since we may choose a_n as in (4.2) with $A = D$ (recall that a finite union of atoms is a special set), it is elementary to check that (4.27) holds, and the uniformity of the convergence stems from the fact that the left side in (4.27) takes at most q different values on \tilde{D} . Applying Proposition 3.8.7 in Aaronson (1997), we obtain

$$\hat{\mu}(\tau_D \leq n) \sim \frac{1}{\Gamma(1+\beta)\Gamma(2-\beta)} \frac{n}{a_n}.$$

However, by duality,

$$\mu(\tau_D \leq n) = \hat{\mu}(\tau_D \leq n).$$

Since (a_n) is regularly varying with exponent β , the exponent of regular variation of the wandering sequence is, obviously, $1 - \beta$. \square

Remark 4.5. Referring to the proof of Lemma 4.4, it should be noted that using in (4.26) a set $D \in \mathcal{X}_0^+$ different from a finite union of atoms, may still define a set \tilde{D} that is a Darling-Kac set. For example, suppose that (Z_k) is a random walk on \mathbb{R} with standard Gaussian steps; that is,

$$P(x, B) = \mathbf{P}(G \in B - x), \quad x \in \mathbb{R}, \quad B \text{ Borel.}$$

Here $G \sim N(0, 1)$. In this case the Lebesgue measure π on \mathbb{R} is an invariant measure. It is not hard to see that $D = [0, 1]$ is a special set that is not a finite union of atoms. Further, $a_n \sim \sqrt{n/2\pi}$ as $n \rightarrow \infty$. We claim that the set \tilde{D} of paths starting in $[0, 1]$, is a Darling-Kac set for the conservative measure-preserving shift operator on E . To see this note that, in this case, there is no difference between the semigroup P and the dual semigroup Q , and, hence,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n \hat{T}^k 1_{\tilde{D}} = \frac{1}{\sqrt{n}} \sum_{k=1}^n P^k 1_{[0,1]} \rightarrow \frac{1}{\sqrt{2\pi}}$$

uniformly on $[0, 1]$.

We can view the sums $S_n(f)$ as being defined on the path space E by setting $h(\mathbf{x}) := f(x_1)$, $\mathbf{x} = (x_1, x_2, \dots) \in E$, and then writing

$$\hat{S}_n(f)(\mathbf{x}) = \sum_{k=1}^n h \circ T^k(\mathbf{x}) = \sum_{k=1}^n f(x_k).$$

This defines the notation used in Theorem 4.6 below. Define a sequence of probability measures (μ_n) on E by

$$\mu_n(A) := \frac{\mu(A \cap \{\tau_D \leq n\})}{\mu(\tau_D \leq n)}, \quad A \in \mathcal{E}.$$

Theorem 4.6. *Suppose that, in addition to the hypotheses of Theorem 4.1, f is supported by a finite union of atoms $D = \bigcup_{i=1}^q \mathbf{a}_i \in \mathcal{X}_0^+$. Then for every $L > 0$, under the measures μ_{nL} ,*

$$\left(\frac{1}{\sqrt{a_n}} \hat{S}_{nt}(f), 0 \leq t \leq L \right) \xrightarrow{n \rightarrow \infty} \left((\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t - T_\infty^L)), 0 \leq t \leq L \right),$$

where T_∞^L is independent of the process $B(M_\beta(t))$ and $\mathbf{P}(T_\infty^L \leq x) = (x/L)^{1-\beta}$ for $x \in [0, L]$ (in particular, $T_\infty^L = 0$ a.s. if $\beta = 1$). The convergence is weak convergence in $\mathcal{C}[0, L]$ for $0 < \beta \leq 1$ and convergence in finite-dimensional distributions if $\beta = 0$. Furthermore, for all $0 \leq \beta \leq 1$,

$$(4.28) \quad \sup_{n \geq 1} \int_E \left(\frac{\hat{S}_n(f)}{\sqrt{a_n}} \right)^2 d\mu_n < \infty.$$

Remark 4.7. By the similar argument to that in Remark 4.3, it is not hard to check that, under the assumptions of Theorem 4.6, if for some $p > 2$, $\mathbf{E}_\mathbf{a} |\xi_1(D)|^p < \infty$ for any atom \mathbf{a} constituting D , then we, correspondingly, have

$$\sup_{n \geq 1} \int_E \left(\frac{|\hat{S}_n(f)|}{\sqrt{a_n}} \right)^p d\mu_n < \infty.$$

Proof. We prove convergence of the finite-dimensional distributions first. For typographical convenience we will only consider one-dimensional distributions. In the case of more than one dimension, the argument is similar, but the notation is more cumbersome. During this proof we may and will modify the definition of $S_{nt}(f)$ to have the sum starting at $k = 0$. Suppose first that $L = 1$.

Set for $m = 1, 2, \dots$, $x \in \mathbb{X}$ and $i = 1, \dots, q$

$$p_m(x, i) := \mathbf{P}_x(Z_{\tau_D} = \mathbf{a}_i | \tau_D = m).$$

Then for $\lambda \in \mathbb{R}$ and a large $K = 1, 2, \dots$,

$$(4.29) \quad \begin{aligned} & \mu_n \left(\frac{\hat{S}_{nt}(f)}{\sqrt{a_n}} > \lambda \right) \\ &= \sum_{m=0}^n \frac{1}{\mu(\tau_D \leq n)} \int_{\mathbb{X}} \mathbf{P}_x \left(\frac{S_{nt}(f)}{\sqrt{a_n}} > \lambda, \tau_D = m \right) \pi(dx) \\ &= \sum_{m=0}^n \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} \sum_{i=1}^q p_m(x, i) \mathbf{P}_{\mathbf{a}_i} \left(\frac{S_{(nt-m)_+}(f)}{\sqrt{a_n}} > \lambda \right) \pi(dx) \\ &= \sum_{k=1}^K \sum_{m=\lfloor (k-1)n/K \rfloor}^{\lfloor kn/K \rfloor - 1} \sum_{i=1}^q \mathbf{P}_{\mathbf{a}_i} \left(\frac{S_{(nt-m)_+}(f)}{\sqrt{a_n}} > \lambda \right) \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} p_m(x, i) \pi(dx). \end{aligned}$$

The second equality uses the fact that f is supported on D and the strong Markov property. In the last equality, we merely partition $\{0, \dots, n-1\}$ into K parts.

Suppose first that $0 < \beta \leq 1$. Working backwards through an argument similar to (4.29), we obtain

$$(4.30) \quad \begin{aligned} & \sum_{k=1}^K \sum_{m=\lfloor (k-1)n/K \rfloor}^{\lfloor kn/K \rfloor - 1} \sum_{i=1}^q \mathbf{P}_{a_i} \left(\frac{S_{n(t-k/K)_+}(f)}{\sqrt{a_n}} > \lambda \right) \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} p_m(x, i) \pi(dx) \\ &= \mu_n \left(\frac{\hat{S}_{T_n^{K,t}}(f)}{\sqrt{a_n}} > \lambda \right), \end{aligned}$$

where

$$T_n^{K,t}(\mathbf{x}) := (nt - (nk/K - \tau_D(\mathbf{x})))_+ \quad \text{if } \tau_D(\mathbf{x}) \in [(k-1)n/K, kn/K).$$

Clearly,

$$\frac{|nt - T_n^{K,t}(\mathbf{x})|}{n} \leq 1/K.$$

By Theorem 4.1 and Lemma 4.4, we see that

$$(4.31) \quad \begin{aligned} & \sum_{k=1}^K \sum_{m=\lfloor (k-1)n/K \rfloor}^{\lfloor kn/K \rfloor - 1} \sum_{i=1}^q \mathbf{P}_{a_i} \left(\frac{S_{n(t-k/K)_+}(f)}{\sqrt{a_n}} > \lambda \right) \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} p_m(x, i) \pi(dx) \\ & \sim \sum_{k=1}^K \mathbf{P} \left((\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t - k/K)_+) > \lambda \right) \frac{\mu(\lfloor (k-1)n/K \rfloor \leq \tau_D < \lfloor kn/K \rfloor - 1)}{\mu(\tau_D \leq n)} \\ & \rightarrow \sum_{k=1}^K \left((k/K)^{1-\beta} - ((k-1)/K)^{1-\beta} \right) \mathbf{P} \left((\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t - k/K)_+) > \lambda \right). \end{aligned}$$

Combining (4.31) with (4.30) implies that

$$\frac{\hat{S}_{T_n^{K,t}}(f)}{\sqrt{a_n}} \implies (\Gamma(\beta + 1))^{1/2} \sigma_f B(M_\beta(t - \hat{T}_{\infty, K})_+),$$

where $\hat{T}_{\infty, K}$ is a discrete random variable independent of B and M_β such that

$$\mathbf{P}(\hat{T}_{\infty, K} = k/K) = (k/K)^{1-\beta} - ((k-1)/K)^{1-\beta}, \quad k = 1, \dots, K.$$

We claim that for every $\varepsilon > 0$

$$(4.32) \quad \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu_n \left(\frac{\sup_{0 \leq s_1, s_2 \leq t, |s_1 - s_2| \leq 1/K} |\hat{S}_{ns_1}(f) - \hat{S}_{ns_2}(f)|}{\sqrt{a_n}} > \varepsilon \right) = 0.$$

Then, since $\hat{T}_{\infty, K} \Rightarrow T_\infty^1$ as $K \rightarrow \infty$, once we prove (4.32), the claim of the theorem in the case $L = 1$ will follow from Theorem 3.2 in Billingsley (1999).

To see that (4.32) is true, repeat the steps in (4.29) and bound the probabilities $p_m(x, i)$ from above by 1. We conclude that (4.32) is bounded from above by

$$\begin{aligned} & \lim_{K \rightarrow \infty} \sum_{i=1}^q \limsup_{n \rightarrow \infty} \mathbf{P}_{a_i} \left(\frac{\sup_{0 \leq s_1, s_2 \leq t, |s_1 - s_2| \leq 1/K} |S_{ns_1}(f) - S_{ns_2}(f)|}{\sqrt{a_n}} > \varepsilon \right) \\ &= \lim_{K \rightarrow \infty} q \mathbf{P} \left(\sup_{0 \leq s_1, s_2 \leq t, |s_1 - s_2| \leq 1/K} (\Gamma(\beta + 1))^{1/2} \sigma_f |B(M_\beta(s_1)) - B(M_\beta(s_2))| > \varepsilon \right), \end{aligned}$$

where at the second step we used Theorem 4.1. Now (4.32) follows from the sample continuity of the process $(B(M_\beta(t)), t \geq 0)$.

This proves the required convergence for $L = 1$. For general L we replace n by nL , t by t/L and use the regular variation of (a_n) . Using the already considered case $L = 1$ we see that

$$\mu_{nL} \left(\frac{\hat{S}_{nt}(f)}{\sqrt{a_n}} > \lambda \right) \rightarrow \mathbf{P} \left((\Gamma(\beta + 1))^{1/2} \sigma_f L^{\beta/2} B(M_\beta(t/L - T_\infty^1)_+) > \lambda \right).$$

Since $LT_\infty^1 \stackrel{d}{=} T_\infty^L$ and the process $B(M_\beta)$ is $\beta/2$ -self-similar, the claim of the theorem in the case $0 < \beta \leq 1$ has been established.

In the case $\beta = 0$ and $L = 1$, we proceed as in (4.29), but stop before breaking the sum into K parts. Consider the case $\lambda \geq 0$; the case $\lambda < 0$ can be handled in a similar manner. Fix $t > 0$ and choose $\varepsilon > 0$ smaller than t . Next, split the sum over m into two sums; the first over the range $m \leq n(t - \varepsilon)$, and the second over the range $n(t - \varepsilon) < m \leq nt$. Denote the first sum

$$\Sigma_{n,1}(\lambda) := \sum_{m \leq n(t-\varepsilon)} \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} \sum_{i=1}^q p_m(x, i) \mathbf{P}_{\mathbf{a}_i} \left(\frac{S_{(nt-m)_+}(f)}{\sqrt{a_n}} > \lambda \right) \pi(dx)$$

and denote the second sum, over the range $n(t - \varepsilon) < m \leq nt$, by $\Sigma_{n,2}(\lambda)$. Let $0 < \rho < 1$. By the slow variation of the sequence (a_n) there is n_ρ such that for all $n > n_\rho$ and for all $m \leq n(t - \varepsilon)$, $a_{nt-m}/a_n \in (1 - \rho, 1 + \rho)$. By Theorem 4.1 there is \hat{n}_ρ such that for all $n > \hat{n}_\rho$,

$$\frac{\mathbf{P}_{\mathbf{a}_i} \left(\frac{S_n(f)}{\sqrt{a_n}} > (1 \pm \rho)^{-1/2} \lambda \right)}{\mathbf{P}(\sigma_f B(E_{\text{st}}) > (1 \pm \rho)^{-1/2} \lambda)} \in (1 - \rho, 1 + \rho),$$

for each $i = 1, \dots, q$, where E_{st} is a standard exponential random variable independent of the Brownian motion. For notational simplicity, we identify n_ρ and \hat{n}_ρ . We see that for $n > n_\rho$,

$$\begin{aligned} (1 - \rho) \frac{\mu(\tau_D \leq n(t - \varepsilon))}{\mu(\tau_D \leq n)} \mathbf{P}(\sigma_f B(E_{\text{st}}) > (1 + \rho)^{-1/2} \lambda) &\leq \Sigma_{n,1}(\lambda) \\ &\leq (1 + \rho) \frac{\mu(\tau_D \leq n(t - \varepsilon))}{\mu(\tau_D \leq n)} \mathbf{P}(\sigma_f B(E_{\text{st}}) > (1 - \rho)^{-1/2} \lambda). \end{aligned}$$

Furthermore,

$$\Sigma_{n,2}(\lambda) \leq \frac{\mu(n(t - \varepsilon) < \tau_D \leq nt)}{\mu(\tau_D \leq n)}.$$

Letting first $n \rightarrow \infty$, then $\varepsilon \rightarrow 0$, and, finally, $\rho \rightarrow 0$, we conclude, by the continuity of the law of $B(E_{\text{st}})$ that

$$\mu_n \left(\frac{\hat{S}_{nt}(f)}{\sqrt{a_n}} > \lambda \right) \rightarrow t \mathbf{P}(\sigma_f B(E_{\text{st}}) > \lambda),$$

which is the required limit in the case $\beta = 0$ and $L = 1$. The extension to the case of a general $L > 0$ is the same as in the case $0 < \beta \leq 1$.

It remains to prove tightness in the case $0 < \beta \leq 1$. We will prove tightness in $\mathcal{C}[0, 1]$. Since we are dealing with a sequence of processes starting at zero, it is enough to show that for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $n = 1, 2, \dots$,

$$(4.33) \quad \mu_n \left(\sup_{0 \leq s, t \leq 1, |t-s| \leq \delta} \frac{1}{\sqrt{a_n}} |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)| > \varepsilon \right) \leq \varepsilon.$$

However, by the tightness part of Theorem 4.1, we can choose $\delta > 0$ such that for every $i = 1, \dots, q$ and $n = 1, 2, \dots$,

$$\mathbf{P}_{\alpha_i} \left(\sup_{0 \leq s, t \leq 1, |t-s| \leq \delta} \frac{1}{\sqrt{a_n}} |S_{nt}(f) - S_{ns}(f)| > \varepsilon \right) \leq \varepsilon.$$

Therefore, arguing as in (4.29), we obtain

$$\begin{aligned} & \mu_n \left(\sup_{0 \leq s, t \leq 1, |t-s| \leq \delta} \frac{1}{\sqrt{a_n}} |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)| > \varepsilon \right) \\ &= \sum_{m=0}^n \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} \sum_{i=1}^q p_m(x, i) \mathbf{P}_{\alpha_i} \left(\sup_{0 \leq s, t \leq 1, |t-s| \leq \delta} \frac{1}{\sqrt{a_n}} |S_{(nt-m)_+}(f) - S_{(ns-m)_+}(f)| > \varepsilon \right) \pi(dx) \\ &\leq \sum_{m=0}^n \int_{\mathbb{X}} \frac{\mathbf{P}_x(\tau_D = m)}{\mu(\tau_D \leq n)} \sum_{i=1}^q p_m(x, i) \mathbf{P}_{\alpha_i} \left(\sup_{0 \leq s, t \leq 1, |t-s| \leq \delta} \frac{1}{\sqrt{a_n}} |S_{nt}(f) - S_{ns}(f)| > \varepsilon \right) \pi(dx) \leq \varepsilon, \end{aligned}$$

proving (4.33).

Finally, we prove (4.28). We have

$$\begin{aligned} \int_E \left(\frac{\hat{S}_n(f)}{\sqrt{a_n}} \right)^2 d\mu_n &= \frac{1}{a_n \mu(\tau_D \leq n)} \int_E (\hat{S}_n(f))^2 d\mu \\ (4.34) \quad &= \frac{1}{a_n \mu(\tau_D \leq n)} \left[n \int_{\mathbb{X}} f^2(x) \pi(dx) + 2 \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \int_{\mathbb{X}} f(x) P^k f(x) \pi(dx) \right], \end{aligned}$$

where in the first step we used that f is supported by D , and in the second step we used the invariance of the measure π . By Lemma 4.4 and conditions (4.5) and (4.6), the supremum over $n \geq 1$ of the right side of (4.34) is finite. \square

5. A MEAN-ZERO FUNCTIONAL CLT FOR HEAVY-TAILED INFINITELY DIVISIBLE PROCESSES

We now define precisely the class of infinitely divisible stochastic processes $\mathbf{X} = (X_1, X_2, \dots)$ for which we will prove a functional central limit theorem. Those processes are given in the form (1.3) of a stochastic integral.

Let (E, \mathcal{E}) be the path space of a Markov chain on \mathbb{X} , as in Section 4. Let $f : \mathbb{X} \rightarrow \mathbb{R}$ be a measurable function satisfying (4.5) and (4.6). We will assume that f is supported by a finite union of atoms (4.3). Let $h(\mathbf{x}) := f(x_1)$, $\mathbf{x} = (x_1, x_2, \dots) \in E$ be the extension of the function f to the path space E defined above.

Let M be a homogeneous symmetric infinitely divisible random measure M on (E, \mathcal{E}) with control measure μ given by (4.25). We will assume that the local Lévy measure ρ of M has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$:

$$(5.1) \quad \rho(\cdot, \infty) \in RV_{-\alpha} \text{ at infinity.}$$

Let

$$(5.2) \quad X_k = \int_E h \circ T^k(\mathbf{x}) dM(\mathbf{x}) = \int_E f(x_k) dM(\mathbf{x}), \quad k = 1, 2, \dots,$$

where T is the left shift on the path space E . Since the function f is supported by a set of a finite measure, it is straightforward to check that the integrability condition (2.1) is satisfied, so (5.2) presents a well defined stationary symmetric infinitely divisible process. Furthermore, we have

$$\mathbf{P}(X_1 > \lambda) \sim \int_{\mathbb{X}} |f(x)|^\alpha \pi(dx) \rho(\lambda, \infty), \quad \lambda \rightarrow \infty,$$

see e.g. Rosiński and Samorodnitsky (1993). That is, the heaviness of the marginal tail of the process \mathbf{X} is determined by the exponent α of regular variation in (5.1). On the other hand, we will assume that the underlying Markov chain is β -regular, $0 \leq \beta \leq 1$, and we will see that the parameter β determines the length of memory in the process \mathbf{X} .

The main result of this work is the following theorem. Its statement uses the tail constant C_α of an α -stable random variable; see Samorodnitsky and Taqqu (1994). We also use the inverse of the tail of the local Lévy measure defined by

$$\rho^\leftarrow(y) := \inf\{x \geq 0 : \rho(x, \infty) \leq y\}, \quad y > 0.$$

Theorem 5.1. *Let $0 < \alpha < 2$ and $0 \leq \beta \leq 1$. Suppose that (Z_n) is a β -regular Harris chain on $(\mathbb{X}, \mathcal{X})$ with an invariant σ -finite measure π . If $\beta = 1$, assume that $a_n = o(n)$. Let f be a measurable function supported on a finite union of atoms $D = \cup_{i=1}^q \mathbf{a}_i \in \mathcal{X}_0^+$. We assume that f satisfies (4.5) and (4.6). If $\beta = 1$, we assume also that $f \in L^{2+\varepsilon}(\pi)$ for some $\varepsilon > 0$. If $\alpha \geq 1$, we also assume that for some $\varepsilon > 0$, $\mathbf{E}_{\alpha_1} |f(Z_{\tau_{\alpha_2}})|^{2+\varepsilon} < \infty$ for any two atoms, α_1, α_2 , constituting D .*

Let $\mathbf{X} = (X_1, X_2, \dots)$ be a stationary symmetric infinitely divisible stochastic process defined in (5.2), where the local Lévy measure of the symmetric homogeneous infinitely divisible random measure M is assumed to satisfy (5.1). We assume, furthermore, that

$$(5.3) \quad x^{p_0} \rho(x, \infty) \rightarrow 0 \quad \text{as } x \downarrow 0$$

for some $p_0 \in (0, 2)$. Then the sequence

$$c_n = C_\alpha^{-1/\alpha} a_n^{1/2} \rho^\leftarrow(\mu(\tau_D \leq n)^{-1}), \quad n = 1, 2, \dots,$$

satisfies

$$(5.4) \quad c_n \in RV_{\beta/2+(1-\beta)/\alpha}.$$

Let $0 < \beta \leq 1$. Then

$$(5.5) \quad \frac{1}{c_n} \sum_{k=1}^n X_k \Rightarrow (\Gamma(\beta + 1))^{1/2} \sigma_f Y_{\alpha, \beta, 2}(\cdot) \quad \text{in } \mathcal{C}[0, \infty),$$

where $(Y_{\alpha, \beta, 2}(t))$ is the process in (3.1), with the usual understanding that the sum in the left hand side is defined by linear interpolation.

If $\beta = 0$, then (5.5) holds in the sense of convergence of finite-dimensional distributions.

Proof. The fact that (5.4) holds follows from the assumption of β -regularity and Lemma 4.4, taking into account that the regular variation of ρ at infinity implies $\rho^\leftarrow \in RV_{-1/\alpha}$ at zero. For later use we also record now that

$$(5.6) \quad \rho(c_n a_n^{-1/2}, \infty) \sim C_\alpha \mu(\tau_D \leq n)^{-1} \quad \text{as } n \rightarrow \infty,$$

which follows directly from the definition of the inverse and the regular variation of the tail of ρ in (5.1).

We start with proving convergence of the finite-dimensional distributions. It is enough to show that

$$\frac{1}{c_n} \sum_{j=1}^J \theta_j \sum_{k=1}^{nt_j} X_k \Rightarrow (\Gamma(\beta + 1))^{1/2} \sigma_f \sum_{j=1}^J \theta_j Y_{\alpha, \beta, 2}(t_j)$$

for all $J \geq 1$, $0 \leq t_1 < \dots < t_J$, and $\theta_1, \dots, \theta_J \in \mathbb{R}$. We use an argument similar to that in Owada and Samorodnitsky (2015).

The standard theory of convergence in law of infinitely divisible random variables (e.g., Theorem 15.14 in Kallenberg (2002)), says that we only have to check the following: in the notation of Theorem 4.6 and of Section 3, for every $r > 0$,

$$(5.7) \quad \int_E \left(\frac{1}{c_n} \sum_{j=1}^J \theta_j \hat{S}_{nt_j}(f) \right)^2 \int_0^{rc_n |\sum \theta_j \hat{S}_{nt_j}(f)|^{-1}} v \rho(v, \infty) dv d\mu \\ \rightarrow \frac{r^{2-\alpha} C_\alpha}{2-\alpha} (\Gamma(\beta + 1))^{\alpha/2} \sigma_f^\alpha \int_{[0, \infty)} \int_{\Omega'} \left| \sum_{j=1}^J \theta_j B(M_\beta((t_j - x)_+, \omega'), \omega') \right|^\alpha \mathbf{P}'(d\omega') \nu_\beta(dx)$$

and

$$(5.8) \quad \int_E \rho \left(rc_n \left| \sum_{j=1}^J \theta_j \hat{S}_{nt_j}(f) \right|^{-1}, \infty \right) d\mu \\ \rightarrow r^{-\alpha} C_\alpha (\Gamma(\beta + 1))^{\alpha/2} \sigma_f^\alpha \int_{[0, \infty)} \int_{\Omega'} \left| \sum_{j=1}^J \theta_j B(M_\beta((t_j - x)_+, \omega'), \omega') \right|^\alpha \mathbf{P}'(d\omega') \nu_\beta(dx).$$

The proof of (5.8) is very similar to that of (5.7), so we only prove (5.7).

We keep $r > 0$ fixed for the duration of the argument. Fix also an integer L so that $t_J \leq L$ and define

$$\psi(y) := y^{-2} \int_0^{ry} x \rho(x, \infty) dx, \quad y > 0,$$

so that the left-hand side in (5.7) can be expressed as

$$\int_E \psi \left(\frac{c_n}{|\sum_{j=1}^J \theta_j \hat{S}_{nt_j}(f)|} \right) d\mu.$$

By Theorem 4.6 and the Skorohod embedding theorem, there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ and random variables Y, Y_1, Y_2, \dots defined on $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ such that

$$\mathbf{P}^* \circ Y_n^{-1} = \mu_{nL} \circ \left(\frac{1}{\sqrt{a_n}} \sum_{j=1}^J \theta_j \hat{S}_{nt_j}(f) \right)^{-1}, \quad n = 1, 2, \dots, \\ \mathbf{P}^* \circ Y^{-1} = \mathbf{P}' \circ \left((\Gamma(\beta + 1))^{1/2} \sigma_f \sum_{j=1}^J \theta_j B(M_\beta(t_j - T_\infty^L)_+) \right)^{-1},$$

$$Y_n \rightarrow Y, \quad \mathbf{P}^*\text{-a.s.}$$

Then

$$\int_E \psi \left(\frac{c_n}{\left| \sum_{j=1}^J \theta_j \hat{S}_{nt_j}(f) \right|} \right) d\mu = \int_{\Omega^*} \mu(\tau_D \leq nL) \psi \left(\frac{c_n}{\sqrt{a_n} |Y_n|} \right) d\mathbf{P}^*.$$

First, we will establish convergence of the quantity inside the integral. By Karamata's theorem (see e.g. Theorem 0.6 in Resnick (1987)),

$$(5.9) \quad \psi(y) \sim \frac{r^{2-\alpha}}{2-\alpha} \rho(y, \infty) \quad \text{as } y \rightarrow \infty.$$

Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} \mu(\tau_D \leq nL) \psi \left(\frac{c_n}{\sqrt{a_n} |Y_n|} \right) &\sim \frac{r^{2-\alpha}}{2-\alpha} \mu(\tau_D \leq nL) \rho(c_n a_n^{-1/2} |Y_n|^{-1}, \infty) \\ &\sim \frac{r^{2-\alpha}}{2-\alpha} |Y_n|^\alpha \mu(\tau_D \leq nL) \rho(c_n a_n^{-1/2}, \infty), \quad \mathbf{P}^*\text{-a.s.}, \end{aligned}$$

where the last line follows from the uniform convergence of regularly varying functions of negative index; see e.g. Proposition 0.5 in Resnick (1987). By (5.6) and the regular variation of the wandering rate in Lemma 4.4, we conclude that

$$\mu(\tau_D \leq nL) \psi \left(\frac{c_n}{\sqrt{a_n} |Y_n|} \right) \rightarrow \frac{r^{2-\alpha}}{2-\alpha} C_\alpha L^{1-\beta} |Y|^\alpha, \quad \mathbf{P}^*\text{-a.s.}$$

It is straightforward to check that

$$\int_{\Omega^*} L^{1-\beta} |Y|^\alpha d\mathbf{P}^* = (\Gamma(\beta+1))^{\alpha/2} \sigma_f^\alpha \int_{[0, \infty)} \int_{\Omega'} \left| \sum_{j=1}^J \theta_j B \left(M_\beta((t_j - x)_+, \omega'), \omega' \right) \right|^\alpha \mathbf{P}'(d\omega') \nu_\beta(dx),$$

so it now remains to show that the convergence discussed so far can be taken under the integral sign. For this, we will use the Pratt lemma (see Exercise 5.4.2.4 in Resnick (1987)). The lemma requires us to find a sequence of measurable functions G_0, G_1, G_2, \dots defined on $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ such that

$$(5.10) \quad \mu(\tau_D \leq nL) \psi \left(\frac{c_n}{\sqrt{a_n} |Y_n|} \right) \leq G_n \quad \mathbf{P}^*\text{-a.s.},$$

$$(5.11) \quad G_n \rightarrow G_0 \quad \mathbf{P}^*\text{-a.s.},$$

$$(5.12) \quad \mathbf{E}^* G_n \rightarrow \mathbf{E}^* G_0 \in [0, \infty).$$

Throughout the rest of the proof C is a positive constant which may change from line to line. Note that by (5.6), $\mu(\tau_D \leq nL) \psi(c_n a_n^{-1/2})$ tends to a positive finite constant, therefore

$$\mu(\tau_D \leq nL) \psi \left(\frac{c_n}{\sqrt{a_n} |Y_n|} \right) \leq C \frac{\psi(c_n a_n^{-1/2} |Y_n|^{-1})}{\psi(c_n a_n^{-1/2})}.$$

Since $\psi \in RV_{-\alpha}$ at infinity, Potter's bounds (see Proposition 0.8 in Resnick (1987)) allow us to write, for $0 < \xi < 2 - \alpha$:

$$\frac{\psi(c_n a_n^{-1/2} |Y_n|^{-1})}{\psi(c_n a_n^{-1/2})} \mathbf{1}\{c_n \geq \sqrt{a_n} |Y_n|\} \leq C (|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi})$$

for sufficiently large n . Further, by (5.3), $y^2\psi(y) \rightarrow 0$ as $y \downarrow 0$, which gives us

$$\psi(y) \leq C y^{-2} \quad \text{for all } y \in [0, 1].$$

Thus,

$$\frac{\psi(c_n a_n^{-1/2} |Y_n|^{-1})}{\psi(c_n a_n^{-1/2})} \mathbf{1}\{c_n < \sqrt{a_n} |Y_n|\} \leq C a_n c_n^{-2} \frac{|Y_n|^2}{\psi(c_n a_n^{-1/2})}.$$

Summarizing, for sufficiently large n ,

$$\mu(\tau_D \leq nL) \psi\left(\frac{c_n}{\sqrt{a_n} |Y_n|}\right) \leq C \left(|Y_n|^{\alpha-\xi} + |Y_n|^{\alpha+\xi} + a_n c_n^{-2} \frac{|Y_n|^2}{\psi(c_n a_n^{-1/2})} \right).$$

If we define G_n to be the right-hand side of the above, then (5.10) is automatic.

Let $G_0 := C(|Y|^{\alpha-\xi} + |Y|^{\alpha+\xi})$. It follows by the definition of c_n and Lemma 4.4 that

$$c_n a_n^{-1/2} \geq C \rho^{\leftarrow}(a_n/n) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

because $a_n/n \rightarrow 0$ (this follows from regular variation considerations if $\beta < 1$, and it is assumed to hold if $\beta = 1$.) Since $y^2\psi(y) \rightarrow \infty$ as $y \rightarrow \infty$ by (5.9) and the fact that $\alpha < 2$, we conclude that

$$a_n c_n^{-2} \frac{|Y_n|^2}{\psi(c_n a_n^{-1/2})} \rightarrow 0$$

\mathbf{P}^* -a.s., so that (5.11) holds.

To show (5.12), recall that by Theorem 4.6, $\sup_{n \geq 1} \mathbf{E}^* |Y_n|^2 < \infty$. This implies uniform integrability of $(|Y_n|^{\alpha \pm \xi}, n \geq 1)$ (with respect to \mathbf{P}^*). Combining these observations,

$$\begin{aligned} \mathbf{E}^* G_n &= C \left(\mathbf{E}^* |Y_n|^{\alpha-\xi} + \mathbf{E}^* |Y_n|^{\alpha+\xi} + a_n c_n^{-2} \frac{\mathbf{E}^* |Y_n|^2}{\psi(c_n a_n^{-1/2})} \right) \\ &\rightarrow C(\mathbf{E}^* |Y|^{\alpha-\xi} + \mathbf{E}^* |Y|^{\alpha+\xi}) = \mathbf{E}^* G_0, \quad n \rightarrow \infty, \end{aligned}$$

as required. This completes the proof of convergence in finite-dimensional distributions.

It remains to prove tightness in the case $0 < \beta \leq 1$. We start by decomposing the process \mathbf{X} according to the magnitude of the Lévy jumps. Denote

$$\begin{aligned} \rho_1(\cdot) &:= \rho(\cdot \cap \{x : |x| > 1\}), \\ \rho_2(\cdot) &:= \rho(\cdot \cap \{x : |x| \leq 1\}), \end{aligned}$$

and let M_i , $i = 1, 2$ denote independent homogeneous symmetric infinitely divisible random measures, with the same control measure μ as M , and local Lévy measures ρ_i , $i = 1, 2$. Then

$$\begin{aligned} (X_k, k = 1, 2, \dots) &\stackrel{d}{=} \left(\int_E f(x_k) dM_1(\mathbf{x}) + \int_E f(x_k) dM_2(\mathbf{x}), k = 1, 2, \dots \right) \\ &:= (X_k^{(1)} + X_k^{(2)}, k = 1, 2, \dots) \end{aligned}$$

in the sense of equality of finite-dimensional distributions. Notice that $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent. Furthermore, since $f \in L^2(\pi)$, we see that

$$(5.13) \quad \mathbf{E}(X_k^{(2)})^2 = \int_{\mathbb{X}} f^2(x) \pi(dx) \int_{-1}^1 y^2 \rho(dy) < \infty.$$

Fix $L > 0$. We will begin with proving tightness of the normalized partial sums of $X_k^{(2)}$ in the space $\mathcal{C}[0, L]$. By Theorem 12.3 of Billingsley (1968), it suffices to show that there exist $\gamma > 1$, $\rho \geq 0$ and $C > 0$ such that

$$(5.14) \quad \mathbf{P} \left(\left| \sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)} \right| > \lambda c_n \right) \leq \frac{C}{\lambda^\rho} (t-s)^\gamma$$

for all $0 \leq s \leq t \leq L$, $n \geq 1$ and $\lambda > 0$.

We dispose of the case $n(t-s) < 1$ first, and, in the sequel, we will assume that $\mu(\tau_D \leq 1) > 0$. If this measure is zero, we will simply replace 1 by a suitable large constant γ and dispose of the case $n(t-s) < \gamma$ first. It follows from (5.13) that

$$\begin{aligned} \mathbf{P} \left(\left| \sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)} \right| > \lambda c_n \right) &\leq \mathbf{P} \left(\max(|X_1^{(2)}|, |X_2^{(2)}|) > \frac{\lambda c_n}{n(t-s)} \right) \\ &\leq C \lambda^{-2} c_n^{-2} n^2 (t-s)^2. \end{aligned}$$

It follows from (5.4) that

$$c_n^{-2} n^2 \in RV_{2-2(\beta/2+(1-\beta)/\alpha)} = RV_{1-(1-\beta)(2/\alpha-1)}.$$

Suppose first that $0 < \beta < 1$. If $(1-\beta)(2/\alpha-1) > 1$, then n^2/c_n^2 is bounded by a positive constant, and we are done. In the case of $0 < (1-\beta)(2/\alpha-1) \leq 1$, since $n(t-s) < 1$, there is $0 < \delta < (1-\beta)(2/\alpha-1)$ such that

$$c_n^{-2} n^2 (t-s)^2 \leq C (t-s)^{1+(1-\beta)(2/\alpha-1)-\delta},$$

which is what is needed for (5.14). If $\beta = 1$, a similar argument works if one uses the stronger integrability assumption on f imposed in the theorem.

Let us assume, therefore, that $n(t-s) \geq 1$. By the Lévy-Itô decomposition,

$$\begin{aligned} &\sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)} \stackrel{d}{=} \int_E (\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) dM_2 \\ &\stackrel{d}{=} \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| \leq \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) d\bar{N}_2 + \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| > \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) dN_2, \end{aligned}$$

where N_2 is a Poisson random measure on $\mathbb{R} \times E$ with mean measure $\rho_2 \times \mu$ and $\bar{N}_2 := N_2 - (\rho_2 \times \mu)$. Therefore,

$$\begin{aligned} &\mathbf{P} \left(\left| \sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)} \right| > \lambda c_n \right) \\ &\leq \mathbf{P} \left(\left| \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| \leq \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) d\bar{N}_2 \right| > \lambda c_n \right) \\ &+ \mathbf{P} \left(\left| \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| > \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) dN_2 \right| > 0 \right). \end{aligned}$$

It follows from (5.3) that,

$$\begin{aligned}
& \mathbf{P}\left(\left|\iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| \leq \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) d\bar{N}_2\right| > \lambda c_n\right) \\
& \leq \frac{1}{\lambda^2 c_n^2} \mathbf{E} \left| \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| \leq \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) d\bar{N}_2 \right|^2 \\
& = \frac{1}{\lambda^2 c_n^2} \iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| \leq \lambda c_n} [y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))]^2 d\rho_2 d\mu \\
& \leq 4 \int_E \left(\frac{\hat{S}_{nt}(f) - \hat{S}_{ns}(f)}{\lambda c_n} \right)^2 \left(\int_0^{\lambda c_n / |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|} y \rho_2(y, \infty) dy \right) d\mu \\
& \leq \frac{C}{\lambda^{p_0}} \frac{1}{c_n^{p_0}} \int_E |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|^{p_0} d\mu.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \mathbf{P}\left(\left|\iint_{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| > \lambda c_n} y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f)) dN_2\right| > 0\right) \\
& \leq \mathbf{P}\left(N_2(\{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| > \lambda c_n\}) \geq 1\right) \\
& \leq \mathbf{E} N_2(\{|y(\hat{S}_{nt}(f) - \hat{S}_{ns}(f))| > \lambda c_n\}) \\
& = 2 \int_E \rho_2(\lambda c_n | \hat{S}_{nt}(f) - \hat{S}_{ns}(f) |^{-1}, \infty) d\mu \\
& \leq \frac{C}{\lambda^{p_0}} \frac{1}{c_n^{p_0}} \int_E |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|^{p_0} d\mu.
\end{aligned}$$

Elementary manipulations of the linear interpolation of the sums and (4.28) show that

$$\begin{aligned}
\mathbf{P}\left(\left|\sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)}\right| > \lambda c_n\right) & \leq \frac{C}{\lambda^{p_0}} \frac{1}{c_n^{p_0}} \max_{n(t-s)-1 \leq m \leq n(t-s)+1} \int_E |\hat{S}_m(f)|^{p_0} d\mu \\
& \leq \frac{C}{\lambda^{p_0}} \frac{1}{c_n^{p_0}} (a_{n(t-s)})^{p_0/2} \mu(\tau_D \leq n(t-s)),
\end{aligned}$$

where at the last step we have used the assumption $n(t-s) \geq 1$, regular variation, and Theorem 4.6.

Suppose first that $0 < \beta < 1$. Choose $\epsilon > 0$ so that $2/\alpha - \epsilon - 1 > 0$. Note that, if (5.3) holds for some $p_0 \in (0, 2)$, it also holds for all larger p_0 . Thus, we may assume that p_0 is close enough to 2 so that

$$(1 - \beta) \left(\frac{p_0}{\alpha} - \frac{\epsilon p_0}{2} - 1 \right) > \beta \left(1 - \frac{p_0}{2} \right).$$

Since we are assuming that $\mu(\tau_D \leq 1) > 0$, we see that

$$\frac{1}{c_n^{p_0}} (a_{n(t-s)})^{p_0/2} \mu(\tau_D \leq n(t-s))$$

$$\begin{aligned}
&\leq C \left(\frac{a_n \mu(\tau_D \leq n)^{2/\alpha - \epsilon}}{c_n^2} \right)^{p_0/2} \left(\frac{a_{n(t-s)}}{a_n} \right)^{p_0/2} \left(\frac{\mu(\tau_D \leq n(t-s))}{\mu(\tau_D \leq n)} \right)^{p_0/\alpha - \epsilon p_0/2} \\
&\leq C \left(\frac{a_{n(t-s)}}{a_n} \right)^{p_0/2} \left(\frac{\mu(\tau_D \leq n(t-s))}{\mu(\tau_D \leq n)} \right)^{p_0/\alpha - \epsilon p_0/2}.
\end{aligned}$$

The first inequality above uses the choice of ϵ and p_0 , while the second inequality follows from the definition of c_n and regular variation of ρ^\leftarrow . Next, we choose $0 < \xi < \min\{\beta, 1 - \beta\}$ such that

$$(1 - \beta) \left(\frac{p_0}{\alpha} - \frac{\epsilon p_0}{2} - 1 \right) - \beta \left(1 - \frac{p_0}{2} \right) - \xi \left(\frac{p_0}{2} + \frac{p_0}{\alpha} - \frac{\epsilon p_0}{2} \right) > 0.$$

By the regular variation of a_n and $\mu(\tau_D \leq n)$,

$$\frac{\mu(\tau_D \leq n(t-s))}{\mu(\tau_D \leq n)} \leq C (t-s)^{1-\beta-\xi}, \quad \frac{a_{n(t-s)}}{a_n} \leq C (t-s)^{\beta-\xi}.$$

Combining these inequalities together, we have

$$\mathbf{P} \left(\left| \sum_{k=1}^{nt} X_k^{(2)} - \sum_{k=1}^{ns} X_k^{(2)} \right| > \lambda c_n \right) \leq \frac{C}{\lambda^{p_0}} (t-s)^\gamma,$$

where $\gamma = (\beta - \xi) p_0/2 + (1 - \beta - \xi)(p_0/\alpha - \epsilon p_0/2)$. Due to the constraints in ϵ , p_0 , and ξ , it is easy to check that $\gamma > 1$. This establishes tightness for the normalized partial sums of $X_k^{(2)}$ in the case $0 < \beta < 1$.

If $\beta = 1$, then the assumption $a_n = o(n)$ and a standard modification of Theorem 12.3 of Billingsley (1968) make the same argument go through.

It remains to prove tightness of the normalized partial sums of $X_k^{(1)}$ in the space $\mathcal{C}[0, L]$, for a fixed $L > 0$. For notational simplicity we take $L = 1$. We will consider first the case $0 < \alpha < 1$. Let $\rho_1^\leftarrow(y) := \inf\{x \geq 0 : \rho_1(x, \infty) \leq y\}$, $y > 0$ be the inverse of the tail of ρ_1 . We will make use of a certain series representation; see Rosiński (1990):

$$(5.15) \quad \left(\sum_{k=1}^{nt} X_k^{(1)}, 0 \leq t \leq 1 \right) \stackrel{d}{=} \left(\sum_{j=1}^{\infty} \epsilon_j \rho_1^\leftarrow \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) \hat{S}_{nt}(f)(V_j^{(n)}), 0 \leq t \leq 1 \right),$$

where (ϵ_j) is an i.i.d. sequence of Rademacher variables (taking $+1, -1$ with probability $1/2$), Γ_j is the j th jump time of a unit rate Poisson process, and $(V_j^{(n)})$ is a sequence of i.i.d. random variables with common law μ_n . Further, (ϵ_j) , (Γ_j) , and $(V_j^{(n)})$ are taken to be independent with each other.

Fix $\xi \in (0, 1/\alpha - 1)$ and for $K > 2(1/\alpha + \xi) - 1$, we split the right-hand side above as follows.

$$\begin{aligned}
T_{n,1}^{(K)}(t) &= \sum_{j=1}^K \epsilon_j \rho_1^\leftarrow \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) \hat{S}_{nt}(f)(V_j^{(n)}), \\
T_{n,2}^{(K)}(t) &= \sum_{j=K+1}^{\infty} \epsilon_j \rho_1^\leftarrow \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) \hat{S}_{nt}(f)(V_j^{(n)}).
\end{aligned}$$

We will prove that the sequence $(c_n^{-1} T_{n,1}^{(K)})$ is, for every K , tight in $\mathcal{C}[0, 1]$, while

$$(5.16) \quad \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq t \leq 1} |T_{n,2}^{(K)}(t)| > \epsilon c_n \right) = 0, \quad \text{for every } \epsilon > 0.$$

Notice that

$$c_n^{-1}T_{n,1}^{(K)}(t) = C_\alpha^{1/\alpha} \sum_{j=1}^K \epsilon_j \rho_1^{\leftarrow} \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) / \rho^{\leftarrow}(\mu(\tau_D \leq n)^{-1}) \frac{\hat{S}_{nt}(f)(V_j^{(n)})}{\sqrt{a_n}}.$$

Since ρ_1^{\leftarrow} and ρ^{\leftarrow} are both regularly varying at zero with exponent $-1/\alpha$,

$$\rho_1^{\leftarrow} \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) / \rho^{\leftarrow}(\mu(\tau_D \leq n)^{-1}) \rightarrow 2^{1/\alpha} \Gamma_j^{-1/\alpha}, \quad n \rightarrow \infty, \quad \text{a.s.}$$

On the other hand, by Theorem 4.6, each $a_n^{-1/2} \hat{S}_{nt}(f)(V_j^{(n)})$ weakly converges in $\mathcal{C}[0, 1]$, and thus, by independence, $c_n^{-1}T_{n,1}^{(K)}$ turns out to be tight in $\mathcal{C}[0, 1]$.

Next, we will turn to proving (5.16). The probability in (5.16) can be estimated from above by

$$\mathbf{P} \left(C_\alpha^{1/\alpha} \sum_{j=K+1}^{\infty} \rho_1^{\leftarrow} \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) / \rho^{\leftarrow}(\mu(\tau_D \leq n)^{-1}) \sup_{0 \leq t \leq 1} |a_n^{-1/2} \hat{S}_{nt}(f)(V_j^{(n)})| > \epsilon \right)$$

Appealing to Potter's bounds and the fact that ρ_1 has no mass on $\{x : |x| \leq 1\}$,

$$\rho_1^{\leftarrow} \left(\frac{\Gamma_j}{2\mu(\tau_D \leq n)} \right) / \rho^{\leftarrow}(\mu(\tau_D \leq n)^{-1}) \leq C \max\{\Gamma_j^{-1/\alpha+\xi}, \Gamma_j^{-1/\alpha-\xi}\}.$$

Combining this bound, Chebyshev's inequality, and the Cauchy-Schwarz inequality, the probability in (5.16) is bounded from above by

$$C\epsilon^{-2} \mathbf{E}(B_n)^2 \mathbf{E} \left(\sum_{j=K+1}^{\infty} \max\{\Gamma_j^{-1/\alpha+\xi}, \Gamma_j^{-1/\alpha-\xi}\} \right)^2,$$

where $B_n = \sup_{0 \leq t \leq 1} |a_n^{-1/2} \hat{S}_{nt}(f)(V_1^{(n)})|$. We know from Remark 4.3 that the sequence $(\mathbf{E}(B_n)^2)$ is uniformly bounded in n . Because of the restriction in K ,

$$\mathbf{E} \left(\sum_{j=K+1}^{\infty} \Gamma_j^{-(1/\alpha \pm \xi)} \right)^2 \leq \left(\sum_{j=K+1}^{\infty} \{\mathbf{E} \Gamma_j^{-2(1/\alpha \pm \xi)}\}^{1/2} \right)^2 \leq C \left(\sum_{j=K+1}^{\infty} j^{-(1/\alpha \pm \xi)} \right)^2,$$

where the rightmost term vanishes as $K \rightarrow \infty$, so the proof of the tightness has been completed.

This proves tightness in the case $0 < \alpha < 1$, and we proceed now to show tightness of the normalized partial sums of $X_k^{(1)}$ in the space $\mathcal{C}[0, 1]$, for the case $1 \leq \alpha < 2$. Recall that, in this case, we impose a stronger integrability assumption on f . We start with the Lévy-Itô decomposition (5.15) and write, for $K > 1$,

$$\begin{aligned} \left(\sum_{k=1}^{nt} X_k^{(1)}, 0 \leq t \leq 1 \right) & \stackrel{d}{=} \left(\sum_{k=1}^{nt} \left(X_k^{(1,K)} + X_k^{(2,K)} \right), 0 \leq t \leq 1 \right) \\ & := \left(\int \int_{|y| \leq Kc_n/\sqrt{a_n}} y \hat{S}_{nt}(f) dN_1 + \int \int_{|y| > Kc_n/\sqrt{a_n}} y \hat{S}_{nt}(f) dN_1, 0 \leq t \leq 1 \right). \end{aligned}$$

Note that the probability that the process $(\sum_{k=1}^{nt} X_k^{(2,K)})$ does not identically vanish on the interval $[0, 1]$ does not exceed

$$\mathbf{P}(N_1\{(\mathbf{x}, y) : |y| > Kc_n/\sqrt{a_n}, \tau_D(\mathbf{x}) \leq n\} \geq 1)$$

$$\begin{aligned}
&\leq \mathbf{E}N_1\{(\mathbf{x}, y) : |y| > Kc_n/\sqrt{a_n}, \tau_D(\mathbf{x}) \leq n\} \\
&= 2\rho_1(Kc_n/\sqrt{a_n}, \infty) \mu(\tau_D \leq n) \\
&\sim 2C_\alpha \frac{\rho(Kc_n/\sqrt{a_n}, \infty)}{\rho(c_n/\sqrt{a_n}, \infty)} \rightarrow 2C_\alpha K^{-\alpha}
\end{aligned}$$

as $n \rightarrow \infty$, and this can be made arbitrarily small by choosing K large. Therefore, we only need to show tightness, for every fixed K , of the normalized partial sums of the process $X_k^{(1,K)}$. As in (5.14), it is enough to prove that there exist $\gamma > 1$, $\rho \geq 0$ and $C > 0$ such that

$$(5.17) \quad \mathbf{P}\left(\left|\sum_{k=1}^{nt} X_k^{(1,K)} - \sum_{k=1}^{ns} X_k^{(1,K)}\right| > \lambda c_n\right) \leq \frac{C}{\lambda^\rho} (t-s)^\gamma$$

for all $0 \leq s \leq t \leq 1$, $n \geq 1$ and $\lambda > 0$. In a manner, similar to the one we employed while proving (5.14), we can dispose of the case $n(t-s) < 1$, so we will look at the case $n(t-s) \geq 1$.

Let $0 < \varepsilon < 1$ be such that $f \in L^{2+\varepsilon}(\pi)$. By Proposition 6.2,

$$\begin{aligned}
(5.18) \quad &\frac{1}{c_n^{2+\varepsilon}} \mathbf{E} \left| \sum_{k=1}^{nt} X_k^{(1,K)} - \sum_{k=1}^{ns} X_k^{(1,K)} \right|^{2+\varepsilon} \\
&\leq \frac{C}{c_n^{2+\varepsilon}} \int_{\mathbb{R} \times E} |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|^{2+\varepsilon} |y|^{2+\varepsilon} \mathbf{1}_{\{|y| \leq Kc_n/\sqrt{a_n}\}} d\rho_1 d\mu \\
&+ \frac{C}{c_n^{2+\varepsilon}} \left(\int_{\mathbb{R} \times E} |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|^2 |y|^2 \mathbf{1}_{\{|y| \leq Kc_n/\sqrt{a_n}\}} d\rho_1 d\mu \right)^{1+\varepsilon/2}.
\end{aligned}$$

By Karamata's theorem,

$$\int_{\mathbb{R}} |y|^{2+\varepsilon} \mathbf{1}_{\{|y| \leq Kc_n/\sqrt{a_n}\}} d\rho_1 \leq C \frac{(c_n/\sqrt{a_n})^{2+\varepsilon}}{\mu(\tau_D \leq n)}.$$

Further, by the fact that f is supported on D , and by the integrability assumption on f , we know that

$$\int_E |\hat{S}_{nt}(f) - \hat{S}_{ns}(f)|^{2+\varepsilon} d\mu \leq C \mu(\tau_D \leq n(t-s)) (a_{n(t-s)})^{(2+\varepsilon)/2};$$

see Remark 4.7. By Lemma 4.4, $\mu(\tau_D \leq n) a_n^{(2+\varepsilon)/2}$ is regularly varying with exponent bigger than 1, so the first term in the right hand side of (5.18) is bounded from above by $C(t-s)^\gamma$, for some $\gamma > 1$. A similar argument produces the same bound for second term in the right hand side of (5.18). Now an appeal to Markov's inequality proves (5.17). \square

6. APPENDIX: FRACTIONAL MOMENTS OF INFINITELY DIVISIBLE RANDOM VARIABLE

In this appendix we present explicit bounds on the fractional moments of certain infinitely divisible random variables in terms of moments of their Lévy measures. These estimates are needed for the proof of Theorem 5.1. We have not been able to find such bounds in the literature. Combinatorial identities for the integer moments have been known at least since Bassanand and Bona (1990). Fractional moments have been investigated, using fractional calculus, by Matsui and Pawlas (2014), but the latter paper does not give general explicit bounds of the type we need.

We will consider fractional moments of nonnegative infinitely divisible random variables and of symmetric infinitely divisible random variables in the ranges needed in the present paper, but our

approach can be extended to moments of all orders. We start with nonnegative infinitely divisible random variables with Laplace transform of the form

$$(6.1) \quad \mathbf{E}e^{-\theta X} = \exp \left\{ - \int_0^\infty (1 - e^{-\theta y}) \nu_+(dy) \right\} := e^{-I(\theta)}, \quad \theta \geq 0,$$

with the Lévy measure ν_+ satisfying

$$\int_0^\infty y \nu_+(dy) < \infty.$$

Proposition 6.1. *Let $1 < p < 2$. Then there is $c_p \in (0, \infty)$, depending only on p , such that for any infinitely divisible random variable X satisfying (6.1),*

$$(6.2) \quad \mathbf{E}X^p \leq c_p \left(\int_0^\infty y^p \nu_+(dy) + \left(\int_0^\infty y \nu_+(dy) \right)^p \right).$$

Proof. If the p th moment of the Lévy measure,

$$\int_0^\infty y^p \nu_+(dy),$$

is infinite, then so is the left hand side of (6.2), and the latter trivially holds. Therefore, we will assume for the duration of the proof that the p th moment of the Lévy measure is finite. We reserve the notation c_p for a generic finite positive constant (that may depend only on p), and that may change from line to line. We start with an elementary observation: there is c_p such that for any $x > 0$,

$$x^p = c_p \int_0^\infty (1 - e^{-xy})^2 y^{-(p+1)} dy.$$

Therefore,

$$(6.3) \quad \mathbf{E}X^p = c_p \int_0^\infty \mathbf{E} (1 - e^{-yX})^2 y^{-(p+1)} dy = c_p \int_0^\infty (1 - 2e^{-I(y)} + e^{-I(2y)}) y^{-(p+1)} dy,$$

where I is defined in (6.1). Denote

$$\theta^+ = \sup\{\theta \geq 0 : I(\theta) \leq 1\} \in (0, \infty].$$

Observe that

$$(6.4) \quad \theta^+ \geq \left(\int_0^\infty y \nu_+(dy) \right)^{-1}.$$

To see that, notice that, if $\theta^+ < \infty$, then

$$1 = I(\theta^+) \leq \theta^+ \int_0^\infty y \nu_+(dy).$$

We now split the integral in (6.3) and write

$$\mathbf{E}X^p = c_p \int_0^{\theta^+} \cdot + c_p \int_{\theta^+}^\infty \cdot := A + B.$$

Note that by (6.4),

$$B \leq c_p \int_{\theta^+}^\infty y^{-(p+1)} dy \leq c_p \left(\int_0^\infty y \nu_+(dy) \right)^p.$$

Next, using the inequality $1 - e^{-2\theta} \leq 2(1 - e^{-\theta})$ for any $\theta \geq 0$, see that $I(\theta) \leq I(2\theta) \leq 2I(\theta)$ for each $\theta \geq 0$. Note also that for $0 \leq b \leq 2a$ we have

$$1 - 2e^{-a} + e^{-b} \leq 2a^2 + (2a - b),$$

and we conclude that

$$A \leq c_p \int_0^{\theta^+} \left(\int_0^\infty (1 - e^{-xy}) \nu_+(dx) \right)^2 y^{-(p+1)} dy + c_p \int_0^{\theta^+} \left(\int_0^\infty (1 - e^{-xy})^2 \nu_+(dx) \right) y^{-(p+1)} dy.$$

Using the fact that for $0 \leq y \leq \theta^+$ we have

$$I(y) \leq \min \left(1, y \int_0^\infty x \nu_+(dx) \right),$$

it follows that

$$\begin{aligned} \int_0^{\theta^+} \left(\int_0^\infty (1 - e^{-xy}) \nu_+(dx) \right)^2 y^{-(p+1)} dy &\leq \int_0^{\left(\int_0^\infty x \nu_+(dx)\right)^{-1}} y^2 \left(\int_0^\infty x \nu_+(dx) \right)^2 y^{-(p+1)} dy \\ &+ \int_{\left(\int_0^\infty x \nu_+(dx)\right)^{-1}}^\infty y^{-(p+1)} dy \leq c_p \left(\int_0^\infty y \nu_+(dy) \right)^p. \end{aligned}$$

Finally,

$$\begin{aligned} &\int_0^{\theta^+} \left(\int_0^\infty (1 - e^{-xy})^2 \nu_+(dx) \right) y^{-(p+1)} dy \\ &\leq \int_0^\infty \left(\int_0^\infty (1 - e^{-xy})^2 y^{-(p+1)} dy \right) \nu_+(dx) = c_p \int_0^\infty y^p \nu_+(dy), \end{aligned}$$

and the proof is complete. \square

We consider next a symmetric infinitely divisible random variable, with characteristic function of the form

$$(6.5) \quad \mathbf{E}e^{i\theta Y} = \exp \left\{ \int_{-\infty}^\infty \left(e^{i\theta y} - 1 - i\theta y \right) \nu(dy) \right\}, \quad \theta \in \mathbb{R},$$

for some symmetric Lévy measure ν , satisfying

$$\int_{|y| \geq 1} y^2 \nu(dy) < \infty.$$

Proposition 6.2. *Let $2 < p < 4$. Then there is $c_p \in (0, \infty)$, depending only on p , such that for any symmetric infinitely divisible random variable Y satisfying (6.5),*

$$(6.6) \quad \mathbf{E}|Y|^p \leq c_p \left(\int_{-\infty}^\infty |y|^p \nu(dy) + \left(\int_{-\infty}^\infty y^2 \nu(dy) \right)^{p/2} \right).$$

Proof. Once again, we may and will assume that the moments of the Lévy measure in the right hand side of (6.6) are finite. We start with the case when $\nu(\mathbb{R}) < \infty$. If (W_j) is a sequence of i.i.d. random variables with the common law $\nu/\nu(\mathbb{R})$, independent of a Poisson random variable N with mean $\nu(\mathbb{R})$, then

$$Y \stackrel{d}{=} \sum_{j=1}^N W_j,$$

and so by the Marcinkiewicz-Zygmund inequality (see e.g. (2.2), p. 227 in Gut (2009)),

$$\mathbf{E}|Y|^p \leq c_p \mathbf{E} \left(\sum_{j=1}^N W_j^2 \right)^{p/2}.$$

The random variable

$$X = \sum_{j=1}^N W_j^2$$

is a nonnegative random variable with Laplace transform of the form (6.1), with Lévy measure ν_+ given by

$$\nu_+(A) = \nu\{y : y^2 \in A\}, \quad A \text{ Borel.}$$

Applying Proposition 6.1 (with $p/2$), proves (6.6) in the compound Poisson case $\nu(\mathbb{R}) < \infty$. In the general case we use an approximation procedure. For $m = 1, 2, \dots$ let ν_m be the restriction of the Lévy measure ν to the set $\{y : |y| > 1/m\}$. Then each ν_m is a finite symmetric measure. If Y_m is an infinitely divisible random variable with the characteristic function given by (6.5), with ν_m replacing ν , then $Y_m \Rightarrow Y$ as $m \rightarrow \infty$, and the fact that (6.6) holds for Y follows from the fact that it holds for each Y_m and Fatou's lemma. \square

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