

# HOW FAST CAN THE CHORD-LENGTH DISTRIBUTION DECAY?

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ABSTRACT. The modelling of random bi-phasic, or porous, media has been, and still is, under active investigation by mathematicians, physicists or physicians. In this paper we consider a thresholded random process  $X$  as a source of the two phases. The intervals when  $X$  is in a given phase, named chords, are the subject of interest. We focus on the study of the tails of the chord-length distribution functions. In the literature, different types of the tail behavior have been reported, among them exponential-like or power-like decay. We look for the link between the dependence structure of the underlying thresholded process  $X$  and the rate of decay of the chord-length distribution. When the process  $X$  is a stationary Gaussian process, we relate the latter to the rate at which the covariance function of  $X$  decays at large lags. We show that exponential, or nearly exponential, decay of the tail of the distribution of the chord-lengths is very common, perhaps surprisingly so.

## INTRODUCTION

Studying porous media, such as human bones, food, rocks, etc., leads naturally to  $D$ -dimensional Boolean models describing presence or absence of material. Mathematically, a Boolean model is a function  $f : \mathbb{R}^D \rightarrow \{0, 1\}$ , when the part of the space where the function  $f$  takes value zero represents the "empty" part (lack of material, or "pore"), while the part of the space where the function  $f$  takes value one represents the "full" part (presence of material, or "matrix"). A Boolean model is often chosen to be stochastic, and a possible stochastic Boolean model is obtained by thresholding a random field  $(X_t)_{t \in \mathbb{R}^D}$  at a given level  $\gamma$  :

$$f(t) = \mathbb{1}_{(\gamma, \infty)}(X_t) = \begin{cases} 1 & \text{if } X_t > \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

This procedure is very commonly considered by physicists (see [16, 14, 3] for instance).

Here we adopt a one dimensional point of view: we draw test lines through the random medium  $(X(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3)$ , and for any line  $\Delta$ , we identify  $(X(\mathbf{x}), \mathbf{x} \in \Delta)$  with a process  $X = (X_t, t \in \mathbb{R})$ .

The successive intervals with  $f(t) = 0$  (respectively  $f(t) = 1$ ) are called *chords*. The chords have been studied in the physics and mathematics literature; see for instance [22, 17, 21] and references therein. In particular, the chord-lengths have been investigated. In a previous paper ([10]) we defined analytically the chord-length distribution functions. In this paper we focus on their rate of decay.

More precisely, let  $X = (X_t)_{t \in \mathbb{R}}$  be a continuously differentiable real strictly stationary process defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . If the process has a finite variance,

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we denote its covariance function by  $\rho$ . All processes considered in this paper have a finite variance and most of them will be Gaussian.

Assuming the derivative of the process does not vanish on intervals of positive lengths, the chord-lengths are well defined by

$$L^0 = \inf\{s > 0 : X_s = \gamma\} \text{ and, for } k \geq 0, L^{k+1} = \inf\{s > 0 : X_{s+L^0+\dots+L^k} = \gamma\}, \quad (1)$$

where  $\gamma$  is a given level (threshold).

Empirically, both exponential-like and power-like rate of decay of the distribution of the chord-lengths have been observed on real data; see for instance the chapter 2 in [12]. Note that this refers to the rate of decay of the probability that a chord is very long. In this paper we investigate the effect of the memory in the thresholded process  $X$  on the rate of decay of this probability. In the case of stationary Gaussian processes the memory is measured by the covariance function  $\rho$ . Section 1 investigates the tail of the distribution of the chord-lengths statistically, on simulated data. We attempt to discriminate between light or heavy tails (of the chord-lengths) both using the Mean Excess Plot method as a graphical method, and estimating the shape parameter of the associated Generalized Pareto Distribution. The numerical results obtained there motivate a probabilistic analysis developed in the next section. When dealing with stationary Gaussian processes having vanishing memory ( $\rho \rightarrow 0$  at infinity), as it is the case in the numerical examples, one of our main results (Theorem 2.4, (ii)) shows that for a thresholded Gaussian process with an exponentially fast decreasing covariance function, the tail of the distribution of the chord-lengths is decaying exponentially fast as well. Perhaps even more surprisingly, we prove that for underlying Gaussian processes whose covariance function is only assumed to decay to zero *at any speed at all* the chord-length distribution decays faster than any negative power function (Theorem 2.4, (i)). This is also true for all  $r$ -mixing processes (Theorem 2.6). These theoretical results are proved in Section 2.

## 1. STATISTICAL ANALYSIS OF SIMULATED DATA

In this section we aim to investigate statistically the behavior of the tail of the distribution of the chord-lengths induced by thresholding a stationary Gaussian process. The purpose here is both to illustrate and motivate the bounds of the next section. To generate samples of chord-lengths, we simulate the underlying Gaussian process. We consider the first two chord-lengths, two different thresholds and three different types of covariance functions of the underlying process.

All Matlab codes, samples and results are available on <http://www.math-info.univ-paris5.fr/~demichel>.

### 1.1. Simulation.

We start with outlining the simulation procedure. Given the covariance function  $\rho$  of a 0 mean unit variance Gaussian process  $X$ , we simulate the process on a discrete subset  $\{t_1, \dots, t_n\}$  of a compact interval  $I$ . These discrete observations will be used to determine approximately the chords completed within the interval  $I$ . Note that we need to simulate a mean 0 and variance 1 Gaussian vector  $(X(t_1), \dots, X(t_n))$  with a covariance matrix  $R = (\rho(|t_i - t_j|))_{1 \leq i, j \leq n}$ . As usual, the key is to get the square root of the covariance matrix.

Since the Cholesky decomposition method is very expensive, we use the Circulant Embedding Matrix method (see [7]). Recall that in this approach the covariance matrix  $R$  is embedded into a circulant matrix  $C$  whose eigenvectors are computed with a Fast Fourier Transform. The square root  $R^{1/2}$  is built from these eigenvectors. This method works if and only if the minimal eigenvalue  $v^-$  of  $C$  is positive and this property is difficult to ascertain *a priori*. In practice we choose an interval  $I$  and a finite grid  $\{t_1, \dots, t_n\} \subset I$  and compute the matrix  $C$  and the minimal eigenvalue  $v^-$ . These choices are crucial since a bad choice may yield an untractable circulant matrix  $C$ . In order to have a good sample of chords, we have to consider both a large interval  $I$  (to ensure that large chords are not missing) and a fine grid  $\{t_1, \dots, t_n\}$  (to ensure that small chords are not missing). In any case, we will miss all the chords that do not fall within the compact interval  $I$ .

Gaussian processes with various covariance functions have been simulated using this method and its extensions; see for instance [11, 18, 19]. We have chosen three types of covariance functions  $\rho$  according to their speed of decay, to see if that would imply different types of chord-length tail behavior. For further illustration, we consider the first two chord-lengths. The different covariance functions  $\rho$  and certain related properties are described in Table 1. We have chosen the time interval  $I = [0, 3]$  and a grid of 6000 points. In each case the Circulant Embedding Matrix method works since the minimum eigenvalue is positive, thus allowing us to obtain samples of the first two chord-lengths  $L^0$  and  $L^1$  for a specified threshold.

TABLE 1. Covariance functions used for simulation.

Example	Speed of decay	Expression $\rho(x)$	Eigenvalue $v^-$
n°1	compact support	$P_{11}(5x)\mathbb{1}_{[0,1]}(5x)$	$6.8426 \cdot 10^{-10}$
n°2	very fast	$\exp(-(5x)^2)$	$1.1827 \cdot 10^{-11}$
n°3	polynomial	$(1 + (6x)^2)^{-4}$	$2.7569 \cdot 10^{-11}$

$$P_{11}(x) = 1 - \frac{22}{3}x^2 + 33x^4 - \frac{77}{2}x^5 + \frac{33}{2}x^7 - \frac{11}{2}x^9 + \frac{5}{6}x^{11}.$$

## 1.2. Statistical analysis.

We perform statistical analysis on the samples of the first two chord-lengths for each of the above three covariance functions. We use two different thresholds,  $\gamma = 0$  and  $\gamma = 1$ . The approach we use is common in the Extreme Value Theory (see for instance [9]). First, we use a graphical method, the Mean Excess Plot (MEP), to try to judge whether the chord-lengths have light (i.e. exponentially fast decreasing) or heavy (i.e. hyperbolically fast decreasing) tails. Next, we fit a Generalized Pareto Distribution (GPD) to the upper part of the chord-length distributions. The estimated from the considered sample parameters of the latter will then indicate either light or heavy tails of the chord-length distributions.

Even though we presently apply these two methods to simulated data, the same approach could be used on real data as well.

### *A graphical method: the Mean-Excess Plot.*

Recall the excess cumulative distribution function (cdf)  $F_u$  of a random variable  $X$  over a threshold  $u \in \mathbb{R}$  is defined in the Peak Over Threshold (POT) approach as the cdf of  $X - u$

conditioned on  $X > u$ , namely

$$F_u(x) = \mathbb{P}(X - u \leq x | X > u), \quad x \geq 0.$$

The corresponding mean excess function  $e$  of  $X$  is defined by  $e(u) = \mathbb{E}(X - u | X > u)$ , whenever it exists.

The plot of the mean excess function  $e$  is a useful graphical tool to help distinguishing between heavy and light tails. For example,  $e(u) = 1/\lambda$  for any  $u$  if  $X$  is exponentially distributed with parameter  $\lambda$ ; heavy-tailed distribution functions have a mean excess function tending to infinity, typically along an asymptotically straight line; distribution functions with tails decaying faster than exponentially fast are characterized by a mean excess function tending to 0.

In practice, one uses the empirical mean excess plot

$$\{(X_{k,n}, e_n(X_{k,n})) : k \in \{1, \dots, n-1\}\},$$

where  $X_{1,n} \leq \dots \leq X_{n,n}$  are the order statistics of an  $n$ -sample  $(X_i)_{1 \leq i \leq n}$ , and  $e_n(u)$  is the empirical mean excess function defined using the empirical cumulative distribution function by

$$\frac{1}{N_u} \sum_{j \in I_n(u)} (X_j - u) \text{ with } I_n(u) = \{j : 1 \leq j \leq n, X_j > u\} \text{ and } N_u = \text{Card}(I_n(u)).$$

In the case of  $u$  equal to one of the order statistics, this is equivalent to

$$e_n(X_{k,n}) = \frac{1}{n-k} \sum_{j=k+1}^n (X_{j,n} - X_{k,n}).$$

***Fitting a GPD to the excesses over a threshold.***

Pickands proved in [15] that, for sufficiently high threshold  $u$ , the excess cdf  $F_u$  of any random variable  $X$  in a domain of attraction of an extreme value distribution can be well approximated by a GPD  $G_{\xi, \sigma(u)}$ , with a shape parameter  $\xi$  and scale parameter  $\sigma = \sigma(u) > 0$ :

$$G(y) = G_{\xi, \sigma(u)}(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\sigma(u)}\right)^{-1/\xi} & \text{if } \xi \neq 0, \\ 1 - \exp\left(-\frac{y}{\sigma(u)}\right) & \text{otherwise,} \end{cases}$$

where  $y \geq 0$  if  $\xi \geq 0$  and  $0 \leq y \leq -\sigma(u)/\xi$  if  $\xi < 0$ . Most of the ‘‘textbook’’ random variables are in the domain of attraction of some extreme value distribution, and so the above approximation of the excess cdf is very general. The shape parameter  $\xi > 0$  arises when  $X$  is heavy tailed, and  $\xi \leq 0$  corresponds to light tails. Therefore, the parameters of the fitted GPD distribution provide information on the tails of  $X$ .

An important question is how to select an appropriate high threshold  $u$ ; we choose it by plotting the empirical mean excess function and choosing  $u$  in the range where the latter appears to be linear or stable.

The parameters of a GPD can be estimated via different methods. We will use the method of moments; see [13]. If  $(Y_j)_{1 \leq j \leq N_u}$  denote the excesses over a given threshold  $u$  in a given

sample, then the moments estimators of the parameters  $\xi$  and  $\sigma(u)$  of the approximating GPD are given, respectively, by

$$\widehat{\xi} = \frac{1}{2} \left( 1 - \frac{\bar{Y}^2}{S_Y^2} \right) \quad \text{and} \quad \widehat{\sigma} = \widehat{\sigma(u)} = \bar{Y} \left( \frac{1}{2} + \frac{\bar{Y}^2}{S_Y^2} \right), \quad (2)$$

where  $\bar{Y}$  and  $S_Y^2$  are the sample mean and variance of the excesses:

$$\bar{Y} = \frac{1}{N_u} \sum_{i=1}^{N_u} Y_i \quad \text{and} \quad S_Y^2 = \frac{1}{N_u - 1} \sum_{i=1}^{N_u} (Y_i - \bar{Y})^2.$$

Provided that the shape parameter satisfies  $\xi < 1/4$ , it can be shown by standard methods that the random vector  $(\widehat{\sigma}, \widehat{\xi})$  is asymptotically normal with covariance matrix  $A$  satisfying, as the sample sizes increases,

$$N_u A \sim \Gamma = \frac{(1 - \xi)^2}{(1 - 2\xi)(1 - 3\xi)(1 - 4\xi)} (a_{ij})_{1 \leq i, j \leq 2},$$

$$\text{with} \quad a_{11} = 2\sigma^2(u)(1 - 6\xi + 12\xi^2), \quad a_{22} = (1 - 2\xi)^2(1 - \xi + 6\xi^2)$$

$$\text{and} \quad a_{12} = a_{21} = \sigma(u)(1 - 2\xi)(1 - 4\xi - 12\xi^2),$$

from which a confidence interval with asymptotic confidence level  $\alpha$  can be deduced:

$$\begin{pmatrix} \widehat{\sigma} \\ \widehat{\xi} \end{pmatrix} + \left( \frac{1}{N_u} \Gamma \right)^{1/2} \begin{pmatrix} q((1 - \alpha)/2) \\ q((1 - \alpha)/2) \end{pmatrix} \leq \begin{pmatrix} \sigma(u) \\ \xi \end{pmatrix} \leq \begin{pmatrix} \widehat{\sigma} \\ \widehat{\xi} \end{pmatrix} + \left( \frac{1}{N_u} \Gamma \right)^{1/2} \begin{pmatrix} q((1 + \alpha)/2) \\ q((1 + \alpha)/2) \end{pmatrix} \quad (3)$$

with  $q(x)$  denoting the  $x$ th quantile of the standard normal distribution.

### 1.3. Application to the chord-lengths.

We generated samples of the first chord-lengths  $(L_i^0)_{1 \leq i \leq n}$  and of the second chord-lengths  $(L_i^1)_{1 \leq i \leq n}$  of the size  $n = 10000$  each.

In each case we started by plotting the empirical mean excess function, in order to judge whether it appears to increase linearly for large levels, or to decay to zero. This was done for each of the three types of covariance and for the two chosen thresholds  $\gamma = 0$  and  $\gamma = 1$ . Next, on each such mean excess plot, we selected a level  $u$  in the range where  $e_n$  looks approximately linear or stable. It is well known that selecting a proper threshold  $u$  is not an easy task as it implies a balance between bias and variance: too high a value of  $u$  leads to too few exceedances and, consequently, high variance of the estimators, whereas too small a value of  $u$  increases the bias of the estimators. The standard practice is to adopt as low a threshold as possible subject to the limiting GPD model providing a reasonable approximation to the empirical tail. We assess this graphically as well.

Having selected a level  $u$ , we estimate the corresponding GPD parameters as in (2), with the associated asymptotic confidence intervals (CI), given in (3), at the confidence level 99%.

The results obtained for the two first chord-lengths,  $L^0$  and  $L^1$ , are summarized in the Table 2 for the threshold  $\gamma = 0$ , and in the Table 3 for the threshold  $\gamma = 1$ . The corresponding empirical mean excess plots and the comparison of the tail of the empirical CDG and the tail of the approximating GPD appear in Figures 1 to 4. The empirical quantile function of order  $\alpha$  is denoted by  $q_n(\alpha)$ .

TABLE 2. Statistical results for  $L^0$  and  $L^1$  with  $\gamma = 0$ .  
 (a) Chord-length  $L^0$  with  $\gamma = 0$ .

Ex.	Chord-length $L^0$				
	$u$	$\alpha$ s.t. $u = q_n(\alpha)$	$N_u$	$\hat{\xi}$ $\hat{\sigma}$	$CI(\xi)$ $CI(\sigma)$
n°1	0.41	90%	910	-0.0955 0.1816	(-0.2531, 0.0621) (0.1047, 0.2585)
n°2	0.90	86%	1374	-0.1530 0.4575	(-0.3301, 0.0241) (0.3169, 0.5981)
n°3	0.49	91%	881	-0.1058 0.2286	(-0.2767, 0.0651) (0.1343, 0.3229)

(b) Chord-length  $L^1$  with  $\gamma = 0$ .

Ex.	Chord-length $L^1$				
	$u$	$\alpha$ s.t. $u = q_n(\alpha)$	$N_u$	$\hat{\xi}$ $\hat{\sigma}$	$CI(\xi)$ $CI(\sigma)$
n°1	0.35	83%	1696	-0.0005 0.1721	(-0.0953, 0.0843) (0.1302, 0.2140)
n°2	1.27	92.5%	746	-0.1911 0.4207	(-0.4526, 0.0704) (0.2238, 0.6176)
n°3	0.46	86%	1380	-0.0063 0.2056	(-0.1088, 0.0962) (0.1530, 0.2582)

TABLE 3. Statistical results for  $L^0$  and  $L^1$  with  $\gamma = 1$ .  
 (a) Chord-length  $L^0$  with  $\gamma = 1$ .

Ex.	Chord-length $L^0$				
	$u$	$\alpha$ s.t. $u = q_n(\alpha)$	$N_u$	$\hat{\xi}$ $\hat{\sigma}$	$CI(\xi)$ $CI(\sigma)$
n°1	0.85	83%	1700	-0.1252 0.5437	(-0.2784, 0.0280) (0.4099, 0.6775)
n°2	1.95	90%	1030	-0.5614 0.5503	(-1.1079, -0.0149) (0.1420, 0.9586)
n°3	1.22	88.25%	1175	-0.2812 0.6561	(-0.5701, 0.0077) (0.3898, 0.9224)

(b) Chord-length  $L^1$  with  $\gamma = 1$ .

Ex.	Chord-length $L^1$				
	$u$	$\alpha$ s.t. $u = q_n(\alpha)$	$N_u$	$\hat{\xi}$ $\hat{\sigma}$	$CI(\xi)$ $CI(\sigma)$
n°1	0.41	90.5%	934	-0.0576 0.5032	(-0.2263, 0.1111) (0.3614, 0.6450)
n°2	0.98	91.3%	866	-0.4243 0.9689	(-0.9412, 0.0926) (0.4357, 1.5421)
n°3	0.52	92.1%	793	-0.1025 0.6391	(-0.3210, 0.1160) (0.4305, 0.8477)

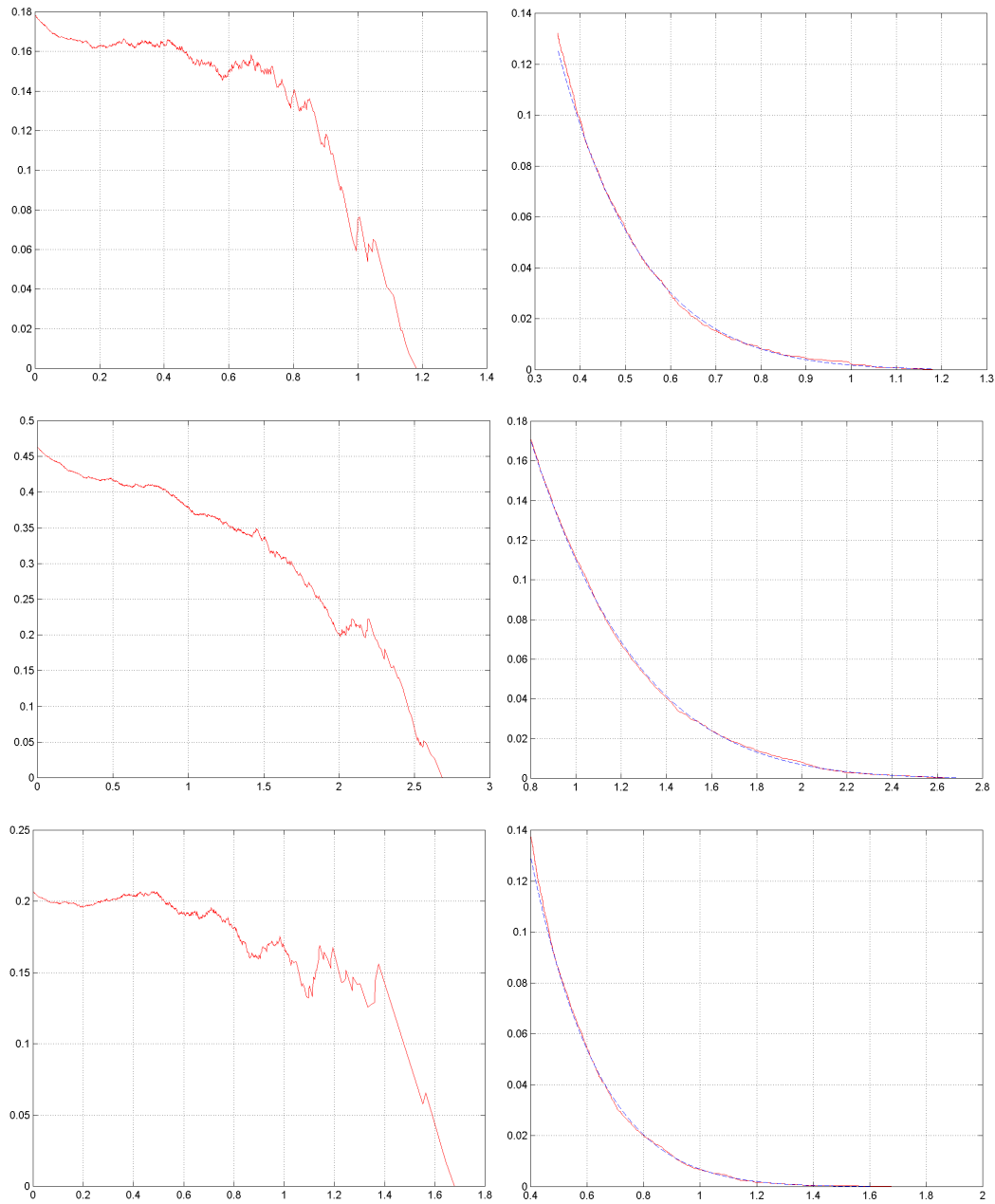


FIGURE 1. Length of the chord  $L^0$  at threshold  $\gamma = 0$  for the three examples of covariance function  $\rho$  (from top to bottom:  $n^{\circ}1$ ,  $n^{\circ}2$ ,  $n^{\circ}3$ ). On the left: the empirical mean excess plots; on the right: the tails of the distribution functions (solid line: sample, dashed line: approximating GPD model). The parameters are given in Table 2(a).



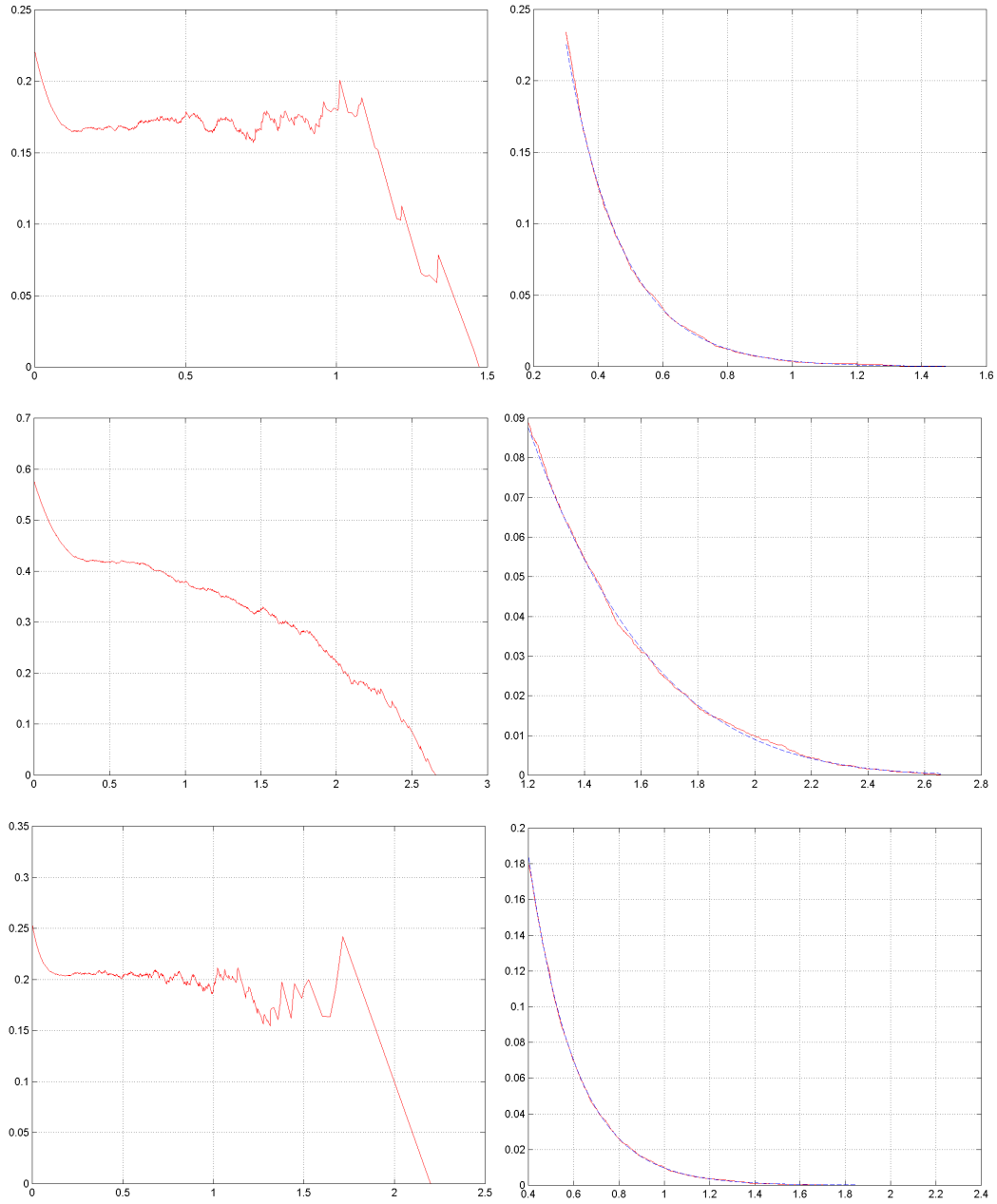


FIGURE 2. Length of the chord  $L^1$  at threshold  $\gamma = 0$  for the three examples of covariance function  $\rho$  (from top to bottom:  $n^{\circ}1$ ,  $n^{\circ}2$ ,  $n^{\circ}3$ ). On the left: the empirical mean excess plots; on the right: the tails of the distribution functions (solid line: sample, dashed line: approximating GPD model). The parameters are given in Table 2(b).

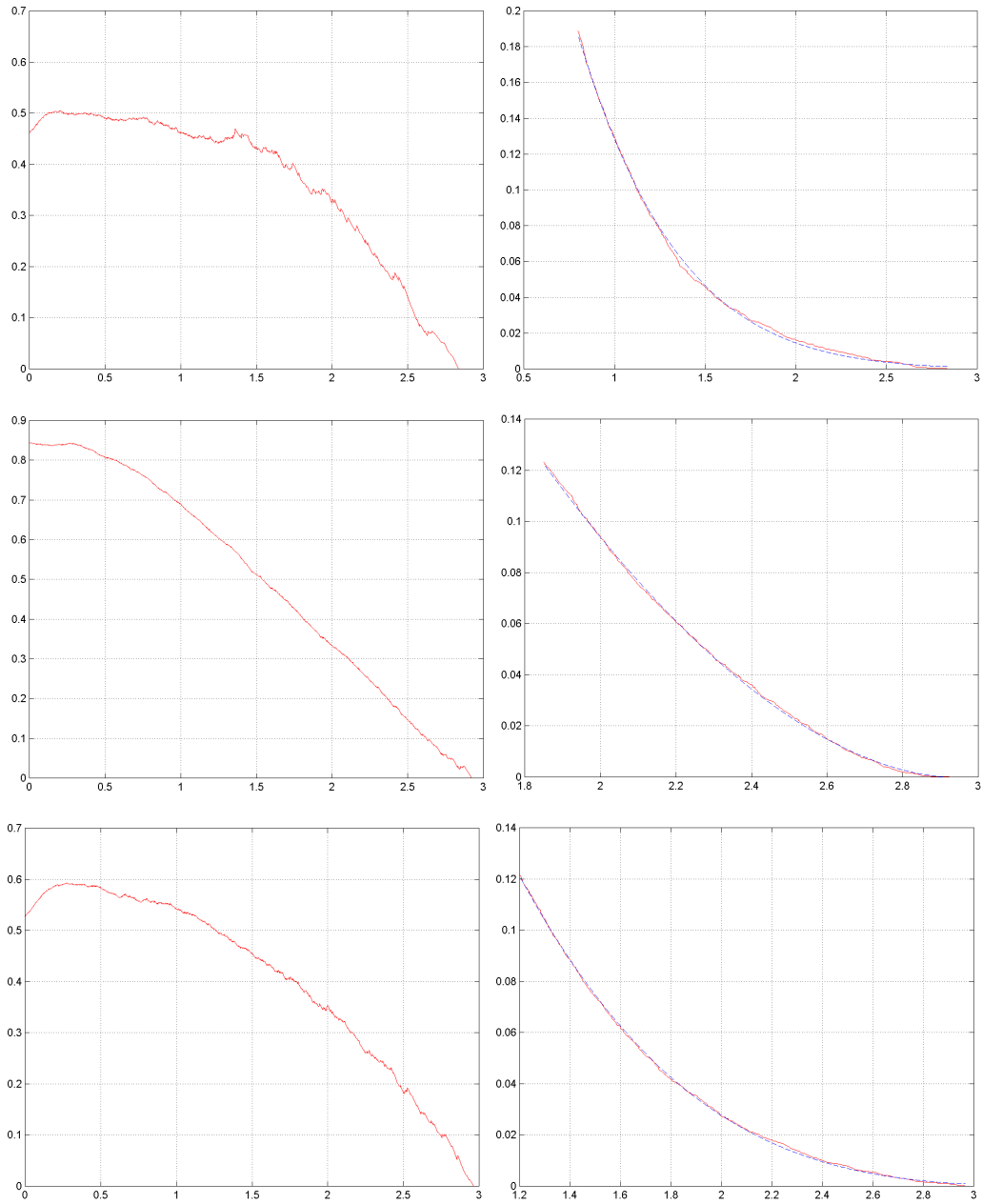


FIGURE 3. Length of the chord  $L^0$  at threshold  $\gamma = 1$  for the three examples of covariance function  $\rho$  (from top to bottom: n°1, n°2, n°3). On the left: the empirical mean excess plots; on the right: the tails of the distribution functions (solid line: sample, dashed line: approximating GPD model). The parameters are given in Table 3(a).

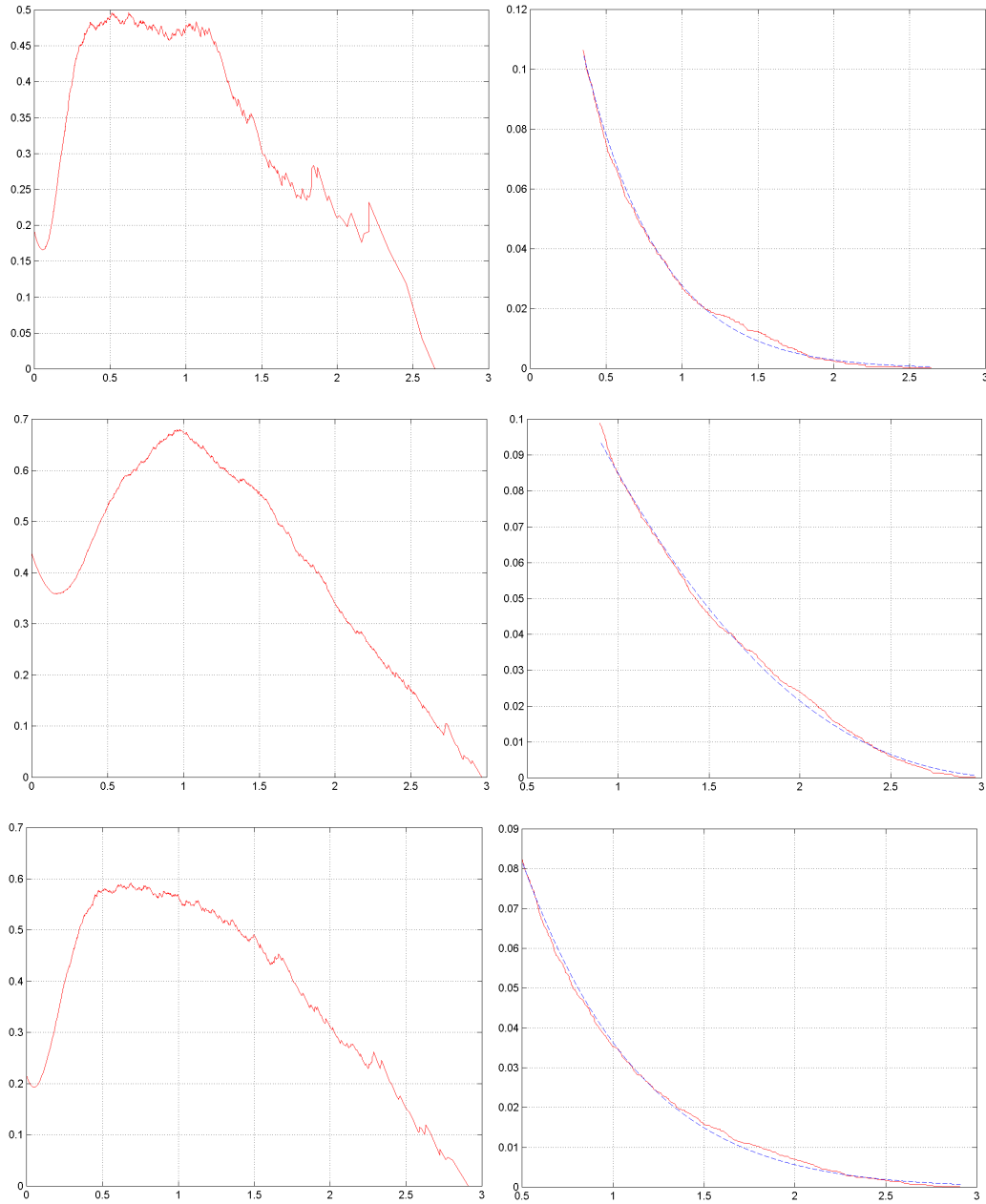


FIGURE 4. Length of the chord  $L^1$  at threshold  $\gamma = 1$  for the three examples of covariance function  $\rho$  (from top to bottom: n°1, n°2, n°3). On the left: the empirical mean excess plots; on the right: the tails of the distribution functions (solid line: sample, dashed line: approximating GPD model). The parameters are given in Table 3(b).

It is obvious that the empirical mean excess functions (the left plots of the above figures) appear to be decaying to zero, which is consistent with light tails of the chord-length distributions. Furthermore, the estimated values of the shape parameter  $\xi$  of the approximating GPD are all negative. Even though most of the 99% confidence intervals for  $\xi$  contain the origin, it is clear that the estimated shape parameter points towards light tails of the chord-length distributions as well. The validity of our procedure is confirmed by the good fit at the selected levels  $u$  that the approximating GPD provides to the sample cdf (see the right plots of figures 1 to 4). The results do not seem to differ significantly for the two chord-lengths, for the two chosen thresholds, and for the three covariance functions  $\rho$ . We note that a statistical test of an exponential decay could also be applied to the tails of the chord-length distributions; see e.g. [4].

**Conclusion:** our empirical analysis appears to indicate light tails of the chord-length, and the result does not seem to be sensitive to the choice of a threshold or a covariance function. To understand this phenomenon we turn now to a probabilistic analysis.

## 2. THEORETICAL BOUNDS FOR THE TAIL OF CHORD-LENGTH DISTRIBUTIONS

In this section we derive upper bounds for the tail of the chord-length distribution for certain families of stochastic processes, mostly stationary Gaussian processes, with appropriate assumptions on the memory of the process (on the covariance function if the process is Gaussian). We will see that under mild assumptions a faster than polynomially fast rate of decay of the chord-length distribution is obtained while stronger assumptions lead to an exponentially fast rate of decay. The bounds we obtain provide an explanation to the numerical results discussed in Section 1.

We start with introducing the terminology related to the speed of decay of the tail of distributions.

**Definition.** Let  $F$  be a mapping from  $[0, \infty)$  into  $[0, \infty)$ .

We say that  $F$  decays exponentially fast if

$$\exists \theta > 0, \exists T_0 > 0, \forall t \geq T_0, \quad F(t) \leq e^{-\theta t} \quad (4)$$

and decays faster than polynomially fast if

$$\forall \beta > 0, \quad \lim_{t \rightarrow \infty} t^\beta F(t) = 0. \quad (5)$$

Clearly, the two notions are not mutually exclusive.

Let  $X = (X_t, t \geq 0)$  be a continuously differentiable strictly stationary process with mean 0 and variance 1.

There is a natural dichotomy between the behavior of chord-lengths if the starting point  $X_0$  is below or above level  $\gamma$ . To account for this, we will write for any chord-length  $L^k$  and any  $t \geq 0$ ,

$$\mathbb{P}(L^k > t) = \mathbb{P}(L^k > t, X_0 > \gamma) + \mathbb{P}(L^k > t, X_0 \leq \gamma).$$

Then one estimates each term in the right hand side, essentially in the same way. Moreover, we invariably use a discretization of time.

## 2.1. General inequalities and example.

We start with two general inequalities. The first inequality will allow us to derive asymptotic upper bounds, under the stationary probability measure  $\mathbb{P}$ , on the tail of the distribution of any chord-length  $L^k$  from the ones obtained for  $L^0$ . The second inequality will make it possible to extend the asymptotic upper bounds on the tail of the chord-length distribution from the stationary probability  $\mathbb{P}$  to the Palm probability measures.

Finally, we exhibit a simple example in which the underlying process  $X$  has memory that does not last longer than  $m$  units of time. This example already contains the key idea used in the sequel.

### 2.1.1. Rate of decay of the tail of distribution of the $k$ -th chord-length.

**Proposition 2.1.** *For any  $k \geq 0$  and any  $t \geq 0$ , one has*

$$\mathbb{P}(L^k > t) \leq 2^k \mathbb{P}\left(L^0 > \frac{t}{(k+1)!}\right).$$

*In particular, if the tail of the distribution of  $L^0$  decays exponentially fast, or faster than polynomially fast, respectively, then so does the tail of the distribution of  $L^k$  for any  $k \geq 0$ .*

*Proof.* The inequality can be easily be proved by induction in  $k$ , beginning with  $k = 0$ , and using the following decomposition:

$$\begin{aligned} \mathbb{P}(L^k > t) &= \mathbb{P}(\exists j \in \{0, \dots, k-1\}, L^j > t/(k+1), L^k > t) \\ &\quad + \mathbb{P}(\forall j \in \{0, \dots, k-1\}, L^j \leq t/(k+1), L^k > t) \\ &\leq \sum_{j=0}^{k-1} \mathbb{P}(L^j > t/(k+1)) + \mathbb{P}\left(\bigcap_{s \in (kt/(k+1), t)} (X_s > \gamma) \cup \bigcap_{s \in (kt/(k+1), t)} (X_s < \gamma)\right) \end{aligned}$$

and noticing that, by stationarity, the last term in the right hand side above is less or equal to  $\mathbb{P}(L^0 > t/(k+1))$ .  $\square$

Therefore, from now on, we will derive bounds only for the tail of the distribution of the initial chord-length  $L = L^0$ .

**2.1.2. Chord-lengths under the Palm probability measures.** In practice it is of special interest to study the chord-length distribution from the moment a level crossing occurs. For that purpose one introduces the so-called *Palm probability measures*. For instance, to study the chord-length distributions occurring after an upcrossing of the level  $\gamma$ , we introduce the Palm probability measure (see for instance [1, 6]) defined by

$$\mathbb{P}_{0+}(A) = \frac{1}{\mu} \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \mathbb{P}(A \cap (U(-\tau, 0) \geq 1)), \quad A \in \mathcal{F},$$

where  $U(s, t)$  denotes the number of upcrossings in the interval  $(s, t)$  of the level  $\gamma$  by the process  $X$ . Further,  $\mu = E[U(0, 1)]$  is assumed to be finite; recall (see [6]) that under our assumptions,

$$\mathbb{P}(U(0, t) \geq 1) = \mu t + o(t) \quad \text{as } t \rightarrow 0. \quad (6)$$

The Palm probability measures  $\mathbb{P}_{0-}$ , describing the behavior of the process after a downcrossing, and  $\mathbb{P}_0$ , describing the behavior of the process after a crossing, are defined analogously.

The distributions of  $L$  under  $\mathbb{P}$  and under  $P_{0+}$  are linked by

$$P_{0+}(L > t) = -\frac{1}{\mu} \frac{\partial}{\partial t} \mathbb{P}(X_0 < \gamma, L > t); \quad (7)$$

see e.g. [1, 6, 21].

Even though the distributions of the chord-lengths under the Palm probability measures are of main interest, they are difficult to evaluate. Fortunately, we have the following:

**Proposition 2.2.** *For any  $t > 0$ , one has*

$$P_{0+}(L > 2t) \leq \frac{1}{t\mu} \mathbb{P}(L > t).$$

*In particular, if the tail distribution of  $L$  under the stationary probability measure  $\mathbb{P}$  decays exponentially fast, or faster than polynomially fast, respectively, then the same is true under the Palm probability measure  $P_{0+}$ .*

*Proof.* Writing  $P_{0+}(L > 2t) \leq \frac{1}{t} \int_t^{2t} P_{0+}(L > s) ds$  and using (7) allow to conclude.  $\square$

A similar argument gives the corresponding bounds for the other Palm probabilities:

$$P_{0-}(L > 2t) \leq \frac{1}{t\mu} \mathbb{P}(L > t) \text{ and } P_0(L > 2t) \leq \frac{1}{2t\mu} \mathbb{P}(L > t).$$

**2.1.3. Example: an  $m$ -dependent process  $X$ .** Recall that a stochastic process  $X$  is  $m$ -dependent if  $X_s$  and  $X_t$  are independent whenever  $|t - s| > m$ . For second order stationary processes, the  $m$ -dependence implies that the covariance function of the process vanishes after lag  $m$ , while for stationary Gaussian processes the converse statement is true as well. Processes with such covariance functions are commonly used in simulation (see e.g. [19]). It is the case for our example n°1 in the simulation part (see §1.1).

**Proposition 2.3.** *Assume that  $X$  is an  $m$ -dependent stationary process such that  $\mathbb{P}(X_0 > \gamma) > 0$  and  $\mathbb{P}(X_0 < \gamma) > 0$ . Then the tail of the distribution of the chord-length  $L$  decays exponentially fast.*

*Proof.* Let  $t > m' > m$  and  $n = \lceil t/m' \rceil \geq 1$ . Clearly,

$$\mathbb{P}(L > t, X_0 < \gamma) \leq \mathbb{P}(L > nm', X_0 < \gamma) \leq \prod_{k=0}^{n-1} \mathbb{P}(X_{km'} < \gamma) = \mathbb{P}(X_0 < \gamma)^{n+1} \leq e^{t \frac{\log \mathbb{P}(X_0 < \gamma)}{m'}}.$$

We deduce that

$$\mathbb{P}(L > t) \leq 2e^{-\theta t} \text{ for all } t > m',$$

with

$$\theta = \frac{1}{m'} \max \{ |\log \mathbb{P}(X_0 < \gamma)|, |\log \mathbb{P}(X_0 \geq \gamma)| \}.$$

This implies the exponentially fast rate of decay of the tail.  $\square$

We mention that the above result remains valid when the  $m$ -dependence property of the process  $X$  is replaced by certain strong mixing properties such as  $\psi$ -mixing (see [2] for a precise definition).

We now consider the case of stationary Gaussian processes under weaker assumptions than the  $m$ -dependence considered above.

## 2.2. Gaussian processes with vanishing memory.

Assume that the covariance function  $\rho$  of a stationary Gaussian process  $X$  tends to 0 at infinity; we will prove that the tail of the distribution of the associated chord-length  $L$  decays faster than polynomially fast. When having a fast rate of decay for  $\rho$ , the distribution  $L$  will decay even faster.

The proof is based on the application of Slepian's lemma, introducing a new Gaussian process whose covariance is compared with  $\rho$ . Without loss of generality, we may and will assume throughout the section that the stationary Gaussian process has zero mean and unit variance.

**Theorem 2.4.** *Let  $X$  be a stationary Gaussian process with covariance function  $\rho$  such that  $\rho(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

- (i) *The tail of the distribution of the chord-length  $L$  decays faster than polynomially fast.*
- (ii) *Moreover if  $\rho$  decays exponentially fast, then the distribution of  $L$  decays exponentially fast as well.*

*Proof.*

(i) Choose  $0 < a < 1$ , and let  $T$  be so large that  $\rho(s) \leq a$  for all  $s \geq T$ . We use the discretization

$$\mathbb{P}(L > t, X_0 > \gamma) \leq \mathbb{P}(X_{jT} > \gamma, j = 0, 1, \dots, [t/T]).$$

By Lemma 2.5 below,

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}(L > t, X_0 > \gamma)}{\log t} \leq \limsup_{t \rightarrow \infty} \frac{\mathbb{P}(X_{jT} > \gamma, j = 0, 1, \dots, [t/T]) \log[t/T]}{\log[t/T]} \leq -\frac{1-a}{a}.$$

Letting  $a \downarrow 0$  we obtain

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}(L > t, X_0 > \gamma)}{\log t} = -\infty.$$

Applying the above to the process  $-X$  we obtain also

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{P}(L > t, X_0 < \gamma)}{\log t} = -\infty,$$

and the two statements together give us the claim of the theorem.  $\square$

**Lemma 2.5.** *Let  $0 < a < 1$  and let  $Y_1, Y_2, \dots$  be a centered unit variance Gaussian process such that  $\text{Cov}(Y_i, Y_j) \leq a$  for all  $i \neq j$ . Then for any  $\gamma \in \mathbb{R}$ ,*

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbb{P}(Y_1 > \gamma, \dots, Y_n > \gamma)}{\log n} \leq -\frac{1-a}{a}. \quad (8)$$

*Proof.* Let  $W_0, W_1, W_2, \dots$  be i.i.d. standard normal random variables, and let  $Z_j = a^{1/2}W_0 + (1-a)^{1/2}W_j$ ,  $j = 1, 2, \dots$ . Then  $Z_1, Z_2, \dots$  is a discrete time centered unit variance Gaussian process such that  $\text{Cov}(Z_i, Z_j) = a$  for all  $i \neq j$ . By the Slepian inequality (see [20]) we know that for any  $n \geq 1$  and  $\gamma \in \mathbb{R}$ ,  $\mathbb{P}(Y_1 > \gamma, \dots, Y_n > \gamma) \leq \mathbb{P}(Z_1 > \gamma, \dots, Z_n > \gamma)$ . Therefore, it is enough to prove (8) with  $(Y_j)$  replaced by  $(Z_j)$ .

Choose any

$$0 < \theta < \frac{1-a}{a}$$

and write

$$\begin{aligned} \mathbb{P}(Z_1 > \gamma, \dots, Z_n > \gamma) &= \int_{-\infty}^{\infty} \phi(x) \left[ \Psi \left( \frac{\gamma - a^{1/2}x}{(1-a)^{1/2}} \right) \right]^n dx \\ &= \int_{x > (2\theta \log n)^{1/2}} + \int_{x \leq (2\theta \log n)^{1/2}} := I_1(n) + I_2(n), \end{aligned}$$

where  $\phi$  and  $\Psi$  are the density and the tail of a standard normal random variable, respectively. We use the bounds

$$\Psi(x) \leq \frac{1}{2} e^{-x^2/2} \quad \text{for } x \geq 0,$$

and

$$\Psi(x) > e^{-(1+\varepsilon)x^2/2} \quad \text{for } x \text{ large enough, and for any } \varepsilon > 0.$$

First of all,

$$I_1(n) \leq \Psi((2\theta \log n)^{1/2}) \leq \frac{1}{2} e^{-\theta \log n} = \frac{1}{2} n^{-\theta}. \quad (9)$$

On the other hand, selecting  $0 < \varepsilon < 1$  so small that

$$\frac{\theta a(1+\varepsilon)}{1-a} < 1, \quad (10)$$

we have for  $n$  large enough,

$$\begin{aligned} I_2(n) &\leq \left[ \Psi \left( \frac{\gamma - a^{1/2}(2\theta \log n)^{1/2}}{(1-a)^{1/2}} \right) \right]^n = \left[ 1 - \Psi \left( \frac{a^{1/2}(2\theta \log n)^{1/2} - \gamma}{(1-a)^{1/2}} \right) \right]^n \\ &\leq \left[ 1 - \exp \left\{ -\frac{(1+\varepsilon)a\theta \log n}{1-a} \right\} \right]^n = \left( 1 - n^{-(1+\varepsilon)a\theta/(1-a)} \right)^n, \end{aligned}$$

and, using (10), we see that for  $n$  large enough,

$$I_2(n) \leq \exp \left\{ -n^{1-(1+\varepsilon)a\theta/(1-a)}/2 \right\} = o(n^{-\theta}).$$

Combining this bound with (9), we conclude that

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbb{P}(Z_1 > \gamma, \dots, Z_n > \gamma)}{\log n} \leq -\theta.$$

Since  $\theta$  can be taken arbitrarily close to  $(1-a)/a$ , we obtain (8) for  $(Z_j)$  replacing  $(Y_j)$ , as required.

(ii) Let  $T > 0$  be the positive number from the definition (4) of the exponentially fast decay of  $\rho$ , and  $\theta > 0$  be the corresponding exponent. We discretize the time parameter of the process  $X$ , defining for  $n \geq 0$ ,  $Y_n = X(nT)$ . Then  $Y = (Y_n)_{n \geq 0}$  is a centered unit variance discrete time stationary Gaussian process, whose covariance function  $\rho_Y$  satisfies

$$\rho_Y(k) = \rho(kT) \leq e^{-\theta kT}, \quad k = 0, 1, 2, \dots \quad (11)$$

For  $t \geq T$ , let  $n = \lceil t/T \rceil \geq 1$ , and note that

$$\mathbb{P}(L > t) \leq \mathbb{P}(Y_j > \gamma, j = 0, \dots, n) + \mathbb{P}(Y_j < \gamma, j = 0, \dots, n).$$

We will only spell out the procedure for obtaining an upper bound for the first term in the right hand side above. We will prove that there exists  $\alpha_0 > 0$  and  $N_0 \geq 0$  such that

$$\text{for all } n \geq N_0, \quad \mathbb{P}(Y_j > \gamma, j = 0, \dots, n) \leq e^{-\alpha_0 n}. \quad (12)$$

This will, clearly, imply that the probability  $\mathbb{P}(L > t)$  decays exponentially fast.



As previously, we will use the Slepian lemma and introduce another centered unit variance discrete time stationary Gaussian process  $Z = (Z_n)_{n \geq 0}$ , with covariance function  $\rho_Z$  equal to the upper bound on the covariance function  $\rho_Y$  in (11), i.e.

$$\rho_Z(k) = r_0^k, \quad k = 0, 1, 2, \dots, \quad \text{with} \quad r_0 = e^{-\theta T}.$$

Such process  $Z$  does exist; in fact, it can be represented as a causal  $AR(1)$  process defined by

$$Z_{n+1} = r_0 Z_n + \xi_{n+1}, \quad n \geq 0,$$

where  $(\xi_n)_{n \geq 0}$  is a Gaussian white noise  $\mathcal{N}(0, 1 - r_0^2)$ . In particular,  $Z$  is a Markov process. This property is important since it allows to proceed in a similar way as in the  $m$ -dependent case. From the Slepian normal comparison lemma, we know it is enough to prove (12) for  $Z$  instead of  $Y$ .

The threshold  $\gamma$  in (12) can be of any sign. Obviously, once we prove the statement for  $\gamma < 0$ , its validity for any other  $\gamma$  will follow. Nonetheless, since there is a particularly simple argument in the case  $\gamma > 0$  that helps to understand the trick when tackling the case  $\gamma \leq 0$ , we present it first.

If  $\gamma > 0$ , we can use the simple bound

$$\mathbb{P}(Z_j > \gamma, j = 0, \dots, n) \leq \mathbb{P}\left(\sum_{j=0}^n Z_j > (n+1)\gamma\right). \quad (13)$$

The random variable  $\sum_{j=0}^n Z_j$  has the normal distribution  $\mathcal{N}(0, \sigma_n^2)$  with  $\sigma_n^2 \leq n \frac{1+r_0}{1-r_0}$ , so we obtain

$$\mathbb{P}(Z_j > \gamma, j = 0, \dots, n) \leq \Psi\left(\left(n\gamma \frac{1-r_0}{1+r_0}\right)^{1/2}\right) \leq e^{-\alpha_0 n}, \quad \text{for } n \text{ large enough,}$$

using once again the standard upper bound for  $\Psi$ .

Note that any  $\alpha_0 < \frac{\gamma(1-r_0)}{2(1+r_0)}$  can be used above.

When  $\gamma \leq 0$ , the estimate (13) is no longer sufficient for our purposes.

Let  $\gamma' = 3 - 2\gamma > 0$  and consider the event

$$A_n = \{\text{At least } \lceil n/3 \rceil \text{ out of } Z_0, Z_2, \dots, Z_{2n-2} \text{ are larger than } \gamma'\},$$

where  $\lceil \cdot \rceil$  denotes the ceiling function.

First, note that

$$\begin{aligned} \mathbb{P}((Z_j > \gamma, j = 0, \dots, 2n-1) \cap A_n) &\leq \mathbb{P}\left(\sum_{j=0}^{n-1} Z_{2j} > (n - \lceil n/3 \rceil)\gamma + \lceil n/3 \rceil \gamma'\right) \\ &\leq \mathbb{P}\left(\sum_{j=0}^{n-1} Z_{2j} > n\right). \end{aligned}$$

This puts us in a situation analogous to the case  $\gamma > 0$  above, and using the same argument we obtain that for any

$$\alpha_1 < \frac{1 - r_0^2}{2(1 + r_0^2)}$$

there is  $N_1$  such that for all  $n \geq N_1$ ,

$$\mathbb{P}((Z_j > \gamma, j = 0, \dots, 2n - 1) \cap A_n) \leq e^{-\alpha_1 n}. \quad (14)$$

In order to estimate  $\mathbb{P}((Z_j > \gamma, j = 0, \dots, 2n - 1) \cap \overline{A_n})$ , let us introduce the sets  $B_n$ ,  $n \geq 1$  defined by

$$B_n = \left\{ (b_1, \dots, b_n) \in \mathbb{R}^n : b_j > \gamma, j = 1, \dots, n, \text{ and at most } \lceil n/3 \rceil - 1 \text{ of the } b_j\text{'s are larger than } \gamma' \right\}.$$

For every  $n \geq 1$  we define a function on  $\mathbb{R}^n$  by

$$Q_n(z_0, \dots, z_{2n-2}) = \mathbb{P}(Z_{2j+1} > \gamma, j = 0, \dots, n-1 \mid Z_{2j} = z_{2j}, j = 0, \dots, n-1),$$

$(z_0, \dots, z_{2k}, \dots, z_{2n-2}) \in \mathbb{R}^n$ , in the sense of the usual continuous conditional probabilities, and write

$$\mathbb{P}(\{Z_j > \gamma, j = 0, \dots, 2n - 1\} \cap \overline{A_n}) = \int_{B_n} Q_n(z_0, \dots, z_{2n-2}) \phi_n(z_0, \dots, z_{2n-2}) dz_0 \dots dz_{2n-2}, \quad (15)$$

where  $\phi_n$  is the joint pdf of  $(Z_0, Z_2, \dots, Z_{2n-2})$ .

Given  $(z_0, z_1, \dots, z_{2n-2}, z_{2n-1})$  such that the vector of the even-numbered coordinates  $(z_0, z_2, \dots, z_{2n-2})$  is in  $B_n$ , the latter vector has at least  $n - \lceil n/3 \rceil + 1$  coordinates which are strictly smaller than  $\gamma'$ . Elementary counting shows that there are at least  $\lceil n/3 \rceil$  odd numbers  $j_1 < j_2 < \dots < j_{\lceil n/3 \rceil}$  in the set  $\{1, 3, \dots, 2n - 3\}$  such that

$$\text{for every } k \in \{1, \dots, \lceil n/3 \rceil\} \quad z_{j_k-1} < \gamma' \quad \text{and} \quad z_{j_k+1} < \gamma'.$$

It follows from the Markov property of the process  $Z$  that

$$\begin{aligned} Q_n(z_0, \dots, z_{2n-2}) &\leq \mathbb{P}\left(Z_{j_1} > \gamma, Z_{j_2} > \gamma, \dots, Z_{j_{\lceil n/3 \rceil}} > \gamma \mid Z_{2j} = z_{2j}, j = 0, \dots, n-1\right) \\ &= \prod_{k=1}^{\lceil n/3 \rceil} \mathbb{P}(Z_{j_k} > \gamma \mid Z_{j_k-1} = z_{j_k-1}, Z_{j_k+1} = z_{j_k+1}). \end{aligned}$$

Recall that all  $z$ -values in the conditions are smaller than  $\gamma'$ .

Given  $(Z_{j_k-1} = z_{j_k-1}, Z_{j_k+1} = z_{j_k+1})$ , the random variable  $Z_{j_k}$  has the normal distribution  $\mathcal{N}(\mu_k, \sigma^2)$  with

$$\mu_k = \frac{r_0}{1 + r_0^2} (z_{j_k-1} + z_{j_k+1}) < \frac{2\gamma' r_0}{1 + r_0^2} \quad \text{and} \quad \sigma^2 = \frac{1 + r_0^4}{(1 + r_0^2)^2} > 0.$$

Therefore

$$Q_n(z_0, \dots, z_{2n-2}) \leq \left( \Psi \left( \frac{\gamma(1 + r_0^2) - 2\gamma' r_0}{\sqrt{1 + r_0^4}} \right) \right)^{\lceil n/3 \rceil},$$

and it follows from (15) that for all  $n \geq 1$ ,

$$\mathbb{P}(\{Z_0 > \gamma, \dots, Z_{2n-1} > \gamma\} \cap \overline{A_n}) \leq e^{-\alpha_2 n},$$

with  $\alpha_2 = \Psi\left(\frac{\gamma(1+r_0^2)-2\gamma'r_0}{\sqrt{1+r_0^4}}\right)$ . Combining this bound with (14), shows that there exists  $N_2$  such that for all  $n \geq N_2$ ,

$$\mathbb{P}(Z_0 > \gamma, \dots, Z_{2n-1} > \gamma) \leq e^{-\alpha_3 n},$$

where we can use any  $\alpha_3 < \min(\alpha_1, \alpha_2)$ . This proves (12) with any  $\alpha_0 < \alpha_3/2$  and some  $N_0$  large enough.  $\square$

Since a stationary Gaussian process is mixing if and only if its covariance function converges to zero (see [5]), Theorem 2.4 (i) states that any mixing stationary Gaussian process has the property that the distribution of  $L$  decays faster than polynomially fast. We do not know at the moment if this property holds also for non-Gaussian mixing stationary processes. Nevertheless, the conclusion holds under a stronger dependence condition when assuming a non-Gaussian process is  $r$ -mixing (see[8]).

### 2.3. $r$ -mixing processes.

Recall that the  $r$ -mixing coefficients of a process  $X$  are defined by

$$r_T = r(A, B_T) = \sup_{\substack{W_1 \in L_X^2(A) \\ W_2 \in L_X^2(B_T)}} |\text{Corr}(W_1, W_2)|, \quad T > 0,$$

where  $A = (-\infty, 0]$  and  $B_T = [T, \infty)$ , and for  $D \subset \mathbb{R}$ ,  $L_X^2(D)$  denotes the closure in  $L^2(\Omega)$  of  $\text{Span}\{X_t, t \in D\}$ . See [8].

In particular, for any  $T > 0$  and events  $C_1, C_2$  such that  $W_1 = \mathbb{1}_{C_1} \in L_X^2(A)$  and  $W_2 = \mathbb{1}_{C_2} \in L_X^2(B_T)$ , we have

$$r_T \geq \frac{|\mathbb{P}(C_1 \cap C_2) - \mathbb{P}(C_1)\mathbb{P}(C_2)|}{(\mathbb{P}(C_1)\mathbb{P}(\overline{C_1})\mathbb{P}(C_2)\mathbb{P}(\overline{C_2}))^{1/2}}. \quad (16)$$

We will say that  $X$  is an  $r$ -mixing process if it satisfies

$$\lim_{T \rightarrow +\infty} r_T = 0. \quad (17)$$

**Theorem 2.6.** *Assume that  $X$  is a stationary  $r$ -mixing process such that  $\mathbb{P}(X_0 > \gamma) > 0$  and  $\mathbb{P}(X_0 < \gamma) > 0$ . Then the tail of the distribution of the chord-length  $L$  decays faster than polynomially fast.*

*Proof.* Fix  $T > 0$  large enough so that  $r_T < \min(\mathbb{P}(X_0 < \gamma), \mathbb{P}(X_0 > \gamma))$ .

Define, for  $n \geq 0$ ,  $Y_n = X(nT)$  and consider the discrete time process  $Y = (Y_n)_{n \geq 0}$ .

For  $n \geq 0$  we apply (16) with  $C_1 = \{X(-(2^n - 1)T) > \gamma, \dots, X(0) > \gamma\}$  and  $C_2 = \{X(T) > \gamma, \dots, X(2^n T) > \gamma\}$ . By stationarity,

$$\begin{aligned} \mathbb{P}(Y_0 > \gamma, \dots, Y_{2^{n+1}-1} > \gamma) &= \mathbb{P}(\{Y_0 > \gamma, \dots, Y_{2^n-1} > \gamma\} \cap \{Y_{2^n} > \gamma, \dots, Y_{2^{n+1}-1} > \gamma\}) \\ &\leq (\mathbb{P}(Y_0 > \gamma, \dots, Y_{2^n-1} > \gamma))^2 + r_T \mathbb{P}(Y_0 > \gamma, \dots, Y_{2^n-1} > \gamma). \end{aligned}$$

Denoting  $p_n = \mathbb{P}(Y_0 > \gamma, \dots, Y_{2^n-1} > \gamma)$  for  $n \geq 0$ , we see that

$$p_{n+1} \leq p_n^2 + r_T p_n, \quad n = 0, 1, \dots \quad (18)$$

which implies that  $p_n \rightarrow 0$  as  $n \rightarrow \infty$  (if  $l = \lim_{n \rightarrow \infty} p_n \neq 0$ , then  $l > p_0$  which would contradict  $(p_n)$  decreasing).

Therefore, there exists  $N_0 \geq 0$  such that for all  $n \geq N_0$  one has  $p_n \leq r_T$ . Thus

$$p_{n+1} \leq 2r_T p_n \quad \text{for all } n \geq N_0$$

and, hence,

$$p_n \leq c_0 (2r_T)^n \quad \text{with } c_0 = p_{N_0} (2r_T)^{-N_0} > 0 \text{ for all } n \geq N_0.$$

Now let  $t > 2^{N_0} T$  be so large that  $n = \lceil \log_2(t/T) \rceil \geq N_0$  and remark that  $r_T < 1/2$ . We have,  $\log_2$  denoting the logarithm in base 2,

$$\mathbb{P}(L > t, X_0 > \gamma) \leq \mathbb{P}(Y_0 > \gamma, \dots, Y_{2^n-1} > \gamma) \leq c_0 (2r_T)^n \leq c'_0 t^{-|\log_2(2r_T)|}$$

for a certain constant  $c'_0 > 0$  depending only on  $N_0$  and  $T$ . We deduce that

$$\overline{\lim}_{t \rightarrow \infty} \frac{\log \mathbb{P}(L > t, X_0 > \gamma)}{\log t} \leq -|\log_2(2r_T)|.$$

Since  $r_T \rightarrow 0$  as  $T \rightarrow \infty$ , we conclude that

$$\lim_{t \rightarrow \infty} \frac{\log \mathbb{P}(L > t, X_0 > \gamma)}{\log t} = -\infty.$$

Hence  $t^\beta \mathbb{P}(L > t, X_0 > \gamma) \rightarrow 0$  as  $t \rightarrow \infty$ , for any  $\beta > 0$ .

An analogous result holds for  $\mathbb{P}(L > t, X_0 \leq \gamma)$ . Therefore the distribution of  $L$  decays faster than polynomially fast.  $\square$

Recall that a stationary Gaussian process with a covariance function which decays in a polynomial way is  $r$ -mixing (see [5]). So an immediate corollary of Theorem 2.6 is the fact that for Gaussian processes with polynomially decaying covariance function, the distribution of the chord-length  $L$  decays faster than polynomially fast, a result obtained in Theorem 2.4 (i) under a weaker assumption.

## CONCLUSION

We have established theoretical results on the rate of decay of the tail of the distribution of the chord-lengths, depending on the memory in the thresholded process. In the case of stationary Gaussian processes, the memory is expressed via the covariance function. The results agree with the empirical results obtained by statistical analysis of simulated processes. We focused on the asymptotic behavior of the tails of the distributions and did not try to derive the best constants in the upper bounds for such tails. It may be possible to refine these bounds and relate them to the threshold. It may further be possible to explore how the bounds change from one chord-length to the next. This will be the subject of a future work. We also plan to investigate what memory properties of the thresholded process imply *lower bounds* on the tails of the chord-length distributions.

We hope that our work will also contribute to study of the  $D$ -dimensional Boolean models for  $D > 1$ . For instance, if  $L^*$  is the spherical contact distance of the phase containing the origin, upper bounds for  $\mathbb{P}(L^* > t)$  can be obtained from our results since

$$L^* = \sup\{R > 0 : B(0, R) \subset 0\text{-phase}\} = \inf_{\alpha \in [0, 2\pi)} L(\alpha),$$

where the 0-phase is the phase containing the origin and  $L(\alpha)$  denotes the initial chord-length in the direction  $\alpha$ .

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