ORIE 6300 Mathematical Programming I October 6, 2014

Problem Set 6

Due Date: October 13, 2016

1. (Bertsimas and Tsitsiklis 3.6) Let x be a basic feasible solution with associated basis B. Prove the following:

   (a) (4 points) If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.

   (b) (4 points) If x is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

2. Consider the linear program $\min(c^T x : x \geq 0, Ax = b)$. Let B denote an optimal basis. Assume that the problem is generic in that each vertex has a unique basis for which it is the corresponding basic solution.

   Assume now that you want to solve a *parametric* problem, i.e., a set of problems of the form $\min((c + \lambda d)^T x : x \geq 0, Ax = b)$, for each possible value of $\lambda \geq 0$. Assume that for any $\lambda \geq 0$ the problem has an optimal solution and that the basis B is a solution for the problem when $\lambda = 0$.

   (a) Prove that the set of values of λ for which basis B is optimal forms an interval $[0, a_1]$. Explain how to compute a_1.

   (b) Show that there is a finite set $a_0 = 0 \leq a_1 \leq \ldots \leq a_k$ and corresponding bases B_i for $i = 0, \ldots, k$ such that $B_0 = B$ and B_i (for $i = 0, \ldots, k$) is the optimal basis if and only if $\lambda \in [a_i, a_{i+1}]$, and B_k is optimal if $\lambda \geq a_k$.

3. Let γ and τ be positive real numbers. Consider the nonlinear operator

$$T : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$
$$T \begin{bmatrix} y \\ x \end{bmatrix} := \begin{bmatrix} y - \gamma\,(Ax - b) \\ \max\{x + \tau\left(A^T y - c\right), 0\} \end{bmatrix},$$

   where $\max\{x + \tau(A^T y - c), 0\} \in \mathbb{R}^n$. In addition, consider the primal-dual pair of linear programs:

$$\min\{c^T x \mid Ax = b, x \geq 0\} \qquad \text{and} \qquad \max\{b^T y \mid A^T y \leq c\}.$$

   Prove that a pair $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ is primal-dual optimal, if, and only if,

$$T \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$