ORIE 6300 Mathematical Programming I September 8, 2016

Problem Set 3

Due Date: September 15, 2016

1. **Alternatives.** The Theorem of Alternatives (proved in lecture 6) states that for a given matrix $A \in \mathbb{R}^{m \times n}$ and a given vector $b \in \mathbb{R}^m$, exactly one of the following two statements holds:

   (a) $\exists x \in \mathbb{R}^n$ such that $Ax \leq b$.

   (b) $\exists y \in \mathbb{R}^m$ such that $y \geq 0$, $A^T y = 0$, and $b^T y < 0$.

   Use the Theorem of Alternatives to prove

   - **Farkas' Lemma:** Given a matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following hold:
     - $\exists x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.
     - $\exists y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.
   - **Mixed Theorem of Alternatives.** Given $A \in \mathbb{R}^{m_1 \times n}$, $C \in \mathbb{R}^{m_2 \times n}$, and $b \in \mathbb{R}^{m_1}, d \in \mathbb{R}^{m_2}$, exactly one of the following hold:
     - $\exists x \in \mathbb{R}^n$ such that $Ax \leq b$ and $Cx = d$.
     - $\exists z \in \mathbb{R}^{m_1}, y \in \mathbb{R}^{m_2}$ such that $z \geq 0$, $A^T z + C^T y = 0$, and $b^T z + d^T y < 0$.

2. **Normal Cones and Strong Duality.** Given $A, \overline{A}, \hat{A}, \widetilde{A}, b, \overline{b}, c, \overline{c}$:

   (a) for all $z = \begin{bmatrix} x & \overline{x} \end{bmatrix}^T$, compute $N_P(z)$, where

   $$P = \left\{ \begin{bmatrix} x \\ \overline{x} \end{bmatrix} \mid Ax + \overline{A}\overline{x} \leq b, \hat{A}x + \widetilde{A}\overline{x} = \overline{b}, x \geq 0 \right\}.$$

   (b) for all $w = \begin{bmatrix} y & \overline{y} \end{bmatrix}^T$, compute $N_D(w)$, where

   $$D = \left\{ \begin{bmatrix} y \\ \overline{y} \end{bmatrix} \mid A^T y + \hat{A}^T \overline{y} \geq c, \overline{A}^T y + \widetilde{A}^T \overline{y} = \overline{c}, y \geq 0 \right\}.$$

   (c) Using the descriptions of the above normal cones, and **without using the strong duality theorem we proved in class**, prove the following variant of the strong duality theorem: suppose that both of the linear programs

   $$\max_{\begin{bmatrix} x \\ \overline{x} \end{bmatrix} \in P} c^T x + \overline{c}^T \overline{x} \qquad\qquad \min_{\begin{bmatrix} y \\ \overline{y} \end{bmatrix} \in D} b^T y + \overline{b}^T \overline{y}$$

   have finite optimal values, which we denote by p^* and d^*, respectively. Then $p^* = d^*$.

3. **Subgradients.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous, convex function, i.e., f satisfies

$$(\forall \lambda \in [0,1])\,(\forall x, y \in \mathbb{R}^n) \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Define the *epigraph* of f:

$$\mathbf{epi}(f) := \{(x,t) \in \mathbb{R}^{n+1} \mid f(x) \leq t\}.$$

- Prove that $\mathbf{epi}(f)$ is closed and convex.
- Prove that $\begin{bmatrix} v & -1 \end{bmatrix}^T \in N_{\mathbf{epi}(f)}(x, f(x))$ if, and only if,

  $$(\forall z \in \mathbb{R}^n) \quad f(z) \geq f(x) + (z-x)^T v.$$

  Such vectors $v \in \mathbb{R}^n$ are called *subgradients* f *at* x, and the set of all such v is denoted by $\partial f(x)$.
- Prove Fermat's rule: x minimizes f if, and only if, $0 \in \partial f(x)$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. Draw the following normal cones: $N_{\mathbf{epi}(f)}(-1,1), N_{\mathbf{epi}(f)}(0,0)$, and $N_{\mathbf{epi}(f)}(1,1)$.

4. **Vertices of Dual Problems.**

- Prove the following lemma: consider the polyhedron $Q := \{x \mid x \geq 0, Cx = d\}$. Then
  - $\overline{x} \in Q$ is an extreme point if

    $$\operatorname{rank}\left(\begin{bmatrix} c_{i_1} & c_{i_2} & \dots & c_{i_k} \end{bmatrix}\right) = k$$

    where c_j is column j of C and $\{i_1, i_2, \dots, i_k\} = \{i \mid \overline{x}_i > 0\}$.
  - any extreme point of Q has at most $\operatorname{rank}(C)$ nonzero elements.
- Suppose that the following linear program has a solution:

  $$\begin{aligned} \text{minimize } & b^T y \\ \text{subject to: } & A^T y = c \\ & y \geq 0 \end{aligned}$$

  Using Problem 4 from homework 2, conclude that at least one solution to this problem is a vertex. Use the first part of this exercise to show that there exists a solution y^* with at most $\operatorname{rank}(A^T)$ nonzero elements.
- Using part 2, prove **Carathéodory's Theorem**: if $x \in \mathbb{R}^n$ is the convex combination of k vectors, $v_1, \dots, v_k$, then x is also the convex combination of at most $n+1$ of the vectors $v_1, \dots, v_k$.