ORIE 6300 Mathematical Programming I	August 25, 2016

Problem Set 1

Due Date: September 1, 2016

1. Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Suppose there exists $d \in \mathbb{R}^n$ such that
   (a) $Ad = 0$; and
   (b) $d > 0$.

   Show that the system of **strict** inequalities $A^T x < 0$ has no solution.

2. Show that the following two statements are equivalent
   (a) Let $A \in \mathbb{R}^{m \times n}$ and let $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$ has a nonnegative solution if, and only if, every $y \in \mathbb{R}^m$ with $y^T A \geq 0$ also satisfies $y^T b \geq 0$.
   (b) Let $\hat{A} \in \mathbb{R}^{\hat{m} \times \hat{n}}$ and let $\hat{b} \in \mathbb{R}^{\hat{m}}$ be a vector. The system $\hat{A}\hat{x} \leq \hat{b}$ has a nonnegative solution if, and only if, every nonnegative $\hat{y} \in \mathbb{R}^{\hat{m}}$ with $\hat{y}^T \hat{A} \geq 0$ also satisfies $\hat{y}^T \hat{b} \geq 0$.

3. Let $A \in \mathbb{R}^{m \times n}$ and let $b \in \mathbb{R}^m$ be a vector. Consider the following compressive sensing problem:

$$\begin{aligned} &\text{minimize } \|x\|_1 \\ &\text{subject to: } Ax = b. \end{aligned}$$

   (a) Reformulate this problem as an equivalent LP and
   (b) take its dual. Simplify the dual LP until the decision variable is a vector $z \in \mathbb{R}^m$.

4. What is the dual of the linear program with variables $x \geq 0$ and an additional single variable λ with constraints $Ax \leq \lambda b$, where the objective is to minimize λ?

5. Consider the following LP:

$$\begin{aligned} \text{Max} \sum_{i=1}^{n} v_i x_i & & & \\ \sum_{i=1}^{n} s_i x_i &\leq B & & \\ x_i &\leq 1 & & i = 1, \ldots, n \\ x_i &\geq 0 & & i = 1, \ldots, n \end{aligned}$$

   (a) Show the dual of the LP above. Use the variable y_0 for the constraint with right-hand side B, and the variables y_i for the $x_i \leq 1$ constraints.

(b) Assume that the numbers v_i and s_i are positive and that

$$\frac{v_1}{s_1} \geq \frac{v_2}{s_2} \geq \cdots \geq \frac{v_n}{s_n}.$$

Let k be the largest index such that $s_1 + s_2 + \cdots + s_{k-1} \leq B$. Show that the following primal and dual LP solutions must be optimal:

$$x_i = \begin{cases} 1 & i < k \\ \frac{B-(s_1+s_2+\cdots+s_{k-1})}{s_k} & i = k \\ 0 & i > k \end{cases}$$

$$y_i = \begin{cases} \frac{v_k}{s_k} & i = 0 \\ s_i\left(\frac{v_i}{s_i} - \frac{v_k}{s_k}\right) & 0 < i < k \\ 0 & i \geq k \end{cases}$$