1 Last Time

- We shifted to the standard form \((A \in \mathbb{R}^{m \times n})\).

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\end{align*}
\]

Also, \(P(A, b) = \{x \mid A^T x = b, x \geq 0\}\).

- **Assumption:** From now on, we assume that the rows of \(A\) are linearly independent \((m \leq n)\).

- **Proposition 1** For \(P(A, b) = \{x \mid A^T x = b, x \geq 0\}\), \(\bar{x} \in P(A, b)\) is a vertex if and only if the columns corresponding to positive components of \(\bar{x}\) are linearly independent.

- **Definition 1** A set \(B\) of \(m\) columns of \(A\) \((A \in \mathbb{R}^{m \times n})\) is a basis if \(\text{rank}(A_B) = m\) \((A_B\) is invertible).

- **Proposition 2** Every vertex of \(P(A, b)\) is a basic feasible solution corresponding to some basis.

2 Verifying optimality

Given a vertex \(\bar{x} \in P(A, b)\), we want to find a “verifying \(\bar{y}\)” which we use to check whether \(\bar{x}\) is optimal. Clearly, any \(\bar{y}\) should satisfy complementary slackness, i.e.

\[\bar{x}_i > 0 \iff (A^T \bar{y})_i = c_i.\]

In order to find \(\bar{y}\), we first prove the following proposition.

**Proposition 3** Suppose \(\bar{x} \in P(A, b)\) has \(m\) non-zero components corresponding to basis \(B\). Then, \(\bar{x}\) is optimal if and only if there exists \(y \in Q(A^T, c)\) such that \(A_B^T y = c_B\).

**Proof:** By theorem from previous lectures, we know that \(\bar{x}\) is optimal if and only if

\[-c \in N_{P(A,b)}(\bar{x}) = \{\hat{c} \mid \exists \hat{y} \in Q(A^T, \hat{c}), \text{s.t.} \quad \bar{x}^T (\hat{c} - A^T \hat{y}) = 0\},\]

which is then equivalent to the condition

\[\exists y \in Q(A^T, c) \text{ s.t. } x_i > 0 \Rightarrow (A^T y)_i = c_i.\]
Note that $x_i > 0$ if and only if $i \in B$, so the last condition can be translated into

$$\exists y \in Q(A^T, c) \text{ s.t. } A_B^T y = c_B.$$  

We also have the following definition regarding non-degeneracy.

**Definition 2** A vertex $\pi \in P(A, b)$ is called non-degenerate if it has exactly $m$ nonzero components and those components correspond to some basis $B$.

For each non-degenerate $\pi$ with basis $B$, we will try to find a $y \in Q(A^T, c)$ such that $A_B^T y = c_B$. Then, $(c - A^T y)_i = 0$ when $i \in B$. For simplicity of notation and computation, we create the following definition.

**Definition 3** For any $y \in \mathbb{R}^m$, the reduced cost $\bar{c}$ with respect to $y$ is $\bar{c} = c - A^T y$.

**Lemma 4** For all $x \in P(A, b)$, $\text{range}(A^T) + N_{P(A, b)}(x) = N_{P(A, b)}(x)$.

**Proof:** Clearly, $N_{P(A, b)}(x) \subseteq \text{range}(A^T) + N_{P(A, b)}(x)$ because $0 \in \text{range}(A^T)$. On the other hand, let $A^T y_0$ be any element in $\text{range}(A^T)$ and $-\hat{c}$ be any element in $N_{P(A, b)}(x)$. Then, there exists $\hat{y}$ such that $A^T \hat{y} \leq \hat{c}$ and $x^T (\hat{c} - A^T \hat{y}) = 0$. Tweaking the equations a bit, we get

$$A^T (\hat{y} - y_0) \leq \hat{c} - A^T y_0$$

and

$$x^T (\hat{c} - A^T y_0 - A^T (\hat{y} - y_0)) = 0.$$  

Thus,

$$A^T y_0 - \hat{c} \in N_{P(A, b)}(x),$$

which means that $\text{range}(A^T) + N_{P(A, b)}(x) \subseteq N_{P(A, b)}(x)$.  

A simple example to illustrate Lemma 5 is when $A = [1, 1]$, $b = 1$ and $c \in \mathbb{R}^2$. $P(A, b)$ in this case is a line segment connecting $(1, 0)$ and $(0, 1)$. If $c$ is in $N_{P(A, b)}$, then $c + \begin{bmatrix} y \\ y \end{bmatrix}$ is still in $N_{P(A, b)}$.  

![Graphical Illustration of a Line Segment and Non-Degenerate Points](image-url)
From Lemma 5, we can easily reach the following conclusion.

**Lemma 5** Consider the two LPs

\[
\begin{align*}
(LP1) \quad & \min \quad c^T x \\
& \text{s.t.} \quad Ax = b \\
& \quad x \geq 0 \\
(LP2) \quad & \min \quad \bar{c}^T x \\
& \text{s.t.} \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

These LPs have the exact same optimal solution.

**Proof:** We know that \( \bar{x} \) solves (1) if and only if \(-c \in N_{P(A, b)}(\bar{x})\). By Lemma 5, we have

\[
N_{P(A, b)}(\bar{x}) = \text{range}(A^T) + N_{P(A, b)}(\bar{x}),
\]

so \( c \in N_{P(A, b)}(\bar{x}) \) if and only if \( \bar{c} = -c + A^T y \in N_{P(A, b)}(\bar{x}) \), which equivalent to \( \bar{x} \) being an optimal solution of (2). \( \square \)

We also have the following observation regarding \( \bar{c} \).

**Observation 1** \( y \in Q(A^T, c) \iff A^T y \leq c \iff c = A^T y \geq 0 \).

Using the conclusions above, we can now work with \( \bar{c} \) instead of \( c \).

### 3 Some Simplex Method-type Computations

Consider a basis \( B \) and the new linear program with reduced cost \( \bar{c} \).

\[
(LP2) \quad \min \quad \bar{c}^T x \\
\text{s.t.} \quad Ax = b \\
\quad x \geq 0.
\]

We reorganize the vectors and matrices so that \( x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, A = [A_B \ A_N] \) and \( \bar{c} = \begin{bmatrix} \bar{c}_B \\ \bar{c}_N \end{bmatrix} \). Note that \( x_B > 0, x_N = 0 \) and

\[
[\begin{bmatrix} A_B \\ A_N \end{bmatrix}] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b.
\]

Multiplying through by \( A_B^{-1} \), we convert LP2 into

\[
\begin{align*}
& \min \quad \bar{c}^T x_B + \bar{c}^T x_N \\
& \text{s.t.} \quad I x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
& \quad x \geq 0.
\end{align*}
\]

Now set \( y = A_B^{-T} c_B \) and \( \bar{c} = c - A^T y \). Then, \( \bar{c}_B = c_B - A_B^T A_B^{-T} c_B = 0 \). Our problem becomes

\[
\begin{align*}
& \min \quad \bar{c}^T x_N \\
& \text{s.t.} \quad x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
& \quad x \geq 0.
\end{align*}
\]

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We can further simplify the problem by discarding $x_B$. Then, the LP looks like

\[ (LP3) \quad \min \; c^T x_N \]
\[ \text{s.t.} \quad A_B^{-1} A_N x_N \leq A_B^{-1} b \]
\[ x_N \geq 0. \]

If $y \in Q(A^T, c)$, then $\pi = c - A^T y \geq 0$. Because $\pi^T x_N \geq 0$, the optimal value of the reduced problem is bounded below by 0. Thus, if $A_B^{-1} b \geq 0$, then $x_N = 0$ is feasible and $\pi^T 0 = 0$, so $x_N = 0$ is an optimal solution. In the original problem, $x = \begin{bmatrix} A_B^{-1} b \\ 0 \end{bmatrix}$ is optimal.

Now suppose $B$ is an arbitrary basis. Let $x_N = 0$ and $x_B = A_B^{-1} b$. Based on the analysis above, we devise the following simplex scheme for finding the optimal solution:

- If $A_B^{-1} b \geq 0$, then $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ is feasible and a vertex.

- If, in addition, $x$ has $m$ nonzero components, then solve $y = A_B^{-1} c_B$ and compute reduced cost $\pi$ with respect to $y$.

- If $\pi \geq 0$, then $x$ and $y$ are optimal by Proposition 3.

**Failure Case**: If there exists $i$ such that $\pi_j < 0$, then $j \in N$ since $c_B = 0$. Increase $x_j$ and we get an improvement in the objective value of LP3. Then, all entries of $x_N$ are zero except for $x_{N(j)}$ which corresponds to $x_j$. Note that we can increase $x_{N(j)}$ as long as

$$ (A_B^{-1} A_N)_{iN(j)} x_j \leq (A_B^{-1} b)_i $$

for all $i$.

We have assumed that $x = A_B^{-1} b$ is feasible so $A_B^{-1} b \geq 0$.

If $(A_B^{-1} A_N)_{iN(j)} \leq 0$ for all $i$, then the constraint is never violated as $x_j \to \infty$, and $c_j x_j \to -\infty$ as $x_j \to \infty$. Therefore, in this case, LP is unbounded.

If there exists $i$ such that $(A_B^{-1} A_N)_{iN(j)} > 0$, then we cannot improve $x_{N(j)}$ over $(A_B^{-1} b)_i / (A_B^{-1} A_N)_{iN(j)}$.

Define

$$ i^* = \arg\min_{i : (A_B^{-1} A_N)_{iN(j)} > 0} \frac{(A_B^{-1} b)_i}{(A_B^{-1} A_N)_{iN(j)}} $$

and

$$ \epsilon = \frac{(A_B^{-1} b)_{i^*}}{(A_B^{-1} A_N)_{i^*N(i)}}. $$

Hence, we can increase $x_j$ to $\epsilon$ and get a new point $\hat{x}_N$. The new point has a strictly smaller cost $\pi_N \hat{x}_N$ than $\pi_N x_N = 0$.

Now we update $x_B$ through the equation

$$ \hat{x}_B = A_B^{-1} b - A_B^{-1} A_N \hat{x}_N. $$
\( \hat{x}_B \geq 0 \) by construction. Clearly, \( \hat{x} = \begin{bmatrix} \hat{x}_B \\ \hat{x}_N \end{bmatrix} \) satisfies \( A\hat{x} = b \).

Notice that \( \hat{x}_{B(i^*)} = 0 \) by construction. Thus, we can update our basis to
\[
\hat{B} = (B \setminus \{i^*\}) \cup \{j\}.
\]

To summarize, the main update step works as follows:

- If \( \overline{c} \not\geq 0 \), find \( j \), s.t. \( \overline{c}_j < 0 \). (\( j \in N \)).
- Check for unboundedness: If \( (A_B^{-1}A_N)_{iN(j)} \leq 0 \) for all \( i \), then LP is unbounded.
- Ratio test: Compute
\[
i^* = \arg \min_{i : (A_B^{-1}A_N)_{iN(j)} > 0} \frac{(A_B^{-1}b)_i}{(A_B^{-1}A_N)_{iN(j)}}
\]
and
\[
\epsilon = \frac{(A_B^{-1}b)_{i^*}}{(A_B^{-1}A_N)_{i^*N(j)}}.
\]
- Update \( x \): \( x_i \leftarrow \epsilon \) and \( x_B \leftarrow A_B^{-1}b - A_B^{-1}A_Nx_N \).
- Update basis: \( B \leftarrow (B \setminus \{i^*\}) \cup \{j\} \). We say that \( j \) enters the basis and \( i^* \) leaves the basis.